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A CONCEPTUAL GUIDE TO CHAOS THEORY

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1 Preface

Recently the theory of chaos has proved to be one of the most rapidly developing fields of research in physics, mathematics, engineering, economy and many other areas. Therefore, there is a real need to present in an understandable way the basic concepts of this theory so that a broad audience of non-specialists can access to this theoretical tool. The aim of this guide is precisely to satisfy this demand. The way to achieve this scope will be, as far as possible, to introduce the concepts rigorously and then to make them clear by means of a non-technical language and appropriate examples. The definitions and theorems will be not formulated in their most general form but rather with a level of abstraction accordant to our needs.

As can be seen in the table of contents, this guide is divided in sections, each section dealing with a major concept. We have preferred to present the theory of chaos within the more general theory of dynamical systems. Although this is not strictly necessary, we feel that the dynamical systems offer the adequate conceptual framework to study chaos. Dynamical systems can be discrete (defined by finite-difference equations) or continuous (defined by differential equations). Conceptually, the discrete dynamical systems are easier than the continuous ones, although most important concepts are common and can be readily reformulated. For simplicity, we use the language of the discrete dynamical systems in all sections except in the last one, which is dedicated to the continuous dynamical systems. Those concepts and results which are specific of continuous dynamics will be treated separately in the last section.

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2 Introduction

The word *chaos* was introduced in the mathematical literature by Li and Yorke in their article [3]. At present, several definitions of chaos are used. The most popular one is connected with the concept of sensitivity or, more precisely, *sensitivity to initial conditions*. This means that initial perturbations are exponentially amplified (in first order) from some time point on, so that all solutions of the evolution equations starting at arbitrarily close initial values eventually diverge. >From a practical point of view, the sensitivity of chaotic systems implies unpredictability in the middle or long term because any inaccuracy in the determination of the initial conditions (i.e. experimental error), however small it might be, will become important as time goes on, rendering the prediction wrong. Since the precision of any measurement is a fortiori limited, it follows that chaotic systems are only predictable in the short term. This conclusion became popular under the name *butterfly effect*.

Therefore, chaos is not in science tantamount of randomness, as it might be thought, but rather of practical unpredictability. The fact that the solution of a system of differential or finite-difference equations exhibits an irregular, erratic, unpredictable or "chaotic" behaviour is not necessary the consequence of the complexity of the equations but, in general, it follows from the non-linearity of the equations, even if their number is small. Therefore, a "chaotic" system, far from being random, is *deterministic*, i.e. its entire evolution is completely determined by mathematical equations. To emphasize this distinction, sometimes one speaks of *deterministic chaos*.

As a matter of fact, deterministic chaos is abundant in nature. For example, chaos can unfold in mechanical devices (double-pendulum, driven oscillators,...), fluid flows (turbulence, cavitation,...), electronic circuits (van der Pohl, Chua,...), chemical reactions (Belousov-Zhabotinsky), just to mention the most traditional fields. But they were not recognized as manifestations of the same principle until the new scientific paradigm of chaos was formulated in the 70's. What before was dismissed as "noise" or "malfunction" and considered a nuisance, appeared now in a new light, turning out to be something familiar. Terms such as bifurcation, period-doubling, intermittency or attractor, which were coined by mathematicians to grasp the subtleties of chaos, began to be uttered in many different scenarios, showing the universality of the new paradigm and the far-reaching power of it's methods.

One important aspect of chaos theory for the practitioner refers to its detection, measurement and control. In fact, it is not necessary at all to know the dynamic (i.e. evolution equation) generating an experimental *time series* to decide whether the observed data are deterministic and, in affirmative case, characterize the "chaoticity" of such a black-box system by means of parameters which can be extracted from its time series alone (Lyapunov exponents, attractor dimension,...). This feature of chaotic systems has allowed to apply chaos theory in a manifold of new research fields where the complexity of the phenomena studied precludes more traditional approaches. To mention a few examples: medicine (heart fibrillation, cardiac arrest, brain activity, epilepsy,...), biology (neuronal firing, prebiotic evolution, population dynamics,...), economy (stock market evolution, economic crisis,...), sociology (social revolutions,...). This being the case, it should not come as a surprise that the theory of chaos and, in particular, the non-linear analysis of time series have become in a short time very popular among scientists and applied mathematicians.

3 Historical Comments

As their name reveals, dynamical systems stem from the classical mechanics of many mass points or, more specifically, from the study of the perfect gas in Statistical Mechanics. Consider N material points moving in a closed volume under known forces. Suppose that the state of this system is determined by the positions $q_{i,0}$ and linear momenta $p_{i,0}$ of each particle at some initial time $t = t_0$. As time goes on, the system evolves according to the Hamilton equations

$$\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, ..., 3N \\ q_i(t_0) &= q_{i,0}, \quad p_i(t_0) = p_{i,0} \end{aligned} \tag{1}$$

where, for simplicity, we suppose that the *Hamiltonian function* $H = H(p_i, q_i)$ does not depend explicitly on time. In this case, H is equal to the total energy of the system (which includes, for example, the kinetic and potential energies) and, therefore, it is a constant of motion,

$$H(q_i, p_i) = E$$

Mechanical systems for which the total energy is constant (or "conserved") are called *conservative systems*.

If, given the initial conditions $q_i(t_0)$, $p_i(t_0)$, $i = 1, ..., 3N$, the Hamilton equations (1) can uniquely be solved for $t \in [t_0, t_1]$, the solution is a curve (with time as parameter) in a bounded set $\Gamma \subset \mathbb{R}^{6N}$ (called *phase* or *state space*) which represents the history of the dynamical system.

Let x be the point in the phase space Γ representing the system considered at time t_0 and define $T_t : \Gamma \to \Gamma$ by setting $T_t(x) := x(t_0 + t)$, $t_0 \le t \le t_1$. The transformation T_t is called the *flow* generated by the differential equations (1).

It fulfills the following two properties:

$$(i)\ T_0 = id \text{ (identity)}, \quad (ii)\ T_{t+s} = T_t \circ T_s = T_s \circ T_t = T_{s+t} \tag{2}$$

In classical mechanics, all physical observables of a system (such as energy, linear momentum, angular momentum, etc.) are smooth functions defined in the phase space. Statistical mechanics, on the other hand, focus on the asymptotic behaviour of those observables,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(T_t x)dt \tag{3}$$

to explain the macroscopic properties of many-particle systems in quasi-steady or steady (equilibrium, stationary,...) states. With other words, given a many-particle system, the scope of tatistical mechanics is to show that the limit (3) (also called "*average over time*") exists and, then, calculate it. The definition of the average over time requires, in principle, that T_t exists for all $t \geq t_0$, or, in practice, for t sufficiently large ($t_1 \gg t_0$). Observe further that alone the calculation of the integrand in (3) requires solving a system of $6N$ differential equations of first order (the Hamilton equations), where N is typically of the order 10^{23}.

When L. Boltzmann studied the rarefied perfect gas at the end of the XIX century trying to derived its macroscopic thermodynamic laws from the Newtonian laws of its microscopic motion, he assumed that the motion of the gas molecules is random and that, as $T \to \infty$, each $x \in \Gamma$ reaches any energetically accessible region of Γ (*ergodic hypothesis*), so that the time- and phase-averages coincide:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(T_t x)dt = \int_\Gamma f(x)d\mu(x)$$

Here μ is the Lebesgue measure of $\mathbb{R}^{6N}$ restricted to the "energy shell" $H^{-1}(E)$ (where E is the total energy of the system) and normalized to 1 (i.e., $\mu(H^{-1}(E)) = 1$). This measure, known in mechanics as the *Liouville measure*, has an important property. Fix $s \in \mathbb{R}$ and let F be a measurable set of $H^{-1}(E)$. Then, denoting as usual by χ_A the characteristic function of the measurable set A (i.e. $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$), it holds

$$\frac{1}{T}\int_0^T \chi_{T_s^{-1}F}(T_t x)dt = \frac{1}{T}\int_0^T \chi_F(T_s T_t x)dt = \frac{1}{T}\int_0^T \chi_F(T_t T_s x)dt$$

where the property (ii) of (2) was used. Therefore, if the thermodynamic limit (with $f = \chi_{T_s^{-1}F}$) exists, it follows

$$\int_\Gamma \chi_{T_s^{-1}F}(x)d\mu(x) = \int_\Gamma \chi_F(T_s x)d\mu(T_s x)$$

so that

$$\mu(T_s^{-1}F) = \mu(F)$$

This means that the Liouville measure is invariant under the action of the transformation T_s (*Liouville's theorem*). The physical interpretation is as follows: any volume in the phase space of a conservative system flows as an incompressible fluid.

Few years after L. Boltzmann had founded statistical mechanics, the French mathematician H. Poincaré started his epoch-making investigations on the three-body problem. In *Les méthodes nouvelles de la mécanique céleste [5]*, Poincaré not only proved that no solution of the type envisioned by Jacobi or Hamilton could exist, but he also laid the foundations for several new branches of mathematics including topology (his *analisis situs*), ergodic theory, homology theory and the qualitative theory of differential equations. Poincaré was probably also the first scientist who understood deterministic chaos in modern sense. In Volume 3 of [5], which is the foundational work of modern dynamical systems theory, he describes his discovery of homoclinic solutions, he writes about sensitivity to initial conditions and about periodic and quasi-periodic trajectories, he uses the surface of section which nowadays is named after him, etc. The depth of Poincaré work was such that, except in topology (which was intensively developed from the very beginning of the XX century), it took several decades to get further major break-throughs in classical mechanics (Kolmogorov 1954, Arnold 1963, Moser 1967), ergodic theory (Birkhoff 1931), homology (Čech 1932, Eilenberg ans Steenrod 1952), qualitative theory of differential equations (Smale 1960) and finally, chaos theory (Lorenz 1963).

In particular, the pioneering results of Poincaré on deterministic chaos remained a kind of scientific curiosity only possible in complex systems (the three-body system is determined by eighteen non-linear differential equations of first order) till the American meteorologist E.N. Lorenz published in 1963 his famous paper on thermal convection in the atmosphere [4] in which he showed that even a simple system of three differential equations of first order in time (one of which is linear) can exhibit sensitivity to initial conditions. So, chaos is not necessarily the consequence of dynamical complexity. Rather, as it turned out, chaos is a consequence of non-linearity. This conclusion was further strengthened by the work of Hénon, Rössler and Feigenbaum, among many others.

The rapid development of this research field in the last two decades is undoubtedly connected with the improvement of the computational capabilities due to the progress in computer technology. Theoretical analysis was supported by the possibility of observation and visualization of strange attractors, calculating Lyapunov exponents, describing and plotting bifurcation diagrams and transition to chaos for many dynamical systems.

4 Dynamical systems and what are they all about

The most general notion of a dynamical system includes three basic ingredients:

1. A "phase space" X, whose elements or "points" x represent possible states of a system.

2. "Time", which may be discrete (n) or continuous (t). It may extend either only into the future (irreversible or noninvertible processes) or into the past as well as the future (reversible or invertible processes). In the first case, $n = 0, 1, 2, ...$ or $t \geq 0$. In the second case, $n = ..., -2, -1, 0, 1, 2, ...$ or $t \in \mathbb{R}$.

3. A time evolution law. This is a rule that allows us to determine the state of the system at each time from its states at previous times. Furthermore, we will assume that the law of time evolution itself does not change with time. In applications, time evolution laws are finite-difference equations if time is discrete and differential equations if time is continuous.

The most characteristic feature of dynamical theories, which distinguishes them from other areas of mathematics, is the emphasis on *asymptotic behaviour*, that is, properties related to the behaviour as time goes to infinity.

