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Time-variant reliability – Computational approaches based on FORM and importance sampling

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Assessing the reliability of randomly excited linear and nonlinear dynamical systems requires the estimation of transient and steady-state response quantities which occur with low probability. Although there are exact solutions available for a certain class of second or higher order nonlinear systems, these solutions are limited to the probability density of the system response in the stationary regime only. Moreover, for the more interesting problem of first-passage failure no exact solutions are known at all. The paper presents an efficient importance sampling approach to the Monte Carlo solution of first-passage problems. It is shown that an appropriate importance sampling density can be constructed utilizing "design point excitations" which are obtained by a procedure analogous to the first order reliability method (FORM). The presented approach can efficiently handle non-white and non-stationary excitations, can be applied to a wide range of nonlinear problems, and yields high levels of confidence at very low numbers of samples.

Key words: *Random vibration, first-passage probability, Monte Carlo simulation, importance sampling, first order reliability method (FORM).*

1. Introduction

Structural reliability analysis has historically been developed to deal with the analysis of time-invariant problems of random variables. Recent developments increasingly focus on aspects concerning the effect of random processes thus requiring the analysis of time-variant responses and consequently, time-variant reliability. However, the number of mathematically exact solutions available for nonlinear dynamical systems under random excitations is still restricted to a narrow class of systems and to the knowledge of stationary probability densities or moments only [1, 2, 3]. For assessing the reliability of

a dynamical system most realistically, in addition to the probabilistic characterization of the response at a certain time point, a reliability measure evolving over a time period is most instrumental. The latter can be achieved by utilizing the first-passage probability, i.e. the probability that the structural response exceeds a given failure boundary in a certain time interval for the first time. It is quite remarkable that up to now a closed-form solution for the first-passage problem does not exist, not even for single-degree-of-freedom linear systems.

For being nevertheless able to assess the reliability of randomly excited dynamical systems, different approximate solution techniques have been proposed, as e.g. cumulant-neglect closure, equivalent linearization or stochastic averaging [3, 4, 5, 6], which are applicable for both stationary and non-stationary problems. Despite their indisputable merits, especially in problems involving the determination of moments of lower order, they lack sufficient accuracy in quantifying structural responses occurring with low probability, as in the case of extreme load/response events, which are of paramount importance in the reliability assessment of engineering structures. The only solution technique not showing any of the above mentioned deficits is presumably Monte Carlo simulation. However, it is most often applied as a last resort only – due to numerical answers of supposedly limited accuracy or prohibitive computational costs.

Whereas these drawbacks of the Monte Carlo simulation technique are an inherent characteristic of its crude form, this does not hold for variance reduction techniques [7, 8]. Although this fact is widely recognized for time-invariant problems, for problems in random vibration the application of variance reduction techniques is still eschewed – despite different, technically sound methods proposed during the last years [9, 10, 11, 12]. The most versatile variance reduction technique appears to be importance sampling utilizing the measure transformation method [13, 14, 15] based on Girsanov's theorem [16].

A different and at first sight much simpler approach are so called (multi-level) splitting methods which can be traced back at least to [17]. These methods have been brought to a wider attention by [18] and [19]. Typical applications to civil engineering systems can be found e.g. in [20, 21]. In multi-level splitting promising sample paths are split into sub-paths at intermediate levels to increase the number of observations of rare events. As has been reported as early as in [19], however, this procedure can result in general settings in an increase of the variance of the estimator. More precarious even is an "apparent bias" [22] in the estimator, when the levels for splitting are not chosen consistently. Indeed, multi-level splitting requires for being effective that the splitting is performed along the most likely path leading to an out-

crossing [22]. However, in this case a more effective sampling method like importance sampling can be used right from start.

In the following we present an importance sampling procedure for randomly excited dynamical systems. Starting by a system of Itô stochastic differential equations modeling the dynamical response, we change, by applying Girsanov's theorem, the drift terms of these equations according to a minimization criterion for the variance of the respective estimators. These changes can be interpreted as (additional) control forces which allow to influence the sample paths in a predetermined way. The unlikeliness of the corresponding sample paths is taken into account by a correction process such that the final estimates become unbiased again. Although theoretically there exists a set of control forces which result in unbiased zero-variance estimators [15], these optimal control forces are in general not accessible (this would require the knowledge of the solution) and have to be replaced by sub-optimal ones.

Such sub-optimal control forces – also called "design point excitations" – can be constructed from an optimization problem analogous to the first order reliability method (FORM). Indeed, when utilizing a time discretization for solving the system of stochastic differential equations, the correspondence between the measure transformation method based on Girsanov's theorem and the importance sampling method utilizing finite dimensional probability distributions – as known from static problems – can be shown. In the latter case the design point excitations are simply the mean values of the importance sampling distribution. Moreover, recognizing the above mentioned correspondence, the importance sampling concept can be straightforwardly extended to dynamical systems under excitations modeled by Poisson processes. The paper concludes with numerical examples of dynamical systems under white noise as well as Poisson-type excitations demonstrating the efficiency of the proposed importance sampling method.

2. Importance sampling concept

Let us assume that the dynamical response of a system at time $t \in [0, T]$ is described by a p-dimensional Itô process $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_p(t))$ in terms of the stochastic differential equations $(j = 1, 2, \ldots, p)$

$$\mathrm{d}X_j(t) = a_j(\mathbf{X}, t)\mathrm{d}t + b_j(\mathbf{X}, t)\,\mathrm{d}W(t), \tag{2.1}$$

with deterministic initial conditions

$$X_j(0) = x_j \tag{2.2}$$

and $W(t)$ as a unit Wiener process. The following derivations can be extended without any restrictions to multiple Wiener processes.

In reliability assessment of dynamical systems we are in general not interested in an exact pathwise representation of $\mathbf{X}(t)$, but only in so called weak solutions of the form

$$v = \mathrm{E}[f(\mathbf{X}, T)]\,, \tag{2.3}$$

whereby $f(\cdot)$ denotes some real-valued function and $\mathrm{E}[\cdot]$ is the expected value. A typical example of a first-passage problem is to determine the probability that the j-th component of $\mathbf{X}(t)$ exceeds a given barrier x_c in the interval $[0, T]$, i.e.

$$v = \mathrm{E}[\mathrm{I}(\max_{0\leq t\leq T} X_j(t) > x_c)], \tag{2.4}$$

with $\mathrm{I}(\cdot)$ denoting an characteristic function that equals one if its argument is true, and zero otherwise.

When we utilize Monte Carlo simulation to evaluate Eq. (2.3), v is replaced by its sample-mean formula [8]

$$\hat{v} = \hat{\mathrm{E}}[f(\mathbf{X}, T)] = \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{X}^{(i)}, T) \tag{2.5}$$

where $\mathbf{X}^{(i)}(t)$ denotes the i-th sample path of $\mathbf{X}(t)$ and n is the number of sample paths. The quantity $\hat{v}$ is an unbiased estimator of v with variance

$$\mathrm{Var}[\hat{v}] = \frac{1}{n}\mathrm{Var}[f(\mathbf{X}, T)]\,. \tag{2.6}$$

The square root of this variance – the so called root mean square error – can be conceived as an indicator of the accuracy of the Monte Carlo estimator. The main concern in Monte Carlo simulation, therefore, is to obtain an estimator with sufficiently small error. Whereas this can always be achieved by simply increasing the number n of sample paths, it is rarely a rewarding procedure, since – as can be seen from Eq. (2.6) – the root mean square error of the estimator is inversely proportional to the square root of the sample size n. Therefore a variance reduction technique like importance sampling becomes mandatory.

