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S. Tokarzewski, J.J. Telega

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Stanisław Tokarzewski and Józef Joachim Telega
Institute of Fundamental Technological Research
Polish Academy of Sciences

S-CONTINUED FRACTION REPRESENTATION FOR EFFECTIVE TRANSPORT COEFFICIENTS OF TWO-PHASE MEDIA

Abstract. The aim of this contribution is to derive and examine the S-continued fraction bounds for the effective dielectric constant ε_e of a two-phase composite for the case where the dielectric coefficients ε_1 and ε_2 of the two components are either complex or real. The starting point for our study is a power expansion of $\varepsilon_e(z)$ at $z=0$, $z=\varepsilon_2/\varepsilon_1-1$. The obtained S-continued fraction bounds on ε_e have an interesting mathematical structure convenient for theoretical and numerical investigations of ε_e. Specific examples of a calculation of bounds on ε_e by S-continued fraction method are also provided.

1. Introduction. Transport coefficients of composite materials may be evaluated effectively by the method of bounds [1-7]. Since the bounds are exact they do not suffer from the uncertainty inherent in effective medium approximations whose validity is difficult to estimate. The bounds become increasingly accurate when more information concerning the geometrical properties of the medium is known.

In the present paper we examine the S-continued fraction representation of the effective dielectric constant ε_e of a two-phase medium for the case where the dielectric constants ε_1

and ε_2 of the components are either complex or real. Milton [1-2] derived an infinite set of narrowing bounds in the complex ε_e-plane. The calculation of the bounds requires the knowledge of successive terms of the power expansion of ε_e in $\varepsilon_1-\varepsilon_2$. The coefficients of the expansion are geometrical in nature and their values are determined by the correlation functions of disordered geometry.

Milton's approach is based on an analytic representation of the effective dielectric constant derived by Bergman [3]. The problem of complex bounds was also discussed by Felderhof [4], who obtained the estimation of ε_e with the help of the four characteristic geometric functions introduced by Bergman [5].

Recently an interesting continued fraction representations for the set of complex bounds on ε_e was obtained by Bergman [6] and also by Clark and Milton [7].

In this paper we construct and examine in detail the S-continued fractions to power expansions of the effective dielectric constants of two-phase composite media. The S-continued fractions are used to produce an infinite set of complex and real bounds on ε_e.

This paper is organized as follows. In Sec.2 we introduce a characteristic geometrical function $zf_1(z)$ representing ε_e and derive the S-continued fraction to $zf_1(z)$. In Sec.3 the definitions of the inclusion regions and bounds for a special class of Stieltjes functions $zf_p(z)$, p=1,2,... are introduced. Explicit expressions for the auxiliary inclusion regions and bounds are derived in Sec.4. In Sec.5 the properties of basic bounds on $zf_1(z)$ are investigated in detail. In Sec.6 we introduce the new characteristic geometrical function $ze_1(z)$ closely related to $zf_1(z)$ and examine its analytical properties. The equivalence between bounds on ε_e determined from $zf_1(z)$ and $ze_1(z)$ by means of S-continued fraction method is shown in Sec.7. As an example of practical application, the bulk dielectric constants for regular arrays of spheres are evaluated in Sec.8. From the general, complex relations we derive in Sec. 9 an infinite set of lower and upper bounds on ε_e valid in real domain. In Sec.10 we carry out the numerical evaluation of real bounds for square and hexagonal arrays of cylinders. In Appendices A,B and C we offer more technical proofs.

2. Basic assumptions, definitions and notations. We consider the effective dielectric constant E_e of a two-phase medium for the case where the dielectric constants ε_1 and ε_2 of both components are complex. The bulk dielectric coefficient E_e is defined by the linear relationship between the volume-averaged electric field $\langle U \rangle$ and the volume-averaged displacement field $\langle D \rangle$

$$\langle D \rangle = E_e \langle U \rangle. \tag{2.1}$$

By φ_1 and φ_2 we denote the matrix and the inclusion volume fractions, respectively. In general E_e will be a second-rank symmetric tensor, even when ε_1 and ε_2 are both scalars. Our study will be focused upon one of the principal value of E_e denoted by $\varepsilon_e(\varepsilon_1,\varepsilon_2)$. The remaining principal values of E_e can be studied similarly.

The analytic properties of the bulk dielectric coefficient $\varepsilon_e(\varepsilon_1,\varepsilon_2)$ were examined by Bergman in [1]. He proved that

$\varepsilon_e(\varepsilon_1,\varepsilon_2)/\varepsilon_1=\varepsilon_e(1,\varepsilon_2/\varepsilon_1)$ is a Stieltjes function of $\varepsilon_2/\varepsilon_1$, analytical except on the negative part of the real axis. Due to this fact we can write

$$\frac{\varepsilon_e(z)}{\varepsilon_1} - 1 = z\, f_1(z), \tag{2.2}$$

where

$$f_1(z)=\int_0^1 \frac{d\gamma_1(u)}{1+zu} \quad , \quad z=h-1,\ h=\varepsilon_2/\varepsilon_1, \tag{2.3}$$

is a Stieltjes function defined in the cut $(-\infty \le z \le -1)$ complex plane by means of the real, bounded and non-decreasing spectrum $\gamma_1(u)$, cf.[12-13]. Consider now the power expansions of (2.2) at z=0

$$zf_1(z) = \sum_{n=1}^{\infty} c_n^{(1)}\, z^n. \tag{2.4}$$

The coefficients $c_n^{(1)}$ are given by the following moments

$$c_n^{(1)} = (-1)^{n+1} \int_0^{\infty} u^{n-1}\, d\gamma_1(u), \qquad n = 1,2,\dots . \tag{2.5}$$

Let us introduce an infinite sequence of functions $f_p(z)$ (p=1,2,...) defined by

$$f_{p-1}(z) = \frac{c_1^{(p-1)}}{1 + z\, f_p(z)}\ ,\ c_1^{(p-1)}=f_{p-1}(0)\ ,\ p=2,3,\dots , \tag{2.6}$$

where $f_1(z)$ given by (2.3) is an input for the relation (2.6), while $c_1^{(p-1)}$ denotes the first coefficient of a power expansion of the function $zf_{p-1}(z)$. We call $zf_1(z)$ the basic function, while $zf_p(z)$, p=2,3,... the auxiliary functions. The functions $f_p(z)$ (p=1,2,...) generated by $f_1(z)$ via the recurrence formula (2.6) are Stieltjes functions of the type (2.3), see [8, Lemma 15.3 and Chap.17A]

$$f_p(z) = \int_0^1 \frac{d\gamma_p(u)}{1+zu} \quad , \quad p=1,2,\ldots . \tag{2.7}$$

The power expansions of $zf_p(z)$ at z=0 can be written as follows

$$zf_p(z) = \sum_{n=1}^{\infty} c_n^{(p)} z^n \quad , \quad p=1,2,\ldots, \tag{2.8}$$

where

$$c_n^{(p)} = (-1)^{n+1} \int_0^{\infty} u^{n-1} \, d\gamma_p(u) \quad , \quad n=1,2,\ldots , \; p=1,2,\ldots \tag{2.9}$$

The fractional transformation (2.6) applied p-1 times to the Stieltjes function $f_1(z)$ lead to the continued fraction relationship between $f_1(z)$ and $f_p(z)$

$$f_1(z) = \frac{g_1}{1} {}_{+} \frac{g_2 \; z}{1} {}_{+} \cdots {}_{+} \frac{g_{p-2} \; z}{1} {}_{+} \frac{g_{p-1} \; z}{1} {}_{+} \frac{zf_p(z)}{1} \quad , \; p=1,2\ldots , \tag{2.10}$$

where

$$g_k = c_1^{(k)} \quad , \quad k=1,2,\dots,p-1. \tag{2.11}$$

Since the first power series coefficients $c_1^{(k)}$ $(k=1,2,\dots,p-1)$ given by (2.9) satisfy the condition $c_1^{(k)}=f_k(0)=\int_0^1 d\gamma_k(u)>0$, hence

$$g_k > 0 \quad , \quad k=1,2,\dots,p-1. \tag{2.12}$$

Now we are in position to propose the algorithm for finding the S-continued fraction coefficients g_k from the coefficients $c_k^{(1)}$ $(k=1,2,\dots,p-1)$ given by (2.5). By applying the linear fractional transformation (2.6) to (2.8) we obtain

$$\frac{g_{k+1}\ z}{1+\sum_{n=1}^{p-2-k} c_n^{(k+2)}\ z^n} = \sum_{n=1}^{p-1-k} c_n^{(k+1)}\ z^n \quad , \quad k=0,1,\dots,p-3, \tag{2.13}$$

where $g_{k+1}=c_1^{(k+1)}$. Simple rearrangements yields

$$c_m^{(1)},\ m=1,2,\dots,p-1 \quad ; \quad g_1=c_1^{(1)}, \tag{2.14}$$

$$k=0,1,\dots p-3 \quad , \quad c_0^{(2+k)}=1 \quad , \quad n=1,2,\dots,p-2-k, \tag{2.15}$$

$$c_n^{(2+k)} = -\frac{1}{c_1^{(1+k)}}\left(\sum_{j=0}^{n-1} c_j^{(2+k)}\ c_{n+1-j}^{(1+k)}\right), \tag{2.16}$$

$$g_{2+k} = c_1^{(2+k)}, \tag{2.17}$$

where

k=0, n=1,2,...,p-2 ; k=1, n=1,2,...,p-3 ; k=p-3, n=1. (2.18)

By starting from (p-1) terms of the Stieltjes series $f_1(z)$ we generate successively, with the aid of (2.15)-(2.17) the power expansions of $f_2(z)$, $f_3(z)$, $f_4(z)$, ..., $f_{p-1}(z)$ with steadily decreasing numbers of terms, see (2.18). At some point we will be left with the function $f_p(z)$, for which no terms of its series expansion are given.

3. Inclusion regions and bounds. Our subsequent developments are based on the assumption that a number of p-1 (p=1,2,...) coefficients of the power expansion of the Stieltjes function $zf_1(z)$, given by (2.4), is known. Let's re-write (2.10) as follows

$$zf_k(z)=\frac{g_k\ z}{1}{}_{+}\frac{g_{k+1}\ z}{1}{}_{+}\cdots{}_{+}\frac{g_{p-2}\ z}{1}{}_{+}\frac{g_{p-1}\ z}{1}{}_{+}\frac{zf_p(z)}{1},\quad 1\le k\le p. \tag{3.1}$$

Now for z given by $(2.3)_{2,3}$ and for fixed parameters u_1,u_2 and v_1,v_2 we introduce a region $\mathfrak{J}_k^{(p)}(z)$ defined in the $\mathfrak{J}_k^{(p)}$-complex plane as follows

$$\mathfrak{J}_k^{(p)}(z) = \left\{\mathfrak{J}_k^{(p)}(z,u,v)\mid u_1\le u\le u_2\ ,\ v_1\le v\le v_2\right\},\quad 1\le k\le p, \tag{3.2}$$

where $\mathfrak{J}_k^{(p)}(z,.,.)$ is a function of u and v.

<u>Definition.1</u>. We call $\mathfrak{J}_k^{(p)}(z)$ the inclusion region for $zf_k(z)$ while $\mathfrak{J}_k^{(p)}(z,u,v)$ the including function, if

$$zf_k(z)\in\mathfrak{F}_k^{(p)}(z)=\left\{\mathfrak{F}_k^{(p)}(z,u,v)\mid u_1\le u\le u_2\ ,\ v_1\le v\le v_2\right\}. \tag{3.3}$$

It is also convenient to define in the $F_k^{(p)}$-complex plane the line $F_k^{(p)}(z,u)$ determined by means of two functions $F_k'^{(p)}(z,u)$ and $F_k''^{(p)}(z,u)$

$$F_k^{(p)}(z,u) = \begin{cases} F_k'^{(p)}(z,u)\ , \text{ if } & 0\le u\le 1, \\ F_k''^{(p)}(z,u)\ , \text{ if } & -1\le u\le 0, \end{cases} \tag{3.4}$$

where as previously z is a fixed complex number given by $(2.3)_{2,3}$.

<u>Definition.2</u>. We call $F_k^{(p)}(z)$ the bound for $zf_k(z)$, while $F_k^{(p)}(z,u)$ the bounding function, if

$$F_k^{(p)}(z) = \partial\mathfrak{F}_k^{(p)}(z) = \left\{F_k^{(p)}(z,u)\mid -1\le u\le 1\right\}. \tag{3.5}$$

Here $\partial\mathfrak{F}_k^{(p)}(z)$ denotes the boundary of the inclusion region $\mathfrak{F}_k^{(p)}(z)$.

