5.6.3. – ośrodki warstwowe
7.3. – transport masy, dyfuzja

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14/1993

WARSZAWA 1993

ISSN 0208-5658

Praca wpłynęła do Redakcji dnia 10 marca 1993 r.

IPPT PAN

Na prawach rękopisu

Instytut Podstawowych Problemów Techniki PAN
Nakład 100 egz. Ark.wyd.4,00 Ark.druk. 4,75
Oddano do drukarni w maju 1993 r.

Wydawnictwo Spółdzielcze sp. z o.o.
Warszawa, ul.Jasna 1

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THERMODIFFUSION IN HETEROGENEOUS ELASTIC SOLIDS AND HOMOGENIZATION

Summary

Overall behaviour of microheterogeneous elastic solids in which thermodiffusion occurs is studied by using homogenization methods. Heterogeneities are characterized by a small positive parameter ε. The material coefficients of the coupled system of linear equations describing thermodiffusion depend on ε. In the periodic case, when heterogeneities are distributed periodically, the homogenization is performed by applying the method of two-scale asymptotic expansions. The convergence and corrector theorems are formulated for a nonperiodic microstructure. It is shown, that the initial conditions for the temperature and chemical potential of the homogenized system are changed in comparison with the primal one provided that the initial conditions for the latter system are nonhomogenous. For a layered composite with periodically distributed layers analytical formulae are derived for all the effective material coefficients. Particular cases are studied by exploiting some available experimental data. The analysis itself and homogenization problem are preceded by an overview of the results concerning thermal problems in microheterogeneous solids and composites.

INTRODUCTION

Overall behaviour of micro heterogeneous materials and composites is justified either mathematically, yet still referring to physical notions [1-12] or rather starting from physical concepts [13-24]. Obviously those two points of view overlap [3,7,8,10,11,12,25,26,27].

Composites are often designed for structures working at elevated temperatures, cf.[13,24,28,29]. Effects of moisture and mass diffusion may also be important [30-32]. Section 1 of the present contribution provides an overview of papers dealing with heat, mass and moisture transfer in heterogeneous media and composite materials. However, the main aim of the paper is a study of the equations of thermodiffusion [33-36] in anisotropic and microheterogeneous solids from the point of view of homogenization. In our previous paper [37] similar problem was investigated under a simplifying assumption that the coupling term in the flow equations may be neglected.

Neither convergence nor corrector theorems were formulated and only periodic microstructure was studied. However, inhomogeneous initial conditions were imposed and the change in the initial temperature and chemical potential was observed. In the note [38] the coupling term was taken into account.

The plan of the present paper, consisting of two parts, is as follows. Part I presents theoretical results concerning homogenization of a three-dimensional elastic solids in which thermodiffusion occurs. Particularly, in Section 2 the basic equations of the thermodiffusion in an anisotropic and nonhomogenous body are formulated. Next, in Section 3 for such a body the initial-boundary value problem is formulated and existence and uniqueness theorem is given. Section 4 deals with the homogenization provided that the microstructure of the body is periodic or quasi-periodic. The method of two-scale asymptotic expansions is used. The local problems, posed on the basic cell Y, and the effective or homogenized material coefficients are examined in Section 5. As in our previous paper [37] the change in the initial conditions for the temperature and chemical potential occurs. This problem is investigated in Section 6. A general convergence theorem, without the assumption of periodicity, is formulated in Section 7.

Two topics are investigated in the second part of the paper. In Section 8 the corrector theorems formulated in [39,40] are extended so as to include the diffusion. Analytical formulae for the effective material coefficients are derived in Section 9, provided that the body consists of periodically distributed layers (one-dimensional homogenization). By using these formulae some specific cases are discussed in Section 10.

PART ONE: THEORETICAL DEVELOPMENTS

1. THERMAL EFFECTS IN HETEROGENEOUS BODIES AND COMPOSITES: OVERVIEW

Before passing to homogenization problems in elastic solids in which thermodiffusion occurs, we will first review the papers dealing with thermal and thermomechanical response of heterogeneous bodies and composites.

1.1. *General treatments*

ABOUDI [13] summarized actually existing averaging methods for finding effective properties of microscopically heterogeneous elastic and inelastic solids, including fibrous composites. Homogenization methods, however, are out-of-scope of his book. Thermal and thermomechanical behaviour has also been included, *cf.* also [17,24,28,41]. The book [4] is intended as a summary of results obtained by applying the method of rotational averaging; thermomechanical problems are also briefly discussed.

