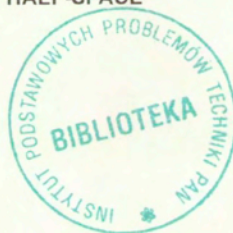
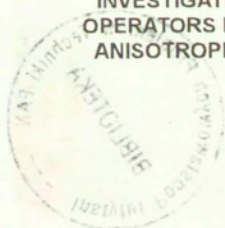


Tadeusz Klecha

INVESTIGATION OF SURFACE WAVE
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ANISOTROPIC ELASTIC HALF-SPACE

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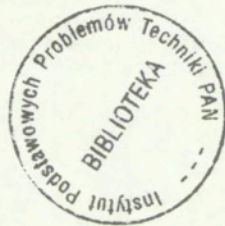
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Investigation of surface wave operators in a nonhomogeneous anisotropic elastic half-space.

Abstract. In the paper new alternative approach to surface wave problem in a nonhomogeneous anisotropic elastic semi-space in terms of stress tensor components vanishing on semi-space boundary is presented. This approach allows to perform analysis of surface wave using Green's function theory similar as is it done for Sturm-Liouville operators in the space of unbounded measure.

Key words. stress surface wave, anisotropic nonhomogeneous elastic semi-space, nonlinear eigenvalue problem.

1. Introduction. The surface wave propagation problem a nonhomogeneous anisotropic elastic semi-space can be formulated in terms of stresses (see [1]) and reduced to generalized eigenvalue problem (see [2]) of the form:

$$A(s)\alpha - \lambda B\alpha = 0 \quad (1.1)$$

where $\alpha = [\alpha_{11} \ \alpha_{22} \ \alpha_{12}]^T$ is a wave amplitude vector from the space $C^2[0, \infty) \times C^3[0, \infty) \times C^2[0, \infty)$ satisfying the condition:

$$\alpha_{22}(0) = \alpha_{12}(0) = 0 \quad (1.2)$$

$$\alpha_{22}(\infty) = \alpha_{12}(\infty) = \alpha_{11}(\infty) = 0$$

The symbol s in (1.1) denotes a wave number ($s > 0$), the operators $A(s)$ and B are 3×3 symmetric - matrices. More precisely $A(s)$ is a selfadjoint matrix - operator and B positively definite matrix - operator, depending on six elastic parameters (see [1]).

In the paper the case of a nonhomogeneous anisotropic semi-space is analysed. Applying the method proposed by Najmark (see [3]) is shown, that the problem can be reduced to equation

$$\ddot{\gamma} + Q\gamma = O \quad (1.3)$$

where $\gamma = [\gamma_1 \ \gamma_2]^T \stackrel{df}{=} [\alpha_{22} \ \alpha_{12}]^T$ is a vector from the space $C^3[0, \infty) \times C^2[0, \infty)$ satisfying compatibility conditions and the relations:

$$\begin{aligned} \dot{\gamma}_1(0) = \gamma_2(0) &= 0 \\ \gamma_1(\infty) = \gamma_2(\infty) &= 0 \end{aligned} \quad (1.4)$$

The dot over any letter denotes differentiation with respect to semi-space depth. Problem (1.3)–(1.4) can be analysed with the help of Green's function theory in the similar way as can be done for Sturm-Liouville and Dirac operators (see [7]).

2. The alternative formulation of surface wave propagation problem in a nonhomogeneous anisotropic elastic semi-space. We consider a nondimensional stress equation for a nonhomogeneous anisotropic elastic semi-space $U = \left\{ (x_1, x_2) : |x_1| < \infty, x_2 \geq 0 \right\}$ (the plane displacement state and arbitrary anisotropy)

$$2\kappa_{ijkl}(x)\ddot{r}_{kl}(x, t) = \left[\varrho^{-1}(x)\tau_{ik,k}(x, t) \right]_{,j} + \left[\varrho^{-1}(x)\tau_{jk,k}(x, t) \right]_{,i} \quad (2.1)$$

where $\tau_{kl} = \tau_{kl}(x, t)$, $(i, j, k, l = 1, 2)$, $x = (x_1, x_2)$ is stress field to be found (see [1]).

The tensor κ_{ijkl} is 4-th order tensor of compliance moduli as a function of a spatial coordinate, satisfying condition:

a) symmetry

$$\kappa_{ijkl} = \kappa_{jikl} = \kappa_{ijlk} = \kappa_{lkij} \quad (2.2)$$

b) strong ellipticity condition

$$\kappa_{ijkl} a_i b_k a_j b_l > 0 \quad (2.3)$$

for arbitrary vectors a, b . Medium density is defined by the function $\varrho(x)$.