Now we are in position to introduce the last definition indispensable for our further investigations.

<u>Definition.3</u>. If k=1 (k=p, p>1) then $\mathfrak{F}_k^{(p)}(z)$ and $F_k^{(p)}(z)$ are called basic inclusion regions and basic bounds, respectively (auxiliary inclusion regions and auxiliary bounds).

Let's assume that the auxiliary including function $\mathfrak{F}_p^{(p)}(z,u,v)$ for $zf_p(z)$ is known. It is obvious that on account of (3.1) the including functions $\mathfrak{F}_k^{(p)}(z,u,v)$ for $zf_k(z)$ has in the $\mathfrak{F}_k^{(p)}$-complex plane the following S-continued fraction representation

$$\mathfrak{F}_k^{(p)}(z,u,v)=\frac{g_k z}{1}{}_+\frac{g_{k+1} z}{1}{}_+\cdots{}_+\frac{g_{p-1} z}{1}{}_+\frac{\mathfrak{F}_p^{(p)}(z,u,v)}{1},\quad 1\le k\le p. \qquad (3.6)$$

Analogously on the basis of (3.1) we can write the S-continued fraction representation for a bounding function $F_k^{(p)}(z,u)$ defined in $F_k^{(p)}$- complex plane.

$$F_k^{(p)}(z,u) = \frac{g_k z}{1}{}_+\frac{g_{k+1} z}{1}{}_+\cdots{}_+\frac{g_{p-1} z}{1}{}_+\frac{F_p^{(p)}(z,u)}{1},\quad 1\le k\le p. \qquad (3.7)$$

For a fixed z and u,v the following recurrence formulae

$$\mathfrak{F}_{p-1-j}^{(p)}(z,u,v)=\frac{g_{p-1}\ z}{1}{}_+\frac{\mathfrak{F}_{p-j}^{(p)}(z,u,v)}{1},\quad j=0,\ldots,p-1-k, \qquad (3.8)$$

$$F_{p-1-j}^{(p)}(z,u)=\frac{g_{p-1} z}{1}{}_+\frac{F_{p-j}^{(p)}(z,u)}{1},\quad j=0,1,\ldots,p-1-k, \qquad (3.9)$$

allow us to calculate practically the values of the including functions $\mathfrak{F}_k^{(p)}(z,u,v)$ represented by (3.8) (bounding function $F_k^{(p)}(z,u)$ represented by (3.9)) from the values of the auxiliary including functions $\mathfrak{F}_p^{(p)}(z,u,v)$ (bounding functions $F_p^{(p)}(z,u)$). On the basis of (2.2)-(2.3) and (3.8)-(3.9) the including functions $\mathfrak{B}^{(p)}(z,u,v)$ and the bounding functions

$B^{(p)}(z,u,v)$ for the effective, dielectric constants $\varepsilon_e(z)/\varepsilon_1$ are as follows:

$$\mathfrak{B}_f^{(p)}(z,u,v)=1+\mathfrak{F}_1^{(p)}(z,u,v) \quad , \quad B_f^{(p)}(z,u)=1+F_1^{(p)}(z,u). \qquad (3.10)$$

According to (3.8)-(3.10) the calculation of the inclusion regions (bounds) for $\varepsilon_e(z)/\varepsilon_1$ by the method of the S-continued fraction requires the knowledge only of the auxiliary regions $\mathfrak{F}_p^{(P)}(z)$ (bounds $F_p^{(P)}(z)$) for a Stieltjes function $zf_p(z)$, see (2.10). In the next Section we derive the analytic expression for both the inclusion regions $\mathfrak{F}_p^{(P)}(z)$ and the bounds $F_p^{(p)}(z)$, $p=1,2,\ldots$.

4. Auxiliary inclusion regions and bounds. To derive an explicit formula for the auxiliary bounds $F_p^{(p)}(z)$ we use, as the available information about a microstructure of a composite, the p-1 coefficients of the power series (2.4) and the inequality

$$\varepsilon_e \le \varepsilon_1 \quad \text{for} \quad \varepsilon_2=0<\varepsilon_1 \qquad (4.1)$$

discussed in [10]. On account of (4.1) and (2.2)-(2.3) we obtain

$$f_1(-1) \le 1. \qquad (4.2)$$

Note that for real z, the inequalities

$$f_p(z)>0 \; , \; \frac{\partial f_p(z)}{\partial z}<0 \; , \; \frac{\partial z f_p(z)}{\partial z}>0, \; z\in[-1,\infty) \; , \; p=1,2,\ldots \qquad (4.3)$$

are a direct consequence of relations (2.7). By using (2.10) we can rewrite (4.2) in the form of the sequence of the following continued fractions

$$\frac{g_1}{1-f_2(-1)} \le 1 \; , \quad \frac{g_1}{1-\dfrac{g_2}{1-f_3(-1)}} \le 1 \; , \quad \frac{g_1}{1-\dfrac{g_2}{1-\dfrac{g_3}{1-f_4(-1)}}} \le 1, \; \dots \; . \tag{4.4}$$

Due to (4.3), relations (4.4) yield

$$f_p(-1) \le V_p \; , \; p=1,2,\dots, \tag{4.5}$$

where

$$V_1 = 1 \quad , \quad V_{p-1} = \frac{g_{p-1}}{1-V_p} \quad , \quad p = 2,3,\dots \; . \tag{4.6}$$

The relations $g_p=\int_0^1 d\gamma_p(u)>0$, $f_p(-1)=\int_0^1 d\gamma_p(u)/(1-u)$ result in

$$f_p(-1) > g_p \quad , \quad g_p > 0. \tag{4.7}$$

Hence, on account of (4.5)-(4.7) we conclude

$$V_1=1 \quad , \; 0 < V_p < 1 \quad , \quad p = 2,3,\dots \; . \tag{4.8}$$

We now pass to the problem of finding the range of $f_p(z)$ subject to (4.5). Relations (2.7) and (4.5) give

$$f_p(-1) = \int_0^1 \frac{d\gamma_p(u)}{1-u} \le V_p, \quad p=1,2,3,\dots . \tag{4.9}$$

Thus

$$d\omega_p(u) = \frac{d\gamma_p(u)}{1-u}, \quad p=1,2,3,\dots . \tag{4.10}$$

are also allowable measures in Stieltjes integrals with

$$\int_0^1 d\omega_p(u) \le V_p. \tag{4.11}$$

Hence we can write

$$f_p(z) = \int_0^1 \frac{1-u}{1+zu}\, d\omega_p(u), \tag{4.12}$$

where $d\omega_p(u)$ is an arbitrary, nonnegative and normalized measure obeying the inequality (4.11). The range of admissible values of $f_p(z)$ resulting from (4.11)-(4.12) forms a convex region obtainable for measures

$$d\omega_p(u)=V_p\delta(u-u_0)du \quad \text{and} \quad d\omega_p(u)=V_p[(1-\alpha)\delta(u)+\alpha\delta(u-1)]du, \tag{4.13}$$

where $u_0\in[0,1]$ and $0\le\alpha\le1$, cf. [8, Chap.17A]. By substituting (4.13) into (4.12) we obtain the auxiliary bounding functions $F_p^{(p)}(z,u)$

$$F_p^{(p)}(z,u) = V_p\, F(z,u), \tag{4.14}$$

where

$$F(z,u) = \begin{cases} \dfrac{z(1-u)}{1+zu}\ , \text{ if } 0 \le u \le 1, \\ (1+u)z\ , \text{ if } -1 \le u \le 0, \end{cases} \tag{4.15}$$

describing in the cut $(-\infty, -1)$ $F_p^{(p)}$-complex plane the boundary of a convex region $\mathfrak{Z}_p^{(p)}(z)$ of admissible values of the Stieltjes function $zf_p(z)$ appearing in (2.10). The curve $F(z,u)$, given by (4.15) and called the elementary bounding function, consists of a circular arc $F'(u,z)$ and of a straight line $F''(u,z)$

$$F'(u,z) = \frac{z(1-u)}{1+zu}\ ,\ 0 \le u \le 1\ ;\ F''(u,z) = (1+u)z\ ,\ -1 \le u \le 0. \tag{4.16}$$

Relations (4.16) were originally derived in a different way by Bergman [14]. For p=1 we have $V_1=1$. Hence the bounding function (4.15)-(4.16) forms in the cut $(-\infty,-1)$ complex plane the convex, lens-shaped region

$$\mathfrak{Z}_1^{(1)}(z) = \left\{ \frac{z\ v\ (1-u)}{1\ +\ z\ u} \middle|\ 0 \le u, v \le 1 \right\}, \tag{4.17}$$

of admissible values of the Stieltjes functions $zf_1(z)$, see Fig.3. Sometimes it is useful to characterize the whole circle $(4.16)_1$ and the straight line $(4.16)_3$ by the triplets of points $\{0,z,-1\}$ and $\{0,z,\infty\}$ obtainable from $(4.16)_1$ for $u=1,0,\infty$ and from $(4.16)_3$ for $u=-1,0,\infty$, respectively. The formulae for determining the circle

$$x = x_0 + R\cos\Phi \quad , \quad y = y_0 + R\sin\Phi \quad , \quad 0 \le \Phi \le 2\pi, \tag{4.18}$$

$$R = \sqrt{(x-x_0)^2+(y-y_0)^2}, \tag{4.19}$$

from triplet of points $\{z_1=x_1+iy_1,\ z_2=x_2+iy_2,\ z_3=x_3+iy_3\}$ are as follows

$$x_{12}=\frac{x_1+x_2}{2}\ ,\ y_{12}=\frac{y_1+y_2}{2}\ ,\ x_{23}=\frac{x_2+x_3}{2}\ ,\ x_{23}=\frac{y_2+y_3}{2}, \tag{4.20}$$

$$m_1=\frac{y_1-y_2}{x_1-x_2}\ ,\ m_2=\frac{y_2-y_3}{x_2-x_3}, \tag{4.21}$$

$$x_0=\frac{\frac{1}{m_2}\ x_{23} - \frac{1}{m_1}\ x_{12}+y_{23}-y_{12}}{\frac{1}{m_2}-\frac{1}{m_1}}\ ,\ y_0=\frac{x_{23}-x_{12}+m_2y_{23}-m_1y_{12}}{m_2-m_1}. \tag{4.22}$$

Here R and (x_0,y_0) denote the radius of the circle and its center.

5. Basic bounds $F_1^{(p)}$. For p=1,2,... the properties of $F_1^{(p)}(z)$- basic bounds determined by means of bounding functions (3.7) will now be investigated. Let's consider the linear fractional transformation (3.9). For j=0 we have

$$F_{p-1}^{(p)}(z,u) = \frac{g_{p-1}\ z}{1 + F_p^{(p)}(z,u)}. \tag{5.1}$$

The relation (5.1) maps a lens-shaped boundary $F_p^{(p)}(z)$, given by Eq. (4.14)-(4.15) into another lens-shaped boundary

$F_{p-1}^{(p)}(z)$. Consequently the basic bounds $F_1^{(1)}(z)$, $F_1^{(2)}(z)$, ..., $F_1^{(p)}(z)$, in which we are particular interested, are also lens-shaped, see Appendix A and Fig.3.

