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# Nonlinear eigenvalue problem in elastodynamics.

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SUMMARY Existence of surface waves in a nonhomogeneous elastic half-space was shown in the paper on the bases of the stress elastodynamics formulation (cf. [1]). Kato's analytical perturbation theory [2] was applied to demonstrate that in the velocity and amplitude of a stress surface wave propagating in a nonhomogeneous anisotropic elastic half-space is an analytic function of the wave number.

## 1 Introduction

In 1971 (cf. [1]) J. Ignaczak showed that the problem of surface wave propagation in a nonhomogeneous isotropic elastic half-space can be reduced to the following eigenvalue problem: find a positive number $\lambda$ and a real symmetric tensor field

$$\alpha_{ij} = \alpha_{ij}(x_2), \quad (\alpha_{ij} \in C^2[0,\infty); \; i,j = 1,2)$$

satysfying following equation:

$$A(s)\boldsymbol{\alpha} - \lambda \boldsymbol{B}\,\boldsymbol{\alpha} = 0, \tag{1.1}$$

together with conditions

$$\alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = \alpha_{11}(\infty) = 0, \tag{1.2}$$

where

$$\boldsymbol{\alpha}(x_2) = [\alpha_{11}(x_2) \;\; \alpha_{22}(x_2) \;\; \alpha_{12}(x_2)]^T \tag{1.3}$$

$$\boldsymbol{A} \equiv \begin{bmatrix} \dfrac{s^2}{\rho} & 0 & \dfrac{s}{\rho}D \\ 0 & -D\dfrac{1}{\rho}D & sD\dfrac{1}{\rho} \\ -sD\dfrac{1}{\rho} & -\dfrac{s}{\rho}D & \dfrac{s^2}{\rho} - D\dfrac{1}{\rho}D \end{bmatrix}, \tag{1.4}$$

$$\boldsymbol{B} \equiv \begin{bmatrix} \dfrac{1-\nu}{2\mu} & \dfrac{-\nu}{2\mu} & 0 \\ \dfrac{-\nu}{2\mu} & \dfrac{1-\nu}{2\mu} & 0 \\ 0 & 0 & \dfrac{1}{\mu} \end{bmatrix}. \tag{1.5}$$

The tensor $\boldsymbol{\alpha}$ defines the stress tensor amplitude, and symbol $D$ denotes differentiation with respect to $x_2$ $\left(D = \dfrac{d}{dx_2}\right)$. The number $s$ is the wave number and $\rho = \rho(x_2)$, $\mu = \mu(x_2)$, $\nu = \nu(x_2)$

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are density, shear modulus, and Poisson ratio, respectively ($0 \leqslant x_2 < +\infty$). The equation is based on stress classical elastodynamics[1].

In an earlier paper [4] J. Ignaczak showed, that the problem of surface wave propagation in a nonhomogeneous isotropic elastic half–space with shear moduls $\mu$ and Poisson's ratio $\nu$ depending on $x_2$, and with constant density, can be reduced to the following one: find a pair $(c_R, \beta(x))$ satisfying the ordinary differential equation of the fourth order:

$$(1.6)\qquad \Big(\frac{1}{s^2}D\frac{1}{1-\Omega}D-1\Big)\frac{1}{1-\kappa}\frac{\Omega}{2-\Omega}[D^2-s^2(1-\Omega\kappa)]\beta \; + \\ +4\Big[\frac{1}{2-\Omega}D^2-D\frac{1}{1-\Omega}D\frac{1-\Omega}{2-\Omega}\Big]\beta \;=\; 0 \quad for \quad x_2\in(0,+\infty)$$

and boundary conditions

$$(1.7)\qquad \beta(0)=\beta(\infty)=0 \\ \frac{1}{s^2(2-\Omega)}D\Big\{\frac{\Omega}{2-\Omega}\frac{1}{1-\kappa}[D^2-s^2(1-\Omega\kappa)]\beta-4s^2\frac{1-\Omega}{2-\Omega}\beta\Big\}\Big|_{\substack{x_2=0\\x_2=+\infty}}=0$$

where

$$(1.8)\qquad \kappa(x_2)=\frac{1-2\nu(x_2)}{2-2\nu(x_2)},\quad \Omega(x)=\frac{c_R^2}{\mu(x_2)}$$

The surface wave velocity $c_R$ is the eigenvalue of the problem ((1.6) – (1.8)). Function $\beta(x_2)$ describing the variation of normal stress is the eigenfunction associated with eigenvalue $c_R$, ($\beta(x_2)=\alpha_{22}(x_2)$). In 1967 C.R.A. Rao [5] extended the formulation (1.6) – (1.7) to the case, when density $\rho$, shear modulus $\mu$, and Poisson's ratio $\nu$ are arbitrary function of $x_2$. In particular case, when

$$(1.9)\qquad \rho(x_2)\equiv 1,\quad \mu(x_2)\equiv const,\quad \varepsilon>0$$

$$(1.10)\qquad \nu_0=\nu(0),\quad \nu_\infty=\nu(\infty),\quad \nu(x_2)=1-(1-\nu_\infty)\Big[1+\frac{\nu_0-\nu_\infty}{1-\nu_0}(1+\varepsilon x_2)^{-2}\Big]^{-1}$$

J. Ignaczak (cf. [4]) obtained an analytical closed form solution. C.R.A. Rao (cf. [6], [7]) investigated the problem in case:

$$(1.11)\qquad \rho(x_2)\equiv 1,\quad \nu(x_2)\equiv\nu_0,\quad \mu(x_2)=\mu_\infty+(\mu_0-\mu_\infty)e^{-\varepsilon x_2}$$

using the power series expansion method.

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[1]The problem (1.1) – (1.2) can be discussed in a class of square itegrable function i.e.:

$$\alpha=[\alpha_{11}\;\;\alpha_{22}\;\;\alpha_{12}]^T\in L^2(0,\infty)\times L^2(0,\infty)\times L^2(0,\infty)=[L^2(0,\infty)]^3\quad A,B\in[L^2(0,\infty)]^3.$$

The problem (1.1) – (1.2) is correctly posed when the condition: $R(A)=R(B)$ is satysfied; $R(A)$, $R(B)$ denote ranges of operators $A, B$ (cf. [2] p.16). From equality $R(A)=R(B)$ it follows that:

$$R(A)=R(B)=\Big\{(\alpha_{11},\alpha_{22},\alpha_{12})\in[C^2[0,\infty)]^3:\; -\Big[\frac{\alpha_{11}-\nu\alpha_{ii}}{2\mu}\Big]^{\cdot\cdot}+\frac{s^2(\alpha_{22}-\nu\alpha_{ii})}{2\mu}-s\Big[\frac{\alpha_{12}}{\mu}\Big]^{\cdot}=0;\quad i=1,2\Big\}.$$

The differential equatin ($^{\cdot}=D$) in bracket corespondends to the compatibility condition (cf. [3] p. 345) for the considered problem.



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The problem (1.6) – (1.7) was also investigated by T. Rożnowski, (cf. [8, 9, 10]).

In [8] the solution was found under the assumptions that density and Poisson's ratio are constant and shear modulus $\mu$ is "weakly" variable exponential function such that the term

$$4\Big(\frac{1}{2-\Omega(x_2)}\frac{d^2}{dx_2^2}-\frac{d}{dx_2}\frac{1}{1-\Omega(x_2)}\frac{d}{dx_2}\frac{1-\Omega(x_2)}{2-\Omega(x_2)}\Big)\beta \tag{1.12}$$

can be neglected.