It is assumed that the functions $\kappa_{ijkl}(x)$ and $\varrho(x)$ ($i, j, k, l = 1, 2$) depend on x_2 and are bounded function of class $C^2[0, \infty)$.

In the elastic semispace:

$$U = \left\{ (x_1, x_2) : |x_1| < \infty, x_2 \geq 0 \right\} \quad (2.4)$$

we are looking for the functions $\tau_{kl}(x, t)$ being the solution of equation (2.1) in the form:

$$\tau_{11} = \alpha_{11}(x_2) \exp[i(sx_1 - t\sqrt{\lambda})] \quad (2.5)$$

$$\tau_{22} = \alpha_{22}(x_2) \exp[i(sx_1 - t\sqrt{\lambda})]$$

$$\tau_{22} = i\alpha_{12}(x_2) \exp[i(sx_1 - t\sqrt{\lambda})] \text{ for } t \in (0, +\infty),$$

where $i = \sqrt{-1}$, $s > 0$, $\lambda > 0$ and the functions $\tau_{kl}(x, t)$ satisfy the conditions

$$\tau_{22}(x_1, 0, t) = \tau_{12}(x_1, 0, t) = 0 \quad \text{for } x_1 \in \mathbb{R}, t \in (0, +\infty) \quad (2.6)$$

$$\tau_{22}(x_1, \infty, t) = \tau_{12}(x_1, \infty, t) = \tau_{11}(x_1, \infty, t) = 0 \quad \text{for } x_1 \in \mathbb{R}, \text{ and } t \in (0, +\infty) \quad (2.7)$$

After substitution (2.5) to (2.1) and applying conditions (2.6), (2.7) and the symmetry of tensor τ the problem of the surface wave propagation can be reduced to generalised eigenvalue problem

$$A(s)\alpha - \lambda B\alpha = 0 \quad (2.8)$$

where $\alpha = [\alpha_{11} \ \alpha_{12} \ \alpha_{22}]^T$,

$$A(s) = \begin{bmatrix} \frac{s^2}{\varrho} & \frac{s}{\varrho}D & 0 \\ -sD\frac{1}{\varrho} & \frac{s^2}{\varrho} - D\frac{1}{\varrho}D & \frac{1}{\varrho}D \\ 0 & -sD\frac{1}{\varrho} & -D\frac{1}{\varrho}D \end{bmatrix} \quad (2.9)$$

$$B = \begin{bmatrix} C_1 & C_2 & C_4 \\ C_2 & C_3 & C_5 \\ C_4 & C_5 & C_6 \end{bmatrix} \quad (2.10)$$

$$\begin{aligned} C_1 &= \kappa_{1111}(x_2), & C_2 &= 2\kappa_{1112}(x_2), \\ C_3 &= 4\kappa_{1212}(x_2), & C_4 &= \kappa_{1122}(x_2), \\ C_5 &= 2\kappa_{1222}(x_2), & C_6 &= \kappa_{2222}(x_2). \\ D &= \frac{d}{dx_2}. \end{aligned} \quad (2.11)$$

The domain of the operators A and B are:

$$\begin{aligned} \text{dom } A &= \left\{ \alpha: \alpha \in C^2[0, \infty) \times C^3[0, \infty) \times C^2[0, \infty); \alpha_{12}(0) = \alpha_{22}(0) = \right. \\ &= \left. \alpha_{12}(\infty) = \alpha_{22}(\infty) = \alpha_{11}(\infty) = 0 \right\} \\ \text{dom } B &= \left\{ \alpha: \alpha \in C^2[0, \infty) \times C^3[0, \infty) \times C^2[0, \infty) \right\} \end{aligned} \quad (2.12)$$

The formulation covers the cases⁽¹⁾ of

- a) isotropy
- b) transverse isotropy
- c) generalised anisotropy (see [4], [5], [6]).

Since $\kappa_{ijkl}(x_2)$ are bounded functions of the class $C^2[0, \infty)$, the functions $C_1(x_2), C_2(x_2), \dots, C_6(x_2)$ are also bounded functions of the class $C^2[0, \infty)$. Similarity medium density $\rho(x_2)$ is also bounded function of class $C^2[0, \infty)$.