Since evaluating v requires weak solutions of Eqs. (2.1) only, the measure transformation method, based on Girsanov's theorem [16, 24], can be applied. Instead of Eqs. (2.1), the stochastic differential equations ($j = 1, 2, \ldots, p$)

$$\begin{aligned} \mathrm{d}\tilde{X}_j(t) &= a_j(\tilde{\mathbf{X}}, t)\mathrm{d}t + b_j(\tilde{\mathbf{X}}, t)\big(u(\tilde{\mathbf{X}}, t)\mathrm{d}t + \mathrm{d}W(t)\big), \\ \mathrm{d}Y(t) &= u(\tilde{\mathbf{X}}, t)Y\mathrm{d}W(t) \end{aligned} \tag{2.7}$$

are evaluated, augmented by the one-dimensional correction process $Y(t)$ with initial conditions

$$\tilde{X}_j(0) = x_j \qquad \text{and} \qquad Y(0) = 1. \tag{2.8}$$

respectively. In Eqs. (2.7) the component $u(\tilde{\mathbf{X}}, t)$ can be interpreted as (additional) control force which allow to influence the structural response. Under certain restrictions for $u(\tilde{\mathbf{X}}, t)$, which are usually met in practical applications [24], the functional v of Eq. (2.3) is evaluated according to Girsanov's theorem as

$$v = \mathrm{E}[f(\mathbf{X}, T)] = \mathrm{E}\big[Y(T) f(\tilde{\mathbf{X}}, T)\big]. \tag{2.9}$$

In other words, the control $u(\tilde{\mathbf{X}}, t)$ will change the Itô process $\mathbf{X}(t)$ to $\tilde{\mathbf{X}}(t)$, whereas this "change" is taken into account by the Radon-Nikodym derivative

$$Y(T) = \exp\left[-\int_0^T u(\tilde{\mathbf{X}}, t)\mathrm{d}W(t) - \frac{1}{2}\int_0^T \big(u(\tilde{\mathbf{X}}, t)\big)^2 \mathrm{d}t\right]. \tag{2.10}$$

The functional v of Eq. (2.9) can be replaced again by its unbiased Monte Carlo estimator

$$\hat{v} = \frac{1}{n}\sum_{i=1}^{n} Y^{(i)}(T) f(\tilde{\mathbf{X}}^{(i)}, T). \tag{2.11}$$

Whereas the mean of the estimator in Eq. (2.11) is not influenced by the choice of the controls – this is exactly what Girsanov's theorem states – the variance of the estimator is affected. Hence, by an appropriate choice of $u(\tilde{\mathbf{X}}, t)$, this control force can be utilized for reducing efficiently the variance of the estimator $\hat{v}$.

Thereby, an optimal control $u^*(\tilde{\mathbf{X}}, t)$ is defined as that control for which $\mathrm{E}[(Y(T) f(\tilde{\mathbf{X}}(T)))^2]$ becomes minimal. By solving the corresponding Hamilton-Jacobi-Bellman equation the optimal control can be derived as [11, 13]

$$u^*(\tilde{\mathbf{X}}, t) = \sum_{j=1}^{p} \frac{b_j(\tilde{\mathbf{X}}, t)}{v(\tilde{\mathbf{X}}, t)} \frac{\partial v}{\partial \tilde{X}_j}(\tilde{\mathbf{X}}, t), \tag{2.12}$$

whereby $v(\tilde{\mathbf{X}}, t)$ denotes the solution of Eq. (2.3) with initial conditions $\tilde{\mathbf{X}}(t)$ at time t.

When utilizing this control $u^*(\tilde{\mathbf{X}}, t)$ in Eqs. (2.7), $\hat{v}$ of Eq. (2.11) will be an unbiased zero-variance estimator of v, as can be shown by Itô's formula. Closer inspection of Eq. (2.12), however, reveals that in order to construct $u^*(\tilde{\mathbf{X}}, t)$ the sought solution v has to be known in advance. Indeed, we not only have to know v for a single set of initial conditions $\mathbf{x}$ at time $t = 0$, but

we have to know v and its derivatives $\partial v/\partial \tilde{X}_i$ for all values of t and $\tilde{\mathbf{X}}(t)$. This is a drawback typical for variance reduction techniques, which can be characterized as methods to utilize known information about a problem at hand. In other words, knowing the solution allows to construct a zero-variance estimator, but renders Monte Carlo simulation superfluous, whereas knowing nothing means also that no variance reduction can be achieved [8].

Nevertheless, Eq. (2.12) can still be utilized rewardingly for variance reduction as long as an approximation $\bar{v}$ for v is readily available. Then, instead of the optimal control force $u^*(\tilde{\mathbf{X}}, t)$, the sub-optimal control force $\bar{u}(\tilde{\mathbf{X}}, t)$ can be constructed. The resulting estimator of v will further on be unbiased, but now with variance [15]

$$\mathrm{Var}[\hat{v}] = \frac{1}{n}\,\mathrm{E}\left[\left(\int_0^T v(\tilde{\mathbf{X}}, t)\big(u^*(\tilde{\mathbf{X}}, t) - \bar{u}(\tilde{\mathbf{X}}, t)\big)dW(t)\right)^2\right]. \tag{2.13}$$

As can be seen from Eq. (2.13), the difference between the two controls $\bar{u}(\tilde{\mathbf{X}}, t)$ and $u^*(\tilde{\mathbf{X}}, t)$ is integrated over time. From this follows that the variance can increase with time as long as the solutions v depend on the initial conditions. Moreover, choosing the controls $\bar{u}(\tilde{\mathbf{X}}, t)$ without taking into account specific characteristics of the dynamical system under investigation can even result in an increase of the variance as compared to crude Monte Carlo simulation.

Solving the stochastic differential Eqs. (2.1) numerically, these equations are replaced by discrete time approximations. Let us subdivide the interval $[0, T]$ into m sub-intervals, such that

$$0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = t_k + \Delta_k < \ldots < t_m = T, \tag{2.14}$$

whereby $\Delta_k = t_{k+1} - t_k$ is the length of the sub-interval $[t_k, t_{k+1}]$. Therewith the stochastic differential Eqs. (2.1) can be written – e.g. in the form of the stochastic Euler scheme – as

$$X_j(t_{k+1}) = X_j(t_k) + a_j(\mathbf{X}, t_k)\Delta_k + b_j(\mathbf{X}_k, t_k)\Delta W_k, \quad X_j(0) = x_j, \tag{2.15}$$

with $\Delta W_k = W(t_{k+1}) - W(t_k)$. It should be noted that for the following it is not necessary to write Eq. (2.15) in form of the Euler scheme. Indeed, any other stochastic integration scheme (cf. [13]) can be utilized. Replacing the k-th increment of the Wiener process by

$$\Delta W_k = \sqrt{\Delta_k}\,\zeta_k, \tag{2.16}$$

with ζ_k being mutually independent standard normal random variates, the functional of Eq. (2.3) is approximated by

$$v = \int_{-\infty}^{+\infty} (m\text{-fold}) \int_{-\infty}^{+\infty} f(\mathbf{X}, t_m)\, \varphi(\boldsymbol{\zeta})\, \mathrm{d}\boldsymbol{\zeta}, \tag{2.17}$$

whereby $\varphi(\cdot)$ is the m-dimensional joint density function of the standard normal random variates $\boldsymbol{\zeta}$.