Example 1 *Consider the (discrete) time evolution law* $f(x) = 4x(1-x)$. *Input a random number* x_0 *between* 0 *an* 1 *and watch the results of the time evolution*

$$x_0 \mapsto x_1 = f(x_0) \mapsto x_2 = f(x_1) = f^2(x_0) \mapsto ... \mapsto x_n := f(x_{n-1}) = f^n(x_0) \mapsto ...$$

where $f^n(x_0) = f(f(...f(x_0)...))$, $n = 2, 3, ...$, *is the n-th iterate of* f. *One gets completely different behaviours depending on* x_0. *Sometimes, very seldom, the values repeats. Most often, they wander "chaotically" about the unit interval with no discernible pattern. Now change the parameter from* 4 *to* 3.839, *i.e. use the function* $f(x) = 3.839x(1-x)$. *For a random entry* x_0 *between* 0 *and* 1, *one observes that the iterates or "successors" of this point,* ..., x_{n-1}, x_n, x_{n-1}, ..., *eventually settle down to a repeating cycle of three numbers,* 0.149888..., 0.489172..., 0.959299..., *repeated over and over again in succession. In the first case, one speaks of chaos; in the second one, of periodicity. To understand this dramatic difference in the fate of the iterates of* $f_a(x) = ax(1-x)$ *depending on the value of the parameter* a *is one example of the kind of issues investigated by the theory of dynamical systems.*

There are many others. Given a (discrete or continuous) time evolution law or, typically, a parametric family of them, such questions as

- is the evolution of the states "chaotic" or regular?,
- does it converge in some sense?,
- which geometric or topological structure has the limit set?
- in which parametric range is the system chaotic?
- how does chaos set in?

constitute the main concern of the theory of dynamical systems from the point of view of its applications.

For simplicity, we focus in this guide on the discrete dynamical systems. This makes things easier from the "technical" point of view without loss of generality at the conceptual level. We return in the last section to the formal parallelism between both frameworks and discuss the peculiarities of the continuous dynamical systems.

5 Discrete dynamical systems

Consider a rule which determines the value of some quantity after a time step knowing its current value. Usually such a rule is a mathematical function and the quantity is a number. In symbols,

$$x_{n+1} = f(x_n) \quad n = 0, 1, 2, ...$$

Observe that

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) = f(f(x_0)) = (f \circ f)(x_0) = f^2(x_0) \end{aligned}$$

and, in general,

$$x_{n+1} = f(x_n) = f(f(x_{n-1})) = f(f^{n-1}(x_0)) = (f \circ f^{n-1})(x_0) = f^n(x_0)$$

where the superscript means "number of iterations" of f with itself,

$$f^n := f \circ f \circ ... \circ f \quad (n \text{ times})$$

The main objective of the theory of discrete dynamical systems is to find out the *asymptotic* (i.e. long-term) behaviour of the quantity in question,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n(x_0)$$

Therefore, knowing the initial value x_0, one has to apply the rule f to x_0 a sufficiently large number of times to obtain its "time" evolution.

Mathematicians model such kind of situations allowing the function f to act on a certain set X endowed with additional properties such as an n-dimensional volume (length, area,...), the notion of continuity (topology), distance (metric),... Depending on this additional structure, the discrete dynamical systems are classified mainly in three classes:

- set-theoretical dynamical systems
- topological dynamical systems
- metrical dynamical systems

Furthermore, as explained in the historical comments, mathematicians (first Liouville and then Poincaré himself) observed that the function f preserved the "volume" defined in X. This discovery led to the generalization of this notion, which now is called an invariant measure. Since the invariant measure was instrumental in the demonstration of the first main results (such as the Poincaré's recurrence theorem or Birkhoff's ergodic theorem), its existence became an essential constituent of the definition of dynamical systems. Before getting to this definition, we need first to introduce some standard nomenclature.

Definition 1 *Let X be a nonempty set and denote by $\mathcal{P}(X)$ the set of its parts. A family of subsets $\mathcal{A} \subset \mathcal{P}(X)$ is called a σ-algebra on X if*

1. $X \in \mathcal{A}$
2. $A \in \mathcal{A}$ *implies* $X \setminus A \in \mathcal{A}$
3. $A_i \in \mathcal{A}$ *for* $i = 1, 2, \ldots$ *implies* $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$

If $\mathcal{A}$ is a σ-algebra on X, the pair $(X, \mathcal{A})$ is called a *measurable space*.

Given X, there are always two trivial σ-algebras on X, namely, $\{\emptyset, X\}$ and $\mathcal{P}(X)$. In general, the elements of a σ-algebra are not known explicitly but rather only a small part of it called its generator. Given $\mathcal{F} \subset \mathcal{P}(X)$, we denote by $\sigma(\mathcal{F})$ the smallest σ-algebra on X such that $\mathcal{F} \subset \sigma(\mathcal{F})$. In this context, $\mathcal{F}$ is called the *generator* of $\sigma(\mathcal{F})$ or, equivalently, $\sigma(\mathcal{F})$ is said to be generated by $\mathcal{F}$. The existence of $\sigma(\mathcal{F})$ follows from the fact that $\mathcal{P}(X)$ is a σ-algebra on X and that the intersection of σ-algebras is again a σ-algebra so that $\sigma(\mathcal{F}) = \cap\{\mathcal{A} \supset \mathcal{F} : \mathcal{A}$ σ-algebra$\}$.

Definition 2 *Let $(X, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \to \mathbb{R}^+ \cup \{+\infty\}$ be a function in the non-negative extended real numbers. It is said that μ is a (positive) measure on $(X, \mathcal{A})$ if*

1. $\mu(\emptyset) = 0$.
2. *If $\{A_i\}_{i=1}^{\infty}$ is a countable family of pairwise disjoint sets of $\mathcal{A}$, then it holds* $\mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

If μ is a measure on $(X, \mathcal{A})$, then $(X, \mathcal{A}, \mu)$ is called a *measure space*. If, furthermore, $\mu(X) < \infty$, $(X, \mathcal{A}, \mu)$ is called a *finite measure space*. In particular, if $\mu(X) = 1$, μ is called a *probability* and $(X, \mathcal{A}, \mu)$ is also called a *probability space* or a *normalized space*. Any finite measure space can be made a probability space by normalizing the measure: $\mu(\cdot) \mapsto \mu(\cdot)/\mu(X)$.

The subsets N of X such that $\mu(N) = 0$ are called *μ-null sets*. If a property of X is true for all $x \in X \setminus N$ where N is a μ-null set, one says that such property holds *almost everywhere* with respect to the measure μ or that it holds *modulo null sets*. In shorthand: μ-a.e. or $\bmod 0$, respectively. The reference to the measure can be omitted if it clear from the context which one is meant.

Example 2 *The* Lebesgue measure λ *of* $\mathbb{R}$ *is first defined on the intervals,*

$$\lambda([a,b]) = \lambda([a,b[) = \lambda(]a,b]) = \lambda(]a,b[) = b-a$$

and then extended to the σ-algebra generated by the intervals by means of Hahn's extension theorem (there is a technical point related to the subsets of null sets, which we do not go into). The definition of the Lebesgue measure in $\mathbb{R}^n$ is mutatis mutandis the same.

Definition 3 *Let $(X, \mathcal{A}, \mu)$ be a measure space. A map $f : X \to X$ is said to be measurable if $f^{-1}(A) \in \mathcal{A}$ for every $A \in \mathcal{A}$. If, additionally, $\mu(f^{-1}(A)) = \mu(A)$ for every $A \in \mathcal{A}$, then one says that f preserves the measure μ or, equivalently, that μ is f-invariant.*

Eventually, one speaks of measure preserving transformations and invariant measures if it is clear what measure space and transformation, respectively, are referred to.

Definition 4 *Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $f : X \to X$ be a μ-preserving transformation. Then, $(X, \mathcal{A}, \mu, f)$ is called a (set-theoretical) dynamical system.*

In the sequel and without loss of generality, we suppose that $\mu(X) = 1$, i.e. that $(X, \mathcal{A}, \mu)$ is a probability space. Therefore, a dynamical system is defined by a measure-preserving transformation acting on a probability space. The points of X are called the *states* of the dynamical system and, correspondingly, X is called the *state space*.

Definition 5 *Let X be a non-empty set and let $\mathcal{O}$ be a family of subsets of X. We say that $(X, \mathcal{O})$ is a topological space if*

1. $X, \emptyset \in \mathcal{O}$
2. $O_1, ..., O_n \in \mathcal{O}$ *implies* $\bigcap_{i=1}^{n} O_i \in \mathcal{O}$
3. $O_\lambda \in \mathcal{O}$ *for* $\lambda \in \Lambda$ *(an arbitrary index set) implies* $\bigcup_{\lambda \in \Lambda} \in \mathcal{O}$

The sets $O_\lambda \in \mathcal{O}$ are called the *open sets* of X and $\mathcal{O}$ is called a *topology* on X. A set $F \subset X$ is said to be *closed* if its complementary $X \backslash F$ is open.

From the stated axioms for open sets and Morgan's laws it follows trivially that the arbitrary intersection and finite union of closed sets are closed. By definition, the total set X and the empty set $\emptyset$ are both open and closed.

If $(X, \mathcal{O})$ is a topological space and $Y \subset X$, it is easy to show that $\mathcal{O}|_Y := \{O \cap Y : O_\lambda \in \mathcal{O}\}$ is a topology on Y called the *relative topology*.

Example 3 *The collection of all open intervals $]a,b[:= \{x \in \mathbb{R} : a < x < b$, $a, b \in \mathbb{R}$, $a < b\}$ together with $\emptyset$ and $\mathbb{R}$ are a topology for the real numbers. The relative topology of, say, the unit interval $[0,1]$ consists of all intervals of the form $[0,a[$ with $0 < a < 1$, $]b,1]$ with $0 < b < 1$ and $]c,d[$ with $0 < c < d < 1$ (besides, of course, $\emptyset$ and $[0,1]$).*

A topology allows to define the concept of neighbourhood. Given a topological space $(X, \mathcal{O})$, a set U is a neighborhood of the point $x \in X$ if there exists an $O \in \mathcal{O}$ such that $x \in O \subset U$. Hence, an open set is a neighbourhood of all its points. A topology also allows to defined the concept of continuity for functions. A function $f : X \to X$ is continuous if $f^{-1}(G)$ is open whenever G is open (or, equivalently, $f^{-1}(F)$ is closed whenever F is closed). Two topological spaces X and Y are called *homeomorphic* if there exists a bijection $f : X \to Y$ such that f and f^{-1} are continuous.

Definition 6 *The σ-algebra generated by the open sets of $(X, \mathcal{O})$ is called the Borel σ-algebra of X and is denoted by $\mathcal{B}$.*

Definition 7 *Let $(X, \mathcal{O})$ be a topological space and μ a measure on $(X, \mathcal{B})$. If the transformation $f : X \to X$ is continuous and μ-preserving, then $(X, \mathcal{B}, \mu, f)$ is said to be a topological dynamical system.*

Depending on X, this definition can be somewhat relaxed to allow, for example, piece- or patch-wise continuous functions.

Definition 8 *Let $\rho : X \times X \to \mathbb{R}^+$ be a function in the non-negative real numbers. It is said that ρ is a metric on X if*

1. $\rho(x, y) = 0$ *if and only if* $x = y$ *for every* $x, y \in X$.
2. $\rho(x, y) = \rho(y, x)$ *for every* $x, y \in X$.
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ *for every* $x, y, z \in X$.

Observe that the metric ρ satisfies all the expected properties of a distance. The pair (X, ρ) is then called a *metric space*.

Definition 9 *Let (X, ρ) be a metric space, $x_0 \in X$ and $r \geq 0$. The open ball of radius r and center in x_0, $B_r(x_0)$, is defined as*

$$B_r(x_0) = \{x \in X : \rho(x, x_0) < r\}$$

The set of all arbitrary unions of open balls define a topology on X. Therefore, any metric space can be made a topological space.

Let K be a subset of X. Any collection of open subsets C_λ of X such that $K \subset \bigcup_\lambda C_\lambda$ is called an *open covering* of K. If from any open covering $\{C_\lambda\}$ of K one can extract a finite (open) subcovering $\{C_{\lambda_i}\}_{i=0}^n$, $K \subset \cup_{i=1}^n C_{\lambda_i}$,then K is called a *compact* set. For sets in $\mathbb{R}^n$, compactness is equivalent to the property of being closed and bounded (a set $B \subset X$ is bounded if there exists a $x_0 \in X$ and $R \in \mathbb{R}$ such that $B_R(x_0) \supset B$).

By construction, the Lebesgue measure of $\mathbb{R}^n$ is invariant under translations and rotations in $\mathbb{R}^n$, but, in general, proving the existence of an invariant measure for a general set X and a general transformation $f : X \to X$ is a difficult task. The following theorem shows that such a measure does exist when X is a compact metrical space and f is continuous, what includes many cases of practical interest.