In [9] T. Rożnowski analysed the equations of motion for a transversely isotropic nonhomogeneous elastic semispace, using the stress motion equations, and formulated the problem of surface wave of the Rayleigh type. He showed that the problem can be reduced to ordinary differential equation of the fourth order with variable coeficients. T. Rożnowski in [10] analysed five particular cases of wave phenomena:

a) transversely isotropic body of a "small nonhomogeneity"

b) "weakly anisotropic" nonhomogeneous body

c) "weakly anisotropic" body with a "small nonhomogeneity"

d) transversely isotropic homogeneous body

e) isotropic nonhomogeneous body.

The surface wave problem can be formulated in an alternative way starting from the displacement equations.

A.G. Alenitsyn (cf. [11, 12, 13, 14]) investigated the equations of motion in the displacement formulation for large wave number using asymptoting methods. As a result, he obtained the aproximate dispersion relation (cf. [15] and [16]).

In this paper some new properties of the surface waves will be presented. The stress formulation will be used. This paper consist of four sections. Sec. 2 is devoted to general formulation of the problem. In Sec. 3 qualitative properties of the solution are discussed. It is demonstrated that for density, shear modulus, Poisson's ratio being bounded of class $C^2[0,\infty)$, the wave velocity and amplitude vector are analytical function of the wave number. In Sec. 4 it is shown that at least one solution exist (and at most finite number of solution) under the assumptions, that density and shear modulus are constant, and Poisson's ratio is bounded function from $C^2[0,\infty)$. The obtained results are limited to the surface waves, propagating in a nonhomogeneous half-space under isothermal conditions.

## 2 Stress formulation of surface waves

Let us consider the two dimensional stress equation for linear elastic theory (cf. [1]) for a nonhomogeneous isotropic medium[2]

$$\mu^{-1}(x)\Big[\frac{\partial^2}{\partial t^2}\tau_{\alpha\beta}(x,t)-\nu(x)\delta_{\alpha\beta}\frac{\partial^2}{\partial t^2}\tau_{\gamma\gamma}(x,t)\Big]-[\rho^{-1}(x)\tau_{\alpha\gamma,\gamma}(x,t)]_{,\beta}- \\ -[\rho^{-1}(x)\tau_{\beta\gamma,\gamma}(x,t)]_{,\alpha}=0 \tag{2.1}$$

[2]see Ignaczak [4], Rao [5]



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where $$\tau_{\alpha\beta} = \tau_{\alpha\beta}(x,t); \quad (\alpha,\beta) = (1,2); \quad [x = (x_1, x_2)]$$
denotes nondimensional stress tensor, $\mu(x)$, $\rho(x)$ are nondimensional shear modulus and density, $\nu(x)$ is Poisson's ratio. Nondimensional time is defined by the formula

$$t = \frac{\tau \mu_0^{1/2}}{x_0 \rho_0^{1/2}} \tag{2.2}$$

where $\tau$ is a real time and $\mu_0$, $\rho_0$ and $x_0$ are units of stress, density and lenght, respectively. Moreover
$$\dot{\tau}_{\alpha\beta} = \frac{\partial \tau_{\alpha\beta}}{\partial t}; \quad \tau_{\alpha\beta,\gamma} = \frac{\partial \tau_{\alpha\beta}}{\partial x_\gamma}$$
It is assumed, that the functions $\rho(x)$, $\mu(x)$ and $\nu(x)$ depend on $x_2$ ($x_2 \in [0,\infty)$) and $\rho(x_2)$, $\mu(x_2)$, $\nu(x_2) \in C^2[0,\infty)$ and

$$\begin{aligned} 0 < \rho_0 \leqslant \rho(x_2) \leqslant \rho_1 < +\infty, \\ 0 < \mu_0 \leqslant \mu(x_2) \leqslant \mu_1 < +\infty, \\ -1 < \nu_0 \leqslant \nu(x_2) \leqslant \nu_1 < +1/2 \qquad for \quad x_2 \in [0, +\infty). \end{aligned} \tag{2.3}$$

The numbers $(\rho_0, \mu_0, \nu_0)$ and $(\rho_1, \mu_1, \nu_1)$ are minimal and maximal values of $(\rho, \mu, \nu)$.

The solution $\tau_{\alpha\beta}$ of the equation (2.1) will be considered in the half-space

$$U = \{(x_1, x_2) : x_2 \geqslant 0, -\infty < x_1 < +\infty\} \tag{2.4}$$

for every $t \in [0, +\infty)$. We shall look for the solution in the form:

$$\begin{aligned} \tau_{11}(x,t) &= \mathbf{Re}\,\{\alpha_{11}(x_2)\exp[i(sx_1 - t\sqrt{\lambda})]\}, \\ \tau_{22}(x,t) &= \mathbf{Re}\,\{\alpha_{22}(x_2)\exp[i(sx_1 - t\sqrt{\lambda})]\}, \\ \tau_{12}(x,t) &= \mathbf{Re}\,\{i\alpha_{12}(x_2)\exp[i(sx_1 - t\sqrt{\lambda})]\}, \end{aligned} \tag{2.5}$$

where $i = \sqrt{-1}$, $s > 0$ wave number, $\lambda > 0$ and Re stands for the real part of the complex-valued function. Moreover it is assumed that, the solution satysfies the conditions

$$\tau_{22}(x_1, 0, t) = \tau_{12}(x_1, 0, t) = 0 \qquad for \; x_1 \in (-\infty, +\infty), \; t \geqslant 0 \tag{2.6}$$

$$\tau_{22}(x_1, \infty, t) = \tau_{12}(x_1, \infty, t) = \tau_{11}(x_1, \infty, t) = 0 \quad for \; x_1 \in (-\infty, +\infty), \; t \geqslant 0 \tag{2.7}$$

The wave velocity, wave period and wave length are $c_R = \frac{\sqrt{\lambda}}{s}$, $T = \frac{2\pi}{\sqrt{\lambda}}$, $l = \frac{2\pi}{s}$. The functions $\alpha_{11}(x,t)$, $\alpha_{22}(x,t)$, $\alpha_{12}(x,t)$, and $c_R$ should be choosen in such a way that tensor field $\tau\ (x,t)$ defined by (2.5) satysfy the field equation (2.1) and condition (2.6) – (2.7)

Introducing (2.5) to (2.1), (2.6), (??) we obtain (cf. [1])

$$\begin{aligned} &\rho^{-1}(s\alpha_{11} + s\dot{\alpha}_{12}) - \lambda(2\mu)^{-1}(\alpha_{11} - \nu\alpha_{\gamma\gamma}) = 0, \\ &-[\rho^{-1}(\dot{\alpha}_{22} - s\alpha_{12})]^{\cdot} - \lambda(2\mu)^{-1}(\alpha_{22} - \nu\alpha_{\gamma\gamma}) = 0, \\ &-[\rho^{-1}(s\dot{\alpha}_{12} + s\alpha_{11})]^{\cdot} - s\rho^{-1}(\dot{\alpha}_{22} - s\alpha_{12}) - \lambda(2\mu)^{-1}2\alpha_{12} = 0 \;\; for \;\; x_2 \in (0, +\infty) \end{aligned} \tag{2.8}$$

and the boundary conditions

$$\alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = \alpha_{11}(\infty) = 0, \tag{2.9}$$



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where the vector $\alpha = [\alpha_{11} \quad \alpha_{22} \quad \alpha_{12}]^T \in [C^2[0,\infty)]^3$.