In order to obtain alternative form of the problem (2.8) – (2.12) we consider eigenvalue problem (2.8) written in the form

$$\begin{aligned} \frac{s^2}{\rho} \alpha_{11} + \frac{s}{\rho} D \alpha_{12} &= \lambda (C_1 \alpha_{11} + C_2 \alpha_{12} + C_4 \alpha_{22}) \\ -sD \left(\frac{1}{\rho} \alpha_{11} \right) + \left(\frac{s^2}{\rho} - D \frac{1}{\rho} D \right) \alpha_{12} - \frac{s}{\rho} D \alpha_{22} &= \lambda (C_2 \alpha_{11} + C_3 \alpha_{12} + C_5 \alpha_{22}) \\ -sD \left(\frac{1}{\rho} \alpha_{12} \right) - D \frac{1}{\rho} D \alpha_{22} &= \lambda (C_4 \alpha_{11} + C_5 \alpha_{12} + C_6 \alpha_{22}) \end{aligned} \quad (2.13)$$

From the first equation of the system (2.13) we obtain

$$\alpha_{11} = \frac{\lambda \rho(x_2) C_2(x_2)}{s^2 - \lambda \rho(x_2) C_1(x_2)} \alpha_{12} - \frac{s}{s^2 - \lambda \rho(x_2) C_1(x_2)} \dot{\alpha}_{12} + \frac{\lambda \rho(x_2) C_4(x_2)}{s^2 - \lambda \rho(x_2) C_1(x_2)} \alpha_{22} \quad (2.14)$$

and after differentiation with respect x_2 we get

⁽¹⁾if $C_1 = \frac{1-\nu}{2\mu}$, $C_2 = \frac{-\nu}{2\mu}$, $C_4 = 0$, $C_3 = \frac{1-\nu}{2\mu}$, $C_5 = 0$, $C_6 = \frac{1}{\mu}$, and $C_1 = -B_1(x_2)$, $C_2 = 0$, $C_3 = 2B_2(x_2)$, $C_4 = -4f(x_2)$, $C_5 = 0$, $C_6 = B(x_2)f(x_2)g(x_2)$ we obtain an isotropy and a transverse isotropy, respectively (see T. Rojnowski [4]).

The parameters: $B(x_2)$, $B_1(x_2)$, $B_2(x_2)$, $f(x_2)$, $g(x_2)$ are defined by T. Rojnowski in the paper [4].

$$\begin{aligned} \dot{\alpha}_{11} = & \left(\frac{\lambda \varrho(x_2) C_2(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} \right)' \alpha_{12}(x_2) + \frac{\lambda \varrho(x_2) C_2(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} \dot{\alpha}_{12}(x_2) + \\ & - \left(\frac{s}{s^2 - \lambda \varrho(x_2) C_1(x_2)} \right)' \alpha_{12}(x_2) - \frac{s}{s^2 - \lambda \varrho(x_2) C_1(x_2)} \ddot{\alpha}_{12}(x_2) + \quad (2.15) \\ & + \left(\frac{\lambda \varrho(x_2) C_4(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} \right)' \alpha_{22}(x_2) + \frac{\lambda \varrho(x_2) C_4(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} \dot{\alpha}_{22}(x_2) \end{aligned}$$

After substituting (2.15) and (2.14) to the second and third equation of the system (2.13) we obtain

$$\begin{aligned} & \left[\frac{s^2}{(s^2 - \lambda \varrho(x_2) C_1(x_2)) \varrho(x_2)} - \frac{1}{\varrho(x_2)} \right] \ddot{\alpha}_{12}(x_2) + \left[\frac{-s \lambda C_2(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} + \right. \\ & \left. \left(\frac{s^2}{(s^2 - \lambda \varrho(x_2) C_1(x_2)) \varrho(x_2)} \right)' - \left(\frac{1}{\varrho(x_2)} \right)' + \frac{s \lambda C_2(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} \right] \dot{\alpha}_{12}(x_2) + \\ & + \left[- \left(\frac{s \lambda C_2(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} \right)' + \frac{s^2}{\varrho(x_2)} - \frac{\lambda^2 \varrho(x_2) C_2^2(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} + \right. \quad (2.16) \\ & \left. - \lambda C_3(x_2) \right] \alpha_{12}(x_2) + \left[- \frac{s \lambda \varrho(x_2) C_4(x_2)}{(s^2 - \lambda \varrho(x_2) C_1(x_2)) \varrho(x_2)} - \frac{s}{\varrho(x_2)} \right] \dot{\alpha}_{22}(x_2) + \\ & + \left[- \frac{s \lambda \varrho(x_2) C_4(x_2)}{(s^2 - \lambda \varrho(x_2) C_1(x_2)) \varrho(x_2)} - \frac{\lambda^2 \varrho(x_2) C_2(x_2) C_4(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} - \lambda C_5(x_2) \right] \alpha_{22}(x_2) = 0 \end{aligned}$$