We pass now to the auxiliary bounding functions $F_{p-1}^{(p-1)}(z)$ given by (4.14)-(4.15):

$$F_{p-1}^{(p-1)}(z,u) = V_{p-1}\ F(z,u)\ ,\ -1\le u\le 1 \tag{5.2}$$

and to $F_{p-1}^{(p)}(z,u)$ determined by (5.1) with the help of (4.6)

$$F_{p-1}^{(p)}(z,u) = \frac{V_{p-1}\ (1-V_p)\ z}{1+\ V_p\ F(z,u)}\ ,\quad -1\le u\le 1, \tag{5.3}$$

where $F(z,u)$ is given by (4.15). By analyzing the formulae (5.2)-(5.3) we conclude that the bounding functions $F_{p-1}^{(p-1)}(z,u)$ and $F_{p-1}^{(p)}(z,u)$ have two intersection points given by

$$F_{p-1}^{(p)}(z,0) = F_{p-1}^{(p-1)}(z,V_p) = \frac{V_{p-1}\ (1-V_p)\ z}{1\ +\ V_p\ z}, \tag{5.4}$$

$$F_{p-1}^{(p)}(z,-1) = F_{p-1}^{(p-1)}(z,-V_p) = V_{p-1}\ (1-V_p)\ z. \tag{5.5}$$

The continued fraction representation (3.7) may be rewritten for $F_1^{(p-1)}(z,u)$ and $F_1^{(p)}(z,u)$ as follows:

$$F_1^{(p-1)}(z,u) = \frac{g_1\ z}{1} {}_{+} \frac{g_2\ z}{1} {}_{+} \cdots {}_{+} \frac{g_{p-2}\ z}{1} {}_{+} \frac{F_{p-1}^{(p-1)}(z,u)}{1}, \tag{5.6}$$

$$F_1^{(p)}(z,u) = \frac{g_1\ z}{1} {}_{+} \frac{g_2\ z}{1} {}_{+} \cdots {}_{+} \frac{g_{p-2}\ z}{1} {}_{+} \frac{F_{p-1}^{(p)}(z,u)}{1}. \tag{5.7}$$

Since $F_{p-1}^{(p-1)}(z,u)$ intersects $F_{p-1}^{(p)}(z,u)$ at two points then it follows from (5.6)-(5.7) that the basic bounding functions $F_1^{(p-1)}(z,u)$ and $F_1^{(p)}(z,u)$ (p=2,3,...) have also two intersections points, say z_1 and z_2 determined by

$$\begin{aligned} z_1 &= F_1^{(p-1)}(z,V_p) = F_1^{(p)}(z,0), \\ z_2 &= F_1^{(p-1)}(z,-V_p) = F_1^{(p)}(z,-1). \end{aligned} \tag{5.8}$$

Now we are in position to examine the sequence of basic inclusion regions $\mathfrak{F}_1^{(p)}(z)$ enclosed in the basic bounds $F_1^{(p)}(z)$. The inequalities (B.18) and (B.24) derived in Appendix B, lead directly to

$$zf_{p-1}(z) \in \mathfrak{F}_{p-1}^{(p)}(z) \subset \mathfrak{F}_{p-1}^{(p-1)}(z). \tag{5.9}$$

On the basis of (3.6) we have

$$\mathfrak{F}_1^{(p-1)}(z,u,v) = \frac{g_1\ z}{1} + \cdots + \frac{g_{p-2}\ z}{1} + \frac{\mathfrak{F}_{p-1}^{(p-1)}(z,u,v)}{1}, \tag{5.10}$$

$$\mathfrak{F}_1^{(p)}(z,u,v) = \frac{g_1\ z}{1} + + \cdots + \frac{g_{p-2}\ z}{1} + \frac{\mathfrak{F}_{p-1}^{(p)}(z,u,v)}{1}. \tag{5.11}$$

Relations (5.9)-(5.11) imply that

$$zf_1(z) \in \mathfrak{F}_1^{(p)}(z) \subset \mathfrak{F}_1^{(p-1)}(z) \subset \ldots \subset \mathfrak{F}_1^{(1)}(z). \tag{5.12}$$

Relations (5.12) was reported by Bergman in [13], however without proof. The main result of this section reads: the successive inclusions regions $\mathfrak{F}_1^{(p)}(z)$ (p=1,2,...) for the Stieltjes function $zf_1(z)$ form a descending sequence such that the smaller bound $F_1^{(p)}(z)$ touches the previous one $F_1^{(p-1)}(z)$ at just two points specified by (5.9). The shape of each bounds is that of a convex lens, i.e., consists of a two circle arcs, see Fig.3. The above conclusion results also from the methods of construction of the complex bounds proposed by Milton [2], Bergman [14] and Felderhof [4].

6. **Basic bounds $E_1^{(p)}$.** In this Section we derive the bounds $E_1^{(p)}(z)$ for the function $ze_1(z)$ defined by

$$1 + z\, e_1(z) = \frac{z + 1}{1 + z\, f_1(z)}, \tag{6.1}$$

where $zf_1(z)$ is given by (2.2)-(2.3). On account of (2.2)-(2.3) the last relation is equivalent to the formula

$$\frac{\varepsilon_2}{\varepsilon_e} - 1 = z\, e_1(z). \tag{6.2}$$

To investigate the properties of $e_1(z)$ it is convenient to rewrite (6.1) in the following form

$$1 + z\, e_1(z) = \frac{1}{e_0(0)}\, (z + 1)\, e_0(z), \tag{6.3}$$

where

$$e_0(z) = \frac{e_0(0)}{1 + z\, f_1(z)}. \tag{6.4}$$

is a Stieltjes function:

$$e_0(z) = \int_0^1 \frac{d\beta_0(u)}{1+zu}. \tag{6.5}$$

The formula (6.5) is a consequence of the linear fractional transformation (6.4) applied to the Stieltjes function $f_1(z)$ [8, Lemma 15.3] and of the inequality $1-f_1(-1)\geq 0$, cf. (4.2). Now we are in position to write the following identity

$$(z + 1) \int_0^1 \frac{d\beta_0(u)}{1 + z\, u} = e_0(0) \left[1 + z \int_0^1 \frac{d\beta_1(u)}{1 + z\, u} \right], \tag{6.6}$$

where

$$e_0(0) = \int_0^1 d\beta_0(u) \quad , \quad d\beta_1(u) = \frac{(1 - u)\ d\beta_0}{e_0(0)}. \tag{6.7}$$

This identity is obtained by expanding both sides of (6.6) in the power series at z=0. By substituting (6.6)-(6.7) via (6.5) into (6.3) we get

$$e_1(z) = \int_0^1 \frac{d\beta_1(u)}{1 + z\, u}. \tag{6.8}$$

On the basis of (6.1) and (4.2) we additionally have

$$e_1(-1) \leq 1. \tag{6.9}$$

By comparing (2.3) with (6.8) and (4.2) with (6.9) we conclude that the function $e_1(z)$, defined by (6.1), has the same analytical properties as the function $f_1(z)$, defined by (2.3). The properties of functions $zf_1(z)$ and $ze_1(z)$ were investigated in detail by Bergman in [5,6]. Consequently the bounding functions $B_e^{(p)}(z,u)$ on the effective, dielectric constant $\varepsilon_e/\varepsilon_1$ calculated from (6.2) can be expressed as follows

$$B_e^{(p)}(z,u) = \frac{z + 1}{1 + E_1^{(p)}(z,u)}. \tag{6.10}$$

Here $E_1^{(p)}(z,u)$ is a bounding function for $ze_1(z)$ analogous to the bounding function $F_1^{(p)}(z,u)$ for $zf_1(z)$, cf. Def.2. The coefficients of the power expansion of the Stieltjes function $ze_1(z)$ are necessary for a calculation of the bounds $B_e^{(p)}$ on $\varepsilon_e/\varepsilon_1$. In order to find them we now propose a simple recurrence formula. The power expansion of $ze_1(z)$:

$$ze_1(z) = \sum_{j=1}^{\infty} d_j^{(1)} z^j, \tag{6.12}$$

is a Stieltjes series with coefficients

$$d_j^{(1)} = (-1)^{j+1} \int_0^1 u^{j-1}\, d\beta_1(u). \tag{6.13}$$

The power series $zf_1(z)$ and $ze_1(z)$ have to satisfy Eq. (6.1), i.e.:

$$1 + c_1^{(1)}z + c_2^{(1)}z + \dots = \frac{z+1}{1 + d_1^{(1)}z + d_2^{(1)}z^2 + \dots} \tag{6.14}$$

Multiplying both sides of (6.14) by $(1+d_1^{(1)}z+d_2^{(1)}z^2+\dots)$ and collecting terms appearing at the same power of z we obtain

$$d_0^{(1)}=1\ ,\ d_1^{(1)}=1-c_1^{(1)}, \tag{6.15}$$

$$d_n^{(1)}=-\sum_{j=0}^{n-1} d_j^{(1)}\ c_{n-j}^{(1)}\ ,\ n=2,3,\dots \tag{6.16}$$

The recurrence formula (6.15) allow us to evaluate (p-1) coefficients of the expansion of $ze_1(z)$, provided that the first (p-1) coefficients of the series $zf_1(z)$ are known.

7. Bounds $B_F^{(p)}$ and $B_E^{(p)}$. In this Section we shall prove that the S-continued fraction method applied to the series $zf_1(z)$, given by (2.4), and to the power expansion $ze_1(z)$ defined by (6.12), lead to bounds $B_f^{(p)}(z)$ and $B_e^{(p)}(z)$, that are exactly the same, i.e.:

$$B_f^{(p)}(z) = B_e^{(p)}(z). \tag{7.1}$$

On account of (2.2), (2.10) and (6.1) the effective dielectric coefficient $\varepsilon_e/\varepsilon_1$ has the following continued fraction representations

$$\frac{\varepsilon_e}{\varepsilon_1} = 1 + \frac{g_1^f\ z}{1} + \frac{g_2^f\ z}{1} + \dots + \frac{g_{p-1}^f\ z}{1} + \frac{v_p^f\left(-1+\dfrac{z+1}{1+ze_p'(z)}\right)}{1} \tag{7.2}$$

and

$$\frac{\varepsilon_e}{\varepsilon_1} = \cfrac{z+1}{1 + \cfrac{g_1^e\, z}{1} + \cfrac{g_2^e\, z}{1} + \dots + \cfrac{g_{p-1}^e\, z}{1} + \cfrac{V_p^e\, z\, e_p(z)}{1}}, \tag{7.3}$$

where, on account of (4.6)

$$g_{p-1}^S = V_{p-1}^S\,(1-V_p^S)\ ,\ s=f,e. \tag{7.4}$$

By applying the elementary bound F(z) to the functions $ze_p'(z)$ and $ze_p(z)$ appearing in (7.2)-(7.3) we obtain

$$B_f^{(p)}(z,u)=1+ \frac{g_1^f z}{1} + \frac{g_2^f z}{1} + \dots + \frac{g_{p-1}^f z}{1} + \cfrac{V_p^f\left(-1+\cfrac{z+1}{1+F(z,u)}\right)}{1}, \tag{7.5}$$

and

$$B_e^{(p)}(z,u)= \cfrac{z+1}{1+ \cfrac{g_1^e\, z}{1} + \cfrac{g_2^e\, z}{1} + \dots + \cfrac{g_{p-1}^e\, z}{1} + \cfrac{V_p^e\, F(z,u)}{1}} \tag{7.6}$$

respectively.

Now we shall demonstrate the following equalities

$$1+ \frac{g_1^f\, z}{1} + \dots + \frac{g_{p-1}^f\, z}{1} + \frac{V_p^f\, z}{1} = \cfrac{z+1}{1 + \cfrac{g_1^e\, z}{1} + \dots + \cfrac{g_{p-1}^e\, z}{1}}, \tag{7.7}$$

$$1+\frac{g_1^f\,z}{1}+\dots+\frac{g_{p-1}^f\,z}{1}=\frac{z+1}{1+\frac{g_1^e\,z}{1}+\dots+\frac{g_{p-1}^e\,z}{1}+\frac{V_p^e\,z}{1}}. \tag{7.8}$$

The rational functions (7.8) and (7.9) are identities, since: 1) they are satisfied for z=-1; 2) the number of p-1 coefficients of power expansions of the left and right sides of (7.7) and (7.8) coincide at z=0. By substituting (7.4) in (7.7) and (7.8) respectively we obtain

$$V_1^e=V_1^f=1\ ,\ V_p^f+V_p^e=1;\ p=2,\ 3,\ \dots\ . \tag{7.9}$$

We pass now to investigating of the bounds (7.5) and (7.6) with the help of the relations (7.7) and (7.8). Bounds (7.5) and (7.6) are uniquely determined by two circles, that are the images of the elementary circle and the straight line

$$F'(z)=\frac{z(1-u)}{1+zu}\ ,\ -\infty\le u\le\infty\quad;\quad F''(z)=z(1+u)\ ,\ -\infty\le u\le\infty, \tag{7.10}$$

following from (4.15), see also (3.4). Replacing F(z,u) in (7.5) and (7.6) successively by $(7.10)_1$ and $(7.10)_2$ we obtain with the help of (7.7)-(7.8).

$$\begin{aligned}
&B_f'^{(p)}(z,0)=B_e'^{(p)}(z,1)\quad;\quad B_f''^{(p)}(z,0)=B_e''^{(p)}(z,-1)\ ,\\
&B_f'^{(p)}(z,1)=B_e'^{(p)}(z,0)\quad;\quad B_f''^{(p)}(z,-1)=B_e''^{(p)}(z,0)\ ,\\
&B_f'^{(p)}(z,\infty)=B_e'^{(p)}(z,\infty)\quad;\quad B_f''^{(p)}(z,-\infty)=B_e''^{(p)}(z,-\infty).
\end{aligned} \tag{7.11}$$

Since the triplets of points $u\in\{0,1,\infty\}$ and $u\in\{0,-1,-\infty\}$ uniquely define the circles given by (7.5) and (7.6) respectively, thus (7.11) is valid for arbitrary $0 \le u \le 1$, cf. (7.1).