Starting from Eg. (2.8) the dot over a symbol will denote the differentiation with respect to $x_2$. We shall also use the symbol $D$ for the operator $D = \frac{d}{dx_2}$ C.R.A. Rao showed (cf. [5]) that the linear eigenvalue problem (2.8)–(2.9) can be further reduced by elimination of $\alpha_{11}$ and $\alpha_{12}$ to nonlinear eigenvalue problem:

$$(2.10)\qquad \Big[\Big\{\Big[D - (H_1 - \frac{2h}{2-\Omega})\Big]\frac{1}{a^2-e^2}[D-(1-2\kappa)H_1] - 1\Big\}\cdot \cdot\Big\{\frac{\Omega}{2-\Omega}\frac{1}{1-\kappa}(D^2+hD-b^2)\Big\}\Big]\alpha_{22}+$$
$$+4\Big\{\frac{1}{2-\Omega}(D^2+hD) - \Big[D-(H_1-\frac{2h}{2-\Omega})\Big]\frac{1}{a^2-e^2}\Big[D-(1-2\kappa)H_1\Big]\frac{a^2}{2-\Omega}\Big\}\alpha_{22} = 0 \quad for \quad x_2 \in (0,\infty)$$

and

$$(2.11)\qquad \alpha_{22}(0) = \alpha_{22}(\infty) = 0$$

$$(2.12)\qquad \Big\{\frac{1}{a^2-e^2}[D-(1-2\kappa)H_1]\frac{\Omega}{2-\Omega}\frac{1}{1-\kappa}\Big[D^2+hD-b^2-\frac{4a^2(1-\kappa)}{\Omega}\Big]\alpha_{22}\Big\}\Big|_{\substack{x_2=0\\ x_2=\infty}} = 0$$

$$(2.13)\qquad \kappa(x_2) = \frac{1-2\nu(x_2)}{2-2\nu(x_2)}, \qquad \nu(x_2) = \frac{1-2\kappa(x_2)}{2-2\kappa(x_2)},$$
$$h = \rho D(\rho^{-1}), \qquad \Omega(x_2) = \frac{c_R^2\rho(x_2)}{\mu(x_2)},$$
$$a^2 = s^2(1-\Omega), \qquad b^2 = s^2(1-\Omega\kappa),$$
$$H_1 = [\Omega/2-\Omega]\cdot[h/2-2\kappa], \qquad e^2 = DH_1 - (1-2\kappa)H_1^2$$

From the solution $(\lambda, \alpha_{22}(x_2))$ of Eqs. (2.10) – (2.12) one can obtained the functions $\alpha_{11}(x_2)$, and $\alpha_{12}(x_2)$ using the formulae

$$(2.14)\qquad \alpha_{11}(x_2) = -\frac{1}{s^2(2-\Omega)}\Big\{[s^2\Omega + 2(D^2+hD)]\alpha_{22} + h\frac{1}{a^2-e^2}\frac{1}{1-\kappa}[D-(1-2\kappa)H_1]\frac{2}{2-\Omega}\frac{1}{1-\kappa}\Big[D^2+hD-b^2-\frac{4a^2(1-\kappa)}{\Omega}\Big]\alpha_{22}\Big\}$$

$$(2.15)\qquad -2s\alpha_{12}(x_2) = \frac{1}{a^2-e^2}[D-(1-2\kappa)H_1]\frac{\Omega}{2-\Omega}\frac{1}{1-\kappa}\Big[D^2+hD-b^2-\frac{4a^2(1-\kappa)}{\Omega}\Big]\alpha_{22}$$

For constant density $\rho$ Eqs. (2.10) – (2.12) take the form (cf. [4])

$$(2.16)\qquad \Big(\frac{1}{s^2}D\frac{1}{1-\Omega}D - 1\Big)\frac{1}{1-\kappa}\frac{\Omega}{2-\Omega}[D^2 - s^2(1-\Omega\kappa)]\alpha_{22} + +4\Big[\frac{1}{2-\Omega}D^2 - D\frac{1}{1-\Omega}D\frac{1-\Omega}{2-\Omega}\Big]\alpha_{22} = 0 \quad for \quad x_2 \in (0,+\infty)$$



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(2.17)
$$\alpha_{22}(0)=\alpha_{22}(\infty)=0$$
$$\left[D\left\{\frac{\Omega}{2-\Omega}\frac{1}{1-\kappa}[D^2-s^2(1-\Omega\kappa)]\alpha_{22}-4s^2\frac{1-\Omega}{2-\Omega}\alpha_{22}\right\}\right]\Bigg|_{\substack{x_2=0\\x_2=+\infty}}=0$$

(2.18)
$$\alpha_{11}(x_2)=-\frac{1}{s^2(2-\Omega)}(s^2\Omega+2D^2)\alpha_{22}$$

(2.19)
$$\alpha_{12}(x_2)=\frac{1}{s^2(1-\Omega)}D\left\{\frac{\Omega}{2-\Omega}\frac{1}{1-\kappa}[D^2-s^2(1-\Omega\kappa)]\alpha_{22}-4s^2\frac{1-\Omega}{2-\Omega}\alpha_{22}\right\}.$$

Cleary, in the eigenvalue problem (2.10) – (2.12) (or (2.16) – (2.17)) only the eigenvalue $\lambda$ enters in a nonlinear way. Due nonlinearity the standard spectral theory cannot be applied. It should be noted that also the problem (2.1) is nonregular[3]. Indeed, writing (2.1) more explicitly we have:

(2.20)
$$\begin{bmatrix}\frac{1-\nu}{\mu} & \frac{-\nu}{\mu} & 0\\ \frac{-\nu}{\mu} & \frac{1-\nu}{\mu} & 0\\ 0 & 0 & \frac{1}{\mu}\end{bmatrix}\begin{bmatrix}\frac{\partial^2}{\partial t^2}\tau_{11}\\ \frac{\partial^2}{\partial t^2}\tau_{22}\\ \frac{\partial^2}{\partial t^2}\tau_{12}\end{bmatrix}=$$
$$=\begin{bmatrix}2\frac{\partial}{\partial x_1}\rho^{-1}\frac{\partial}{\partial x_1} & 0 & 2\frac{\partial}{\partial x_1}\rho^{-1}\frac{\partial}{\partial x_2}\\ 0 & 2\frac{\partial}{\partial x_2}\rho^{-1}\frac{\partial}{\partial x_2} & 2\frac{\partial}{\partial x_2}\rho^{-1}\frac{\partial}{\partial x_1}\\ \frac{\partial}{\partial x_2}\rho^{-1}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_1}\rho^{-1}\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_2}\rho^{-1}\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_1}\rho^{-1}\frac{\partial}{\partial x_1}\end{bmatrix}\begin{bmatrix}\tau_{11}\\ \tau_{22}\\ \tau_{12}\end{bmatrix}$$

The characteristic determinat

(2.21)
$$\begin{bmatrix}-2\rho^{-1}\xi_1^2 & 0 & -2\rho^{-1}\xi_1\xi_2\\ 0 & -2\rho^{-1}\xi_1^2 & -2\rho^{-1}\xi_2\xi_1\\ -\rho^{-1}\xi_2\xi_1 & -\rho^{-1}\xi_1\xi_2 & -\rho^{-1}(\xi_1^2+\xi_2^2)\end{bmatrix}$$

of the operator on the R.H.S of Eq. (2.20) is equal to zero for any point $(\xi_1,\xi_2)$. It can be shown that the solution of Eq. (2.1) satisfies the compatibility conditions of the two–dimensional elasticity theory[4]

(2.22)
$$\{\mu^{-1}[(1-\nu)\tau_{11}-\nu\tau_{22}]\}_{22}+\{\mu^{-1}[(1-\nu)\tau_{22}-\nu\tau_{11}]\}_{11}-2\{\mu^{-1}\tau_{12}\}_{12}=0$$
$$for\quad (x,t)\in U\times[0,\infty).$$

So, the system (2.20) with conditions (2.22) can be considered as a regular one.