Theorem 1 (Krylov-Bogoliubov) *Let (X, ρ) be a compact metric space and let $f : X \to X$ a continuous transformation. Then there exists an f-invariant probability measure μ_f on $(X, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of X.*

In general, the invariant measure μ_f is not unique and its structure can be very complex. This explains why μ_f is explicitly known only in exceptional cases.

Definition 10 *Given a compact metric space (X, ρ), its Borel σ-algebra $\mathcal{B}$, a probability measure μ on $(X, \mathcal{B})$ and a continuous, μ-preserving transformation $f : X \to X$, we call $(X, \mathcal{B}, \mu, f)$ a metric dynamical system.*

Thus, as it was said before, the theorem of Krylov-Bogoliubov guarantees the existence of an invariant measure in many situations occurring in practice, although one generally does not know how it looks like –nor is interested in it. In fact, what matters in applications (especially concerning chaos theory) is not the invariant measure but other issues such as periodic orbits, attractors, etc. In other words: the focus is on X and f rather than on $\mathcal{B}$ and μ. Therefore, one usually refers to (X, f) as a metric dynamical system, without caring about the invariant measure.

Remark 1 *Metric dynamical system are particular cases of the topological ones and these, in turn, are particular cases of the set-theoretical dynamical systems. In the following, when it is clear from the context which one we are talking about, we refer to it by just "dynamical system".*

Remark 2 *In many applications, X is an interval (or homeomorphic to an interval) of $\mathbb{R}^n$. In this case one speaks of n-dimensional (dynamical) systems.*

Example 4 *Consider the maps*

$$\begin{array}{lll} f_a(x) = ax(1-x), & x \in [0,1], & a \in [0,4] \\ f_c(x) = x^2 + c, & x \in [-1,1], & c \in [-1,0] \\ f_\mu(x) = 1 - \mu x^2, & x \in [0,1], & \mu \in [0,2] \\ f_\lambda(x) = \lambda x^2 \sin(\pi x), & x \in [0,1], & \lambda \in [-1,1] \end{array}$$

They define some classical one-dimensional discrete dynamical systems. The function $f_a(x) = ax(1-x)$ is called the logistic map.

Example 5 *The two-dimensional dynamical system $(x_n, y_n) \mapsto f(x_n, y_n) =: (x_{n+1}, y_{n+1})$, $x = 0, 1, \ldots$, defined by the equations*

$$\left.\begin{array}{l} x_{n+1} = y_n + 1 - ax_n^2 \\ y_{n+1} = bx_n \end{array}\right\}$$

is called the Hénon system. *Hénon studied numerically the solutions of this discrete evolution system for $a = 1.4$ and $b = 0.3$. Observe that, if $b = 0$, then $y_n = 0$ for all n and $x_n \mapsto 1 - ax_n^2$, which is the one of the previous one-dimensional maps extended to the whole real line.*

One could argue that, since $\mathbb{R}^2$ is not compact, the Hénon system is not a bona fide dynamical system. Nevertheless, one is only interested in what happens in a neighbourhood of the "attractor" (assuming that for the actual values of a and b there exists any) and then, there is a compact set $\mathcal{R}$ in the plane (called the *trapping region*) containing the attractor and such that $f(\mathcal{R}) \subset \mathcal{R}$. The dynamics is thought to be restricted to $\mathcal{R}$.

Example 6 *The so-called* standard map *is defined on* $\mathbb{R}^2/\mathbb{Z}^2$ *by means of the general transformation*

$$\left.\begin{array}{ll} x_{n+1} = x_n + f(y_n) & \pmod 1 \\ y_{n+1} = x_n + y_n + f(y_n) & \pmod 1 \end{array}\right\}$$

where f is a smooth periodic function of period one. Choice of different functions f leads to several important dynamical systems for which some basic questions (such as the "coexistence problem") are still open .

6 Orbits of a system and ω-limits

Let (X, f) be a dynamical system and $x_0 \in X$. Define the *forward orbit* or *trajectory* of x_0 to be the sequence

$$O^+(x_0) = \{x_0, f(x_0), f^2(x_0), f^3(x_0), ...\} \equiv \{f^n(x_0)\}_{n=0}^{\infty}$$

where $f^0(x_0) \equiv x_0$.

Similarly, if f is invertible, the *backward orbit* or *trajectory* of x_0 is the sequence

$$O^-(x_0) = \{x_0, f^{-1}(x_0), f^{-2}(x_0), f^{-3}(x_0), ...\} \equiv \{f^{-n}(x_0)\}_{n=0}^{\infty}$$

As already explained in previous pages, one of the main scopes of the theory of dynamical systems is to study the typical long-term behaviour of orbits. One way to characterize forward and backward orbits is via the concept of ω- and α-limits, respectively.

Definition 11 *The ω-limit of $x_0 \in X$ is the set*

$$\omega(x_0) = \{x \in X : \textit{exist a subsequence } \{n_i\} \textit{ such that } \lim_{i\to\infty} f^{n_i}(x_0) = x\}$$

If $x \in \omega(x_0)$, x is said to be an ω-limit point for x_0. Intuitively, the ω-limit of $x_0 \in X$ consists of all those points x of X such that each neighbourhood of x is visited an infinite number of times by the forward orbit of x_0.

Definition 12 *If $f : X \to X$ is invertible, the α-limit of $x \in X$ is the set*

$$\alpha(x) = \{y \in X : \textit{exist a subsequence } \{n_i\} \textit{ such that } \lim_{i\to\infty} f^{-n_i}(x) = y\}$$

If $y \in \alpha(x)$, y is said to be an α-limit point for x. Intuitively, the α-limit of $x \in X$ consists of all those points y of X such that each neighbourhood of y is visited an infinite number of times by the backward orbit of x.

The set of all ω-limit (α-limit) points for x is denoted by $\omega(x)$ (correspondingly, $\alpha(x)$) and it is called its *ω-limit (α-limit) set.*

Under general conditions, it turns out that X can be divided in subsets, $X = \cup X_k$ such that $f(X_k) \subset X_k$ for every k and $\omega(x)$ is independent of the initial state within each subsets, i.e. $\omega(x)$ is the same for all $x \in A_k \subset X_k$ with $\mu(X_k \setminus A_k) = 0$.

7 Fixed, periodic and hyperbolic points

Definition 13 *We say that $x \in X$ is a fixed point of $f : X \to X$ if $f(x) = x$. A point $x \in X$ is said to be a periodic point of f of period $p \in \mathbb{N}$ if $f^p(x) = x$ and $f^j(x) \neq x$ for $1 \leq j \leq p-1$.*

In particular, a fixed point of f is a periodic point of period 1.

Let $x \in X$ be a periodic point of f of period p. The ordered set

$$\{x, f(x), f^2(x), ..., f^{p-1}(x)\}$$

is called a *periodic orbit* or a *periodic cycle of length p*.

Definition 14 *Let (X, f) be a one-dimensional dynamical system with f differentiable. A periodic point $x \in X$ of period p is called hyperbolic if $|(f^p)'(x)| \neq 1$.*

In the important case $X = \mathbb{R}^n$ and f differentiable at the hyperbolic point x of period p, we say

1. x is a *sink* or *attracting periodic point* if all of the eigenvalues of the derivative

$$Df^p(x) := \begin{pmatrix} \partial f_1^p(x)/\partial x_1 & \cdots & \partial f_1^p(x)/\partial x_n \\ \cdots & \cdots & \cdots \\ \partial f_n^p(x)/\partial x_1 & \cdots & \partial f_n^p(x)/\partial x_n \end{pmatrix}$$

   are less than 1 in absolute value,

2. x is a *source* or *repelling periodic point* if all of the eigenvalues of $Df^p(x)$ are greater than 1 in absolute value,

3. x is a *saddle point* otherwise, i.e. if some eigenvalues of $Df^p(x)$ are larger and some are less than 1 in absolute value.

Still other usual denominations in the literature for attracting and repelling periodic points are *stable* and *unstable periodic points*, respectively.

In particular, if $X \subset \mathbb{R}$, f is differentiable and the hyperbolic point $x \in \mathbb{R}$ has period p, then x is stable or unstable depending on $|(f^p)'(x)|$ being less or greater than one, respectively. Periodic points x such that $|(f^p)'(x)| = 1$ are sometimes called indifferent..

One also speaks of attractive (or stable) and repelling (or unstable) periodic orbits or cycles in the obvious sense.

The dynamics around an hyperbolic fixed point is simple. The following Proposition illustrates this claim for one-dimensional systems.

Proposition 2 *Let x be a hyperbolic fixed point of a one dimensional system (X, f).*

1. *If $|f'(x_0)| < 1$, then there is an open interval U about x such that*

$$x \in U \;\Rightarrow\; \lim_{n\to\infty} f^n(x) = x_0$$

2. *If $|f'(x_0)| > 1$, then there is an open interval U about x such that*

$$x \in U,\, x \neq x_0 \;\Rightarrow\; \exists k > 0 \text{ such that } f^k(x) \notin U$$

8 Attractors

Let (X, f) be a dynamical system. The set $A \subset X$ is called *invariant* if $f(A) \subset A$. If, furthermore, there is no proper subset of A with this property, A is called an *indecomposable* invariant set.

If (X, ρ) is a metric space and $S \subset X$, the open set

$$U_\varepsilon := \{x \in X : \rho(x, S) < \varepsilon\} \quad (\varepsilon > 0)$$

(where $\rho(x, S) = \inf\{\rho(x, s) : s \in S\}$) is called an ε-neighbourhood of S.

Definition 15 (Holmes, Guckenheimer *[2]*) *An attractor is an indecomposable closed invariant set Λ with the property that, given $\varepsilon > 0$, there is a set U of positive Lebesgue measure in an ε-neighborhood of Λ such that $x \in U$ implies that the ω-limit set of x is contained in Λ and the forward orbit of x is contained in U.*

Definition 16 (Devaney *[1]*) *A set Λ is called an attractor for f if there is a neighborhood N of Λ for which the closure of N is a trapping region and*

$$\Lambda = \bigcap_{n=0}^{\infty} f^n(N)$$

Although there are several technical definitions of attractors (including the foregoing ones), the basic idea is the same. First of all, invariant means that if you start from one point of the set, you will stay in the set forever. An attractor is an invariant set which attracts all orbits starting from some surrounding set called its trapping region. Usually, the word trapping region is reserved for a connected closed set containing the attractor whereas the basin of attraction comprises all points of X which ω-limit set is contained in Λ. Different authors put additional topological or measure-theoretical conditions upon these definitions in order to assure that the basin of attraction of Λ is large enough compared to X.

9 Stable and Unstable Sets

We consider first the one-dimensional case.

Let $x_0 \in X \subset \mathbb{R}$ be a repelling fixed point or, more generally, a repelling periodic point, i.e. there is an open interval around x_0 on which f is one-to-one and expanding,

$$|f(x) - x_0| > |x - x_0|$$

We define the *local unstable set* at x_0 to be the maximal such an open interval about x_0 and denote it by $W^u_{\text{loc}}(x_0)$.

In the higher dimensional case, $X \subset \mathbb{R}^n$, suppose that $x_0 \in X$ is a hyperbolic periodic point and that $f : X \to X$ is a local (C^1) diffeomorphism about x_0, i.e. there exists neighbourhoods $U(x_0)$ and $V(f(x_0))$ such that (i) $f : U(x_0) \to V(f(x_0))$ is continuously differentiable and has an inverse $g : V(f(x_0)) \to U(x_0)$, (ii) g is continuously differentiable. Then, the *local stable* and *unstable sets* of x_0 are defined as follows:

$$\begin{aligned} W^s_{\text{loc}}(x_0) &= \{x \in U_s : \lim_{n\to\infty} f^n(x) = f^n(x_0) \text{ and } f^n(x) \in U_s, \forall n \geq 0\} \\ W^u_{\text{loc}}(x_0) &= \{x \in U_u : \lim_{n\to\infty} g^n(x) = g^n(x_0) \text{ and } g^n(x) \in U_u, \forall n \geq 0\} \end{aligned}$$

Notice that U_u is the maximal neighbourhood of x_0 where f is invertible and that it is invariant under g. Furthermore,

$$W^s(x_0) := \bigcup_{n=0}^{\infty} g^n(W^s_{\text{loc}}(x_0)) \quad \text{and} \quad W^u(x_0) := \bigcup_{n=0}^{\infty} f^n(W^u_{\text{loc}}(x_0)) \tag{4}$$

are the *global stable* and *unstable sets* of x_0, respectively.