Describing now the boundary of the area which will be out-crossed by the Itô process $\mathbf{X}(t_l)$ at time t_l (with $l = 1, 2, \ldots, m$) as

$$\begin{aligned} \bar{g}(\mathbf{X}(t_l)) &= \bar{g}\big(\mathbf{X}(\zeta_0(t_l), \ldots, \zeta_{m-1}(t_l))\big) \\ &= g(\zeta_0(t_l), \ldots, \zeta_{m-1}(t_l)) = 0, \end{aligned} \tag{2.18}$$

then the most likely excitation leading to an out-crossing of this boundary is defined by $\bar{\boldsymbol{\zeta}}(t_l) = (\bar{\zeta}_0(t_l), \ldots, \bar{\zeta}_{m-1}(t_l))$ which minimizes

$$\beta(t_l) = \left[\sum_{k=0}^{m-1} (\bar{\zeta}_k(t_l))^2\right]^{1/2} \tag{2.19}$$

subject to

$$g(\bar{\zeta}_0(t_l), \ldots, \bar{\zeta}_{m-1}(t_l)) = 0. \tag{2.20}$$

To solve Eq. (2.19) standard techniques from the first order reliability method can be applied (cf. [25, 26]). It should also be emphasized that an analogous formulation is given in [27, 28] for determining the mean out-crossing rate of randomly excited systems.

With the above given most likely excitation $\bar{\boldsymbol{\zeta}}(t_l)$, equation (2.15) is modified in such a way that

$$\tilde{X}_j(t_{k+1}) = \tilde{X}_j(t_k) + a_j(\tilde{\mathbf{X}}, t_k)\Delta_k + b_j(\tilde{\mathbf{X}}, t_k)\big(\bar{u}(t_k; t_l)\Delta_k + \Delta W_k\big), \tag{2.21}$$

in which the above utilized control $\bar{u}(t_k; t_l)$ is defined by

$$\bar{u}(t_k; t_l) = \frac{1}{\sqrt{\Delta_k}}\, \bar{\zeta}_k(t_l) \quad (k = 0, 1, \ldots, m-1;\ l = 1, 2, \ldots, m). \tag{2.22}$$

Taking into account the unlikeliness of such a modification of equation (2.21), the functional of equation (2.17) is evaluated by the importance sampling integral [8]

$$v = \int_{-\infty}^{+\infty} (m\text{-fold}) \int_{-\infty}^{+\infty} f(\tilde{\mathbf{X}}, t_l)\, \frac{\varphi(\tilde{\boldsymbol{\zeta}})}{h(\tilde{\boldsymbol{\zeta}})}\, h(\tilde{\boldsymbol{\zeta}}) \mathrm{d}\tilde{\boldsymbol{\zeta}}. \tag{2.23}$$

The importance sampling density $h(\tilde{\boldsymbol{\zeta}})$ in equation (2.23) is defined as follows

$$h(\tilde{\boldsymbol{\zeta}}) \propto \exp\left[-\frac{1}{2}\sum_{k=0}^{m-1} (\tilde{\zeta}_k - \bar{\zeta}_k(t_l))^2\right]. \tag{2.24}$$

Therewith, the likelihood ratio is given by:

$$\frac{\varphi(\tilde{\zeta})}{h(\tilde{\zeta})} = \exp\left[-\sum_{k=0}^{m-1} \bar{\zeta}_k(t_l)\zeta_k - \frac{1}{2}\left(\beta(t_l)\right)^2\right] = \exp\left[-\sum_{k=0}^{m-1} \bar{u}(t_k;t_l)\Delta W_k - \frac{1}{2}\sum_{k=0}^{m-1}\left(\bar{u}(t_k;t_l)\right)^2\Delta_k\right]. \tag{2.25}$$

It should be noted that Eq. (2.25) is a discrete version of the Radon-Nikodym derivative given by Eq. (2.10).

When determining the first-passage probability in the time interval $[0, T]$, there are – according to the time discretization – m most likely excitations leading to an out-crossing. Weighting these excitations by $\Phi(-\beta(t_l))$, i.e. their probability of occurrence, results in the m-modal importance sampling density [23, 29] given by:

$$h(\tilde{\zeta}) \propto \sum_{l=1}^{m} \Phi(-\beta(t_l)) \exp\left[-\frac{1}{2}\sum_{k=0}^{m-1}\left(\tilde{\zeta}_k - \bar{\zeta}_k(t_l)\right)^2\right]. \tag{2.26}$$

For linear systems the sampling density can be improved as proposed in [30] or [31]. However, these methods fail, in general, for nonlinear systems.

3. Extension to Poisson processes

Taking into account the above shown correspondence between the measure transformation method based on Girsanov's theorem and the importance sampling method utilizing finite dimensional probability distributions as in the case of Eqs. (2.25) and (2.26), the importance sampling concept can be straightforwardly extended to Poisson-type loadings of quite arbitrary distribution type without recurring to an extension of Girsanov's theorem as done e.g. in [32]. Let us assume that the time-variant reliability of a structure or structural element can be determined from the weak solution of a system of differential equations (with $j = 1, 2, \ldots, p$) of type

$$\mathrm{d}Y_j(t)/\mathrm{d}t = \alpha_j(\mathbf{Y}, t)\,\gamma_j\big(\Psi(t)\big), \qquad Y_j(0) = y_j \tag{3.1}$$

where the external action (or load) $\Psi(t)$ is modeled as a random pulse train (Poisson process) of the form

$$\Psi(t) = \sum_{k=1}^{q} \Psi_k\,\delta(t - T_k). \tag{3.2}$$

In Eq. (3.2) the quantities Ψ_k are independent and identically distributed random variables obeying the cumulative probability distribution function

$F_\Psi(\psi)$, q is the number of random pulses arriving at random times T_k and $\delta(\cdot)$ is the Dirac delta function. The arrival times T_k are defined by

$$T_k = \sum_{j=1}^{k} \Theta_j \tag{3.3}$$

where Θ_j, $j = 1, \ldots, k$, are the waiting times between consecutive pulses. The waiting times Θ_j are assumed to be independent and identically distributed random variables with cumulative probability distribution function $F_\Theta(\theta)$.

In Eq. (3.1) the functions $\alpha_j(\cdot)$ and $\gamma_j(\cdot)$ describe, respectively, the state of the structure and the influence of external random actions $\Psi(t)$. Particular cases of Eq. (3.1) are random fatigue crack growth equations

$$\mathrm{d}A/\mathrm{d}t = c\, g(A)\big(\Psi(t)\big)^m, \tag{3.4}$$

whereby A denotes the crack length and c, m are material parameters [33], or the equation of motion of randomly excited oscillators

$$\mathrm{d}X_j/\mathrm{d}t = a_j(\mathbf{X}, t) + b_j(\mathbf{X}, t)\Psi(t), \tag{3.5}$$

with $a_j(\cdot)$ and $b_j(\cdot)$ as drift and diffusion terms, respectively [3].

For convenience, the random variables Ψ_k and Θ_j are represented in terms of standard normal random variables ζ_i. In fairly general situations this can be achieved utilizing the elementary transformation (with $i = 1, 2, \ldots, q+k$)

$$\zeta_i = \Phi^{-1}\big(F_\Psi(\Psi_i)\big) \qquad \text{and} \qquad \zeta_{q+i} = \Phi^{-1}\big(F_\Theta(\theta_i)\big), \tag{3.6}$$

respectively, whereby $\Phi(\cdot)$ denotes the standard normal integral. Having represented the excitation process again by mutually independent normal random variates Eqs. (2.19) and (2.20) can be invoked to determine the design point excitations. And, consequently, Eq. (2.26) can be utilized for importance sampling – as will be shown in the following examples among others.