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[3]see: [17][18][19][20]

[4]The compatibility conditions restricted to the field $\tau$ have the form:

$$\{\mu^{-1}[(1-\nu)\alpha_{11}-\nu\alpha_{22}]\}^{\cdot\cdot}-s\{\mu^{-1}[(1-\nu)\alpha_{22}-\nu\alpha_{11}]\}+2s\{\mu^{-1}\alpha_{12}\}^{\cdot}=0;\quad (\cdot=\frac{d}{dx_2})$$



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The condition (2.22) follows from (2.20), if stress fied $\tau_{\alpha\beta}$ is sufficiently smooth on $U\times[0,\infty)$ and L.H.S. of equation (2.22), together with the first time derivative vanishes for $t=0$. The last conditions are equivalent to the assumption, that deformation and its velocity fullfil the compatibility conditions. Vanishing of the determinant (2.21) implies that the operator

$$\sum\tau_{\alpha\beta}(x,t)\equiv\begin{bmatrix}2\frac{\partial}{\partial x_1}\rho^{-1}\frac{\partial}{\partial x_1} & 0 & 2\frac{\partial}{\partial x_1}\rho^{-1}\frac{\partial}{\partial x_2}\\ 0 & 2\frac{\partial}{\partial x_2}\rho^{-1}\frac{\partial}{\partial x_2} & 2\frac{\partial}{\partial x_2}\rho^{-1}\frac{\partial}{\partial x_1}\\ \frac{\partial}{\partial x_2}\rho^{-1}\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1}\rho^{-1}\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_2}\rho^{-1}\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_1}\rho^{-1}\frac{\partial}{\partial x_1}\end{bmatrix}\begin{bmatrix}\tau_{11}\\ \tau_{22}\\ \tau_{12}\end{bmatrix} \tag{2.23}$$

defined on the domain:

$$D_1(\Sigma)=\{(\tau_{11},\tau_{22},\tau_{12})\in[C^2(U\times[0,\infty))]^3:$$
$$\tau_{22}(x_1;0,t)=\tau_{12}(x_1;0,t)=\tau_{22}(x_1;\infty,t)=\tau_{12}(x_1;\infty,t)=\tau_{11}(x_1;\infty,t)=0\}.$$

or

$$D_2(\Sigma)=\{(\tau_{11},\tau_{22},\tau_{12})\in[L^2(U\times[0,\infty))]^3:$$
$$\tau_{22}(x_1;0,t)=\tau_{12}(x_1;0,t)=\tau_{22}(x_1;\infty,t)=\tau_{12}(x_1;\infty,t)=\tau_{11}(x_1;\infty,t)=0\}.$$

is not invertible, unless the condition (2.22) is fullfiled.

## 3 On the analytical dependence of velocity and amplitude of the surface wave on the wave number

In the complex Hilbert space $H$ generated by the scalar product[5]

$$(\alpha,\beta)=\int_0^\infty(\bar\alpha_{11}\beta_{11}+\bar\alpha_{22}\beta_{22}+\bar\alpha_{12}\beta_{12})dx_2 \tag{3.1}$$

with norm

$$\|\alpha\|^2=\int_0^\infty(|\alpha_{11}|^2+|\alpha_{22}|^2+|\alpha_{12}|^2)dx_2<+\infty \tag{3.2}$$

Eq. (2.8) can be written in operator form

$$A(s)\alpha-\lambda B\,\alpha=0, \tag{3.3}$$

where

$$\alpha=[\alpha_{11}\quad\alpha_{22}\quad\alpha_{12}]^T\quad A(s)\equiv\begin{bmatrix}\frac{s^2}{\rho} & 0 & \frac{s}{\rho}D\\ 0 & -D\frac{1}{\rho}D & sD\frac{1}{\rho}\\ -sD\frac{1}{\rho} & -\frac{s}{\rho}D & \frac{s^2}{\rho}-D\frac{1}{\rho}D\end{bmatrix}\quad B\equiv\begin{bmatrix}\frac{1-\nu}{2\mu} & \frac{-\nu}{2\mu} & 0\\ \frac{-\nu}{2\mu} & \frac{1-\nu}{2\mu} & 0\\ 0 & 0 & \frac{1}{\mu}\end{bmatrix}$$

[5]In order to be able to apply Kato's perturbation theory we have to extend the problem to the complex plane



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The domain of operators $A$ and $B$ may be defined as follows

(3.4) $$\mathcal{D}(A) = \{\alpha:\ \alpha_{ij} \in C^2[0,\infty);\ \alpha_{12}(0) = \alpha_{22}(0) = \alpha_{12}(\infty) = \alpha_{22}(\infty) = \alpha_{11}(\infty) = 0\}$$

(3.5) $$\mathcal{D}(B) = \{\alpha:\ \alpha_{ij} \in C^2[0,\infty)\};\ \ i,j = 1,2$$

The sets $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are dense in $H$ since the set $C_0^\infty[0,\infty) \times C_0^\infty[0,\infty) \times C_0^\infty[0,\infty)$ is dense in $H$ and is contained in $\mathcal{D}(A)$ and $\mathcal{D}(B)$. We have

THEOREM 3.1 *The operator $A$ is symmetric in the $H$.*

THEOREM 3.2 *The matrix $B$ is positive definite operator.* [6]

Let us consider the forms $\mathcal{U}[\alpha] = (A\,\alpha,\alpha)$, $\mathcal{B}[\alpha] = (B\,\alpha,\alpha)$ described by the formulae

$$(A\,\alpha,\alpha) = \int_0^\infty \frac{1}{\rho}[|\dot\alpha_{22} - s\alpha_{12}|^2 + |\dot\alpha_{12} + s\alpha_{11}|^2]dx_2$$

$$(B\,\alpha,\alpha) = \int_0^\infty (2\mu)^{-1}[(1-\nu)|\alpha_{22}|^2 + (1-\nu)|\alpha_{22}|^2 + 2|\alpha_{12}|^2 - 2\nu \mathrm{Re}\,(\alpha_{11}\bar\alpha_{22})]dx_2$$

In view of (2.3) we have $(A\,\alpha,\alpha) \geqslant 0$. Operators $A$ and $B$ being symmetric, are closable in the space $H$. Let $\bar A(s)$, $\bar B$ denote the closure of operators $A$ and $B$. Let us set in $H$ the form:

(3.6) $$\mathcal{U}[\alpha] = \sum_{i=0}^\infty \mathcal{U}^{(i)}(s_0)[\alpha](z-s_0)^i$$

for $z$ belonging to a certain neighbourhood of the real semi-axis $s$, $s_0 \in (0,\infty)$[7], where

(3.7) $$\mathcal{U}^{(0)}[\alpha] = (A(s_0)\alpha,\alpha) = \int_0^\infty \rho^{-1}(|\dot\alpha_{22} - s_0\alpha_{12}|^2 + |\dot\alpha_{12} + s_0\alpha_{11}|^2)dx_2$$

(3.8) $$\mathcal{U}^{(1)}(s_0)[\alpha] = \int_0^\infty \frac{1}{\rho}\{2s_0|\alpha_{12}|^2 + 2s_0|\alpha_{11}|^2 - 2\mathrm{Re}\,(\alpha_{12}\dot\alpha_{22}) + 2\mathrm{Re}\,(\alpha_{11}\dot\alpha_{12})\}dx_2$$