$$\begin{aligned} & \left[- \frac{s}{\varrho(x_2)} + \frac{\lambda s C_4(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} \right] \dot{\alpha}_{12}(x_2) + \left[-s \left(\frac{1}{\varrho(x_2)} \right)' + \right. \\ & \left. - \frac{\lambda^2 \varrho(x_2) C_2(x_2) C_4(x_2)}{s^2 - \lambda \varrho(x_2) C_1(x_2)} - \lambda C_5(x_2) \right] \alpha_{12}(x_2) + \left(\frac{-1}{\varrho(x_2)} \right) \ddot{\alpha}_{22}(x_2) + \quad (2.17) \\ & + \left[- \left(\frac{1}{\varrho(x_2)} \right)' \right] \dot{\alpha}_{22}(x_2) + \left[\frac{-\lambda^2 \varrho(x_2) C_4^2(x_2)}{s^2 - \lambda C_1(x_2) \varrho(x_2)} - \lambda C_6(x_2) \right] \alpha_{22}(x_2) = 0 \end{aligned}$$

The equation (2.16) and (2.17) can be written in more compact way

$$\mathbf{R}_0 \ddot{\beta} + \mathbf{R}_1 \dot{\beta} + \mathbf{R}_2 \beta = 0 \quad (2.18)$$

where

$$\beta(x_2) = [\alpha_{22}(x_2) \quad \alpha_{12}(x_2)]^T \quad (2.19)$$

$$\mathbf{R}_0(s, \lambda) = \begin{bmatrix} -\frac{1}{\varrho(x_2)} & 0 \\ 0 & \frac{s^2}{(s^2 - \lambda\varrho(x_2)C_1(x_2))\varrho(x_2)} - \frac{1}{\varrho(x_2)} \end{bmatrix} \quad (2.20)$$

$$\mathbf{R}_1(s, \lambda) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (2.21)$$

$$a_{11} = -\left(\frac{1}{\varrho(x_2)}\right) \cdot$$

$$a_{12} = -\frac{s}{\varrho(x_2)} + \frac{\lambda s C_1(x_2)}{s^2 - \lambda\varrho(x_2)C_1(x_2)}$$

$$a_{21} = \left(\frac{-s\lambda\varrho(x_2)C_4(x_2)}{(s^2 - \lambda C_1(x_2)\varrho(x_2))\varrho(x_2)} - \frac{s}{\varrho(x_2)}\right)$$

$$a_{22} = \frac{-s\lambda C_2(x_2)}{s^2 - \lambda C_1(x_2)\varrho(x_2)} + \left(\frac{s^2}{(s^2 - \lambda C_1(x_2)\varrho(x_2))\varrho(x_2)}\right) \cdot \left(\frac{s}{\varrho(x_2)}\right) \cdot +$$

$$+ \frac{\lambda s C_2(x_2)}{s^2 - \lambda\varrho(x_2)C_1(x_2)}$$

$$\mathbf{R}_2(s, \lambda) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad (2.22)$$

$$b_{11} = \frac{-\lambda^2\varrho(x_2)C_4^2(x_2)}{s^2 - \lambda\varrho(x_2)C_1(x_2)} - \lambda C_6(x_2)$$

$$b_{12} = -s\left(\frac{1}{\varrho(x_2)}\right) \cdot -\frac{\lambda^2\varrho(x_2)C_2(x_2)C_4(x_2)}{s^2 - \lambda\varrho(x_2)C_1(x_2)} - \lambda C_5(x_2)$$

$$b_{21} = -\frac{s\lambda C_4(x_2)}{s^2 - \lambda\varrho(x_2)C_1(x_2)} - \frac{\lambda^2\varrho(x_2)C_2(x_2)C_4(x_2)}{s^2 - \lambda\varrho(x_2)C_1(x_2)} - \lambda C_5(x_2)$$

$$b_{22} = -\left(\frac{s\lambda C_2(x_2)}{s^2 - \lambda\varrho(x_2)C_1(x_2)}\right) \cdot + \frac{s^2}{\varrho(x_2)} - \frac{\lambda^2\varrho(x_2)C_2^2(x_2)}{s^2 - \lambda\varrho(x_2)C_1(x_2)} - \lambda C_3(x_2)$$

The equation (2.18) shut by supplemented by compatibility conditions written for the field

$$\beta = \begin{bmatrix} \alpha_{22} \\ \alpha_{12} \end{bmatrix}.$$

Compatibility conditions has the form:

$$\begin{aligned} (C_1\alpha_{11} + C_2\alpha_{12} + C_3\alpha_{22})'' - s^2(C_4\alpha_{11} + C_5\alpha_{12} + C_6\alpha_{22}) + \\ + s(C_2\alpha_{11} + C_3\alpha_{12} + C_5\alpha_{22})' = 0^{(2)} \text{ for } x_2 \in [0, +\infty). \end{aligned} \quad (2.23)$$

Tensor α_{11} is given by (2.14).