To illustrate the equivalence of the bounds $B_f^{(p)}(z)$ and $B_e^{(p)}(z)$ resulting from the power series $zf_1(z)$ and $ze_1(z)$ respectively, we examine the following Stieltjes functions

$$zf_1(z) = \ln(1+\eta z) \quad , \quad ze_1(z) = \frac{z+1}{1+\ln(1+\eta z)} - 1, \tag{7.12}$$

defined in the cut $(-\infty,-1/\eta)$ complex plane, where η is a real, positive number. Functions $(7.12)_{1,2}$ have Stieltjes-integral representations:

$$zf_1(z) = z\int_0^1 \frac{d\gamma_1(u)}{1+zu}\,du \quad , \quad ze_1(z) = z\int_0^1 \frac{d\beta_1(u)}{1+zu}\,du. \tag{7.13}$$

It is worth noting that the spectrum $\gamma_1(u)$ appearing in $(7.13)_1$ has the form

$$\gamma_1(u) = \begin{cases} u \quad , \quad \text{if } 0 \le u \le \eta, \\ \eta \quad , \quad \text{if } \eta \le u. \end{cases} \tag{7.14}$$

Consider now the power expansions of $zf_1(z)$ and $ze_1(z)$:

$$zf_1(z)=(-1)^{n+1}\sum_{n=1}^{\infty} c_n^{(1)} z^n \ , \ ze_1(z)=(-1)^{n+1}\sum_{n=1}^{\infty} d_n^{(1)} z^n, \tag{7.15}$$

where

$$c_n^{(1)} = \frac{1}{n}\ \eta^n, \tag{7.16}$$

while $d_n^{(1)}$ are determined by (7.16) via (6.15). One can easily verify that for $0 \le \eta \le 0.6$ the functions $(7.12)_{1,2}$ satisfy

$$f_1(-1) = -\ln(1-\eta) \le 1 \quad , \quad e_1(-1) = \lim_{z \to -1} \frac{z+1}{z(1+\ln(1+\eta z))} - 1 \le 1. \qquad (7.17)$$

Hence bulk, dielectric constant $\varepsilon_e(z)/\varepsilon_1$ can be represented by $zf_1(z)$ if $(7.12)_1$ is used and by $ze_1(z)$ in the case $(7.12)_2$.

$$\frac{\varepsilon_e(z)}{\varepsilon_1} = 1+zf_1(z) \quad , \quad \frac{\varepsilon_e(z)}{\varepsilon_1} = \frac{z+1}{1+z_1 e(z)} \quad , \quad 0 \le \eta \le 0.6. \qquad (7.18)$$

The starting point for our numerical procedure are the coefficients $c_k^{(1)}$ and $d_k^{(1)}$ of power series $(7.15)_1$ and $(7.15)_2$, see Tab.1. By applying the recurrence formulae (2.14)-(2.20) and (4.6) to the power series $(7.15)_{1,2}$ we get directly the coefficients g_j^f, g_j^e and constants V_p^f, V_p^e appearing in (3.7) and (4.14), see Tabs 2, 3 and Figs 1, 2.

Table 1. The coefficients $c_k^{(1)}$ and $d_k^{(1)}$ of power expansions of the Stieltjes functions $(7.12)_{1,2}$.

η	k	1	2	3	4	5	6	7
0.5	$c_k^{(1)}$	0.5000	0.1250	0.0416	0.0156	0.0062	0.0026	0.0011
0.5	$d_k^{(1)}$	0.5000	1.2500	0.0833	0.0625	0.0484	0.0379	0.0299
0.6	$c_k^{(1)}$	0.6000	0.1800	0.0720	0.0324	0.0155	0.0078	0.0040
0.6	$d_k^{(1)}$	0.4000	0.0600	0.0360	0.0288	0.0255	0.0234	0.0219

Note that the relation (7.9) between the constants V_p^f and V_p^e is satisfied. By using the recurrence formula (3.9) we also calculated the basic bounds $F_1^{(p)}(z)$ and $E_1^{(p)}(z)$ on Stieltjes functions $zf_1(z)$ and $ze_1(z)$ given by $(7.12)_1$ and $(7.12)_2$ respectively.

Table 2. The coefficients g_p^f and g_p^e of the continued fraction expansions of the Stieltjes functions $(7.12)_{1,2}$.

η	g \ p	1	2	3	4	5	6	7
0.5	g_p^f	0.5000	0.2500	0.08333	0.1666	0.1000	0.1500	0.1071
0.5	g_p^e	0.5000	0.2500	0.41666	0.1333	0.1750	0.1035	0.4926
0.6	g_p^f	0.6000	0.3000	0.1000	0.2000	0.1200	0.1800	0.2857
0.6	g_p^e	0.4000	0.1500	0.4500	0.2666	0.2733	0.1404	0.1833

Table.3 The constants V_p^f and V_p^e for the auxiliary bounds $F_p^{(p)}(z)$ and $E_p^{(p)}(z)$.

η	V \ p	1	2	3	4	5	6	7
0.5	V_p^f	1.0000	0.5000	0.5000	0.8333	0.8000	0.8750	0.8286
0.5	V_p^e	1.0000	0.5000	0.5000	0.1667	0.2000	0.1250	0.1714
0.6	V_p^f	1.0000	0.4000	0.2500	0.6000	0.6667	0.8200	0.7805
0.6	V_p^e	1.0000	0.6000	0.7500	0.4000	0.3333	0.1800	0.2195

The sequences of bounds $F_1^{(p)}(z)$ and $E_1^{(p)}(z)$, (p=1,2,3,4) are depicted in Figs 3: for p=1, $F_1^{(p)}(z)$ and $E_1^{(p)}(z)$ coincide: for p=2,3,4 they differ. Fig 4 presents the bounds $B_f^{(p)}(z)$, $B_e^{(p)}(z)$ defined by (3.10) and (6.10). The curves $B_f^{(p)}(z)$ and $B_e^{(p)}(z)$ coincide for all values of p=1,2,3.4,... .

8. **Regular arrays of spheres.** To illustrate a practical application of the S-continued fraction method for evaluation of the complex effective dielectric constant we consider simple, body-centred and face-centred, cubic lattices of spheres embedded in an infinite matrix material. By ε_e, ε_2, ε_1 we denote the dielectric constants of a composite, of spheres and of a matrix respectively. For a macroscopically isotropic, two-component composites the first two coefficients of power series

(2.4) are as follows [3]

$$\frac{\varepsilon_e}{\varepsilon_1} - 1 = zf_1(z) = \varphi_2\, z - \frac{1}{d}\,\varphi_1\,\varphi_2\, z^2 + \dots\,, \tag{8.1}$$

where $z=\varepsilon_2/\varepsilon_1-1$. Here φ_2, φ_1 denote volume fractions of the inclusions and of the matrix respectively, while d=2 for two- and d=3 for three-dimensional systems. On account of (2.14)-(2.20), the S-continued fraction (2.10) corresponding to (8.1) is expressed by

$$zf_1(z) = \frac{\varphi_2\, z}{1} {}_{+} \frac{(\varphi_1/d)\, z}{1} {}_{+} \dots\,. \tag{8.2}$$

On the basis of (3.7) and (2.14)-(2.18), the low order bounding functions $F_1^{(p)}(z,u)$ (p=1,2,3) take the form

$$F_1^{(1)}(z,u) = F(z,u)\quad,\quad F_1^{(2)}(z,u)= \frac{\varphi_2\, z}{1} {}_{+} \frac{V_2^f\, F(z,u)}{1}, \tag{8.3}$$

$$F_1^{(3)}(z,u)=\frac{\varphi_2\, z}{1} {}_{+} \frac{(\varphi_1/d)\, z}{1} {}_{+} \frac{V_3^f\, F(z,u)}{1}, \tag{8.4}$$

where F(z,u) is the elementary bounding function, cf. (4.15). The constants V_p^f (p=2,3) resulting from (4.6) are

$$V_2^f = \varphi_1\quad,\quad V_3^f = \frac{d-1}{d}. \tag{8.5}$$

By substituting (4.14)-(4.15) in (8.3) and (8.4) respectively we can specify low order bounding functions for $\varepsilon_e/\varepsilon_1$:

1) p=1- no coefficients of the series (8.1) are known, then

$$B_f^{(1)}(z,u) = \begin{cases} 1 + \dfrac{\dfrac{1-u}{1+zu}\, z}{1}, & \text{if } 0\le u\le 1, \\[2ex] 1 + \dfrac{(1+u)\, z}{1}, & \text{if } -1\le u\le 0; \end{cases} \tag{8.6}$$

2) p=2- the first coefficient of the series (8.1) is known, then

$$B_f^{(2)}(z,u) = \begin{cases} 1 + \dfrac{\varphi_2\, z}{1} + \dfrac{\varphi_1\, \dfrac{1-u}{1+zu}\, z}{1}, & \text{if } 0\le u\le 1, \\[2ex] 1 + \dfrac{\varphi_2\, z}{1} + \dfrac{\varphi_1\, (1+u)\, z}{1}, & \text{if } -1\le u\le 0; \end{cases} \tag{8.7}$$

3) p=3- two coefficients of the power expansion of (8.1) are given, then

$$B_f^{(3)}(z,u) = \begin{cases} 1 + \dfrac{\varphi_2\, z}{1} + \dfrac{\dfrac{\varphi_1\, z}{d}}{1} + \dfrac{\dfrac{d-1}{d}\, \dfrac{1-u}{1+zu}\, z}{1}, & \text{if } 0\le u\le 1, \\[2ex] 1 + \dfrac{\varphi_2\, z}{1} + \dfrac{\dfrac{\varphi_1\, z}{d}}{1} + \dfrac{\dfrac{d-1}{d}\, (1+u)\, z}{1}, & \text{if } -1\le u\le 0. \end{cases} \tag{8.8}$$

The S-continued fraction bounds (8.6)-(8.8) coincide with the low order complex bounds first derived by Milton [1] and independently by Bergman [14] in a form of a rational functions. Figs 5,6 and 7 present the circular arcs $B_f^{(1)}(z)$, $B_f^{(2)}(z)$ and $B_f^{(3)}(z)$.

The calculation of bounds more narrow then (8.6)-(8.8) requires knowledge of many terms of a power expansion of $\varepsilon_e(z)/\varepsilon_1$ at z=0. For the simple, body-centred and face-centred cubic lattices of spheres McPhedran and Milton [15] evaluated a number of coefficients of a power expansion of $\varepsilon_e(\alpha)/\varepsilon_1$, $\alpha=z/(z+2)$ at $\alpha=0$ and gathered them in the tables as discrete functions of φ_2. In Appendix C we develop a simple formula relating the terms of a power series of $\varepsilon_e(z)/\varepsilon_1$ to the terms of the power expansion of $\varepsilon_e(\alpha)/\varepsilon_1$, $\alpha=z/(z+2)$. From the coefficients given in [15, Tabs 6,7,8] we calculated via the algorithm (C.10)-(C.13) the coefficients $c_n^{(1)}$ of the power series (2.4). The S-continued fraction procedure (2.14)-(2.18) and the recurrence relations (4.6) applied to the coefficients $c_n^{(1)}$ led to the continued fraction coefficients g_n^f and constants V_p gathered in Tab.4.

Table 4. $c_n^{(1)}$-the coefficients of the power expansion of $\varepsilon_e/\varepsilon_1$; g_n^f-the coefficients of the S-continued fraction to $\varepsilon_e/\varepsilon_1$; V_p^f-the coefficients for the auxiliary bounds $F_p^{(p)}(z)$.