(3.9) $$\mathcal{U}^{(2)}(s_0)[\alpha] = \int_0^\infty \frac{2}{\rho}(|\alpha_{12}|^2 + |\alpha_{11}|^2)dx_2,$$

(3.10) $$\mathcal{U}^{(n)}(s_0)[\alpha] = 0, \qquad n = 3,4,\ldots.$$

---

[6]The eigenvalues of matrix $B$ are $\frac{1-2\nu}{2\mu}, \frac{1}{2\mu}, \frac{1}{\mu}$. The symmetric matrix $B$ is positive definite $\Longleftrightarrow$ when all its eigenvalues $\lambda_i$ are positive and $(B\,\alpha,\alpha) \geqslant \min_i \lambda_i(\alpha,\alpha)$

[7]The neighbourhood is a set: $V = \{z:\ |z-s_0| < \frac{1}{b+c}$ *and* $z \notin (-\infty,0]\}$ where $b = \frac{1}{\varepsilon}$, $c = \frac{2}{\varepsilon}$, $\varepsilon > 0$. We can expand the region of holomorphism by choosing a suitable $\varepsilon$. The meaning of $b$, $c$, $\varepsilon$ will be clear in the further reasoning.



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The form $\mathcal{U}^{(1)}(s_0)[\alpha]$ is a derivative of $(A(s)\alpha,\alpha)$ with respect the real parameter $s$ at $s = s_0$,

$$\mathcal{U}^{(1)}(s_0)[\alpha] = \lim_{s\to s_0} \frac{(A(s)\alpha,\alpha) - (A(s_0)\alpha,\alpha)}{s - s_0}$$

*Similarly*

$$\mathcal{U}^{(2)}(s_0)[\alpha] = \lim_{s\to s_0} \frac{\mathcal{U}^{(1)}(s)[\alpha] - \mathcal{U}^{(1)}(s_0)[\alpha]}{s - s_0}$$

..............................................................................

$$\mathcal{U}^{(n)}(s_0)[\alpha] = \lim_{s\to s_0} \frac{\mathcal{U}^{(n-1)}(s)[\alpha] - \mathcal{U}^{(n-1)}(s_0)[\alpha]}{s - s_0}$$

We have the following theorem:

THEOREM 3.3 *The closure $\bar{\mathcal{U}}(z)$ of the form $\mathcal{U}(z)$ generates a family operators $\bar{A}(z)$ which is $B$ - holomorphic*[8]

In order to demonstrate that $\bar{A}(z)$ is a $B$ - holomorphic family of operators, we shall use Kato's $B$ - holomorphism criterion[9] ([2], p. 398).

Let $\mathcal{U}^{(n)}(s_0)[\alpha]$ be a sequence of sesquilinear form in $H$ $(n = 0, 1, 2...)$ and let the form $\mathcal{U}^{(0)}(s_0)[\alpha]$ be sectorial[10] and closable, and with the domain $D(\mathcal{U}^{(0)}) = D$. Assume that the forms $\mathcal{U}^{(n)}(s_0)[\alpha]$ for $n \geqslant 1$ are bounded with respect to $\mathcal{U}^{(0)}[\alpha]$ i.e. $D \subset D(\mathcal{U}^{(n)})$, and

$$(*) \qquad \left|\mathcal{U}^{(n)}(s_0)[\alpha]\right| \leqslant c^{n-1}(a\|\alpha\|^2 + b\,\mathrm{Re}\,\mathcal{U}^{(0)}(s_0)[\alpha]), \quad \alpha \in D, \quad n > 1, \quad a, b \geqslant 0, \quad c > 0$$

Then operators $\bar{A}(z)$ corresponding to the forms $\bar{\mathcal{U}}(z)[\alpha]$ are a $B$ - holomorphic family of operators for $|z - s_0| < \dfrac{1}{b+c}$.

---

[8]The family $\mathcal{U}(z)$ of sesquilinear (unbounded) forms defined in the region $z \in D_0$ is called a holomorphic family of type $(a)$ if: *i)* for each $z \in D_0$ the form $\mathcal{U}[\alpha]$ is sectorial and closed; *ii)* the domain of the form does not depend on $z$ and is dense in $H$; *iii)* for each $\alpha \in D$ the function $\mathcal{U}(z)$ is a holomorphic function on $D_0$ ($D$ denotes the domain of the form $\mathcal{U}(z)[\alpha]$). Let operator $A(z)$ be the maximal sectorial operator related to the form $\mathcal{U}(z)$ for every $z \in D_0$. Such a family of operators associated with the holomorphic family of type $(a)$ of the forms is termed $B$ - holomorphic.

[9]Let $\mathcal{U}^{(n)}$, $n = 0, 1, 2, ...$, be a sequence of sesquilinear forms in $H$. Let $\mathcal{U} = \mathcal{U}^{(0)}$ be densely defined, with $D(\mathcal{U}) = D$, sectorial and closable. Let $\mathcal{U}^{(n)}$, $n \geqslant 1$, be relatively bounded with respect to $\mathcal{U}$ so that $D(\mathcal{U}^{(n)}) \supset D$ and

$$|\mathcal{U}^{(n)}[u]| \leqslant c^{n-1}(a\|u\|^2 + b\mathrm{Re}[u]), \quad u \in D, \quad n \geqslant 1.$$

where $\mathrm{Re} = Re\,\mathcal{U}$ and $a, b \geqslant 0$. Then the form

$$\mathcal{U}(\chi)[u] = \sum_{n=0}^{\infty} \chi^n \mathcal{U}^{(n)}[u], \quad D(\mathcal{U}^n(\chi)) = D,$$

and the associated polar form $\mathcal{U}(\chi)[u, v]$ are defined for $|\chi| < 1/c$. $\mathcal{U}(\chi)$ is sectorial and closable for $|\chi| < 1/(b + c)$. Let $\bar{\mathcal{U}}(\chi)$ be its closure. $\{\bar{\mathcal{U}}(\chi)\}$ is a holomorphic family of forms of type (a) for $|\chi| < 1/(b + c)$. The operator $T(\chi)$ associated with $\bar{\mathcal{U}}(\chi)$ forms a holomorphic family of type $(B)$

[10]Let $\Theta(\mathcal{U})$ denote the set of values of the form $\mathcal{U}$. We call the form $\mathcal{U}$ sectorial when $\Theta(\mathcal{U})$ is a subset of the sector $|arg(\xi - \gamma)| \leqslant \theta$, $0 \leqslant \theta < \frac{\pi}{2}$ ($\gamma$ being a real number). From the definition of a sectorial form it results that every nonnegative form is sectorial (cf. [2], p. 310).



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To show that the assumptions of this criterion are satisfied let us observe that $\mathcal{U}^{(0)} = \mathcal{U}^{(0)}(s_0)[\alpha] = (A(s_0)\alpha, \alpha)$ is a nonnegative, symmetric and hence sectorial form fixed in the dense set $D$. The density of $D$ results from the fact that the set $\mathcal{D}(\boldsymbol{A}) \subset D \subset H$ and $\mathcal{D}(\boldsymbol{A})$ is dense. Thus the form $\mathcal{U}^{(0)}$ is closable.