Remark 3 *According to the* Hadamard-Perron theorem, *if f is a local diffeomorphism around the fixed point x_0, then there is an open neighborhood of x_0, U, such that $U \cap W^s_{loc}(x_0)$ and $U \cap W^u_{loc}(x_0)$ are manifolds (roughly speaking diffeomorphic to some open subset of $\mathbb{R}^n$ for some n).*

Usually the point x_0 is a fixed point of f. This is certainly the case when the discrete dynamical system considered arises from a flow with a periodic orbit via a local Poincaré surface of section, the fixed point corresponding to the intersection with the periodic orbit (see the section dedicated to continuous dynamical systems). As in this special case, the stable and unstable sets use also to be smooth manifolds of low dimensionality.

A fixed point with a stable and an unstable set is called a *saddle node point*. The stable set of an attractive fixed point is its whole basin of attraction.

10 Homoclinic points

Let (X, f) be a dynamical system and $x_0 \in X$ an unstable fixed point. The point $x \in X$ is called *homoclinic* to the fixed point x_0 if $x \in W^s(x_0) \cap W^u(x_0)$ (see (4)). If x belongs to both the stable and unstable sets of two different fixed points, then x is called a *heteroclinic* point.

If x is homoclinic to x_0, then both the forward and the backward orbit of x converges to x_0. Moreover, each point of $O^+(x)$ and $O^-(x)$ is trivially also a homoclinic point for x_0. In this case, the stable and unstable sets of x_0 crosses an infinite number of times building what are called *homoclinic tangles* or *webs*. Also important to understand chaos is the fact that in every neighbourhood of a homoclinic point there is an infinite number of periodic points.

11 Bifurcation of dynamical systems

Let $(X, f_\lambda)_{\lambda \in \Lambda}$ be a family of dynamical systems, where Λ is a set of parameters (usually, Λ is an interval of $\mathbb{R}^n$). We are interested in situations where the orbit structure in the state space X changes qualitatively with λ. Such a qualitative change is called a *bifurcation*. Correspondingly, the critical value λ_0 where the bifurcation occurs is called the bifurcation value of the parameter.

Especially interesting are sequences of bifurcations which lead to a successively more complicated behaviour of orbits in the state space and, finally, to chaotic behaviour (v.g. Feigenbaum cascades).

12 Topological equivalence and other analytical tools

Two dynamical systems (X, f) and (Y, g) are said to be *topologically conjugate* if there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$ or, equivalently, $f = h^{-1} \circ g \circ h$. Topologically conjugate systems are the same system insofar as there is a one-to-one correspondence between the orbits of each system. If h is not a homeomorphism but just a continuous map (i.e. the correspondence between orbits is many-to-one), then (X, f) and (Y, g) are said to be *topologically semiconjugate*.

If two dynamical systems are topological equivalent, then they have the same dynamical properties. For example, if x is a (stable or unstable) periodic point of (X, f), then $h(x)$ is a (stable or unstable, respectively) periodic point of (Y, g). If $O^+(x)$ is dense in X, then $O^+(h(x))$ is dense in Y. Etc.

Remember that a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a C^1 diffeomorphism if both f and f^{-1} are continuously differentiable.. Furthermore, assume that f has a hyperbolic fixed point at x_0. Then,

Theorem 3 (Hartman-Grobman) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 diffeomorphism with a hyperbolic fixed point x_0. There exists a homeomorphism h define on some neighbourhood U of x_0 such that $(h \circ f)(\xi) = (Df(x_0) \circ h)(\xi)$ for all $\xi \in U$.*

The Hartman-Grobman linearization theorem implies that a dynamical system is locally (i.e. in some sufficiently small neighbourhood) topologically conjugate (or dynamically equivalent) to its linearization. Since dynamical systems defined by linear maps are much easier to analyze and classify than the non-linear ones, Theorem (3) provides a very useful tool to study local properties.

13 Entropy

Depending on the dynamical system, one can define two types of entropy: topological or metrical entropy.

Let $(X, \mathcal{O})$ be a compact topological space. Given two open coverings of X, $\alpha = \{A_i\}_{i=1}^n$ and $\beta = \{B_j\}_{j=1}^m$ ($A_i, B_j \in \mathcal{O}$ for $1 \leq i \leq n$, $1 \leq j \leq m$), define a new covering $\alpha \vee \beta$ as follows:

$$\alpha \vee \beta := \{A_i \cap B_j : 1 \leq i \leq n, \ 1 \leq j \leq m\}$$

Further, let denote by $N(\alpha)$ the minimal number of sets of α needed to cover X and set

$$H_0 := \ln N(\alpha)$$

If $\varphi : X \to X$ is a continuous map, then the family of open sets $\varphi^{-1}(\alpha) := \{\varphi^{-1}(A_i)\}_{i=1}^n$ is again an open covering of X and, thus, $\varphi^{-k}(\alpha) := \{\varphi^{-k}(A_i)\}_{i=1}^n$ ($k = 1, 2, ...$) are also open coverings of X, where $\varphi^{-k}(A) = (\varphi^{-1} \circ ... \circ \varphi^{-1})(A) = \varphi^{-1}(\varphi^{-(k-1)}(A))$ (k times) for any $A \in \alpha$.

Finally, define

$$h(\varphi, \alpha) := \lim_{n \to \infty} \frac{1}{n} H_0(\alpha \vee \varphi^{-1}(\alpha) \vee ... \vee \varphi^{-n+1}(\alpha))$$

It can be easily shown that this limit exists.

Definition 17 *The topological entropy $h(\varphi)$ of the topological dynamical system $(X, \mathcal{B}, \varphi)$ is defined as*

$$h(\varphi) := \sup_{\alpha} h(\varphi, \alpha)$$

where the supremum is taken over all open coverings of X.

The topological entropy is a measure of the rate of "mixing" due to the repeated action of φ on X.

In case that $(X, \mathcal{B}, \mu, \varphi)$ is a metrical dynamical system (i.e. there is metric ρ defined on X and the σ-algebra $\mathcal{B}$ is generated by the open balls). one can define the metric entropy. Suppose ξ is a partition of the space X (modulo null sets) into measurable subsets $C_1, ..., C_r$, i.e.

$$X = \bigcup_{i=1}^{r} C_i \ (\mathrm{mod}\, 0), \ \ C_i \in \mathcal{B}, \ \ C_i \cap C_j = \emptyset \text{ for } i \neq j, \ \ 1 \leq i, j \leq r$$

Definition 18 *The entropy of the (measurable) partition $\xi = \{C_i\}_{i=1}^r$ is the number*

$$H(\xi) = -\sum_{i=1}^{r} \mu(C_i) \ln \mu(C_i)$$

If $\mu(C_i) = 0$, we assume that $\mu(C_i) \ln \mu(C_i) = 0$.

Analogously to the definition for coverings, set $\varphi^{-k}(\xi) := \{\varphi^{-k}(C_i)\}_{i=1}^r$, where now ξ is a partition of X and $\varphi : X \to X$ is μ-preserving, and $\xi \vee \eta := \{C_i \cap D_j : 1 \le i \le r, 1 \le j \le s\}$ for any two partitions $\xi = \{C_i\}_{i=1}^r$ and $\eta = \{D_j\}_{j=1}^s$ of X. Furthermore, set similarly as before,

$$h_\mu(\varphi, \xi) := \lim_{n\to\infty} \frac{1}{n} H(\xi \vee \varphi^{-1}(\xi) \vee ... \vee \varphi^{-n+1}(\xi))$$

where, again, it can be shown that this limit exists.

Definition 19 *The metrical entropy $h_\mu(\varphi)$ of the metrical dynamical system $(X, \mathcal{B}, \mu, \varphi)$ is defined as*

$$h_\mu(\varphi) := \sup_\xi h_\mu(\varphi, \xi)$$

where the supremum is taken over all measurable partitions of X.

The following theorem establish the relation between the topological and metrical entropies.

Theorem 4 (Goodman) *Let (X, ρ) be a compact metrical space and $\varphi : X \to X$ a continuous map. Then*

$$h(\varphi) = \sup_\mu h_\mu(\varphi)$$

where the supremum is taken over all probabilistic φ-invariant measures on $(X, \mathcal{B})$.

14 Conservative and non-conservative systems

We say that the dynamical system $(X, \mathcal{A}, \mu, f)$ is *conservative* if $\mu(f(A)) = \mu(A)$ for every $A \in \mathcal{A}$. This means that, as the iteration of f on any region R of phase space goes on, the "volume" $\mu(R)$ does not shrink.

Historically, the name conservative comes from the classical mechanics of N particles. If no dissipation of energy occurs (i.e. energy is conserved), it was proved by Liouville that the Lebesgue measure of the phase space restricted to the energy surface ("Liouville measure") is conserved in time. See the section "Historical comments" for a more detailed discussion.

Systems which are not conservative are called *non-conservative*. In particular, if $\mu(f(A)) < \mu(A)$ for every $A \in \mathcal{A}$, the dynamical system is called *dissipative*.

In order to decide whether a given dynamical system $(X, \mathcal{A}, \mu, f)$ (with $X \subset \mathbb{R}^n$ and f continuously differentiable) is conservative, one evaluates the *volume variation factor*, which is the absolute value of $\det Df(x)$. If $|\det Df(x)| \neq 1$ for some $x \in X$, then the system is non-conservative.

>From the point of view of chaos theory, dissipative systems are the most interesting ones. In fact, since the (fractal) dimension of the attractors is less than the phase space dimension, attractors are only possible if the phase space volume shrinks with each iteration, i.e. if the dynamics is dissipative. But not only that. Other distinctive phenomena of chaos theory such as bifurcations and intermittence are also exclusive of dissipative systems.

Example 7 *The Jacobian of the Hénon transformation*

$$f(x,y) = (y + 1 - ax^2, bx)$$

is

$$Df(x,y) = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

Thus, the volume variation factor is

$$|\det Df(x,y)| = |b|$$

independently of a and x. For $b = 0.3$, the Hénon system is dissipative and, not surprisingly, it has an attractor, known as the Hénon attractor.

15 Sensitive dependence on initial conditions (sensitivity)

Definition 20 *Let (X, ρ) be a metric space and let μ be a measure on X. We say that $f : X \to X$ has sensitivity (or sensitive dependence on initial conditions) if there is a set $Y \subset X$ with $\mu(Y) > 0$ and an $\varepsilon > 0$ such that for every $x \in Y$ and every neighborhood U of x, there exists $y \in U$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \varepsilon$.*

In practice, sensitivity means that there is a sizeable set of starting points which neighbouring orbits diverge from each other at some time (although they can join again later).

16 Lyapunov exponents

Consider, to begin with, a one-dimensional dynamical system (X, f), $X \subset \mathbb{R}$. Suppose additionally that f is differentiable in X. Then one defines the Lyapunov exponent $\lambda(x)$ as

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \ln |(f^n)'(x)| \tag{5}$$

Applying the chain rule of derivation, one obtains

$$\begin{aligned}(f^n)'(x) &= (f^{n-1} \circ f)'(x) = (f^{n-1})'(f(x)) \cdot f'(x) = ... \\ &= f'(f^{n-1}(x)) \cdot f'(f^{n-2}(x)) \cdot ... \cdot f'(x) \\ &= \prod_{k=1}^{n} f'(f^{n-k}(x))\end{aligned}$$

so that

$$\begin{aligned}\lambda(x) &= \lim_{n\to\infty} \frac{1}{n} \ln \left|\prod_{k=1}^{n} f'(f^{n-k}(x))\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln \left|f'(f^{n-k}(x))\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left|f'(f^{k}(x))\right|\end{aligned}$$

Observe that f' is calculated at the first $n-1$ points of the orbit of x.

To justify this definition, it can be shown from (5) and the Mean Value Theorem for f^n that, if at some point $x \in X$ the Lyapunov exponent $\lambda(x)$ is positive, then for every $\varepsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ and a neighbourhood $U_{n_1 n_2}$ of x such that, for every $n_1 < n < n_2$ and every $x_1, x_2 \in U_{n_1 n_2}$ it holds

$$e^{(\lambda(x)-\varepsilon)n} |x_1 - x_2| < |f^n(x_1) - f^n(x_2)| < e^{(\lambda(x)+\varepsilon)n} |x_1 - x_2|$$

where f is supposed to be differentiable with continuity. Hence $e^{\lambda(x)}$ measures, in first order, the rate of divergence of the orbits starting from x_1 and x_2.