4. Numerical examples

4.1. Linear oscillator

The principle of the above outlined importance sampling procedure is demonstrated firstly for a linear oscillator, described by the non-dimensional equation of motion

$$\ddot{X}(t) + 2\eta\dot{X}(t) + X(t) = \sqrt{4\eta}\,\xi(t), \qquad X(0) = \dot{X}(0) = 0, \tag{4.1}$$

whereby $\xi(t)$ is a zero-mean Gaussian white noise with $\mathrm{E}[\xi(t)\xi(t+\tau)] = \delta(\tau)$ and $\eta = 0.05$ is a viscous damping coefficient. In the following we want to estimate the probability that $X(t)$ crosses up the level $x_c = 4$ for the first time in the interval $(0 \leq t \leq T = 50)$, i.e.

$$v(0, \mathbf{x}) = \mathrm{E}[\mathrm{I}(\max_{0 \leq t \leq T} X(t) > x_c)]. \tag{4.2}$$

For estimating $v(0, \mathbf{x})$, Eq. (4.1) is written in form of the Itô stochastic differential equations

$$\begin{aligned} \mathrm{d}X_1(t) &= X_2 \mathrm{d}t, \\ \mathrm{d}X_2(t) &= (-2\eta X_2 - X_1)\mathrm{d}t + \sqrt{4\eta}\,\mathrm{d}W(t). \end{aligned} \tag{4.3}$$

Utilizing an equidistant time discretization with step size $\Delta t = T/m$, the k-th increment ΔW_k (with $t_k = k\Delta t$) of the Wiener process is replaced by

$$\Delta W_k = \sqrt{\Delta t}\,\zeta_k, \qquad (k = 0, 1, \ldots, m-1), \tag{4.4}$$

whereby ζ_k are standard normal random variates. Therewith, the response of the oscillator is given as

$$X(t) = \sum_{k=0}^{m-1} \sqrt{4\eta\Delta t}\,\zeta_k\, h(t - k\Delta t), \qquad 0 \leq t < m\Delta t, \tag{4.5}$$

with $h(\cdot)$ denoting the unit-impulse response function

$$h(t) = \frac{1}{\omega} \exp(-\eta t) \sin(\omega t), \qquad \omega = \sqrt{1 - \eta^2}. \tag{4.6}$$

For determining the control force $\bar{u}(t; t_l)$ for up-crossing the level x_c at time $t_l = l\Delta t$, the quantities $\bar{\zeta}_k(t_l)$ in

$$\bar{u}(t; t_l) = \sum_{k=0}^{m-1} \frac{1}{\sqrt{\Delta t}}\, \bar{\zeta}_k(t_l) \delta(t - t_k) \tag{4.7}$$

have to be chosen now in such a way that, on the one hand, $X(t)$ reaches the level x_c at time t_l, i.e.

$$x_c - \sum_{k=0}^{m-1} \sqrt{4\eta\Delta t}\,\bar{\zeta}_k(t_l) h(t_l - t_k) = 0 \tag{4.8}$$

and, on the other hand, the β-index

$$\beta(t_l) = \left[\sum_{k=0}^{m-1} \left(\bar{\zeta}_k(t_l)\right)^2 \right]^{-1/2} \tag{4.9}$$

becomes minimal. As is well known from FORM, the solution to Eqs. (4.8) and (4.9), respectively, are

$$\bar{\zeta}_k(t_l) = \frac{(\beta(t_l))^2}{x_c}\sqrt{4\eta\Delta t}\, h(t_l - t_k) \tag{4.10}$$

and

$$\beta(t_l) = x_c\left[\sum_{k=0}^{m-1} 4\eta\Delta t\, h^2(t_l - t_k)\right]^{-1/2} = \frac{x_c}{\sigma(t_l)}, \tag{4.11}$$

with $\sigma(t_l)$ denoting the standard deviation of the response $X(t)$ at time t_l. Therewith, the control is given (in its continuous form) as

$$\bar{u}(t;t_l) = \frac{\sqrt{4\eta}\, x_c}{\sigma^2(t_l)}\, h(t_l - t), \qquad 0 \le t < t_l. \tag{4.12}$$

In Fig. 1 the control $\bar{u}(t;T)$ according to Eq. (4.12) for an up-crossing of the level $x_c = 4$ at time $T = 50$ is displayed. In Fig. 2 a sample path of $X(t)$ is compared with a sample path of $\tilde{X}(t)$ utilizing this control. Only small differences in the trajectories can be observed until time $t = 30$. From there on, however, the control $\bar{u}(t;T)$ becomes dominant and excites the oscillator clearly in its resonant frequency such that $\tilde{x}(t)$ crosses up the threshold x_c when approaching the end of the time interval. Moreover, by comparing the values of the control $\bar{u}(t;T)$ for two times $t_1 = 1.2$ and $t_2 = 1.7$ in Fig. 3,

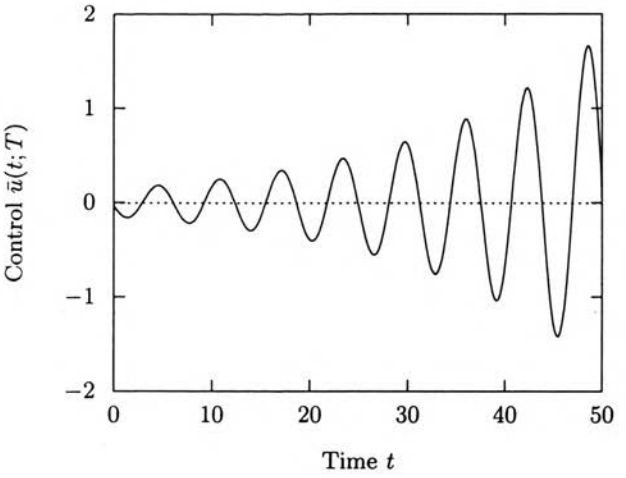

FIGURE 1. Control $\bar{u}(t,T)$ of the linear oscillator for an up-crossing of the level $x_c = 4$ at time $T = 50$.

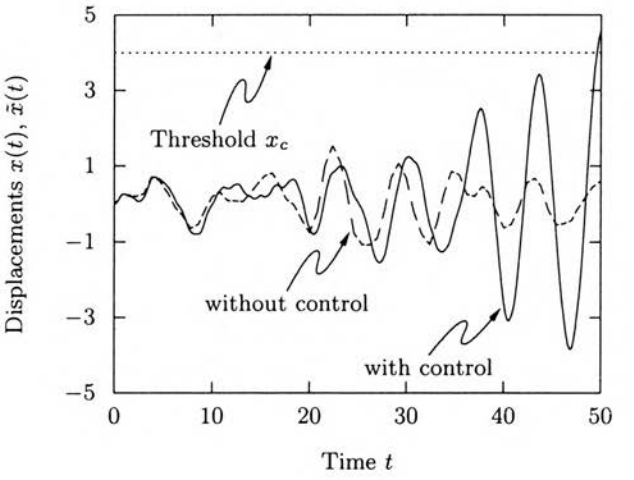

FIGURE 2. Sample paths of linear oscillator without and with control $\bar{u}(t;T)$ from Fig. 1 ($T = 50$, $x_c = 4$).