From the inequalities[11]

$$(3.11)\qquad \left|\mathcal{U}^{(1)}(s_0)[\alpha]\right| \leqslant \varepsilon \max_{x_2\in[0,\infty)} \frac{1}{\rho}\Big(\int_0^\infty (|\alpha_{11}|^2 + |\alpha_{22}|^2 + |\alpha_{12}|^2)dx_2\Big) + \\ +\frac{1}{\varepsilon}\int_0^\infty \frac{1}{\rho}(|\dot\alpha_{22} - s_0\alpha_{12}|^2 + |\dot\alpha_{12} + s_0\alpha_{11}|^2)dx_2$$

and

$$(3.12)\qquad \left|\mathcal{U}^{(2)}(s_0)[\alpha]\right| = \int_0^\infty \frac{2}{\rho}(|\alpha_{11}|^2 + |\alpha_{22}|^2)dx_2 \leqslant \max_{x_2\in[0,\infty)} \frac{2}{\rho}\int_0^\infty (|\alpha_{11}|^2 + |\alpha_{22}|^2 + |\alpha_{12}|^2)dx_2 + \\ +\frac{2}{\varepsilon^2}\int_0^\infty \frac{1}{\rho}(|\dot\alpha_{22} - s_0\alpha_{12}|^2 + |\dot\alpha_{12} + s_0\alpha_{11}|^2)dx_2$$

it follows that $D(\mathcal{U}^{(n)}) \supset D(\mathcal{U}^{(0)})$, $n = 1, 2, 3, \dots$, and that there exist $a = \dfrac{\varepsilon}{\rho_0}, b = \dfrac{1}{\varepsilon}, c = \dfrac{2}{\varepsilon}$. Thus the operator $\bar{A}(z)$ forms a holomorphic family of type $(B)$. From Th. 3.3 it follows that the following theorem is valid.

THEOREM 3.4 *The form $\mathcal{U}(z)$ given by (3.6) is defined for $|z - s_0| < \varepsilon/2$ and for $|z - s_0| < \varepsilon/3$ is sectorial and closable. The closure $\tilde{\mathcal{U}}(z)$ of the form $\mathcal{U}(z)$ generetes a B-holomorphic family of operator $\bar{\boldsymbol{A}}(z)$ where $\bar{\boldsymbol{A}}(z)$ is maximal and closed operator.*

Now we shall consider eigenvalue problem qiven by

$$(3.13)\qquad \bar{\boldsymbol{A}}(z)\boldsymbol{\alpha} - \lambda\bar{\boldsymbol{B}}\boldsymbol{\alpha} = 0$$

where $\bar{\boldsymbol{A}}(z)$ is the operator defined in Th. 3.4 and $\bar{\boldsymbol{B}}$ is the closure of $B$. From Kato theorems (cf. [2] p. 416–423) it follows:

THEOREM 3.5 *If the pair $(\lambda(z), \boldsymbol{\alpha}(z))$ is a solution of the eigenvalue problem (3.13), then it is an analytical function with respect to $z$ for $z \in V = \{z : |z - s_0| < \varepsilon/3$ and $z \notin (-\infty, 0]\}$.*

---

[11] To prove inequalities (3.11), (3.12) we use the inequalities

$$\int \sum u_i v_i dx \leqslant \Big(\int \sum |u_i|^2 dx\Big)^{1/2} \Big(\int \sum |v_i|^2 dx\Big)^{1/2}$$

$$2ab \leqslant \varepsilon a^2 + \frac{1}{\varepsilon}b^2$$

where $v_i$ and $u_i$ are complex function, $a$ and $b$ are real function, and $\varepsilon > 0$.



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THEOREM 3.6 *If the pair* $(\lambda(s), \alpha(x_2, s))$ *is a solution of the eigenvalue problem* (3.3), *then it is an analytical function of the wave-number* $s$.

It means that

$$(\lambda(s), \alpha(x_2, s)) \equiv \Big(\sum_{n=0}^{\infty} \lambda_n (s - s_0)^n, \quad \alpha = \sum_{n=0}^{\infty} \alpha_n(x_2)(s - s_0)^n\Big)$$

where $\lambda_n = \frac{1}{n!}(\frac{d^n \lambda}{ds^n})_{s=s_0}, \quad \alpha_n(x_2) = \frac{1}{n!}(\frac{\partial^n \alpha}{\partial s^n})_{s=s_0}, \quad s_0 \in (0, \infty) \quad x_2 \geqslant 0.$

The proof of the theorem 4.1 follows directly from Theorem 3.5 and from the fact, that each solution of (3.3) is also a solution of (3.13).

Natural approach to the considered eigenvalue problem

$$A\alpha - \lambda B\alpha = 0$$

is investigation of the generalized resolvent

$$(A - \xi B)^{-1}.$$

Let us introduce the spaces $X$ and $Y$ defined by

$$X = \Big\{(\alpha_{11}, \alpha_{22}, \alpha_{12}) \in [L^2(0,\infty)]^3, \ [C^2[0,\infty)]^3 : \ -\Big[\frac{\alpha_{11} - \nu\alpha_{ii}}{2\mu}\Big]^{\cdot\cdot} + $$
$$+ s^2 \frac{\alpha_{22} - \nu\alpha_{ii}}{2\mu} - s\Big[\frac{\alpha_{12}}{\mu}\Big]^{\cdot} = 0, \quad i = 1,2 \quad for\ every\ x_2 \geqslant 0\Big\}$$

$$Y = \Big\{(g_{11}, g_{22}, g_{12}) \in [L^2(0,\infty]]^3, \ [C^2(0,\infty]]^3 : -\ddot{g}_{11}(x_2) + s^2 g_{22}(x_2) - s\dot{g}_{12}(x_2) = 0,$$
$$for\ every\ x_2 \geqslant 0\Big\}$$

It is easy to check that the spaces $X$, $Y$ are linear subspaces of $[L^2(0,\infty]]^3$ and $[C^2(0,\infty]]^3$.
Let $\mathcal{C}(X,Y)$ be a space of closed operators from $X$ to $Y$ and $\mathcal{B}(X,Y)$ be space of bounded operators from $X$ to $Y$. Since $\bar{A} \in \mathcal{C}(X,Y)$, $\bar{B} \in \mathcal{B}(X,Y)$ and $\widetilde{B^{-1}} \in \mathcal{B}(X,Y)$, thus $\widetilde{B^{-1}A} \in \mathcal{C}(X,X) = \mathcal{C}(X)$, $\widetilde{AB^{-1}} \in \mathcal{C}(Y,Y) = \mathcal{C}(Y)$ and the eigenvalue problems

$$\bar{A}\,\alpha - \lambda\bar{B}\,\alpha = 0, \qquad \widetilde{B^{-1}A}\,\alpha - \lambda\alpha = 0, \qquad \widetilde{AB^{-1}}\,\alpha - \lambda\alpha = 0$$

are equivalent [cf. [2] p. 417, 418].

For simplicity let us take homogeneous case $\rho = const$, $\mu = const$, $\nu = const$.