Remark 4 *If $\lambda(x) > 0$ this means that neighbouring orbits diverge exponentially (for short times) from each other and, therefore, a positive Lyapunov exponent implies sensitive dependence on initial conditions.*

In general, if X is an n-dimensional surface (more rigorously, an n-dimensional manifold) and $x \in X$, let $\boldsymbol{\nu}$ be a tangent vector to X at the point x. Denote by $Df^n(x)(\boldsymbol{\nu})$ the (Frechet) derivative of f^n at the point x in the direction $\boldsymbol{\nu}$. Then, the Lyapunov exponent is defined as

$$\lambda(x, \boldsymbol{\nu}) := \lim_{n\to\infty} \frac{1}{n} \ln \|Df^n(x)(\boldsymbol{\nu})\| \tag{6}$$

where $\|\cdot\|$ is the norm (i.e. the generalization of length) in the tangent space.

The number of different Lyapunov exponents at x is at most equal to the dimension of the tangent space.

In the usual case that $X \subset \mathbb{R}^n$ and f is continuously differentiable, it follows from (6) that the n Lyapunov exponents of f at the fixed point x are the logarithm of the absolute values of the n (in general complex) eigenvalues μ_i, $1 \leq i \leq n$, of the Jacobian matrix

$$Df(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{1 \leq i,j \leq n}$$

i.e.

$$\lambda_i(x) = \ln|\mu_i| \quad \text{and} \quad |\mu_i| = e^{\lambda_i(x)}$$

Therefore, if x is a fixed point, it will be

- a sink (i.e. $|\mu_i| < 1$, $1 \leq i \leq n$) if and only if all Lyapunov exponents $\lambda_i(x)$ are negative.
- a source (i.e. $|\mu_i| > 1$, $1 \leq i \leq n$) if and only if all Lyapunov exponents $\lambda_i(x)$ are positive.
- non-hyperbolic (i.e. $|\mu_i| = 1$ for some i) if and only if some Lyapunov exponents $\lambda_i(x)$ are zero.

Nonhyperbolic points fixed points for which all the Lyapunov exponents $\lambda_i(x)$ vanish are called *centers*.

Remark 5 *The* multiplicative ergodic theorem of Osedelec *implies that the Lyapunov exponents of f exist in great generality if f is C^1 and its derivative Df is continuous (Hölder continuity is enough). Indeed, for any f-invariant measure μ, almost all points with respect to μ have Lyapunov exponents.*

17 Definition of chaos

There is no universally accepted definition of chaos. The two most popular proposals are the following.

Definition 21 (Devaney [1]) *The topological dynamical system (X, f) is chaotic if*

1. *It has sensitivity to initial conditions.*
2. *It is transitive.*
3. *Its periodic points build a dense subset of X.*

A subset $D \subset X$ is dense if for any open set $G \subset X$ it holds $D \cap G \neq \emptyset$, i.e. in any neigbourhood of X points of D can be found. Transitiveness means that there exists a point $x_0 \in X$ such that its forward orbit is dense in X. Roughly speaking, each region of the phase is visited by the orbit of x_0 an infinite number of times. From this follows that, given any two sets of the phase space, there is always a trajectory starting in one of them and piercing the other one. So to say, all regions of phase space are connected among them.

Definition 22 (Li-Yorke [3]) *Let (X, f) be a dynamical system. We say that this system is chaotic in the Li-Yorke sense if*

1. *for every $k = 1, 2, ...$ there is a periodic point in X having period k*
2. *there is an uncountable subset $S \subset X$ (containing no periodic points) which satisfies the following conditions:*
   (a) *For every $p, q \in S$ with $p \neq q$,*

$$\lim_{n\to\infty} \sup |f^n(p) - f^n(q)| > 0 \quad \textit{and} \quad \lim_{n\to\infty} \inf |f^n(p) - f^n(q)| = 0$$

   (b) *For every $p \in S$ and every periodic point $q \in X$*

$$\lim_{n\to\infty} \sup |f^n(p) - f^n(q)| > 0$$

These conditions mean that the system has a huge number of completely different orbits with an involved attracting set. In fact, the limit conditions in Definition (22) spell that the orbits starting at points p and q come from time to time arbitrarily close whereas, at other times, they are at a finite distance from each other.

From a more practical point of view, people associate chaos with the positivity of the Lyapunov exponents (what guarantees sensitivity) in a domain of positive measure of the state space.

18 Bernoulli system

Consider a finite set of symbols $\{0, 1, .., j\}$, a corresponding set of probabilities $p_0, p_1, ..., p_j$ (with $p_0 + ... + p_j = 1$) and let X be the set of all sequences $(x_1, x_2, ..., x_n, ...)$ with $x_n \in \{0, 1, .., j\}$ for $n \in \mathbb{N}$, i.e.

$$X = \prod_{n=1}^{\infty} \{0, 1, .., j\}$$

The so-called *product σ-algebra* $\mathcal{P}$ is generated by the sets of the form

$$\begin{aligned} {}_k[i_1, i_2, ..., i_l] &= \{x \in X : x_k = i_1, x_{k+1} = i_2, ..., x_{k+l-1} = i_l\} \\ &= {}_k[i_1] \cap ... \cap {}_{k+l-1}[i_l] \end{aligned}$$

which are called *elementary blocks* or *cylinders* of length $l = 1, 2, ...$ and initial coordinate k. Further, define the *product measure* first on the elementary blocks,

$$\mu({}_k[i_1, i_2, ..., i_l]) = p_{i_1} \cdot p_{i_2} \cdot ... \cdot p_{i_l}$$

and then extend it to the whole $\mathcal{P}$ in the natural way (making use of Kolmogorov Theorem) to get the measure space $(X, \mathcal{P}, \mu)$.

The *Bernouilli shift* $\sigma : X \to X$ is defined as

$$\sigma((x_1, x_2, ..., x_n, ...)) = (x_2, x_3, ..., x_{n+1}, ...)$$

that is, σ shifts the entries of the sequence $x = (x_1, x_2, ..., x_n, ...)$ one place to the left. Observe that σ is a non-invertible function since it is j-to-1.

Furthermore, the product measure μ is σ-invariant. We prove it for the elementary blocks. Indeed, for $k, l = 1, 2, ...$

$$\sigma^{-1}({}_k[i_1, i_2, ..., i_l]) = {}_{k+1}\, [i_1, i_2, ..., i_l]$$

and, since the product measure does not depend on the initial coordinate k,

$$\mu\left(\sigma^{-1}({}_k[i_1, i_2, ..., i_l])\right) = p_{i_1} \cdot p_{i_2} \cdot ... \cdot p_{i_l} = {}_k\, [i_1, i_2, ..., i_l])$$

The dynamical system $(X, \mathcal{P}, \mu, \sigma)$ is called a *Bernouilli system*. For simplicity the set of symbols is usually chosen to be $\{0, 1\}$. In this case, there is an obvious one-to-one correspondence between X and the real numbers of the interval $[0, 1]$ in binary notation.

Despite its apparent simplicity, the Bernouilli system embodies all the complexity of a typical chaotic system. In fact, most properties of dynamical systems are proved by showing that they are topologically conjugate to a Bernouilli dynamical system.

Bernouilli systems can readily be made a metrical system by means of the metric

$$\rho(x, y) := \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$$

Let us see next that Bernouilli systems are chaotic.

- Choose two points which differ in the position $k > 1$,

  $$x = (x_1, x_2, ..., x_{k-1}, x_k, ...) \quad \text{and} \quad y = (x_1, x_2, ..., x_{k-1}, x_k + 1, ...)$$

  where, without loss of generality, the entries are binary digits (arithmetic operations are meant therefore modulo 2). Then, the shifted points σx, σy differ in the position $k - 1$, the points $\sigma^2 x$, $\sigma^2 y$ in the position $k - 2$ and, in general, the points $\sigma^j x$, $\sigma^j y$ differ in the position $k - j$ (as long as $j < k$) so that

  $$\rho((\sigma^j x, \sigma^j y) = 2^j \rho(x, y) \quad (j = 1, ..., k - 1)$$

  This implies sensitivity to initial conditions.

- To prove the transitivity of the Bernouilli system we will explicitly exhibit a point which orbit is dense in X. For binary systems, such a point can be constructed by listing all the possible strings of length $l = 1, 2, 3, ...$ in some arbitrary order. For example,

  $$\bar{x} = (0, 1, 0, 0, 0, 1, 1, 0, 1, 1, ...)$$

  where we have explicitly listed first the two strings of length 1 (0 and 1) and then the four strings of length 2 $(00, 01, 10, 11)$.

- The periodic points of period $l \geq 1$ corresponds to those $x \in X$ such that

$$x_{n+l} = x_n,\ x_{n+l+1} = x_{n+1}, \dots\ x_{n+2l-1} = x_{n+l}$$

for $n = 1, 2, \dots$ Periodic points of period l are represented by

$$(x_1, x_2, \dots, x_l, x_1, x_2, \dots, x_l, \dots) = (\overline{x_1, x_2, \dots, x_l})$$

Given a point an arbitrary point $x = (x_1, x_2, \dots, x_k, \dots)$, then

$$\rho((x_1, x_2, \dots, x_k, \dots), (\overline{x_1, x_2, \dots, x_k})) \leq \frac{1}{2^k}, \quad k \in \mathbb{N}$$

and the density of the periodic points follows.

19 Ornstein theorem

Let $\mathcal{R}$ be an equivalence relation for dynamical systems $(X, \mathcal{A}, \mu)$. We say that a function ϕ defined for any dynamical system is an *invariant* for the equivalence relation $\mathcal{R}$ if $\phi(X_1) = \phi(X_2)$ whenever $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ are $\mathcal{R}$-equivalent. If, furthermore, from $\phi(X_1) = \phi(X_2)$ it follows that $(X_1,\ \mathcal{A}_1,\ \mu_1)$ and $(X_2,\ \mathcal{A}_2,\ \mu_2)$ are $\mathcal{R}$-equivalent, then ϕ is called a *complete invariant.*

Example 8 *The topological entropy is an invariant for topological conjugacy.*

Theorem 5 (*Ornstein*) *Two Bernoulli systems which have the same metrical entropy are metrically isomorph.*

Therefore, the metrical entropy is a complete invariant for isometry between Bernouilli shifts.

In many cases, to study the properties of a given dynamical system, one resorts to the corresponding symbolic dynamics. Complete invariants for dynamical properties are then used to distinguish non-equivalent systems from each other. In this sense, Ornstein's theorem classify all metrically isomorphic Bernouilli systems according to their entropy, two such systems being metrically isomorphic if and only if they have the same metrical entropy. In the general case that no complete invariant is known for a given equivalence relation, several non-complete invariants can be used in order to reduce the number of possible equivalent systems.

20 Hausdorff dimension

The *Hausdorff dimension* of a metric space X is the infimum of the numbers α with the following property: for any $\varepsilon > 0$, there exists a number $\delta > 0$ and a covering $\mathcal{U}$ of X such that the sets $B \in \mathcal{U}$ all have diameter smaller than δ and $\sum_{B \in \mathcal{U}} (\operatorname{diam} B)^\alpha < \varepsilon$.

The Hausdorff dimension agrees with the usual notion of integral dimension for the usual Euclidean spaces (v.g. smooth curves and surfaces).

We pass to explain the rationale for this definition. First, notice that for an interval in $\mathbb{R}^1$ of length d, the length is trivially proportional to d; in $\mathbb{R}^2$ the area of a disk of diameter d is proportional to d^2; in $\mathbb{R}^3$ the volume of a ball of diameter d is proportional to d^3, etc. Moreover we observe that the area of the interval in $\mathbb{R}^2$ is zero and that the volume of the disk in $\mathbb{R}^3$ is also zero, so these sets are negligible in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, as far as the measure is concerned. The definition of Hausdorff dimension embodies this idea. If we have some non-typical set at hand, we can ask for which power α the volume of this set is positive.

In practice, the set $S \subset \mathbb{R}^n$ which dimension has to be determined is covered by a regular grid of linear side s and the number of boxes with a non-empty intersection with S is counted; let $N(s)$ be this count. Then we define the *box-counting dimension* D_b (or capacity) of S by means of the power law

$$N(s) \sim s^{-D_b}$$

i.e.

$$D_b = \lim_{s \to 0} \frac{\log N(s)}{\log(1/s)}$$

In the simplest cases, D_b coincides with the Hausdorff dimension.