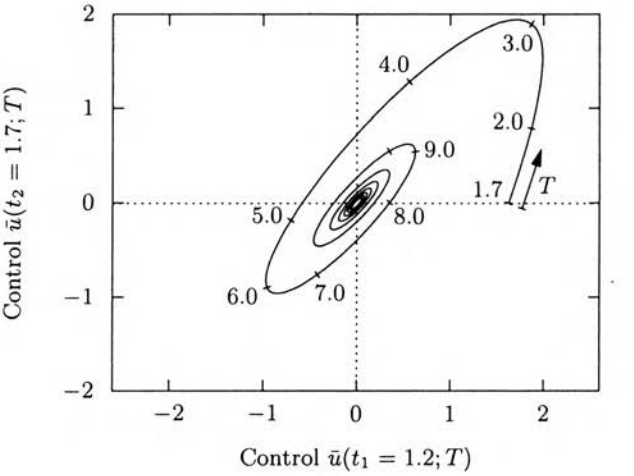

FIGURE 3. Controls $\bar{u}(t_1;T)$ and $\bar{u}(t_2;T)$ with $t_1 = 1.2$ and $t_2 = 1.7$, respectively, for different up-crossing times T of the linear oscillator ($x_c = 4.0$).

it becomes evident from the elliptically inward spiraling form that, on the one hand, there exists a considerable dependency between the controls and, on the other hand, there is a manifest periodicity of the contributions of the controls in time, which diminishes with increasing distance from the chosen up-crossing time T.

In Fig. 4 the importance sampling estimators ($n = 10^3$) of the up-crossing probability of the threshold $x_c = 4$ are compared, respectively, with the results from crude Monte Carlo simulation (solid line, $n = 10^6$) and an approximate solution from linear random vibration theory (dashed line). As can be seen, there is an excellent agreement between the estimators from importance sampling and crude Monte Carlo simulation for up-crossing probabilities greater than $1.0 \cdot 10^{-5}$. Beyond this value, however, crude Monte Carlo simulation breaks down – despite the enormous sample size. The importance sampling procedure, on the other hand, is capable to provide estimates for any absolute value of the first-passage probability. Moreover, as indicated in Fig. 4 by the 99 % confidence intervals (cf. [7]), in case of the importance sampling estimators there is only a small statistical error present which is indeed independent of the estimated absolute value – as should be expected from a variance reduction technique.

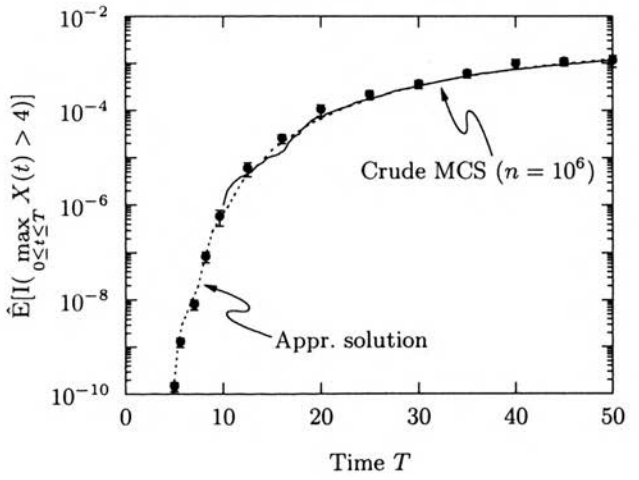

FIGURE 4. Importance sampling estimators (•) and 99 % confidence intervals (I) of the first-passage probability of the linear oscillator ($n = 10^3$, $x_c = 4$).

4.2. Linear oscillator under exponentially distributed random noise

As next example we investigate again the linear single-degree-of-freedom oscillator, however, it is now subjected to exponentially distributed random noise, i.e. its dimensionless equation of motion is

$$\ddot{X}(t) + 2\eta\dot{X}(t) + X(t) = \psi(t), \qquad X(0) = \dot{X}(0) = 0, \tag{4.13}$$

where $\eta = 0.05$ is a viscous damping coefficient and $\psi(t)$ is the forcing term assumed to be a random pulse train with an exponential probability density function with zero mean, i.e.

$$F(\psi) = 1 - \exp\left[-\lambda^{-1}(\psi + \lambda)\right], \qquad \psi \geq -\lambda. \tag{4.14}$$

Assuming equidistant waiting times $\Delta t = 0.5$ between consecutive pulses, we want to estimate the probability that the response $X(t)$ reaches or exceeds the limit $x_c = 15$ in the time interval $0 \leq t \leq T = 50$. In Fig. 5 the

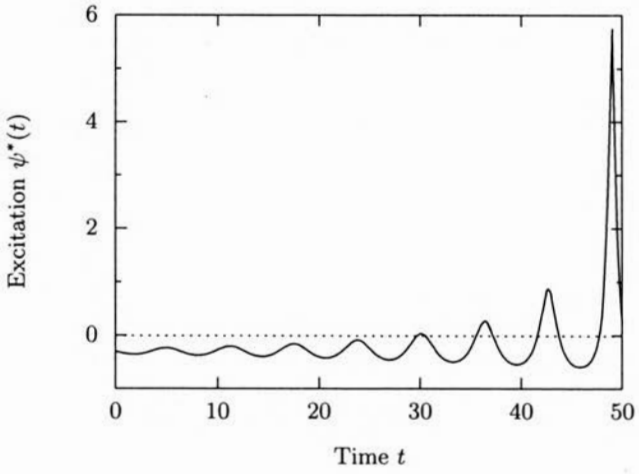

FIGURE 5. Design point excitation $\psi^*(t)$ for $x_c = 15$ and $t = T$.

determined design point excitation $\psi^*(t)$ for an up-crossing of the level x_c at time $T = 50$ is depicted. As can be seen, the design point excitation shows an oscillatory as well as non-stationary behavior. In fact, utilizing the design point excitation the oscillator gets excited in its resonant frequency such that $X(t)$ up-crosses the level x_c at the end of the time interval – as can be seen from the corresponding response $x^*(t)$ shown in Fig. 6.

In Fig. 7 the estimated first-passage probabilities utilizing the importance sampling method ($n = 3 \cdot 10^3$) and the crude Monte Carlo simulation approach (solid line, $n = 10^7$) are compared for different levels x_c. There is again an excellent agreement of the results of the two approaches. This can also seen by the small statistical error as indicated by the 99 % confidence intervals, which demonstrates the general – i.e. independent of the size of the estimated quantity – applicability of the proposed importance sampling concept.

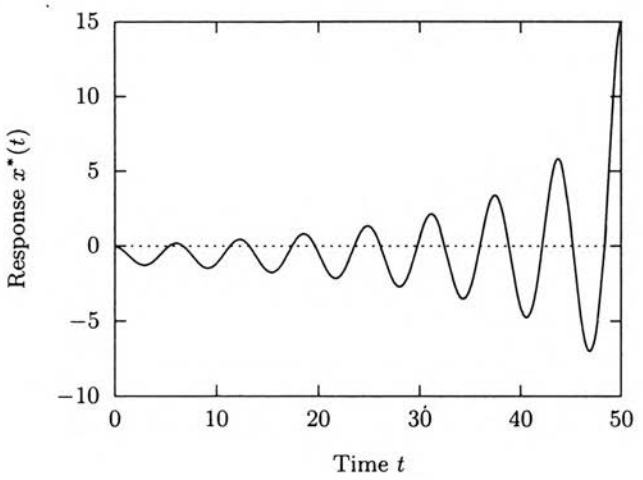

FIGURE 6. Response of linear oscillator due to design point excitation $\psi^*(t)$ from Fig. 5 ($T = 50$, $x_c = 15$).