By means of the substitution

$$\beta = U\gamma \quad (2.24)$$

where U is any solution of differential matrix equation

$$\dot{U} + \frac{1}{2}P_1U, \quad \det U \neq 0 \quad (\text{see [3] p. 118}) \quad (2.25)$$

the equation

$$P_0\ddot{\beta} + P_1\dot{\beta} + P_2\beta = 0 \quad (2.26)$$

$$\left(P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_1 = R_0^{-1}R_1, \quad P_2 = R_0^{-1}R_2 \right)$$

can be reduced to the following form

$$\ddot{\gamma} + Q\gamma = 0 \quad (2.27)$$

where

$$\gamma(0) = (U^{-1}\beta)(0) = 0$$

$$\gamma(\infty) = (U^{-1}\beta)(\infty) = 0$$

$$Q = -\frac{1}{2}\dot{P}_1 - \frac{1}{4}P_1^2 + P_2 \quad (2.28)$$

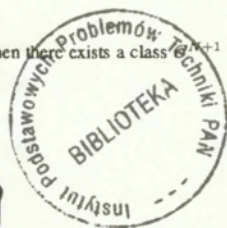
⁽²⁾Compatibility theorem for plane displacement (c.f. [8]). Assume that B is a simply-connected open region in the (x_1, x_2) -plane. Let u_i be a class C^3 field on B , and let

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad i, j = 1, 2 \quad (a)$$

Then

$$2\varepsilon_{12,12} = \varepsilon_{11,22} + \varepsilon_{22,11} \quad (b)$$

Conversely, let $\varepsilon_{ij} (= \varepsilon_{ji})$ be a class C^N ($N \geq 2$) field on B that satisfies (b). Then there exists a class C^{N+1} field u_i on B such that (a) holds.



The problem (2.27) can be investigated with the help of the Green's function theory or iterative method used for Sturm-Liouville and Dirac operators in the space of unbounded measure (see [7]). Compatibility theorem for plane displacement implies that the problem of surface wave propagation in a nonhomogeneous, anisotropic semispace can be reduced to finding a solution Eq. (2.27), which satisfying plane compatibility condition in zero:

$$\begin{aligned} & \left\{ \left[C_1(x_2)\alpha_{11} + C_2(x_2)\alpha_{12} + C_3(x_2)\alpha_{22} \right] + \right. \\ & -s^2 \left[C_4(x_2)\alpha_{11} + C_5(x_2)\alpha_{12} + C_6(x_2)\alpha_{22} \right] + \\ & \left. +s \left[C_2(x_2)\alpha_{11} + C_3(x_2)\alpha_{12} + C_5(x_2)\alpha_{22} \right] \right\} (0) = 0 \end{aligned} \quad (2.29)$$

Tensor α_{11} is defined by (2.14)

Remark 1. In the case of isotropic and homogeneous semi-space equation (2.26) has the following solution:

$$\beta = \begin{bmatrix} \beta_1(x_2) \\ \beta_2(x_2) \end{bmatrix} := C \begin{bmatrix} e^{-x_2 h_2} - e^{-x_2 h_1} \\ -\frac{2}{s} \frac{h_1}{2-\omega} (e^{-x_2 h_2} - e^{-x_2 h_1}) \end{bmatrix} \quad (2.30)$$

Where

$$\begin{aligned} h_1 &= s\sqrt{1-\omega\kappa}, & h_2 &= s\sqrt{1-\omega}, \\ \kappa &= \frac{1-2\nu}{2-2\nu}, & 0 < \omega < 1. \end{aligned}$$

From (2.30) and from the plane compatibility condition one obtains

$$\begin{aligned} & \frac{Cs^2}{2\mu(2-\omega)} e^{-x_2 s\sqrt{1-\omega}} \left[(2-\omega)^2 - 4\sqrt{(1-\omega)(1-\omega\kappa)} \right] + \\ & + \frac{Cs^2(1-\kappa)}{2\mu(2-\omega)} e^{-x_2 s\sqrt{1-\omega\kappa}} [0] = 0 \end{aligned} \quad (2.31)$$

Since $C \neq 0$ one obtains the classical equation determining velocity of the surface wave in homogeneous isotropic elastic semi-space

$$(2-\omega)^2 - 4\sqrt{(1-\omega)(1-\omega\kappa)} = 0. \quad (2.32)$$

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