Arrays of spheres		n=1	n=2	n=3	n=4	n=5	n=6	n=7
φ_2=0.5	$c_n^{(1)}$	0.50	0.0833	0.0235	0.0019	0.0043	0.0023	
Simple	g_n^f	0.50	0.1667	0.1158	0.2693	0.1170	0.3203	
cubic	V_n^f	1.00	0.5000	0.6667	0.8261	0.6741	0.8263	0.6162
φ_2=0.6	$c_n^{(1)}$	0.60	0.0800	0.0149	0.0042	0.0016	0.0008	
Body-	g_n^f	0.60	0.1333	0.0530	0.3376	0.0710	0.3856	
centered	V_n^f	1.00	0.4000	0.6667	0.9204	0.6331	0.8878	0.5657
φ_2=0.7	$c_n^{(1)}$	0.70	0.0700	0.0144	0.0056	0.0028	0.0016	
Face-	g_n^f	0.70	0.1000	0.1064	0.3458	0.1046	0.3381	
centered	V_n^f	1.00	0.3000	0.6667	0.8403	0.5884	0.8221	0.5872

The sequence of higher order bounds $B_f^{(p)}(z)$ (p=4,5,6) of the effective dielectric constants $\varepsilon_e/\varepsilon_1$ of regular arrays of spheres are depicted in Figs 5,6 and 7. The complex bounds obtained here can be improved by using the inequality

$$\frac{\varepsilon_e(x)}{\varepsilon_1}\frac{\varepsilon_e(y)}{\varepsilon_1} \geq 1, \text{ if } y = -\frac{x}{x+1}, \; x\geq -1 \tag{8.9}$$

derived by Schulgasser for isotropic and cubic systems ([19], [5]).

9. Real effective dielectric constants. In this section we study a two-phase composite material for the case where the dielectric constants of both components ε_1 and ε_2 are real. For z=x the elementary inclusion region $\mathfrak{J}(z)$ enclosed in the bound F(z) given by (4.14)-(4.15) reduces to the interval [0,x] occupying the real axis of a $\varepsilon_e/\varepsilon_1$-complex plane:

$$\mathfrak{J}(x) = \left\{x(1+u) \mid -1\leq u\leq 0\right\}. \tag{9.1}$$

Therefore the inclusion function $\mathfrak{B}_f^{(p)}(x,u)$ for $\varepsilon_e(x)/\varepsilon_1$ take, on the basis of (3.7) and (3.10), the following S-continued fraction form

$$\mathfrak{B}_f^{(p)}(x,u)=1+\frac{g_1^f\ x}{1}{}_{+}\frac{g_2^f\ x}{1}{}_{+\cdots+}\frac{g_{p-1}^f\ x}{1}{}_{+}\frac{V_p^f(1+u)x}{1}, \; -1\leq u\leq 0. \tag{9.2)}$$

Since the coefficients g_{p-1}^f satisfy (2.12) one can easily prove via recurrence formula (3.9), that the bounding function (9.2) obeys the following inequalities

$$\frac{\partial B_f^{(p)}(x,u)}{\partial u}\begin{cases}\le 0 \text{ , if p is odd and } x\ge 0,\\ \ge 0 \text{ , if p is odd and } -1\le x\le 0,\\ \ge 0 \text{ , if p is even and } x\ge 0,\\ \le 0 \text{ , if p is even and } -1\le x\le 0,\end{cases} \tag{9.3}$$

for every $u\in[-1,0]$. Due to (9.3) $B_f^{(p)}(x,u)$ takes the extremal values for u=0 and u=-1. Therefore we can write

$$\begin{aligned}&B_f^{(p)}(x,0) \le \frac{\varepsilon_e}{\varepsilon_1} \le B_f^{(p)}(x,-1), \text{ if p is even and } x\ge 0,\\ &B_f^{(p)}(x,0) \le \frac{\varepsilon_e}{\varepsilon_1} \le B_f^{(p)}(x,-1), \text{ if p is even and } -1\le x\le 0,\\ &B_f^{(p)}(x,-1) \le \frac{\varepsilon_e}{\varepsilon_1} \le B_f^{(p)}(x,0), \text{ if p is odd and } x\ge 0,\\ &B_f^{(p)}(x,0) \le \frac{\varepsilon_e}{\varepsilon_1} \le B_f^{(p)}(x,-1), \text{ if p is odd and } -1\le x\le 0.\end{aligned} \tag{9.4}$$

The relations (9.4) allow us to construct lower and upper bounds on $\varepsilon_e/\varepsilon_1$, which is now real in the form of S-continued fractions given by (9.2).

10. Regular arrays of cylinders. To illustrate the theoretical results (9.4) we consider the square and hexagonal lattices of cylinders, with a dielectric constant ε_2, embedded in an infinite matrix made of a material with the dielectric coefficient ε_1. By $\varepsilon_e(x)/\varepsilon_1$ ($x=h-1$, $h=\varepsilon_2/\varepsilon_1$) we denote as previously the effective dielectric constants of a composite. By substituting z=x, d=2, u=0 and u=-1 in $(8.5)_2$-$(8.8)_2$ respectively, we get the following low order S-fraction bounds on $\varepsilon_e/\varepsilon_1$:

1) p=1- no coefficients of the series (8.1) are known, then

$$1 \le \frac{\varepsilon_e}{\varepsilon_1} \le 1+x, \quad \text{if } x\ge 0 \quad \text{and} \quad 1+x \le \frac{\varepsilon_e}{\varepsilon_1} \le 1, \quad \text{if } -1\le x\le 0; \qquad (10.1)$$

2) p=2- the first coefficient of the series (8.1) is known, then

$$1 + \frac{\varphi_2\, x}{1} + \frac{x\, \varphi_1}{1} \le \frac{\varepsilon_e}{\varepsilon_1} \le 1 + \frac{\varphi_2\, x}{1}, \text{ if } -1\le x\le 0 \qquad (10.2)$$

3) p=3- two coefficients of the power expansion (8.1) are given, then

$$1+ \frac{\varphi_2 x}{1} + \frac{0.5\varphi_1 x}{1} \le \frac{\varepsilon_e}{\varepsilon_1} \le 1+ \frac{\varphi_2 x}{1} + \frac{0.5\varphi_1 x}{1} + \frac{0.5x}{1}, \text{ if } x\ge 0, \qquad (10.3)$$

$$1+ \frac{\varphi_2 x}{1} + \frac{0.5\varphi_1 x}{1} + \frac{0.5x}{1} \le \frac{\varepsilon_e}{\varepsilon_1} \le 1+ \frac{\varphi_2 x}{1} + \frac{0.5\varphi_1 x}{1}, \text{ if } -1\le x\le 0. \qquad (10.4)$$

One can readily verify that (10.2) is equivalent to

$$\left(\varphi_1/\varepsilon_1 + \varphi_2/\varepsilon_2\right)^{-1} \le \varepsilon_e \le \varphi_1\, \varepsilon_1 + \varphi_2\, \varepsilon_2, \qquad (10.5)$$

while the formulae (10.3)-(10.4) reduce to the inequalities valid for:

(i) $h=\varepsilon_2/\varepsilon_1\ge 1$

$$\varepsilon_1 + \frac{\varphi_2}{1/(\varepsilon_2-\varepsilon_1)+\varphi_1/(d\cdot\varepsilon_1)} \le \varepsilon_e \le \varepsilon_2 + \frac{\varphi_1}{1/(\varepsilon_1-\varepsilon_2)+\varphi_2/(d\cdot\varepsilon_2)}, \qquad (10.6)$$

(ii) $h=\varepsilon_2/\varepsilon_1\le 1$

$$\varepsilon_1 + \frac{\varphi_2}{1/(\varepsilon_2-\varepsilon_1)+\varphi_1/(d\cdot\varepsilon_1)} \ge \varepsilon_e \ge \varepsilon_2 + \frac{\varphi_1}{1/(\varepsilon_1-\varepsilon_2)+\varphi_2/(d\cdot\varepsilon_2)}. \qquad (10.7)$$

The relations (10.5) and (10.6)-(10.7) are well known as the Wiener [16] and Hashin-Shtrikman bounds [17], respectively. The calculation of higher order, improved bounds requires more then two coefficients of a power expansion of $\varepsilon_e/\varepsilon_1$. For square and hexagonal arrays of cylinders the terms $c_n^{(1)}$ (n=1,2,...) of a power series of $\varepsilon_e/\varepsilon_1$ were calculated by the method presented in [18] and gathered in Tab.5. The S-continued fraction procedure (2.14)-(2.18) and (4.16) applied to $c_n^{(1)}$ (Table 5) allowed us to get continued fraction terms g_n^f and constants V_p (p=1,2,...). The values of the first six of g_n^f and seven of V_p are given in Tab.5. Figs 8 and 9 present the sequences of upper and lower bounds on the effective dielectric constant for square and hexagonal arrays of cylinders respectively. Note that the results depicted in these figures obey the inequalities (9.4).

Table 5. $c_n^{(1)}$-the coefficients of the power expansion of $\varepsilon_e/\varepsilon_1$; g_n^f-the coefficients of the S-continued fraction to $\varepsilon_e/\varepsilon_1$; V_P^f-the coefficients for the auxiliary bounds $F_p^{(p)}(z)$.

Arrays of cylinders		n=1	n=2	n=3	n=4	n=5	n=6	n=7
φ=0.75	$c_n^{(1)}$	0.75	0.0937	0.0301	0.0153	0.0092	0.0061	
Square	g_n^f	0.75	0.1250	0.1962	0.3037	0.1441	0.3558	
array	V_n^f	1.00	0.2500	0.5000	0.6075	0.5000	0.7117	0.5000
φ=0.88	$c_n^{(1)}$	0.88	0.0528	0.0108	0.0050	0.0030	0.0021	
Hexagonal	g_n^f	0.88	0.0600	0.1460	0.3539	0.1693	0.3307	
array	V_n^f	1.00	0.1200	0.5000	0.7079	0.5000	0.6614	0.5000

For odd numbers of n the coefficient V_n (n=1,3,...) take, according to Dykhne-Mendelson-Keller relation [20-22]

$$\frac{\varepsilon_e(x)}{\varepsilon_1}\,\frac{\varepsilon_e(y)}{\varepsilon_1} = 1, \text{ if } y = -\frac{x}{x+1}. \tag{10.8}$$

the constant value V_n=0.5 (Tab.5). Identity (10.8) is valid for two-dimensional composites, that have either isotropic, or cubic, or triangular, or hexagonal symmetry under rotations. In the present paper we will not analyze the identity (10.8) in the context of an improvement of the bounds for two-dimensional composites.

11. Discussion and summary. Starting from a partial information about power expansions of the characteristic, geometrical functions $zf_1(z)$ and $ze_1(z)$ we have studied theoretically and numerically the S-continued fraction method of a calculation of complex and real bounds on ε_e.

By investigating the properties of linear fractional transformations (5.2) and (5.3) the general relations between the successive inclusion regions for $zf_1(z)$ has been derived, see (5.12). It is worth noting that until now the set of estimation (5.12) was obtained mainly by employing the methods of constructing complex bounds.

Relation (5.12) has also enable us to derive an infinite set of lower and upper bounds on the real, effective dielectric constant ε_e in dependence on the ratio of physical properties of components of composite and on a number of the available power series coefficients, cf. (9.4).

We have also proved that $zf_1(z)$ and $ze_1(z)$ are equally useful for a construction of bounds on ε_e, since the S-fraction method applied to $zf_1(z)$ and $ze_1(z)$ yields exactly the same results.

The S-continued fraction method presented here is based on the numerical algorithms given by (2.14)-(2.18),(3.9) and (4.6), which are simply recursive and do not involve the solution of a large number of coupled equations.

As an example of practical application we have studied in complex and real domains the effective dielectric constants for regular arrays of spheres and regular lattices of cylinders. For spheres we have found sequences of lens-shaped, narrowing bounds estimating $\varepsilon_e/\varepsilon_1$ (Figs 5,6,7), while for cylinders an upper and lower bounds on $\varepsilon_e/\varepsilon_1$ (Figs 8,9).

The S-continued fraction method applies also to the analysis of other, mathematically identical quantities like the overall effective electrical or thermal conductivity of a two-component composite medium. It can also be extended to other properties like elastic moduli of a two-component composite.