PEDERSEN [29] examines the problem how the modified Brown-Stobbs model (the mean field model) contributes to, we cite: *a rigorously based and realistic theory of the thermoelasticity and plasticity of composites.*

BENVENISTE and DVORAK [42] considered binary composites with general anisotropic constituents and arbitrary phase geometry, subjected to combined thermomechanical loading. The established results are used to prove a consistency property of the Mori-Tanaka micromechanics model in the context of thermomechanical problems. Moreover the correspondence relations between thermomechanical problems and pure mechanical problems are obtained in a closed and simple form. In [43] a micromechanical model is proposed for predicting the effective thermomechanical model of multiphase composites and fibrous composites.

The paper by NODA [44] reviews the achievements in modelling materials with temperature-dependent material coefficients, cf. also [42, Section 5].

1.2. *Thermal conduction and thermal expansion*, (see also Subsections which follow)

In the paper [27] the effective conductivity of composites with periodic structure is discussed from the viewpoint of some analytical and approximate methods, cf. also [13, 17, 45, 46].

BENVENISTE and MILOH [47] solved the problem of the determination of the effective conductivities of composite media in which a thermal boundary resistance exists at constituent interfaces.

The paper [48] is concerned with the effective thermal conductivity of cracked bodies containing oriented elliptical cracks. In [49] a micro mechanical model of composites containing coated fibers which are nonaligned and possess cylindrical orthotropy is investigated. The mean field concept of Mori and Tanaka is employed to find the effective thermal conductivity.

BOWLES and TOMPKINS [50] first review briefly some of available analyses for the prediction of the coefficients of thermal expansion (CTE) of unidirectional composites. Next, these authors assess the validity of these different analyses by comparing the predicted CTE values with experimental data.

In the paper by ALLEN and LEE [51], a simple analytical model is presented for the effective moduli and thermal expansion coefficients of a short fiber or whisker composite subjected to a number of different material forming processes.

LEWIŃSKI and KUCHARSKI [52] proposed an alternative effective model for analysing distribution of temperature in composites with periodic microstructure. The model encompasses the corrector-type terms of asymptotic expansions and hence involves some extra effective moduli.

1.3. *Thermoelasticity, thermoviscoelasticity and thermodiffusion* (see also Subsections 1.6, 1.9)

The earlier mathematical results on some aspects of the homogenization theory are reported in the papers [53-55] by the eminent representatives of the Italian School of the Calculus of Variations, and precede those of the French School; the earlier results of the latter -in what regards homogenization- are summarized in the books [5,10]. By using the results presented in [53-55], McConnell performed homogenization of linear elastic [56] and thermoelastic [57] media in the static case, provided that they are constructed of laminated materials, cf. also Section 9 of the present contribution. Three-dimensional, static homogenization problems of thermoelastic media were next considered in the papers [58-60] by employing various techniques.

FRANCFORT'S contributions [61,62] are crucial in the homogenization of the linear equations of coupled thermoelasticity with periodic material coefficients. This author proved that homogenization does not change the type of the system of equations, though the heat equation is parabolic. The second surprising result is that the initial temperature for the homogenized problem changes, provided that the initial conditions of the primal problem are inhomogeneous, cf. also [8]. Homogeneous initial conditions are imposed in the papers [63-65].

Bloch expansion techniques [66] are used by TURBÉ [67] to perform homogenization of a linear thermoelastic solid with two relaxation times (the initial conditions are also homogeneous).

The results due to FRANCFORT [61,62] were further elaborated in [39,40]. Particularly, the convergence theorem for $\varepsilon \longrightarrow 0$ and corrector results are given in the case of a general, not necessarily periodic microstructure. The mathematically rigorous results obtained by BRAHIM-OTSMANE, FRANCFORT and MURAT contradict those due to BUISSON, MOLINARI and BERVEILLER [68]. The authors of the last paper claim that due to thermoelastic coupling the overall behaviour of the body is viscoelastic (the self-consistent scheme was applied).

Macroscopic viscous behaviour, with long time memory, was observed by FLEURY [69] in his study of a mixture composed of a thermoelastic material and a viscous compressible fluid.

As we have noted in the Introduction, in our paper [37] homogenization of equations of the linear thermodiffusion in an elastic body was performed under a simplifying assumption that the thermodiffusive coupling may be neglected. Only the shift in the temperature was discussed. Our present paper is complete in that both the thermodiffusive coupling term is taken into account and the formulae for the initial value of the temperature and the chemical potential are derived.

FRANCFORT and SUQUET [70] investigated macroscopic mechanical and thermal behaviour of microperiodically heterogeneous materials (Kelvin-Voigt type vicoelasticity). At the macroscopic level the behaviour is non-local in time (viscoelasticity with fading memory). In the paper [70] it is assumed that the thermal expansion tensor vanishes. Terms involving this tensor were taken into account in the formal homogenization procedure employed by WOJNAR [71], under the assumption of homogeneous initial conditions.