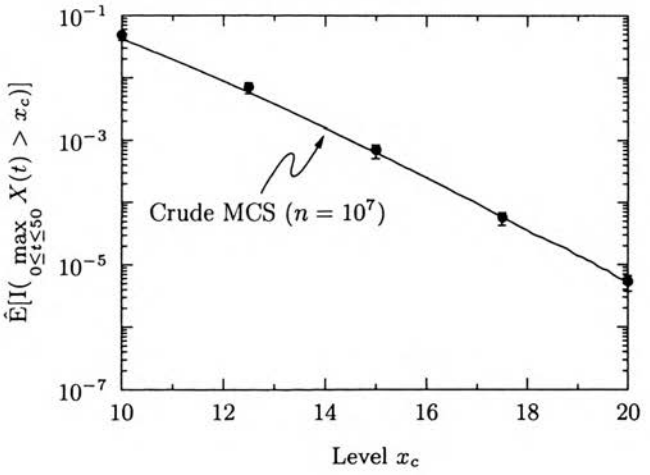

FIGURE 7. Importance sampling estimators (•) and 99 % confidence intervals (I) of the first-passage probability of the linear oscillator under exponentially distributed random noise($n = 3 \cdot 10^3$, $T = 50$).

4.3. Duffing oscillator

Given is a Duffing oscillator under external white noise excitation $\xi(t)$ with $E[\xi(t)\xi(t+\tau)] = \delta(\tau)$, described by the dimensionless equation of motion

$$\ddot{X} + 2\eta\dot{X} + X + \varepsilon X^3 = \sqrt{4\eta}\,\xi(t), \qquad X(0) = \dot{X}(0) = 0, \tag{4.15}$$

whereby $\eta = 0.05$ denotes a viscous damping coefficient and ε is the degree of non-linearity. In Figs. 8 and 9, respectively, the control for an up-crossing of the level $x_c = 2$ at time $T = 50$ and the influence of this control on the deflection $X(t)$ of the Duffing oscillator are depicted for a degree of non-linearity $\varepsilon = 1$. As can be seen, the utilized control $\bar{u}(t;T)$ results in a strong

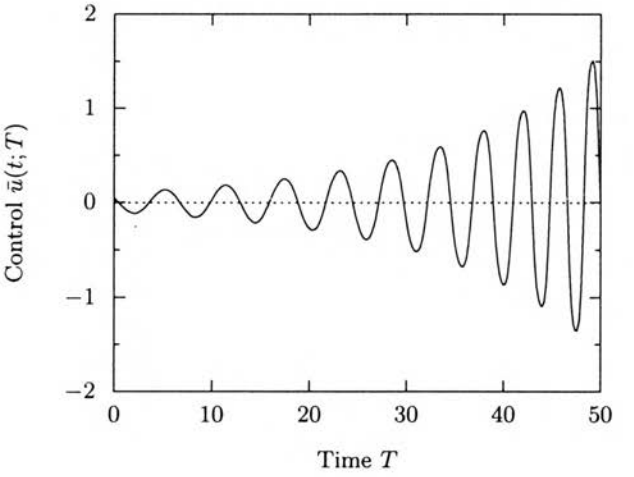

FIGURE 8. Control $\bar{u}(t;T)$ of Duffing oscillator for an up-crossing of level $x_c = 2$ at time $T = 50$ ($\varepsilon = 1$).

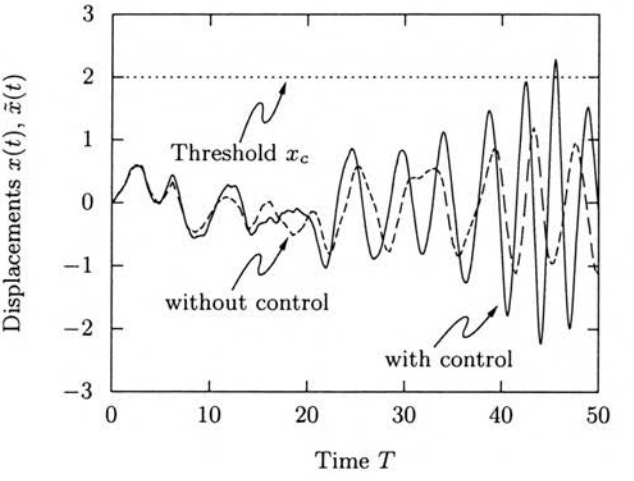

FIGURE 9. Sample paths of Duffing oscillator without and with control $\bar{u}(t;T)$ from Fig. 8 ($T = 50$, $x_c = 2$, $\varepsilon = 1$).

amplification of the response of the Duffing oscillator such that it crosses up the threshold $x_c = 2$ near the specified up-crossing time T.

In Fig. 10 the importance sampling estimators ($n = 10^3$) of the first up-crossing probability are compared with the results from crude Monte Carlo simulation (solid line, $n = 10^6$) for the degree of non-linearity $\varepsilon = 2$. As can be seen, there is a very good agreement between the two different estimators. Moreover, and maybe even more worthwhile to be pointed out, almost independent of the absolute value of the estimated first-passage probability, there exists a sufficient confidence in the importance sampling estimators as indicated by the 99 % confidence intervals.

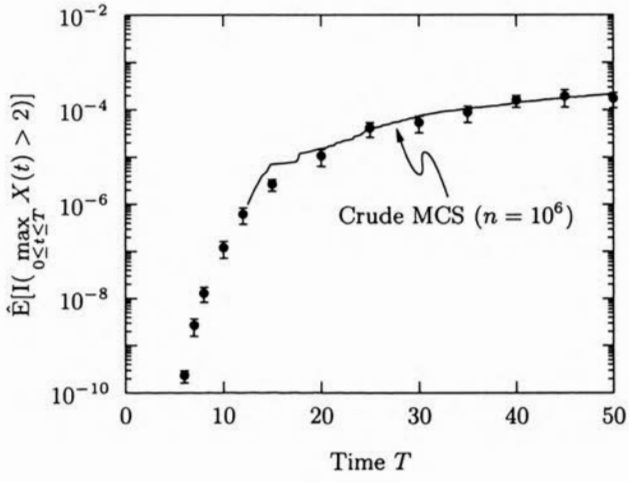

FIGURE 10. Importance sampling estimators (•) and 99 % confidence intervals (I) of first-passage probability of the Duffing oscillator ($n = 10^3$, $\varepsilon = 2$, $x_c = 2$).

4.4. Oscillator with hysteretic restoring force

Given is a hysteretic oscillator described by the non-dimensional equation of motion [34]

$$\begin{aligned} &\ddot{X} + 2\eta\dot{X} + \alpha X + (1-\alpha)Z = \sqrt{2\pi S_0}\,\xi(t), && X(0) = \dot{X}(0) = 0, \\ &\dot{Z} = -\gamma|\dot{X}|Z|Z|^{\nu-1} - \beta\dot{X}|Z|^{\nu} + A\dot{X}, && Z(0) = 0. \end{aligned} \tag{4.16}$$

whereby η denotes a viscous damping coefficient, α is the the post- to pre-yielding stiffness ratio, $\xi(t)$ is a Gaussian white noise with $E[\xi(t)\xi(t+\tau)] = \delta(\tau)$, and ν, γ, β and A are adjustable parameters to describe the hysteretic behavior. (In the following we choose $\eta = 0.05$, $S_0 = (16\pi)^{-1/2}$, $\nu = 1$, $\gamma = \beta = 0.5$, $\alpha = 0.05$ and $A = 1$, respectively.)