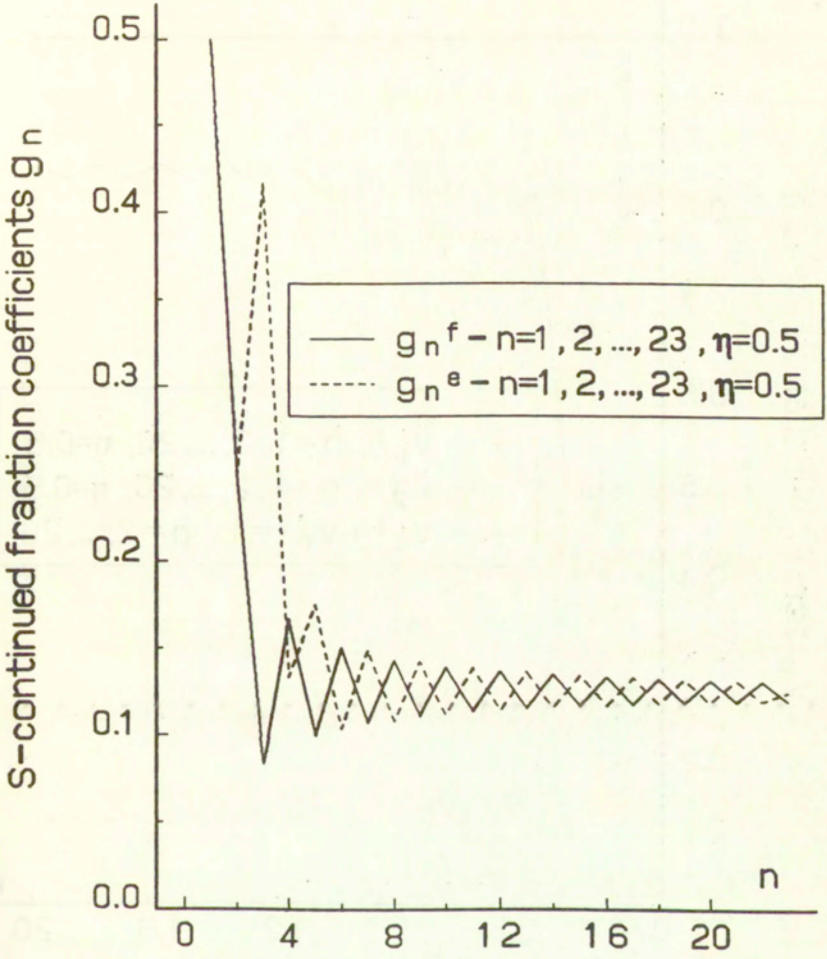

Fig. 1 The S-continued fraction coefficients $g_n^{(f)}$ and $g_n^{(e)}$ for the Stieltjes functions $zf_1(z)=\ln(1+0.5z)$ and $ze_1(z)=(z+1)/(zf_1(z)-1)$ respectively.

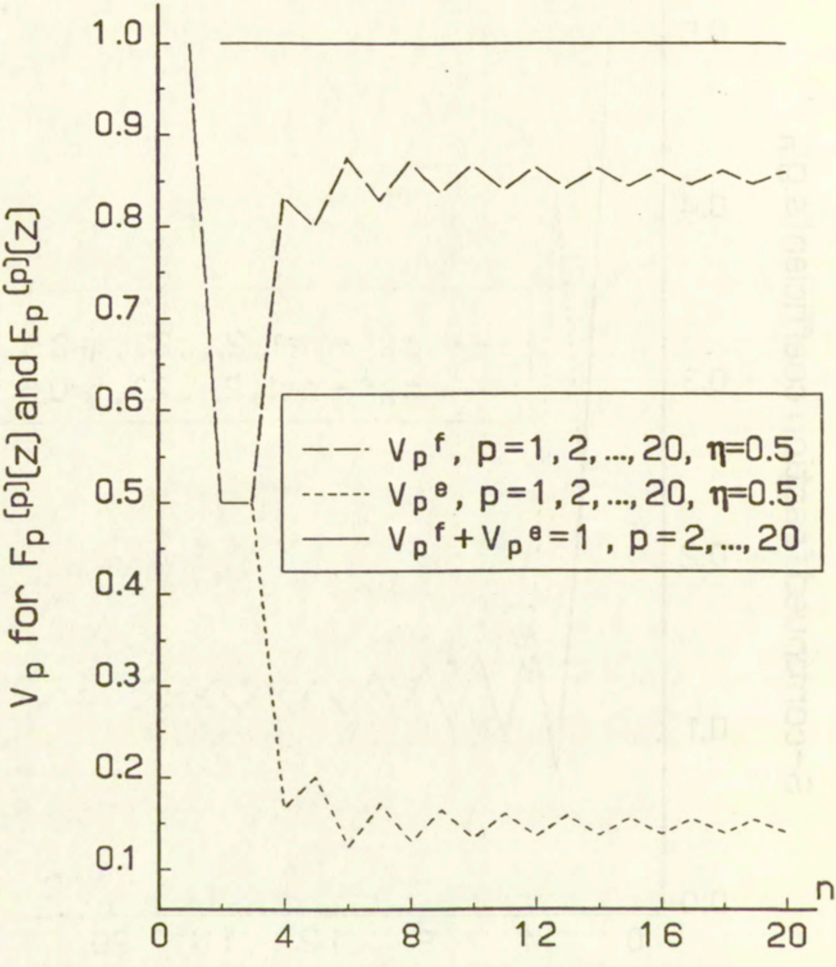

Fig.2 The constants $V_p^{(f)}$ and $V_p^{(e)}$ calculated from the S-continued fraction expansions of the Stieltjes functions for $zf_1(z)=\ln(1+0.5z)$ and $ze_1(z)=(z+1)/(zf_1(z)-1)$ respectively.

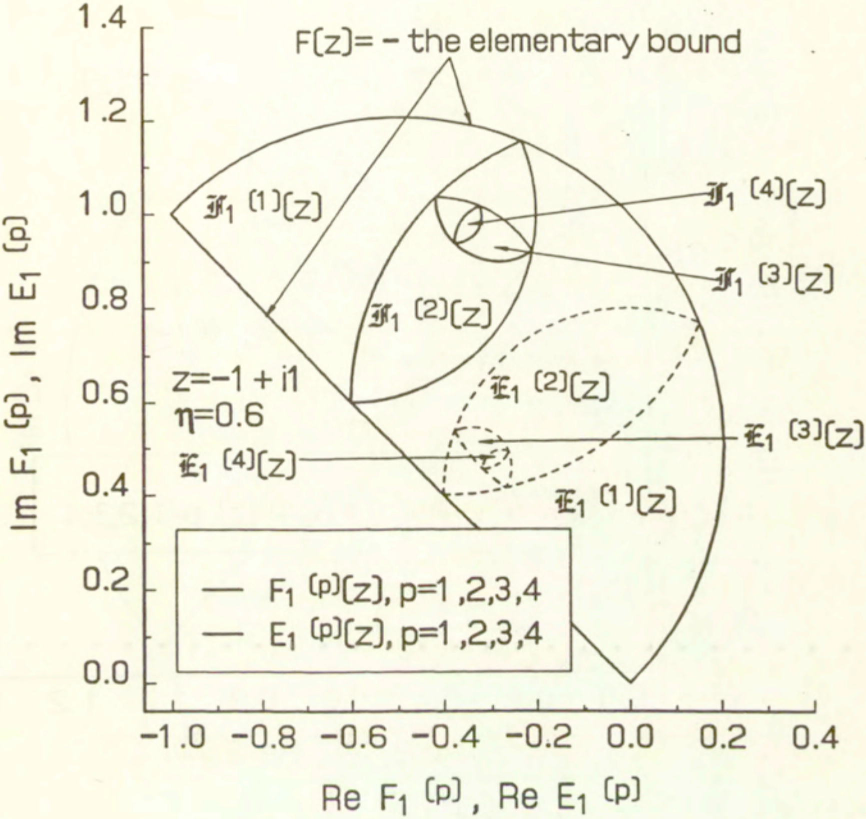

Fig. 3 The sequences of the inclusion regions $\mathfrak{F}_1^{(p)}(z)$, $\mathfrak{E}_1^{(p)}(z)$ and of the basic bounds $F_1^{(p)}(z)$, $E_1^{(p)}(z)$ for $zf_1(z)=\ln(1+0.5z)$ and $ze_1(z)= (z+1)/(zf_1(z)-1)$ respectively. The elementary bound $F(z)$ coincides with the basic bounds $F_1^{(1)}(z)$ and $E_1^{(1)}(z)$.

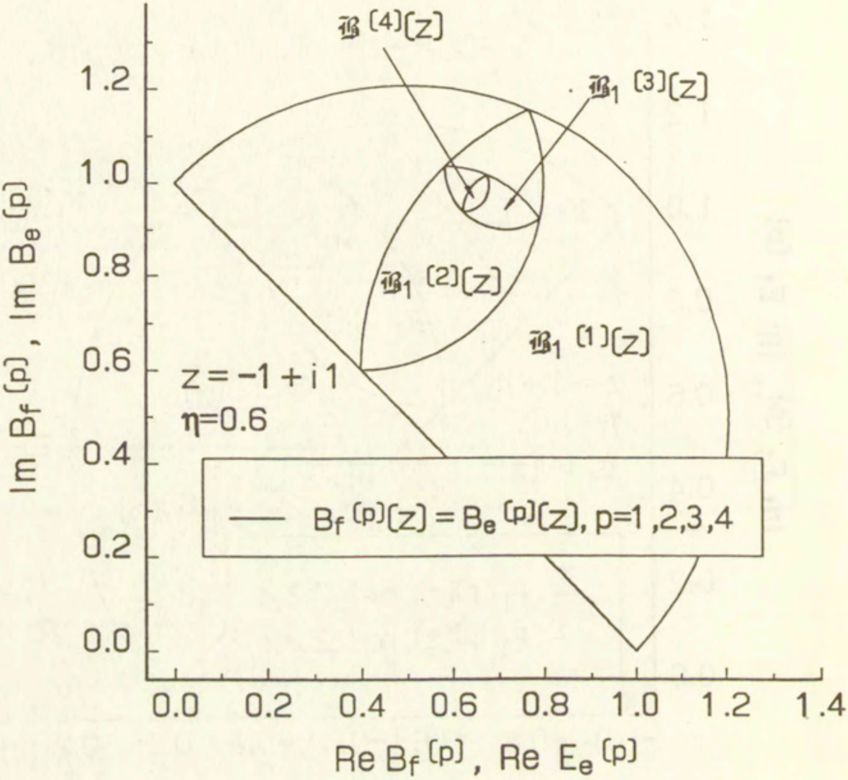

Fig. 4 The sequences of the lens-shaped bounds $B_f^{(p)}(z)=B_e^{(p)}(z)$ on the effective dielectric constant $\varepsilon_e/\varepsilon_1$ represented by $1+\ln(1+0.5z)$.

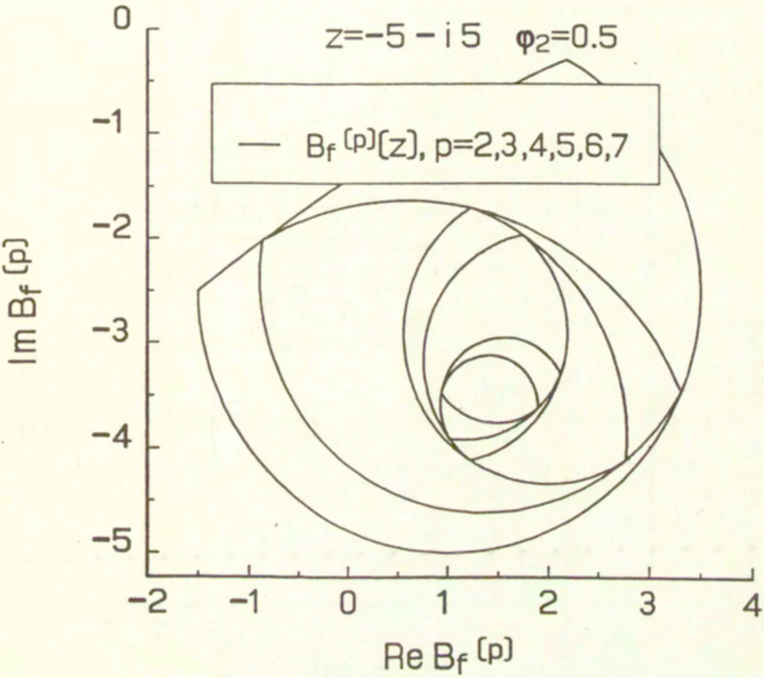

Fig.5 The sequence of the lens-shaped bounds $B_f^{(p)}(z)$ ($p=2,3,4,5,6,7$) on the effective dielectric constant $\varepsilon_e(z)/\varepsilon_1$ for the simple cubic lattice of spheres of volume fraction $\varphi_2=0.5$, $z=-5-i5$.