21 Fractal geometry

A set of points is said to be *fractal* (or a fractal set) if its Hausdorff dimension is not integral.

A typical example of fractals in dynamical systems are the attractors.

The property of being fractal appears in dynamical systems very often associated to self-similarity. To explain what is a self-similar object, we need first to define the concept of similarity.

Definition 23 *A map $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is called a similarity transformation if it is the composition of the following three operations:*

1. *Rescaling:* $\mathbf{x}' = \Lambda_\lambda \mathbf{x}$, *where*

$$x'_i = \lambda x_i, \quad \lambda > 0, \quad i = 1, ..., n$$

   (if $\lambda < 1$, Λ is a contraction by a factor λ; if $\lambda > 1$, it is an expansion by a factor λ)

2. *Rotation:* $\mathbf{x}'' = R\mathbf{x}'$, *where R is a $n \times n$ invertible matrix such that $R^{transpose} = R^{-1}$ (such matrices are called orthogonal).*

3. *Translation:* $\mathbf{x}''' = T_{\mathbf{a}}\mathbf{x}''$, *where*

$$x_i''' = x_i'' + a_i, \quad a_i \in \mathbb{R}, \quad i = 1, ..., n$$

Thus, applying a similarity transformation $\sigma = T_{\mathbf{a}} \circ R \circ \Lambda_\lambda$ to all points $\mathbf{x}$ of a set S we get a set $\sigma(S)$ which, by definition, is similar to S. We say that S is self-similar if

$$S = \sigma_1(S) \cup \sigma_2(S) \cup ... \cup \sigma_n(S)$$

where $\sigma_1, \sigma_2, ..., \sigma_n$ are similarity transformations.

Example 9 *Let C be the ternary Cantor set. Then $C = \sigma_1(C) \cup \sigma_2(C)$ where $\sigma_1 = \Lambda_{1/3}$ is a contraction by a factor $1/3$ and $\sigma_2 = \Lambda_{1/3} \circ T_{2/3}$ is the same contraction as before followed by a translation to the right by a vector of length* $2/3$.

22 Strange attractors

Definition 24 *The set $\Lambda \subset \mathbb{R}^n$ is said to be a* strange attractor *if the following conditions hold:*

1. *Λ is an attractor.*
2. *Orbits starting at the trapping region of Λ have sensitivity to initial conditions.*
3. *Λ has fractal structure.*

In the literature there is often assumed that the strange attractor has to contain a transversal homoclinic orbit.

Example 10 *The Hénon attractor is a strange attractor since its fractal dimension is* 1.28.

23 Route to chaos

Consider a parametric family of dynamical systems and suppose that, for some fixed values $\mathbf{p}_{rg}$ of the parameters, the system exhibits regular behaviour, i.e. the generic orbits of the system are asymptotically periodic (converging to periodic cycle of length l, where $l = 1, 2, ...$), whereas for a different set of parameter values $\mathbf{p}_{ch}$, the behaviour is chaotic. The question now is, how does the system make the transition from order to chaos?

To investigate this question, let us vary continuously the parameters from the regular setting $\mathbf{p}_{rg}$ to the chaotic one $\mathbf{p}_{ch}$ and observe what happens. Typically the system evolves smoothly till, at some "critical" parametric values $\mathbf{p}_1, \mathbf{p}_2, ...$, qualitative changes occur. This qualitative changes can be of different type. We consider the most important ones.

1. *Period-doubling* (or *Feigenbaum scenario*): the period of the limiting periodic orbits doubles when crossing the critical values $\mathbf{p}_1, \mathbf{p}_2, \ldots$ At the same time, the distance between critical values shortens as the period increases, resulting in an infinite cascade of period-doublings. The system enters chaos at the point $\mathbf{p}_\infty = \lim_{n\to\infty} \mathbf{p}_n$. This scenario is typical of low-dimensional systems.

2. *Intermittency*: the (quasi-) periodic orbits become occasionally unstable and show during short time intervals irregular motion. As the parameter keeps varying, the short bursts of chaos get longer and longer till the regular (or *laminar*) phase completely disappears..

3. *Hopf bifurcation*: This kind of bifurcation occurs in higher dimensional dynamical systems. Typically, an attracting fixed point (sink) becomes unstable (source) and gives birth to a stable invariant limit cycle which attracts both the orbits emerging from the source and from outside.

All these phenomena are observed experimentally in many physical systems, even if their number of degrees of freedom is large.

In order to understand quantitatively what happens at the different critical parameter values, consider for simplicity the logistic family $f_a(x) = ax(1-x)$, $0 \le a \le 4$. The bifurcation value for the transition from period 2^n to period 2^{n+1} corresponds (for increasing a) to the parameter value a^* for which $f_a^{2^{n+1}}$ passes from having only 2^{n+1} fixed points (2^n stable and 2^n unstable) to having 2^{n+2} fixed points (2^{n+1} stable and 2^{n+1} unstable). Remember that the fixed points of $f_a^{2^{n+1}}$ are the real roots of the polynomial $f_a^{2^{n+1}}(x) - x$ of degree 2^{n+2}. At the left of a^*, half of the roots of $f_a^{2^{n+1}}(x) - x = 0$ are real (and also fixed points of $f_a^{2^n}$, 2^n of them being stable and the other 2^n unstable) and the other half are complex. At a^*, these 2^{n+1} complex roots become (real) stable fixed points of $f_a^{2^{n+1}}$ (building a periodic cycle of length 2^{n+1}), whereas the 2^n former stable fixed points of $f_a^{2^n}$ become unstable. So to say, at the bifurcation point a^*, each stable fixed point of $f_a^{2^n}$ becomes unstable, giving birth to two new stable fixed points of $f_a^{2^{n+1}}$.

Intermittency can be also found in the logistic family just before entering any of the periodic windows immersed in the "chaos band" ($3.5699... < a \le 4$). The most prominent one is the period-3 window, which starts at $a = 1 + \sqrt{8}$.

As for the Hopf bifurcation, consider the family of maps

$$F_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = (\lambda + \beta(x^2+y^2)) \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\lambda > 0$ and β are parameters. Note that 0 is a fixed point for every λ and that the Jacobian matrix of F_λ satisfies

$$DF_\lambda(0) = \lambda \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus, at $\lambda = 1$ the eigenvalues of $DF_\lambda(0)$ cross the unit circle in the complex plane. It follows that 0 is an attracting fixed point for $\lambda < 1$ and a repelling one for $\lambda > 1$.

To study the bifurcation which occurs at $\lambda = 1$, we change coordinates. In polar coordinates, F_λ takes the form $(r, \theta) \mapsto (r_1, \theta_1)$ with

$$r_1 = r\left|\lambda + \beta r^2\right|$$
$$\theta_1 = \theta + \alpha$$

Suppose that $\lambda + \beta r^2 > 0$ so that $r_1 = \lambda r + \beta r^3$. This map has an invariant circle of radius $r = \sqrt{(1-\lambda)/\beta}$, provided $(1-\lambda)/\beta > 0$. When (i) $\beta < 0$, it must hold $\lambda > 1$ and it can be graphically shown that all points in a neighborhood of the circle are attracted to it. This means that at the bifurcation point $\lambda = 1$, an invariant circle is born at the origin as the attracting fixed point becomes repelling. When (ii) $\beta > 0$, $\lambda < 1$ and, hence, 0 is attracting while the invariant circle is repelling. As λ goes to zero, the invariant circle and the origin coalesce and the fixed point at 0 becomes repelling.

24 The Feigenbaum constant

Consider the so called logistic family of maps, $f_a : [0,1] \to [0,1]$

$$f_a(x) := ax(1-x), \quad 0 \le a \le 4$$

Geometrically, this is a family of inverted parabolas through the points $(0,0)$ and $(1,0)$, symmetric with respect to the line $x = 1/2$ and with their maximum at $(1/2, a/4)$. First we determine the fixed points and their stability intervals. From the fixed point equation

$$x = f_a(x) = ax(1-x)$$

it follows

$$x = 0 \equiv p_0 \quad \text{and} \quad x = \frac{a-1}{a} \equiv p_a$$

Observe that p_a is the intersection point of the parabola and the line $y = x$ (bisector) and it only exists for $1 \le a \le 4$.

Next, the derivative of f_a is

$$f'(x) = a(1-2x)$$

so that

$$f'(0) = a \quad \text{and} \quad f'(p_a) = 2 - a$$

Thus,

$$|f'(0)| < 1 \quad \text{if } 0 \le a < 1$$
$$|f'(0)| = 1 \quad \text{if } a = 1$$
$$|f'(0)| > 1 \quad \text{if } a > 1$$

and

$$\begin{array}{ll} |f'(p_a)| < 1 & \text{if } 1 < a < 3 \\ |f'(p_a)| = 1 & \text{if } a = 1 \text{ or } a = 3 \\ |f'(p_a)| > 1 & \text{if } a < 1 \text{ or } a > 3 \end{array}$$

Therefore, the stability interval of p_0 is $[0, 1)$ and for p_a is $(1, 3)$. So to say, at the value $a = 1$ the fixed point p_0 becomes unstable whereas p_a becomes stable. This is a first qualitative change as a increases from 0 and provides a first example of bifurcation.

At $a = 3$ also the fixed point p_a becomes unstable. Let call b_1 this second bifurcation point.

After crossing $a = 3$ from the left, both fixed points p_0 and p_a are unstable. Instead of orbits converging to p_a as just before crossing b_1, one gets now for each a a single stable cycle of length 2 (for almost all starting conditions). This remains so till arriving at the next critical value of the parameter $b_2 = 1 + \sqrt{6} \approx 3.449$, where the 2-cycle becomes unstable, giving birth to a stable cycle of length 4.

This so called "period-doubling scenario" repeats again and again: a stable 2^n-cycle becomes unstable at the critical value b_{n-1} of the parameter, giving birth to a stable cycle of length 2^{n+1}, and, at the same time, the length of the corresponding stability interval (b_n, b_{n-1}) gets shorter with increasing n. Call

$$\delta_n := b_{n+1} - b_n, \quad n = 1, 2, ...$$

Then, Feigenbaum showed that

$$\lim_{n \to \infty} \frac{\delta_n}{\delta_{n+1}} = 3.5699456... \equiv s_\infty$$

This so-called *Feigenbaum constant* s_∞ is universal in the sense that it is the same for a wide family of one-parametric maps called S-unimodal maps. An *S-unimodal map* f fulfills the following assumptions:

1. $f : [a, b] \to [a, b]$ is twice continuously differentiable.

2. f has a unique maximum at x_{crit} such that $f''(x_{\text{crit}}) \neq 0$

3. f is monotone in $[0, x_{\text{crit}})$ and $(x_{\text{crit}}, 1]$.

4. the *Schwarzian derivative* of f,

$$S_f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

is negative for every $x \in [a, b]$.

If f is an S-unimodal map, then we also call *S-unimodal* the one-parametric family of maps $\{f_\lambda \equiv \lambda \cdot f : \lambda \in \mathbb{R} \text{ such that } f_\lambda([a, b]) \subset [a, b]\}$.

25 Intermittence phenomenon

An orbit is said to be *intermittent* (or exhibit *intermittence*) if it consists of intervals of quasiperiodic motion (*laminar phase*) interrupted by comparatively short intervals of irregular motion (*chaotic phase*).

Intermittence sets in when a dynamical system approaches a *saddle-node bifurcation*. For simplicity, consider a one dimensional map f_a depending on one parameter a and suppose that for $a = a^*$ the map f_a has a saddle-node point (i.e. a fixed point with a stable and an unstable set) at $x = x^*$. For, say, $a \precsim a^*$, f_a has no fixed point about x^* while for $a \succsim a^*$, two fixed points (one stable and one unstable) emerge from x^*. Graphically, if $a = a^* - \varepsilon$, $\varepsilon \ll 1$, and $f_a^n(x_0)$ falls on the attractive side of x^*, it moves slowly towards it along the narrow channel left by the curve $y = f_a(x)$ and the bisector $y = x$ and it takes a great number of iterations for it to pass to the repulsive side and scape. During this "channeling", that part of the orbit of x_0 remains close to $f_{a^*}(x^*)$ so that it is practically constant or quasiperiodic. Once $f_a^{n+p}(x_0)$ $(p \gg 1)$ is through, it can eventually return to the attractive side of x^*. This is related to the fact that there are an infinite number of homoclinic points in any neighborhood of x^* and, in turn, each of them has an infinite number of periodic points around.