In Figs. 11 and 12 the control $\bar{u}(t;T)$ for an up-crossing of the level $x_c = 15$ at time $T = 50$ and its influence on the restoring force-deformation relation is shown, respectively. As can be seen, the control results mainly in permanent deformations in the direction of the threshold x_c, with the major

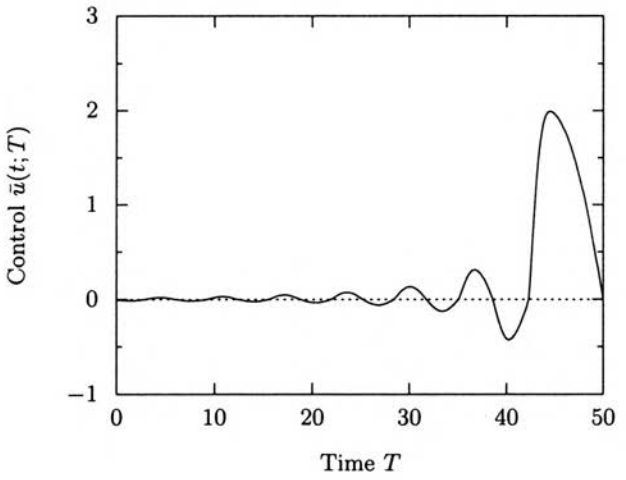

FIGURE 11. Control $\bar{u}(t;T)$ of hysteretic oscillator for an up-crossing of the level $x_c = 15$ at time $T = 50$.

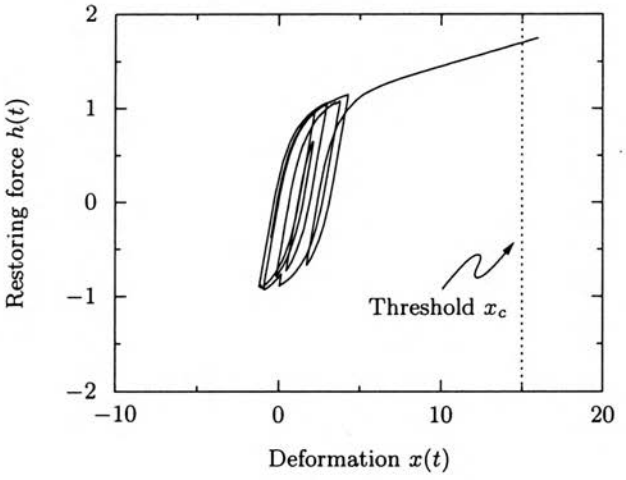

FIGURE 12. Sample path of the restoring force-deformation relation for the hysteretic oscillator when utilizing the control $\bar{u}(t;T)$ from Fig. 11 for an up-crossing of the level $x_c = 15$ at time $T = 50$.

contribution from the last half-period of vibration. It should also be noted that the control for the hysteretic oscillator shows thereby a substantially different behavior as e.g. the control for the Duffing oscillator, reflecting the likewise different nature of the respective dynamic systems.

In Fig. 13 the importance sampling estimators ($n = 10^3$) are compared with the crude Monte Carlo simulation results (solid line, $n = 10^6$). Again, there is an excellent agreement between the estimators for the strongly nonlinear behavior when utilizing the stiffness ratio $\alpha = 0.05$ ($\alpha = 1$ describes a linear system), which is also implicitly depicted by the small 99 % confidence intervals (see Fig. 13).

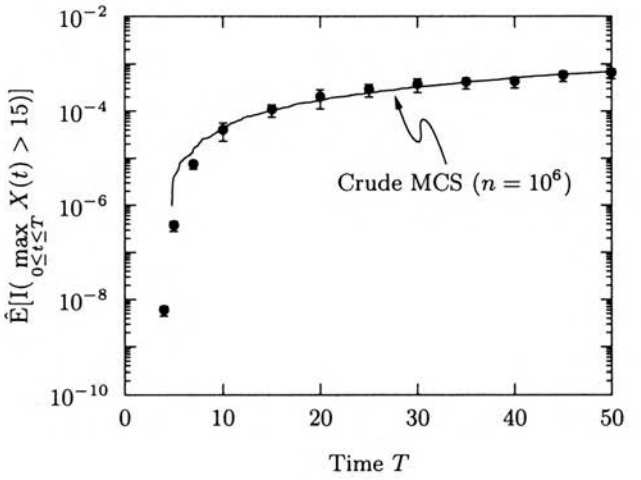

FIGURE 13. Importance sampling estimators (•) and 99 % confidence intervals (I) of first-passage probability of the hysteretic oscillator ($n = 10^3$, $x_c = 15$).

4.5. Compound Poisson process

As next example a so called compound Poisson process $X(t)$ is investigated, i.e.

$$X(t) = \sum_{k=1}^{q} \Psi_k H(t - t_k), \qquad X(0) = 0 \tag{4.17}$$

where $H(\cdot)$ denotes the Heaviside step function with $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ otherwise. For the ease of presentation, we assume that the waiting times Δt and, therewith, also the number q of random pulses are deterministic. The random pulses Ψ_k are exponentially distributed with cumulative distribution function $F(\psi) = 1 - \exp(-\psi/\lambda)$. The probability that

the process $X(t)$ up-crosses the level x_c in the time interval $0 \leq t \leq q\Delta t = T$ is given by

$$P(\max_{0\leq t\leq T} X(t) > x_c) = \exp(-\psi/\lambda) \sum_{k=0}^{q-1} \frac{\psi^k}{\lambda^k k!} \tag{4.18}$$

In the following we choose $q = 50$ and $\lambda = 5$.

TABLE 1. First-passage probability of compound Poisson process for the time interval $0 \leq t \leq T = 50$ ($\lambda = 5$).

Level x_c	Exact solution	Importance sampling method	
		$\hat{\mathrm{E}}[\mathrm{I}(\max_{0\leq t\leq 50} X(t) > x_c)]$	Mean square error
250	$4.8 \cdot 10^{-1}$	$4.3 \cdot 10^{-1}$	8.6 %
300	$8.4 \cdot 10^{-2}$	$7.8 \cdot 10^{-2}$	19.2 %
350	$5.1 \cdot 10^{-3}$	$3.8 \cdot 10^{-3}$	25.0 %
400	$1.3 \cdot 10^{-4}$	$1.4 \cdot 10^{-4}$	34.3 %
450	$1.6 \cdot 10^{-6}$	$1.8 \cdot 10^{-6}$	41.7 %
500	$1.2 \cdot 10^{-8}$	$8.1 \cdot 10^{-9}$	33.3 %

Note: Number of samples for importance sampling is $n = 3 \cdot 10^3$.