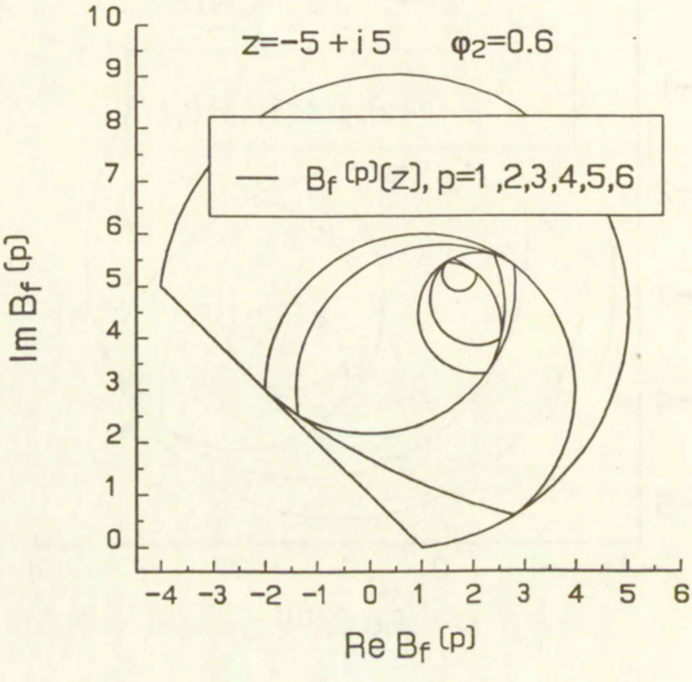

Fig. 6 The sequence of the lens-shaped bounds $B_f^{(p)}(z)$ (p=1,2, 3,4,5,6) on the effective dielectric constant $\varepsilon_e(z)/\varepsilon_1$, for the body-centred cubic lattice of spheres of volume fraction φ =0.6, z=-5+i5.

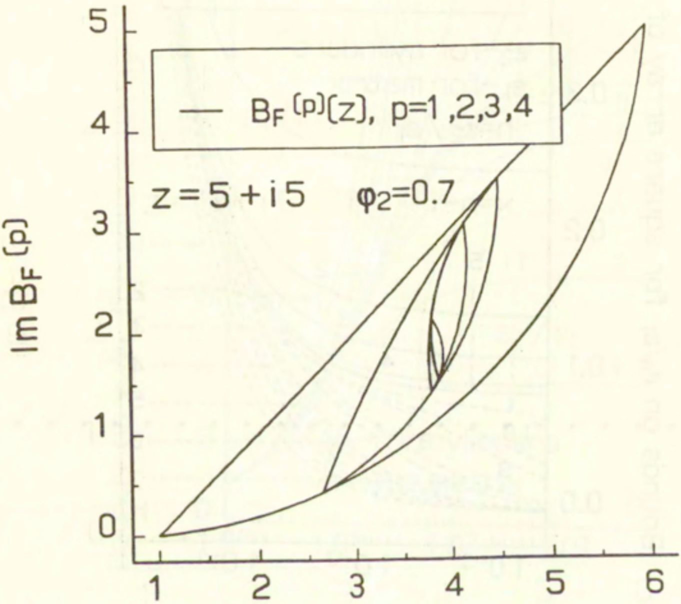

Fig.7 The sequence of the lens-shaped bounds $B_f^{(p)}(z)$ on the effective dielectric constant $\varepsilon_e(z)/\varepsilon_1$, for the face-centred cubic lattice of spheres of volume fraction $\varphi_2=0.7$, $z=5+i5$.

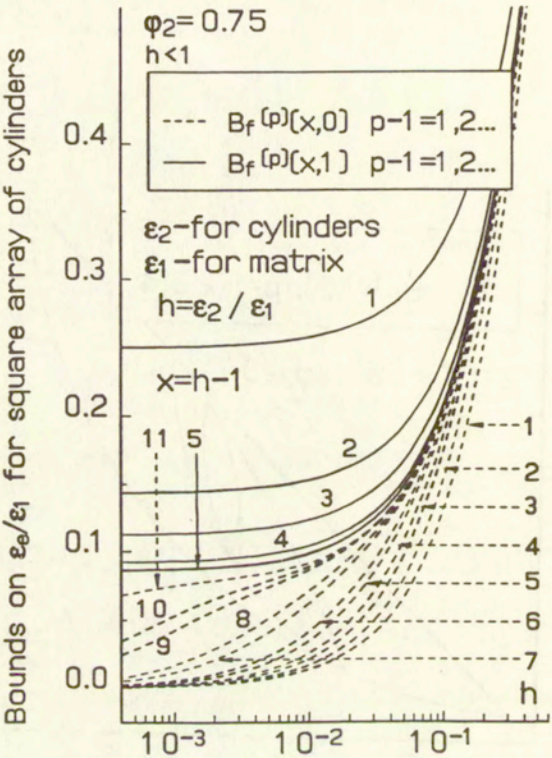

Fig. 8 The sequence of upper and lower bounds on the effective dielectric constant $\varepsilon_e(x)/\varepsilon_1$ for a square array of cylinders of volume fraction $\varphi_2=0.75$ for $\varepsilon_e/\varepsilon_1<1$.

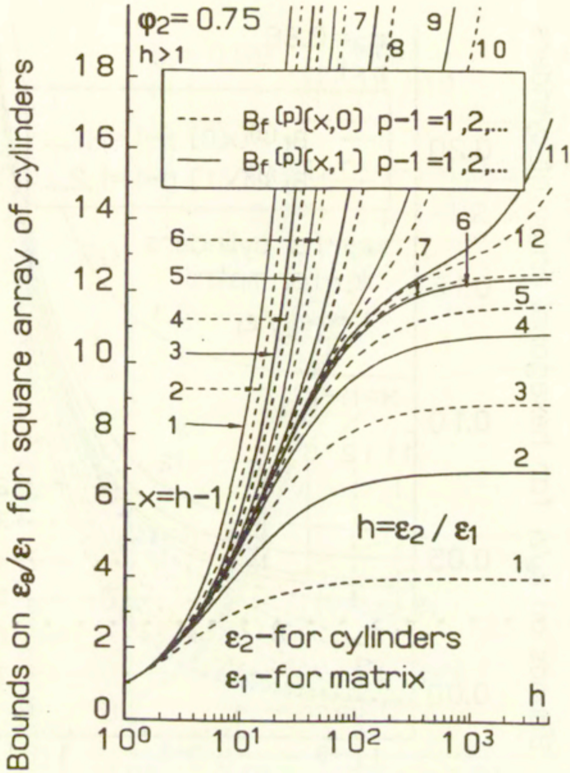

Fig.9 The sequence of upper and lower bounds on the effective dielectric constant $\varepsilon_e(x)/\varepsilon_1$ for a square array of cylinders of volume fraction φ_2=0.75 for $\varepsilon_e/\varepsilon_1$>1.

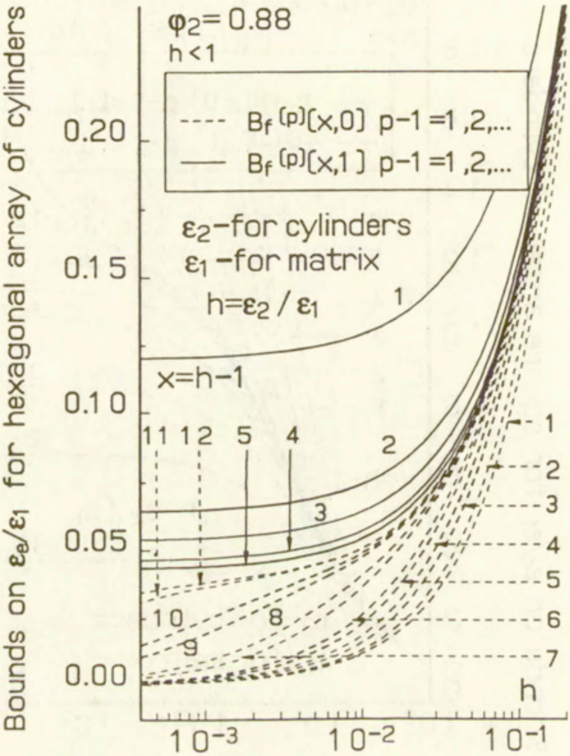

Fig.10 The sequence of upper and lower bounds on the effective dielectric constant $\varepsilon_e(x)/\varepsilon_1$ for a hexagonal array of cylinders of volume fraction $\varphi_2=0.88$ for $\varepsilon_e/\varepsilon_1<1$

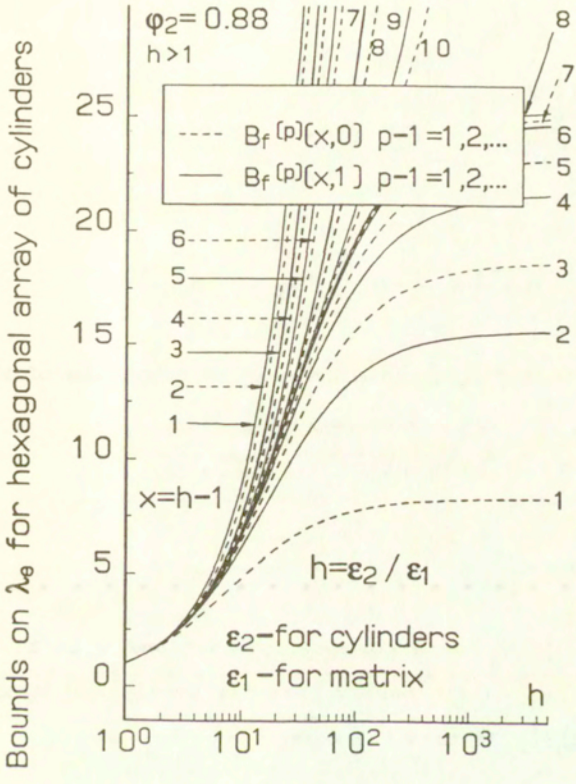

Fig.11 The sequence of upper and lower bounds on the effective dielectric constant $\varepsilon_e(x)/\varepsilon_1$ for a hexagonal array of cylinders of volume fraction $\varphi_2=0.88$ for $\varepsilon_e/\varepsilon_1>1$.

Appendix A. In this appendix we examine the linear fractional transformation

$$S = \frac{g\,z}{1+W}\,, \tag{A.1}$$

applied to:

1) a circle equation

$$W\,W^{*} - W_0\,W^{*} - W_0^{*}\,W + W_0\,W_0^{*} - \rho_W^{2} = 0, \tag{A.2}$$

where W^{*}, W_0 and ρ_W denote the complex conjugate of W, the circle center and its radius respectively;

2) a straight line equation

$$W_0'\,W^{*} + W_0'^{*}\,W = w, \tag{A.3}$$

where W_0' stands for a complex constant and w is a real constant. One can easily verify that for $W=x+iy$ and $W_0=x_0+iy_0$ relations (A.2)-(A.3) take the form

$$(x-x_0)^2 + (y-y_0)^2 = \rho_W^2 \quad , \quad x_0'\,x + y_0'\,y = \frac{w}{2}. \tag{A.4}$$

It is convenient to rewrite the linear fractional transformation (A.1) as follows

$$S = gzT \quad , \quad T = \frac{1}{U} \quad , \quad U = 1 + W \tag{A.5}$$

and examine simpler transformations $(A.5)_{1,2,3}$ successively.