26 Symbolic dynamics

Let (X, f) be a dynamical system. Let $(X_i)_{i=1}^I$ be a *partition* of X, i.e.

$$\bigcup_{i=1}^{I} X_i = X \quad \text{and} \quad X_{i_1} \cap X_{i_2} = \emptyset \quad \forall\, i_1 \neq i_2$$

Let X_π be the Cartesian product of $\{1, 2, ..., I\}$, i.e.

$$X_\pi = \prod_{i=1}^{I} \{1, 2, ..., I\}$$

To each point x of X we associate a *symbol sequence* $s = (s_n)_{n=0}^{\infty} \in X_\pi$ in the following way:

$$s_n = i \;\text{ iff }\; f^n(x) \in X_i,\; 1 \leq i \leq I$$

This association is a map $\phi : X \to X_\pi$ such that $s = \phi(x)$.

We say that the dynamical system (X, f) admits *symbolic dynamics* (or a symbolic representation) with respect to the partition $(X_i)_{i=0}^I$ if different points of X have different symbol sequences associated to them, i.e. if ϕ is one-to-one (but not necessarily onto). Furthermore, notice that the sequence associated to $f(x)$, $\phi(f(x))$, is obtained from the sequence associated to x by dropping the first entry, i.e. $\phi(f(x)) = \sigma(\phi(x))$ for every $x \in X$.

Moreover, the symbol sequences are defined so as to reflect the dynamics (orbit structure) of f. In this way, the study of the dynamical system is reduced to an essentially combinatorial problem involving the symbols $\{1, 2, ..., I\}$.

In the case that (X, f) is a metrical system, one can require that (X, f) is in addition metrically isomorphic to $(X_\pi, \mathcal{P}, \mu_\pi, \sigma)$, where $\mathcal{P}$ and μ_π are the product σ-algebra and the product measure, respectively. This means that there exists a map $\Phi : X \to X_\pi$ called a *metrical isomorphism* such that for almost every $x \in X$ (with respect to μ),

$$\Phi(f(x)) = \sigma(\Phi(x))$$

and Φ is measure preserving.

$$\mu(\Phi^{-1}(A)) = \mu_\pi(A) \quad \text{for every } A \in \mathcal{P}$$

Example 11 *Consider the function*

$$f(x) = 2x \pmod 1$$

in the unit interval and the partition $\{A_0, A_1\} = \{[0, 1/2), [1/2, 1]\}$. *It can be shown that this map preserves the Lebesgue measure* λ *of the unit interval. Define* $T : [0,1] \to \prod_{j=0}^{\infty}\{0,1\}$ *(the space of all sequences of the form* $(s_0, s_1, ..., s_n, ...)$ *with* $s_n \in \{0,1\}$*),*

$$(T(x))_n = s \in \{0, 1\} \quad \Leftrightarrow \quad f^n(x) \in A_s \quad (n = 0, 1, 2, ...)$$

Then T *is a topological conjugacy between* $f : [0,1] \to [0,1]$ *and* $\sigma : \prod_{j=0}^{\infty}\{0,1\} \to \prod_{j=0}^{\infty}\{0,1\}$, *where* σ *denotes the shift*

$$\sigma((s_0, s_1, ..., s_n, ...)) = (s_1, s_2, ..., s_{n-1}, ...)$$

Therefore, all conjugacy-invariant properties of the original dynamical system can be studied (more easily) in the "symbolic space" $\left(\prod_{j=0}^{\infty}\{0,1\}, \sigma\right)$ *and, afterwards, transferred to* $([0,1], f)$. *Moreover, it can be shown that* $([0,1], \mathcal{B}, \lambda, f)$ *is metrically isomorphic to* $(\prod_{j=0}^{\infty}\{0,1\}, \mathcal{P}, \mu, \sigma)$, *where the product measure* μ *is defined by*

$$\mu({}_n[0]) = \mu({}_n[1]) = \frac{1}{2} \quad (n = 0, 1, 2, ...)$$

The resulting deterministic symbolic dynamics is equivalent to the random process describing the tossing of an unbiased coin.

27 Complex dynamics

Iteration of a function defined on the complex plane $\mathbb{C}$ or, rather, on the compact *Riemann sphere* $\mathbb{S} = \mathbb{C} \cup \{\infty\}$ leads to a *complex dynamical system.*

Example 12 *Consider the function* $f(z) = z^2$ *in* $\mathbb{C}$ *and the dynamics defined by it. Notice that, in polar coordinates,*

$$|z| \to |z|^2, \ \ \theta \to 2\theta \pmod{2\pi}$$

so that f amounts to the composition of a rescaling (contraction for all points inside the unit circle and stretching for all points outside the unit circle) and a duplication of the polar angle. Therefore, the unit circle is an invariant set for f and $\{0,1\}$ is the set of fixed points. Furthermore, since

$$|f'(0)| = 0, \quad |f'(1)| = 2$$

the origin is attracting (its basin of attraction is the whole open unit disk, $|z| < 1$) while 1 is repelling. The orbits of all points outside the unit circle scape to infinity. On the unit circle, the dynamics is chaotic. In fact, (i) the length of any small arc on the unit circle duplicates with each iteration (as long as it is less than π), what proves sensitivity to initial conditions, (ii) any orbit starting at a point on the unit circle with $\theta/2\pi$ irrational is dense (Jacobi theorem), and (iii) all points with $\theta/2\pi = k/(2^n - 1)$ for any $k \in \mathbb{Z}$, $n \in \mathbb{N}$, are periodic and they are certainly dense in $[0,1]$.

Usually, the functions considered in complex dynamics are analytic (v.g. polynomials) or, more generally, rational (i.e. quotients of polynomials). In this latter case, the function is thought to be defined in the Riemann sphere.

Definition 25 *Let f be a polynomial. The Julia set J_f is defined as the closure of all the preimages of the repelling periodic points z_p,*

$$J_f = \overline{\bigcup_{z_p} \bigcup_{k=0}^{\infty} f^{-k}(z_p)}$$

The definition of J_f for f being a rational function is more complicate and will be not considered here.

In the previous example ($f(z) = z^2$), $J_f = \{z : |z| = 1\}$ since all periodic points on the unit circle are repelling ($|f'(z)| = 2\,|z| = 2$) and they are already dense in the unit circle.

The following Proposition summarizes the main properties of J_f.

Proposition 6 *It holds*

- *J_f is compact and non-empty.*
- *$f(J_f) = J_f = f^{-1}(J_f)$*
- *If J_f has interior points, then $J_f = \mathbf{S}$.*
- *For every $z \in J_f$, the set $\{f^{-n}(z) : n \in \mathbb{N}\}$ is dense in J_f.*
- *If G is an open set such that $G \cap J_f \neq \emptyset$, then there exists an $m \in \mathbb{N}$ such that $f^m(G \cap J_f) = J_f$.*
- *If f has two attractive fixed points and both basins of attraction are connected, then J_f is a Jordan curve.*

The complex dynamics divides the Riemann sphere in several invariants subsets, namely, the basins of attraction of the stable periodic points and the Julia set. In the basins of attraction, the dynamics is trivial in the sense that all orbits converge to the corresponding periodic cycle, while the chaotic dynamics is restricted to the (compact) Julia set, which usually has a fractal structure.

The map $f_c : \mathbb{C} \to \mathbb{C}$ defined by $f_c(z) = z^2 + c$ is called the *Mandelbrot transformation* and it is probably the most famous one of complex dynamics. Notice that, on the real line and for real parameter c, the Mandelbrot transformation reduces to the map $f_c(x) = x^2 + c$, which is topologically conjugate to the logistic map.

Definition 26 *Let f_c be the Mandelbrot transformation, $f_c(z) = z^2 + c$. The Mandelbrot set M is the set of all parameter values $c \in \mathbb{C}$ such that the Julia set J_{f_c} is connected.*

Observe that the Julia set is define in the plane of states ("z-plane"), whereas the Mandelbrot set is defined in the parameter space ("c-plane").

It can be shown that the definition of M is equivalent to the following one:

$$c \in M \Leftrightarrow (f_c^n(0))_{n=0}^{\infty} = (0, c, c^2 + c, (c^2 + c)^2 + c, ...) \nrightarrow \infty$$

This means that to decide whether $c \in M$ all we need is to study the orbit of 0 under iteration of f_c. If $O^-(0)$ remains bounded, then $c \in M$, otherwise $c \notin M$. On the other hand, it can easily be shown that

$$|z| \geq |c| \text{ and } |z| \geq 2 \Rightarrow f_c^n(z) \to \infty$$

Therefore, to calculate M numerically, pick up an $N \in \mathbb{N}$ sufficiently large and set

$$M = \{c \in \mathbb{C} : |f_c^n(0)| < 2, \ 1 \leq n \leq N\}$$

The Mandelbrot set is fractal although not self-similar. Repeated magnifications of its border reveal an infinity of copies of it at all scales which seem to be "floating around". But these copies do not reproduce exactly the whole set nor two such reduced copies are identical to each other. Moreover, Douday and Hubbard proved that M is connected. This means that all its parts, however dispersed they may appear, are actually connected to the central core by extremely fine filaments.

For the transformations $f_c(z) = z^n + c$, $n = 3, 4, ...$, one can define the corresponding Mandelbrot sets M_n in an analogous way.

28 Nonlinear analysis of time series

Given a *time series* (at uniformly separated times)

$$x(0), x(\tau), ..., x(n\tau), ...$$

how to determine whether it has been generated randomly other rather by a nonlinear deterministic system?

If the numbers $x_k = x(k\tau), ...$ are not random, it means that they follows some rule,

$$x_k = f(x_{k-1}, x_{k-2}, ..., x_0), \quad k = 1, 2, ...$$

where f is, in principle, an unknown function. In the most simple case, x_k depends only on the preceding data or, at least, the dependence on the other data is negligible, so that

$$x_k = f(x_{k-1})$$

To corroborate this hypothesis, we plot all pairs (x_{k-1}, x_k) on the plane. If the hypotesis is true, the points (x_{k-1}, x_k) will fall on the graph of the function f. If the hypothesis is wrong, then all these points will form a more or less uniformly distributed cloud of points.

In the general case, one proceed as follows. Define a *time delay* $T = m\tau$ ($m = 1, 2, ...$), an *embedding dimension* d, and define the following d-dimensional vectors:

$$\begin{array}{l}(x(0), x(T), ..., x((d-1)T)\\(x(\tau), x(\tau + T), ..., x(\tau + (d-1)T)\\(x(2\tau), x(2\tau + T), ..., x(2\tau + (d-1)T)\\. . . . .\\(x(k\tau), x(k\tau + T), ..., x(k\tau + (d-1)T)\end{array}$$

They define a polygonal curve in an d-dimensional space called the *embedding space*. The mathematician Takens proved that, if d is sufficiently large and T is chosen adequately, then there exists (under some additional conditions) an invertible map Φ between the trajectory in the embedding space and the trajectory in state space. Therefore, the map Φ can be thought of as a change of variables. This means that all properties which are independent of the coordinates (as, for example, Lyapunov exponents and Hausdorff dimension) can be calculated in the embedding space.

29 Continuous dynamical systems

Let $f : X \to \mathbb{R}^n$ be a smooth function defined on some subset $X \subset \mathbb{R}^n$ and associate to each $x_0 \in X$ the Cauchy problem

$$\left.\begin{array}{l}dx/dt = f(x)\\x(t_0) = x_0\end{array}\right\} \qquad (7)$$

According to the general theory of ordinary differential equations, (7) has a unique maximal smooth solution $\phi_{x_0}(t)$ defined for all $x_0 \in X$ and t in some interval $I(x_0) =]a, b[\subset \mathbb{R}$. The independent variable t is usually *time* and so it will be called in the following. Observe that the (system of) differential equation(s) in (7) does not depend explicitly on time. Such systems are called *autonomous*.

We say that the *vector field* f generates the *flow* $\Phi_t : X \to \mathbb{R}^n$,

$$\Phi_t(x) = \phi_x(t)$$

where t belongs to some interval $I \subset \mathbb{R}$. So to say, in going from the solution $\phi_x(t)$ of (7) to the corresponding flow $\Phi_t(x)$, the stress switches from the individual behaviour of solutions ($x \mapsto \phi_x(t)$ with fixed x and variable t) to the global behaviour of the whole family of solutions ($x \mapsto \phi_x(t)$ with fixed t and variable x).