In Tab. 1 the importance sampling estimators of the first passage probability utilizing $n = 3 \cdot 10^3$ sample functions are compared with the exact results for different up-crossing levels x_c. As can be seen, the importance sampling results show a good agreement with the exact results, although there is still a considerable statistical error present. The reason therefore can be seen in the fact, that the first order reliability method provides a relatively poor approximation of the first passage probability in the present case. This can be easily visualized by considering the sum of two exponentially distributed random variables Ψ_1 and Ψ_2, i.e.,

$$Z = \Psi_1 + \Psi_2. \tag{4.19}$$

Although this function is linear in original space, there is a quite significant non-linearity after transformation into standard normal space by means of

$$\zeta_k = \Phi^{-1}\big(1 - \exp[-\psi_k/\lambda]\big). \tag{4.20}$$

This is seen from Fig. 14. Moreover, there is not one clearly distinguishable most likely combination of u_1 and u_2 leading to the value of $z = 10$. This causes problems when determining the importance sampling density based on the design point as determined from the first order reliability method.

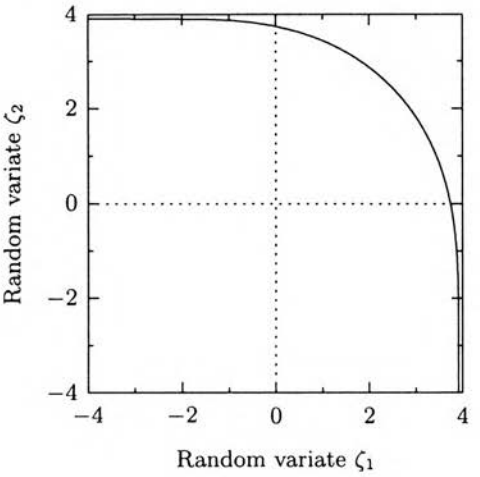

FIGURE 14. Contour of $z = \psi_1 + \psi_2 = 10$ in standard normal space.

4.6. Fatigue crack growth

As a last example a massless, homogeneous, isotropic plate of width $w = 200$ with a central crack of length $2a$ under uniform tension loading $\Psi(t)$ defined as

$$\Psi(t) = \sum_{k=1}^{q} \Psi_k \delta(t - t_k) \tag{4.21}$$

is investigated. The dimensionless crack growth law is given as

$$\mathrm{d}A/\mathrm{d}t = c\left(\pi A \sec(\pi A/w)\right)^{m/2}\left(\Psi(t)\right)^m, \qquad A(0) = a_0 \tag{4.22}$$

where $a_0 = 3$ is the initial crack length. The material parameters c and m governing the crack growth are chosen as $c = 1$ and $m = 3$, respectively. The random pulses Ψ_k are exponentially distributed with cumulative distribution function $F(\psi) = 1 - \exp(-\psi/\lambda)$ (with $\lambda = 0.0275$).

The original nonlinear Eq. (4.22) can be conveniently transformed into a linear equation by introducing a new variable z [35]

$$z = \int_{a_0}^{a} \frac{\mathrm{d}a}{f(a)} \tag{4.23}$$

leading to the differential equation

$$\mathrm{d}z = \frac{\mathrm{d}a}{f(a)} = \left(\Psi(t)\right)^m \mathrm{d}t \tag{4.24}$$

which is easily integrated. The relation between the physical crack length a and the transformed crack length z is depicted in Fig. 15.

The first-passage probability that the crack length $A(t)$ exceeds the critical crack length $a_c = 100$ (corresponding to $z_c = 0.15391$) is calculated by importance sampling. Applying first order reliability analysis reveals that the

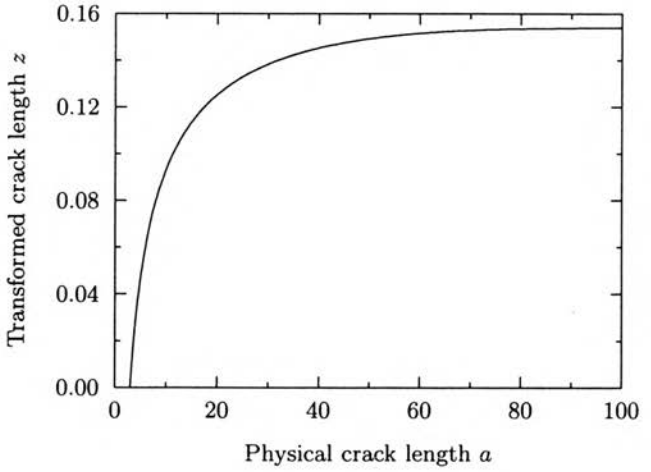

FIGURE 15. Transformation from physical crack length a to transformed crack length z.

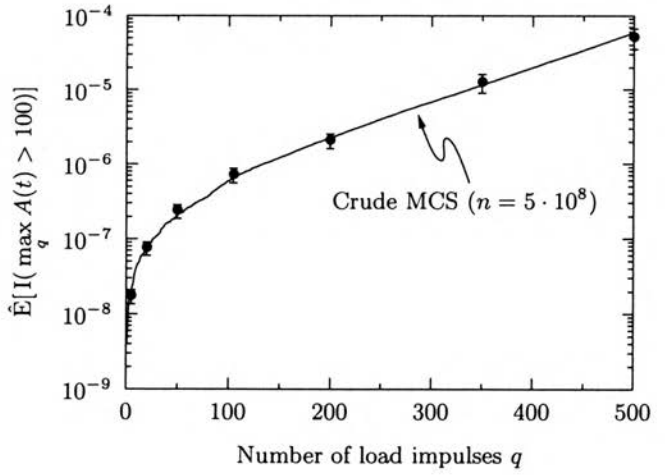

FIGURE 16. Importance sampling estimators (•) and 99 % confidence intervals (I) of the first-passage probability of crack propagation ($n = 3 \cdot 10^3$, $a_c = 100$)

design points have only one significantly non-zero coordinate. This indicates that only one random load at a time is most likely to cause failure. In Fig. 16 the estimated first-passage probabilities are shown for the importance sampling method ($n = 3 \cdot 10^3$) and the crude Monte Carlo simulation approach (solid line, $n = 5 \cdot 10^8$). As can be seen, there is an excellent agreement between the two approaches. Inspecting the 99% confidence intervals shows that the efficiency of the importance sampling method is independent of the level of the first passage probability determined – as should be expected – for a variance reduction technique.

5. Concluding remarks

A generally applicable importance sampling procedure for linear and nonlinear dynamical systems under random excitations has been presented. The procedure allows – at least theoretically – to construct optimal, i.e. unbiased zero-variance estimators of the system response. Nevertheless, by utilizing sub-optimal control forces constructed via the solution of an optimization problem analogous to the one from first order reliability method, the variance of the estimators can be decreased drastically as compared to crude Monte Carlo simulation. This is achieved by spending additional computational effort in the determination of a set of sub-optimal controls (or "design point excitations") which are utilized as importance sampling mean excitations. While the effort required for the computation of these controls is not negligible, it is primarily dominated by the number of random variables used for the time-discretization of the random excitation process. This is due to the numerical gradient calculations required for the solution of the optimization problem. In this form, it is acceptable for a sufficiently small number of random variables, i.e. in significantly non-stationary or transient situations. The number of analyses required is not influenced by the number of degrees of freedom of the system. On the other hand, in the stationary situation the determination of the design point excitations turns out to be trivial for some cases. In these cases only one nonlinear dynamic analysis needs to be carried out as shown in [36]. This means that for a wide range of problems the suggested approach can be considered to be extremely efficient. The advantages become especially clear when considering the system response in the low-probability regions (distribution tails) which are not accessible to crude Monte Carlo simulation. Finally, it should be pointed out that due to the application of standard concepts from FORM it is fairly easy to incorporate randomness of system parameters (described e.g. in terms of random fields) into the analysis (see e.g. [28]).

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