(i) Transformation U=1+W . By substituting W=U-1 to (A.2) we obtain

$$UU^{\bullet}-U_0U^{\bullet}-U_0^{\bullet}U+U_0U_0^{\bullet}-\rho_W^2=0 \quad , \quad U_0=1+W_0 \quad , \quad \rho_U^2=\rho_W^2; \qquad (A.6)$$

Similarly , (A.3) yields

$$U_0' \ U^{\bullet} + U_0'^{\bullet}U = u \quad , \quad U_0'=W_0' \quad , \quad u=w+W_0'+W_0'^{\bullet} \qquad (A.7)$$

(ii) Transformation T=1/U. By substituting U=1/T into $(A.6)_1$ we get the circle equation, provided that $U_0U_0^{\bullet}-\rho_u^2\neq 0$

$$TT^{\bullet}-T_0T^{\bullet} - T_0^{\bullet}T + T_0^{\bullet}T^{\bullet} -\rho_T^2 =0, \qquad (A.8)$$

$$T_0= \frac{U_0^{\bullet}T^{\bullet}}{U_0U_0^{\bullet}-\rho_u^2} \quad , \quad \rho_T^2= \frac{\rho_u^2}{(U_0U_0-\rho_u^2)^2} \qquad (A.9)$$

and the straight line equation, provided that $U_0U_0^{\bullet}-\rho_u^2=0$

$$T_0^{\bullet}T^{\bullet} + T_0 \ T = t \quad , \quad T_0 = U_0^{\bullet} \quad , \quad t=1; \qquad (A.10)$$

Similarly for u≠0, $(A.7)_1$ yields

$$TT^{\bullet}-T_0'T^{\bullet}-T_0'^{\bullet}T+T_0'T_0^{\bullet}-\rho_T^2=0 \quad , \quad T_0' = \frac{U_0'^{\bullet}}{u} \ , \ \rho_T^2 = \frac{U_0' \ U_0'^{\bullet}}{u^2}; \qquad (A.11)$$

and for u=0 we have

$$T_0' \ T^{\bullet} + T_0'^{\bullet} \ T = t \quad , \quad T_0' = U_0'^{\bullet} \quad , \quad t=0. \qquad (A.12)$$

(iii) Transformation $S=gzT$. The substitution of $T=S/(gz)$ into (A.8) leads to

$$SS^{*}-S_0S^{*}-S_0^{*}S+S_0S_0^{*}-\rho_s^{2}=0, \tag{A.13}$$

$$S_0=gzT_0 \quad , \quad \rho_s^{2}=g^2zz^{*}\rho_T^{2}; \tag{A.14}$$

and to

$$S_0'S^{*}+S_0'^{*}S = s \quad , \quad S_0' = \frac{T_0'}{g\ z} \quad , \quad s=t \tag{A.15}$$

respectively. By analyzing (A.6)-(A.15) one readily concludes that linear fractional transformation (A.1) maps the family of circles and straight lines into itself.

Appendix B. We shall now examine the relationship between the bounding functions $F_{p-1}^{(p-1)}(z,u)$ and $F_{p-1}^{(p)}(z,u)$ given by (5.2) and (5.3), respectively. Let us rewrite $F_{p-1}^{(p-1)}(z,\tau)$ and $F_{p-1}^{(p)}(z,u)$ as follows

$$F_{p-1}^{(p-1)}(z,\tau) = V_{p-1}\ z\ \zeta_{p-1}^{(p-1)}(z,\tau) \quad , \quad -1\le\tau\le1, \tag{B.1}$$

$$F_{p-1}^{(p)}(z,u) = V_{p-1}\ z\ \zeta_{p-1}^{(p)}(z,u) \quad , \quad -1\le u\le1, \tag{B.2}$$

where

$$\zeta_{p-1}^{(p-1)}(z,\tau)=\begin{cases} \zeta'^{(p-1)}_{p-1}(z,\tau)= \dfrac{1-\tau}{1+z\tau} \ , \text{ if } 0\le\tau\le1, \\ \zeta''^{(p-1)}_{p-1}(z,\tau)= 1+\tau \quad , \text{ if } -1\le\tau\le0, \end{cases} \tag{B.3}$$

$$\zeta_{p-1}{}^{(p)}(z,u)= \begin{cases} \zeta'_{p-1}{}^{(p)}(z,u)= \dfrac{1-V_p}{1+zV_p\,\dfrac{1-u}{1+zu}}\,, & \text{if } 0\le u\le 1, \\[2ex] \zeta''_{p-1}{}^{(p)}(z,u)= \dfrac{1-V_p}{1+zV_p(1+u)}\,, & \text{if } -1\le u\le 0. \end{cases} \tag{B.4}$$

By substituting $z=r_0e^{i\varphi_0}$ into $(B.3)_1$ we get

$$\zeta'_{p-1}{}^{(p-1)}(r_0,\varphi_0,\tau) = \rho_{p-1,p-1}\; e^{i\Phi_{p-1,p-1}}\;,\quad 0\le\tau\le 1, \tag{B.5}$$

where

$$\rho_{p-1,p-1} = \frac{1-\tau}{\sqrt{(1+\tau\, r_0\cos\varphi_0)^2+(\tau\, r_0\sin\varphi_0)^2}}, \tag{B.6}$$

$$\Phi_{p-1,p-1}=-\text{arc tg}\frac{\tau\, r_0\sin\varphi_0}{1+\tau\, r_0\cos\varphi_0}. \tag{B.7}$$

For our further investigations the function $(B.3)_2$ is not interesting. Let us examine the formula $(B.4)_1$. Setting $z=r_0e^{i\varphi_0}$ into $\zeta'_{p-1}{}^{(p)}(z,u)$ we obtain

$$\zeta'_{p-1}{}^{(p)}(r_0,\varphi_0,u) = \rho'_{p,p-1}\; e^{i\Phi'_{p,p-1}}\;,\quad 0\le u\le 1 \tag{B.8}$$

where

$$\rho'_{p,p-1} = \frac{1-V_p}{\sqrt{[1+R(u)\cos(\varphi(u)+\varphi_0)]^2+[R(u)\sin\varphi(u)]^2}}, \tag{B.9}$$

$$\Phi'_{p,p-1} = -\text{arc tg}\,\frac{R(u)\sin[\varphi(u)+\varphi_0]}{1+R(u)\cos[\varphi(u)+\varphi_0]}. \tag{B.10}$$

and

$$R(u)=\frac{(1-u)r_0V}{\sqrt{(1+ur_0\cos\varphi_0)^2+(ur_0\sin\varphi_0)^2}}, \tag{B.11}$$

$$\varphi(u) = -\text{arc tg}\,\frac{r_0u\sin\varphi_0}{1+r_0u\cos\varphi_0} \tag{B.12}$$

Relations (B.5)-(B.7) and (B.8)-(B.12) describe in the complex planes $(\rho_{p-1,p-1},\ \Phi_{p-1,p-1})$ and $(\rho'_{p,p-1},\ \Phi'_{p,p-1})$ the curves $\zeta'^{(p)}_{p-1}(r_0,\varphi_0,u)$ and $\zeta'^{(p)}_{p-1}(r_0,\varphi_0,u)$ respectively. Of interest is only the case where $\Phi_{p-1,p-1}=\Phi'_{p,p-1}$. By comparing (B.7) and (B.10) we obtain

$$\frac{R(u)\sin[\varphi(u)+\varphi_0]}{1+R(u)\cos[\varphi(u)+\varphi_0]}=\frac{\tau r_0\sin\varphi_0}{1+\tau r_0\cos\varphi_0}\ ,\quad 0\le u\le 1. \tag{B.13}$$

Hence

$$\tau(u) = \frac{R(u)\sin(\varphi(u)+\varphi_0)}{r_0[\sin\varphi_0+R(u)\sin(\varphi(u))]}\ ,\quad 0\le u\le 1, \tag{B.14}$$

and consequently

$$0\le\tau\le V_p. \tag{B.15}$$

We can now compare the radii $\rho_{p-1,p-1}(\tau(u))$ and $\rho'_{p,p-1}(u)$ provided that the parameter $\tau(u)$ is given by (B.14). On the basis of (B.6), (B.9) and (B.13) we obtain

$$\frac{\rho'_{p,p-1}(u)}{\rho_{p-1,p-1}[\tau(u)]} = \frac{(1 - V_p) r_0 \tau(u) \sin\varphi_0}{(1 - \tau(u)) R(u) \sin(\varphi(u)+\varphi_0)}, \quad 0 \le u \le 1, \tag{B.16}$$

or

$$\frac{\rho'_{p,p-1}(u)}{\rho_{p-1,p-1}[\tau(u)]} = \frac{(1 - V_p) \sin\varphi_0}{(1 - \tau(u))[\sin\varphi_0 + R(u)\sin(\varphi(u))]}. \tag{B.17}$$

On account of (B.12) and (B.15) we infer that: 1) the ratio $(1-V_p)/(1-\tau) \le 1$; 2) if $\sin\varphi_0 \le 0$ $(\sin\varphi_0 \ge 0)$ then $\varphi(u) \ge 0$ $(\varphi(u) \le 0)$. Thus

$$\frac{\rho'_{p,p-1}(u)}{\rho_{p-1,p-1}[\tau(u)]} \le 1 \quad , \quad 0 \le u \le 1. \tag{B.18}$$

We pass now to the study of $(B.4)_2$. By rewriting $\zeta''^{(p)}_{p-1}(z,u)$ in polar coordinates $(\rho''_{p-1,p-1}, \Phi''_{p-1,p-1})$ we obtain

$$\zeta''^{(p)}_{p-1}(r_0,\varphi_0,u) = \rho''_{p-1,p-1}\, e^{i\Phi''_{p-1,p-1}} \tag{B.19}$$

where

$$\rho''_{p-1,p-1} = \frac{1-V_p}{\sqrt{(1 + V_p u' r_0 \cos\varphi_0)^2 + (V_p u' r_0 \sin\varphi_0)^2}}, \tag{B.20}$$

$$\Phi''_{p-1,p-1} = -\mathrm{arc\,tg}\, \frac{V_p u' r_0 \sin\varphi_0}{1 + V_p u' r_0 \cos\varphi_0}, \tag{B.21}$$

$$u'=1+u \quad , \quad -1 \le u \le 1. \tag{B.22}$$

As previously of interest is only the case, where

$\Phi_{p-1,p-1}=\Phi''_{p-1,p-1}$. Hence due to (B.7) and (B.21) we obtain

$$\tau(u')=V_p u' \quad , \quad 0 \leq u' \leq 1. \tag{B.23}$$

By substituting (B.22) in (B.6) we can compare the radii $\rho_{p-1,p-1}(u')$ and $\rho''_{p-1,p-1}(\tau(u'))$:

$$\frac{\rho''_{p-1,p-1}(u')}{\rho_{p-1,p-1}[\tau(u')]} = \frac{1-V_p}{1-V_p u'} \leq 1 \quad , \quad 0 \leq u' \leq 1. \tag{B.24}$$

The inequalities (B.18) and (B.24) imply that the bounds $F_{p-1}{}^{(p)}(z)$ are always enclosed in the bounds $F_{p-1}{}^{(p-1)}(z)$, $p=2,3,\ldots$.

Appendix C. In this appendix we shall derive the recurrence formula for determining the coefficients c_n of the power expansion of $\varepsilon_e(z)/\varepsilon_1$

$$\xi(z) = \varepsilon_e(z)/\varepsilon_1 = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{C.1}$$

for the case, where

$$\alpha = \frac{z}{z+2}, \tag{C.2}$$

provided that the coefficients a_n of a series $\varepsilon_e(\alpha)/\varepsilon_1$

$$\xi(z) = \varepsilon_e(\alpha)/\varepsilon_1 = 1 + \sum_{n=1}^{\infty} a_n \alpha^n \tag{C.3}$$

are known. To this end we consider the infinite system of equations

$$\xi_z^{(o)} = \xi_\alpha^{(0)}, \tag{C.4}$$

$$\xi_z^{(1)} = \xi_\alpha^{(1)} \frac{A_1^{(1)}}{(z-2)^2}, \quad A_1^{(1)}=2, \tag{C.6}$$

$$\xi_z^{(2)}=\xi_\alpha^{(2)} \frac{A_2^{(2)}}{(z-2)^4} +\xi_\alpha^{(1)} \frac{A_1^{(2)}}{(z-2)^3}, \quad A_1^{(2)}=4, \ A_1^{(2)}=-4, \tag{C.7}$$

....

relating the successive derivatives

$$\xi_z^{(n)} = \frac{d^n\xi}{dz^n}, \quad \xi_\alpha^{(n)} = \frac{d^n\xi}{d\alpha^n} \tag{C.8}$$

By substituting z=0 in (C.4)-(C.8) and taking into account the relations $c_n=\xi_z^{(n)}(0)/n!$ and $a_n=\xi_\alpha^{(n)}(0)/n!$ we obtain after lengthy calculations the following recurrence formula

$$c_n = \sum_{n=0}^{j-1} \frac{1}{2^{2j-k}} a_{j-k} A_{j-k}^{(j)}, \quad j=1,2,\ldots, \ A_1^{(1)}=2, \tag{C.9}$$

$$A_{j-k}^{(j)}= -(2j-1-k)A_{j-k}^{(j-1)}+2A_{j-k-1}^{(j-1)}, \quad k=1,2,\ldots,j-2, \tag{C.10}$$

$$A_1^{(j)}= -j \ A_1^{(j-1)} \tag{C.11}$$

allowing us to calculate the coefficients c_n, provided that the coefficients a_n are known, see (C.1)-(C.2).

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