The flow verifies (in its domain of definition) the group properties:

1. $\Phi_0(x) = \phi_x(0) = x$, i.e. $\Phi_0 =$ identity.
2. $\Phi_{t+s}(x) = \phi_x(t+s) = \phi_{\phi_x(s)}(t) = \Phi_t(\phi_x(s)) = \Phi_t(\Phi_s(x))$, i.e. $\Phi_{t+s} = \Phi_t \circ \Phi_s$

Definition 27 *If the flow Φ_t is defined for all $t \in \mathbb{R}$, the pair (Ω, Φ_t) is called a continuous dynamical system. Ω is its phase space.*

Roughly speaking, a continuous dynamical system is defined by an autonomous system of differential equations. The states $x \in X$ evolve with time along the solution curve that at the initial time $t = 0$ goes through it,

$$x \mapsto \Phi_t(x)$$

Remark 6 *There is no loss of generality in assuming that the flow Φ_t is generated by a system of autonomous differential equation since otherwise the system can be made autonomous by adding a new dependent variable (i.e. increasing the phase space dimension by one). Indeed,*

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t) \quad \Leftrightarrow \quad \begin{cases} d\mathbf{x}/dt &= \mathbf{F}(\mathbf{x}, \theta) \\ d\theta/dt &= 1 \end{cases}$$

with $\theta(t_0) = t_0$ ($\Rightarrow \theta(t) = t + \theta_0 - t_0$).

Given the point $x \in X$, the forward and backward orbit of x is defined as

$$O^+(x) = \{\Phi_t(x) : t \geq 0\}, \quad O^-(x) = \{\Phi_t(x) : t \leq 0\}.$$

Most of the concepts defined in previous sections for discrete dynamical systems can be transferred to the continuous case just by substituting "f^n" by "Φ_t" and other minor changes. For example, $x \in X$ is a *fixed, stationary* or *equilibrium point* if $x = \Phi_t(x)$ for $\forall t \in \mathbb{R}$, while it is a *periodic point* of period T if $\Phi_t(x) = \Phi_{t+T}(x)$, for $\forall t \in \mathbb{R}$.

A point p is an *ω-limit point* of x if there are $t_1 < t_2 < \ldots$ such that

$$\lim_{t_i \to \infty} \Phi_{t_i}(x) = p$$

A point q is an *α-limit point* of x if

$$\lim_{t_i \to -\infty} \Phi_{t_i}(x) = q$$

The union of all α- and ω-limit points of a dynamical system define its α- and ω-limit sets, respectively.

One defines the *local stable* and *unstable manifolds* of the fixed point $x_0 \in X$ as

$$\begin{aligned} W^s_{loc}(x_0) &= \{x \in U : \lim_{t\to\infty} \Phi_t(x) = x_0 \text{ and } \Phi_t(x) \in U, \forall t \geq 0\} \\ W^u_{loc}(x_0) &= \{x \in U : \lim_{t\to-\infty} \Phi_t(x) = x_0 \text{ and } \Phi_t(x) \in U, \forall t \leq 0\} \end{aligned}$$

where $U \subset \mathbb{R}^n$ is a neighbourhood of the fixed point x_0, and the global ones as

$$W^s(x_0) := \bigcup_{t\leq 0} \Phi_t(W^s_{loc}(x_0)) \quad \text{and} \quad W^u(x_0) := \bigcup_{t\geq 0} \Phi_t(W^u_{loc}(x_0))$$

Although discrete and continuous dynamical systems seem so far formally analogue, there are however some important differences:

1. A necessary condition for a continuous systems to be chaotic is that $\dim X \geq 3$. This means that there is no space in the line nor in the plane for a flow to be chaotic. For example, since the phase space of the simple pendulum

   $$\ddot{\theta} + \omega_0^2 \sin\theta = 0 \quad \Leftrightarrow \quad \begin{cases} d\theta/dt &= \dot{\theta} \\ d\dot{\theta}/dt &= -\omega_0^2 \sin\theta \end{cases}$$

   ($\theta \in \mathbb{R}$ is the azimuthal angle and $\dot{\theta} \in \mathbb{R}$ is the angular velocity) has dimension 2, its motion cannot be chaotic, although its flow is non-linear. One-dimensional discrete dynamical systems can though be chaotic.

2. Continuous systems are not the "limit" in any sense of discrete systems. If this were the case, there would be chaotic one-dimensional continuous system.

Example 13 (The driven oscillator) *Its equation is*

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 \sin x = F\cos\omega t \tag{8}$$

where $\gamma > 0$ is the damping coefficient, ω_0 is the natural frequency of the oscillator, and F and ω ($\neq \omega_0$ in general) are the amplitude and frequency, respectively, of the external (periodic) driving force. Define the adimensional coefficients

$$\tau := \omega_0 t, \quad b := \frac{2\gamma}{\omega_0}, \quad \mathcal{F} := \frac{F}{\omega_0^2}, \quad a := \frac{\omega}{\omega_0}$$

so that (8) becomes

$$\left.\begin{aligned} dx/d\tau &= \mu \\ d\mu/d\tau &= -b\mu - \sin x + \mathcal{F}\cos\phi \\ d\phi/d\tau &= a \end{aligned}\right\}$$

The existence of periodic or chaotic solutions depend, of course, on the values of b, $\mathcal{F}$ *and* a *and also on the initial condition* (x_0, μ_0, ϕ_0).

Example 14 (The Duffing oscillator) *Duffing's equation*

$$\ddot{x} + 2\gamma\dot{x} + \alpha x + \beta x^3 = F\cos\omega t$$

describes in mechanics the motion of an (externally driven) damped nonlinear spring. The following special cases have been extensively studied in the literature:

1. *Hard spring:* $\alpha > 0$, $\beta > 0$.
2. *Soft spring:* $\alpha > 0$, $\beta < 0$.
3. *Inverted Duffing oscillator:* $\alpha < 0$, $\beta > 0$.
4. *Nonharmonic Duffing oscillator:* $\alpha = 0$, $\beta > 0$.

If the term $\sin x$ *in the driven oscillator equation is replaced by its* 3*th order approximation* $x - x^3/6$, *one gets the equation of the soft spring. Duffing's equation can be rewritten as an autonomous system of first order in the following way:*

$$\left.\begin{aligned} dx/dt &= y \\ dy/dt &= -2\gamma y - \alpha x - \beta x^3 + F\cos z \\ dz/dt &= \omega \end{aligned}\right\}$$

with $z(0) = 0$.

Example 15 (The Lorenz System) *The first continuous chaotic system of dimension* 3 *was proposed by the American meteorologist E.N. Lorenz in 1963. It is the flow defined by the following system of differential equations:*

$$\left.\begin{aligned} dx/dt &= -\sigma x + \sigma y \\ dy/dt &= rx - y - xz \\ dz/dt &= -bz + xy \end{aligned}\right\} \qquad (9)$$

The numbers σ, b *and* r *are parameters. The system (9) describes the Rayleigh-Bénard convection although the physical interpretation of the variables* x, y, z *is not direct.*

The ω-limit set of the Lorenz system with $\sigma = 10$, $b = 8/3$ and $r = 28$ (the values chosen by Lorenz in his seminal paper [4]) is the famous *Lorenz attractor*, which geometric appearance slightly reminds of a butterfly with extended wings.

The unitary volume-variation rate in continuous dynamical system is given by the divergence of the vector field which generates the flow. For the Lorenz system,

$$f(x,y,z) = (-\sigma x + \sigma y, rx - y - xz, -bz + xy) \;\Rightarrow\; \operatorname{div} f(x,y,z) = -\sigma - 1 - b$$

(independently of r and x) and, therefore,

$$\frac{dV(t)/dt}{V(t)} = \operatorname{div} f(x,y,z) \;\Rightarrow\; V(t) = V(t_0)e^{-(\sigma+b+1)(t-t_0)}$$

where $dV(t) = dx(t)dy(t)dz(t)$ is the volume element of $\mathbb{R}^3$ at time t. In particular, if $\sigma = 10$ and $b = 8/3$, then

$$V(t_0+1) = V(t_0)e^{-13.67} \approx 10^{-6}V(t_0)$$

what shows how fast the orbits of the Lorenz system are compressed onto the attractor. The fractal dimension of the Lorenz attractor is 2.06.

Finally, let us return to the connection between discrete and continuous systems. There is an obvious way to derive a discrete system from a continuous one just by advancing time in finite steps, i.e. instead of considering the continuous evolution of the states, $x \mapsto \Phi_t(x)$, take "snapshots" of its evolution at equally separated times,

$$x \mapsto \Phi_{t_n}(x), \quad t_n := t_0 + n\Delta t$$

But the most important way, both theoretically and practically, to obtain from a continuous dynamic a discrete one is the technique of the *Poincaré surface of section*.

Let γ be a periodic orbit of some flow Φ_t in $\mathbb{R}^n$ arising from a non-linear vector field $f(x)$ and let $\Sigma \subset \mathbb{R}^n$ be a local cross section of dimension $n-1$. The hypersurface Σ need not be planar but must be everywhere *transverse* to it, i.e. $f(x) \cdot n(x) \neq 0$ (scalar product) for all $x \in \Sigma$, where $n(x)$ is the unit normal to Σ at x. Denote the (unique) point where γ intersects Σ by p and let $U \subset \Sigma$ be some neighbourhood of p (if γ has multiple intersections with Σ, then shrink Σ until there is only one intersection). Then the first *return* or *Poincaré map* $P : U \to \Sigma$ is defined for a point $q \in U$ by

$$P(q) = \Phi_\tau(q)$$

where $\tau = \tau(q)$ is the time taken for the orbit $\Phi_t(q)$ based at q to first return to Σ. Note that $\tau(q)$ need not be equal to $T = T(p)$, the period of γ.

Clearly p is a fixed point for the Poincaré map P and it is not difficult to see that the stability of p for P reflects the stability of γ for the flow Φ_t. In particular, if p is hyperbolic and $DP(p)$, the linearized map, has n_s eigenvalues with modulus less than one and n_u with modulus greater then one ($n_s + n_u = n-1$), then $\dim W^s(p) = n_s$ and $\dim W^u(p) = n_u$ for the map. Since the orbits of P lying in W^s and W^u are formed by intersections of orbits (solution curves) of Φ_t with Σ, $\dim W^s(\gamma) = \dim W^s(p) + 1$ and $\dim W^s(\gamma) = \dim W^s(p) + 1$.

Example 16 *Consider the planar system*

$$\left.\begin{aligned} \dot{x} &= x - y - x(x^2+y^2) \\ \dot{y} &= x + y - y(x^2+y^2) \end{aligned}\right\} \tag{10}$$

and take as cross section

$$\Sigma = \{(x,y) \in \mathbb{R}^2 : x > 0, y = 0\}$$

In polar coordinates, (10) becomes

$$\left.\begin{aligned} \dot{r} &= r(1-r^2) \\ \dot{\theta} &= 1 \end{aligned}\right\} \tag{11}$$

and

$$\Sigma = \{(r,\theta) \in \mathbb{R}^+ \times S^1 : \theta = 0\}$$

Solving (11), we obtain the global flow

$$\Phi_t(r_0, \theta_0) = \left(\left(1 + \left(\frac{1}{r_0^2} - 1 \right) e^{-2t} \right)^{-1/2}, t + \theta_0 \right)$$

The time of flight τ for any $q \in \Sigma$ is simply $\tau = 2\pi$ and thus the Poincaré map is given by

$$P(r_0) = \left(1 + \left(\frac{1}{r_0^2} - 1 \right) e^{-4\pi} \right)^{-1/2}$$

Notice that P has a fixed point at $r_0 = 1$, reflecting the circular closed orbit γ of radius 1 of (11). Here P is a one-dimensional map and its linearization is given by

$$\begin{aligned} DP(1) &= P'(1) = -\frac{1}{2} \left(1 + \left(\frac{1}{r_0^2} - 1 \right) e^{-4\pi} \right)^{-3/2} \cdot \left(-\frac{2e^{-4\pi}}{r_0^3} \right) \Bigg|_{r_0=1} \\ &= e^{-4\pi} < 1 \end{aligned}$$

Hence, $p = 1$ is a stable fixed point and γ is a stable or attracting closed orbit.

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