The solution of equation

$$A\alpha - \xi B\alpha = 0, \ \ \alpha \in D(A) \cap D(B) \subset X \ \ is \ \ \alpha = [0,0,0]^T \ \ if \ \ \xi \notin \{\omega_1, \omega_2, \omega_3\}$$

where $\omega_1$, $\omega_2$, $\omega_3$ are the roots of equation

$$(2-\omega)^2 - 4\sqrt{(1-\omega)(1-\omega\kappa)} = 0, \quad \kappa = (1-2\nu)(2-2\nu)^{-1}.$$



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A solutions of the equation $A\alpha - \xi B\alpha = 0$
take the from

$$\begin{aligned}
\alpha_{11} &= -\beta_0\left[e^{-x_2h_2} - \frac{2+\xi(1-2\kappa)}{2-\xi}e^{-x_2h_1}\right] \\
\alpha_{22} &= \beta_0\left[e^{-x_2h_2} - e^{-x_2h_1}\right] \\
\alpha_{12} &= -\frac{2}{s}\frac{\beta_0}{2-\xi}h_1\left[e^{-x_2h_2} - e^{-x_2h_1}\right] \\
h_1 &= s\sqrt{1-\xi\kappa}, \qquad h_2 = s\sqrt{1-\xi}
\end{aligned}$$

Introducing such $\boldsymbol{\alpha}$ to the compability conditions ([6][p. 7]) we get

$$\frac{\beta_0 s^2}{2\mu(2-\xi)}e^{-x_2s\sqrt{1-\xi}}[(2-\xi)^2 - 4\sqrt{(1-\xi)(1-\xi\kappa)}] + \frac{\beta_0 s^2}{2\mu(2-\xi)(1-\nu)}e^{-x_2s\sqrt{1-\xi\kappa}}[\,0\,] = 0$$

Therefore if $\xi \notin \{\omega_1, \omega_2, \omega_3\}$ then $(2-\xi)^2 - 4\sqrt{(1-\xi)(1-\xi\kappa)} \neq 0$ and $\beta_0 = 0$. In this case $(\boldsymbol{A} - \xi\boldsymbol{B})^{-1}$ exists.

Let us now consider multiplicity of eigeuvalue $\lambda = 0$. This problem can be written in the form:

$$\boldsymbol{A}(s)\alpha = 0.$$

As the domain of the operator $A$ we take the set:

$$D(A) = \Big\{\alpha = [\alpha_{11}, \alpha_{22}, \alpha_{12}]^T \in [L^2(0,\infty)]^3,\ \ [C^2[0,\infty)]^3 : \alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = \alpha_{11}(\infty) = 0\Big\}$$

We have:

$$\boldsymbol{A}(s)\alpha = \boldsymbol{0} \Leftrightarrow \left\{\begin{array}{c} s\alpha_{11} + \dot{\alpha}_{12} = 0 \\ -s\alpha_{12} + \dot{\alpha}_{22} = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \alpha_{11}(x_2) = C_1\varphi''(x_2) \\ \alpha_{22}(x_2) = -s^2C_1\varphi(x_2) \\ \alpha_{12}(x_2) = -sC_1\varphi'(x_2) \end{array}\right.$$

We obtain the solution in the form

$$\begin{aligned}
ker\ \boldsymbol{A}: \quad \alpha_{11}(x_2) &= C_1(2 - 4\alpha_k x_2 + \alpha_k^2 x_2^2)e^{-\alpha_k x_2} \\
\alpha_{22}(x_2) &= -s^2x_2^2C_1e^{-\alpha_k x_2} \\
\alpha_{12}(x_2) &= -sC_1(2x_2 - \alpha_k x_2^2)e^{-\alpha_k x_2}
\end{aligned}$$

where $C_1 \in R$, $x_2 \in [0, +\infty)$, $\alpha_k > 0$.

It is clear that in this case

$$dim\ ker\boldsymbol{A} = \infty$$

Note then in the case, when the domain of the operator is a subspace of the function satysfying the compability conditions;

$$dim\ ker\boldsymbol{A} = 0.$$



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## 4 On finite number of the solutions in case of constant density and shear modulus, and with Poisson ratio being bouded function of class $C^2[0,\infty)$ (cf. [27])

In the paper [27] it was demonstrated that in the case when $\varrho(x_2) = const$, $\mu(x_2) = const$ and Poisson ratio $\nu(x_2)$ is bounded function of class $C^2[0,\infty)$ the equation (2.16) – (2.17) reduces in the class

$$K := \{\alpha_{22} = \alpha_{22}(x_2) \in C^4[0,\infty), \quad \alpha_{22}(\infty) = 0\}$$

to

$$\frac{1}{1-\kappa(x_2)}\Big[D^2 - s^2(1-\Omega_0\kappa(x_2))\Big]\alpha_{22} = C_1 \exp\Big(-s\sqrt{1-\Omega_0}x_2\Big) \text{ for } x_2 \in (0,\infty) \tag{4.1}$$

$$\alpha_{22}(0) = 0, \tag{4.2}$$

$$D\Big\{\frac{\Omega_0}{1-\kappa(x_2)}\Big[D^2 - s^2(1-\Omega_0\kappa(x_2))\Big]\alpha_{22} - 4s^2(1-\Omega_0)\alpha_{22}\Big\}_{|x_2=0} = 0, \tag{4.3}$$

where $C_1$ is arbitrary constant.

The equation (4.1) with the conditions (4.2) i (4.3) has a solution in the form

$$\alpha_{22}(x_2,\Omega_0,s) = A\exp\Big(\int_0^{x_2}\xi_1(\tau,s,\Omega_0)\Big) - \frac{1}{\Omega_0 s^2}C_1\exp(-sx_2\sqrt{1-\Omega_0}) \tag{4.4}$$

where $(\Omega_0, s) \in (0,1)\times(0,\infty)$,

$$\kappa(x_2) = \frac{1-2\kappa(x_2)}{2-2\kappa(x_2)} \qquad \Omega_0 = \frac{c_R^2}{\mu_0)} \tag{4.5}$$

$$-s\sqrt{1-\Omega_0\kappa_0} \leqslant \xi_1(\tau,\Omega_0,s) \leqslant s\sqrt{1-\Omega_0\kappa_1}$$

for every $0 \leqslant \tau \leqslant x_2$, and $A$ arbitrary constant.

In the paper [27] that dispersion equation takes form

$$f(\Omega_0,s) := (2-\Omega_0)^2 + \frac{4\sqrt{1-\Omega_0}\xi_1(0,\Omega_0,s)}{s} = 0, \tag{4.6}$$

where $\xi_1(0,\Omega_0,s)$ is analytical function of variables $(\Omega_0,s) \in (0,1)\times(0,\infty)$. Hence the function $f(\Omega_0,s)$ is analytical for $(\Omega_0,s) \in (0,1)\times(0,\infty)$ and the following inequality

$$\begin{aligned} -4\sqrt{(1-\Omega_0)(1-\Omega_0\kappa_0)} + (2-\Omega_0)^2 &\leqslant \frac{4\sqrt{1-\Omega_0}\xi_1(0,\Omega_0,s)}{s} + (2-\Omega_0)^2 \\ &\leqslant -4\sqrt{(1-\Omega_0)(1-\Omega_0\kappa_1)} + (2-\Omega_0)^2 \end{aligned} \tag{4.7}$$

is valid for every $s \in (0,\infty)$.

The functions

$$\begin{aligned} f_0(\Omega_0) &:= -4\sqrt{(1-\Omega_0)(1-\Omega_0\kappa_0)} + (2-\Omega_0)^2 \\ f_1(\Omega_0) &:= -4\sqrt{(1-\Omega_0)(1-\Omega_0\kappa_1)} + (2-\Omega_0)^2 \end{aligned}$$



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vanish for $\Omega_0 = 0$ and $\Omega_0 = C_0^2$ ($\Omega_0 = C_1^2$) respectively, where $C_0^2$, $C_1^2$ are the squares of surface wave velocities with $\mu \equiv 1$, $\varrho \equiv 1$, $\kappa(x_2) \equiv \kappa_0$ ($\mu \equiv 1$, $\varrho \equiv 1$, $\kappa(x_2) \equiv \kappa_1$). Since $\kappa_0 < \kappa_1$, then $C_1^2 < C_0^2$.

The analyticity of $f(\Omega_0, s)$ for every $(\Omega_0, s) \in (0,1) \times (0, \infty)$ together with the inequalities (4.7) imply that there exists at least one root (or at most, a countable number of roots) of the equation $f(\Omega_0, s) = 0$ for every $(\Omega_0, s) \in [C_1^2, C_0^2] \times (0, \infty)$. This completes the proof of existence of at least one solution to the eigenproblem discussed in the present section.

We have following theorem:

THEOREM 4.1 *For every $s > 0$, the equation $f(\Omega_0, s) = 0$ has at most a finite numer of solutions.*

P r o o f. If the number of the solutions of the equation $f(\Omega_0, s) = 0$ for a given $s > 0$ is infinite, then the set $S = \{f(\Omega_0, s) = 0\}$ has an accumulation point in $[C_1^2, C_0^2]$. Since the function $f(\Omega_0, s)$ is analytical in the domain $(\Omega_0, s) \in (0,1) \times (0, \infty)$, $f$ vanishes in the interval $[C_1^2, C_0^2]$ which contradicts the inequality (4.5).

REMARK. If the branches of the dispersion relation (4.6) intersect, then the intersection points are algebraic branch-points (cf. [24] p. 119 part II), (cf. [25] p. 174 - 181).

## 5 Surface stress waves in anisotropic nonhomogeneous elastic semi-space

Let us consider two dimensional stress equation for nonhomogeneous anisotropic semi-space (cf. J. IGNACZAK [26]):

$$2\chi_{ijkl}(x)\frac{\partial^2}{\partial t^2}\tau_{kl}(x,t) = \left[\varrho^{-1}(x)\tau_{ik,k}(x,t)\right]_{,j} + \left[\varrho^{-1}(x)\tau_{jk,k}(x,t)\right]_{,i}{}^{1} \tag{5.1}$$

where $\tau_{kl} = \tau_{lk}(x,t)$, $(i,j,k,l = 1,2)$, $x = (x_1, x_2)$.

The tensor $\chi_{ijkl}(x)$ is a tensor of compliance satysfying the following conditions:

(5.2) symmetry : $\chi_{ijkl} = \chi_{jikl} = \chi_{ijlk} = \chi_{lkij}$

(5.3) strong elipticity : $\chi_{ijkl} \cdot a_i\, b_k\, a_j\, b_l > 0$ for each pair of vector a, b

The function $\varrho(x)$ denotes density. We shall use the similar approach as for isotropic case. Let us assume that the functions $\chi_{ijkl}(x)$, $\varrho(x)$, $(i,j,k,l = 1,2)$ depend on $x_2$ ($x_2 \in [0, \infty)$) and $\chi_{ijkl}(x_2)$, $\varrho(x_2)$ are bounded functions of class $C^2[0, \infty)$. We seek a solution $\tau_{kl}$ of the equation (5.1) in the half-space

$$U = \{(x_1, x_2): \quad x_2 \geqslant 0, \quad -\infty < x_1 < +\infty\}$$

[1] If $\tau_{ij}(x,t)$ is a solution to Eq. (5.1), then the displacement vector $u_i$ will be expressed by equation:

$$u_i(x,t) = \varrho^{-1}\int_0^t (t-\tau)\tau_{ij,j}(x,t)d\tau + u_i\Big|_{t=0} + t\frac{\partial u_i}{\partial t}\Big|_{t=0}$$

(cf. [8] p. 385 - 359).



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for each $t \in [0, +\infty)$ in the form

$$
(5.4) \qquad \begin{aligned}
\tau_{11}(x,t) &= \alpha_{11}(x_2)\exp[i(sx_1 - t\sqrt{\lambda})],\\
\tau_{22}(x,t) &= \alpha_{22}(x_2)\exp[i(sx_1 - t\sqrt{\lambda})],\\
\tau_{12}(x,t) &= i\alpha_{12}(x_2)\exp[i(sx_1 - t\sqrt{\lambda})],
\end{aligned}
$$

where $i = \sqrt{-1}$, $s > 0$, $\lambda > 0$ and

$$
(5.5) \qquad \tau_{22}(x_1, 0, t) = \tau_{12}(x_1, 0, t) = 0 \quad \text{for } x_1 \in (-\infty, +\infty),\ t \geqslant 0
$$

$$
(5.6) \qquad ,\tau_{22}(x_1, \infty, t) = \tau_{12}(x_1, \infty, t) = \tau_{11}(x_1, \infty, t) = 0 \quad \text{for } x_1 \in (-\infty, +\infty),\ t \geqslant 0.
$$

Introducing the relation (5.4) to (5.1) and using (5.6), (5.6) and the symmetry of the tensor $\tau$ ($\tau_{12} = \tau_{21}$) we reduced the problem to finding the number $\lambda$ and real symmetric field $\alpha_{kl}$ $(k, l = 1, 2)$ from equation

$$
(5.7) \qquad \mathbf{A}(s)\boldsymbol{\alpha} - \lambda\mathbf{B}\boldsymbol{\alpha} = \mathbf{0},
$$

$$
(5.8) \qquad \boldsymbol{\alpha} = [\alpha_{11}\ \alpha_{12}\ \alpha_{22}]^T \qquad \mathbf{A} \equiv \mathbf{A}(s) \equiv \begin{bmatrix} \dfrac{s^2}{\varrho} & \dfrac{s}{\varrho}D & 0 \\ -sD\dfrac{1}{\varrho} & \dfrac{s^2}{\varrho} - D\dfrac{1}{\varrho}D & -\dfrac{s}{\varrho}D \\ 0 & -sD\dfrac{1}{\varrho} & -D\dfrac{1}{\varrho}D \end{bmatrix},
$$

$$
(5.9) \qquad \mathbf{B} \equiv \mathbf{B}(\mu, \nu) \equiv \begin{bmatrix} c_1 & c_2 & c_4 \\ c_2 & c_3 & c_5 \\ c_4 & c_5 & c_6 \end{bmatrix}.
$$

The domains of the operators $\mathbf{A}$, $\mathbf{B}$ can be defined as:

$$
(5.10) \qquad \begin{aligned}
\mathcal{D}(\mathbf{A}) &= \{\boldsymbol{\alpha}:\ \alpha_{ij} \in C^2[0,\infty);\ \alpha_{12}(0) = \alpha_{22}(0) = \alpha_{12}(\infty) = \alpha_{22}(\infty) = 0\}\\
\mathcal{D}(\mathbf{B}) &= \{\boldsymbol{\alpha}:\ \alpha_{ij} \in C^2[0,\infty)\}.
\end{aligned}
$$

The functions $c_1(x_2) = \chi_{1111}(x_2)$, $c_2(x_2) = 2\chi_{1112}(x_2)$, $c_3(x_2) = 4\chi_{1212}(x_2)$, $c_4(x_2) = \chi_{1122}(x_2)$, $c_5(x_2) = 2\chi_{1222}(x_2)$, $c_6(x_2) = \chi_{2222}(x_2)$ are bounded functions of the class $C^2[0,\infty)$ satysfying condition (5.2) and(5.3). Additionally we assume, that density $\varrho = \varrho(x_2)$ is bounded function of the class $C^2[0,\infty)$ and $0 < \varrho_0 \leqslant \varrho(x_2) \leqslant \varrho_1$.

In complex Hilbert space $H$ introduced in Sec. 2 with norm and scalar product defined by (3.1) and (3.2) respectively, the conditions of Kato theorems for $B$-holomorphic eigenvalue problem (cf. [5] p. 416 - 426) are satisfied for the problem (5.6) – (3.8). This implies that Th (3.1) – Th (3.5) are valid also in case of the anisotropic nonhomogeneous semi-space.



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