Homogenization of the equations of thermodiffusion in a viscoelastic body was announced in [72]. MAKSIMOV and KOCHETKOW [73] reported their results of theoretical and experimental investigation of thermal deformation of hybrid polymer composites over a range of temperatures that includes a transition of the polymer matrix from a glassy state to a viscoelastic one.

By exploiting the notion of a representative elementary volume and Hill's condition, in the paper [74] the macroscopic thermomechanical behaviour of heterogeneous media is studied.

1.4. *Thermopiezoelectric composites* (see also Subsection 1.3)

The study of piezoelectric composites, with periodic structure, by homogenization methods started with the contribution by BIELSKI and BYTNER [75], and the paper by TELEGA [76]. The energy method was used in [75], while in [76] the Γ-convergence method was applied. In the paper by DUNN and TAYA [77], the effective moduli of piezoelectric composites are found by using the dilute, self-consistent and differential micromechanics theories. In the papers [78,79] the Bloch expansion method was used to study the dynamic equations of microperiodic piezoelectricity. The macroscopic material coefficients coincide with those obtained by TELEGA [76].

In the paper by the present authors [80], the equations of the linear thermopiezoelectricity are homogenized (for a periodic microstructure).The coupled system of dynamic equations is examined by the two-scale asymptotic method. It is shown that the initial condition for the temperature of the homogenized body also changes, in comparison with the initial temperature of the primal body with microperiodic structure.

TAUCHERT [81] investigated the response of thin composite plates constructed of thermopiezoelectric layers and subject to stationary thermal and electric fields.

1.5. *Thermoelastic contact problem and homogenization*

Homogenization techniques were used in [82,83] to study the heat conduction between two solids, provided that the distribution of asperities is microperiodic. In our opinion, the approach used in these two theses seems to be applicable for modelling a rough contact between matrices and fibers in composites.

1.6. *Influence of moisture on the overall behaviour of composites* (see also Subsection 1.3)

Moisture deteriorates mechanical properties of composites since in micropores usually mechanically stimulated hydrolysis occurs [84]. Thermal fields can contribute to an activation of this process. Simplified models of thermal expansion and moisture swelling are reviewed by HASHIN [18].

The effects of moisture and thermal fields on the behaviour of periodically microheterogeneous linear elastic solids are investigated in [8, 31]. The basic equations are similar to those examined by us in Ref. [37], though the initial conditions were not taken into account.

Thermal and hygroscopic effects in laminates and fibrous composites are discussed in the monograph by DATOO [28].

BASI *et al.* [30] investigated inhomogeneous anisotropic plates, including hygrothermal effects. The governing equations are presented and boundary conditions are specified. The implications for the special case of laminated plates, in which each layer consists of an orthotropic material with constant elastic moduli are shown.

The hygrothermal-mechanical behaviour of composite cylinders was investigated in the papers [32, 41]. It was assumed that hygrothermal and mechanical loads may vary in the radial and circumferential directions, but must be independent of axial coordinate. The problems considered are stationary ones.

JONES *et al.* [85] discussed the *criteria* for failure due to delamination damage in the case of thermomechanical behaviour of composites provided that absorption of moisture occurs.

1.7. *Porous materials*

One of the subjects investigated in the impressive monograph by KAVIANY [19] are various methods suitable for averaging of transport equations (mainly heat transport) in single-phase and two-phase flows in porous media, *cf.* also Refs [86,87]. In the paper by AURIAULT *et al.* [88] the macroscopic description of metal powders under compaction at high temperatures is obtained from consideration at the microscopic level. The material is composed of metal particles and pores, and the two-scale asymptotic method is employed to derive the overall relations.

The authors of the series of papers [89-92] discussed practical aspects and applications of the homogenization theory to stationary thermomechanical problems of porous materials. Possible applications to materials like mortars and concretes with cellular structure, *etc.*, were also suggested.

1.8 *Damage* (see also Subsection 1.6)

ALLEN and HARRIS [93] developed a model for predicting the thermomechanical behaviour of initially elastic composites subjected to both monotonic and cyclic fatigue loading. The thermodynamics with internal state variables is constructed. Moreover, it is shown that suitable definitions of the locally averaged field variables will lead to useful thermodynamic constraints on a local scale containing statistically homogeneous damage. Next, in Ref. [94] the three-dimensional tensor equations obtained in [93] are simplified using symmetry constraints. A specific constiutive model is developed for the case of matrix cracks only, *cf.* also Refs [95, 96].

1.9. *Fibrous composites* (see also Subsections 1.1, 1.2, 1.6, 1.8 and 1.11)

Fibrous composites are often used as structural elements, *cf.* Ref. [28]. Hence the practical interest for understanding their response to thermomechanical loadings. To achieve this aim one must obviously know macroscopic properties of such composites, *cf.* Refs [97-102].

ABOUDI [103] derived a set of constitutive relations for the prediction of the overall behaviour of fiber-reinforced materials which are thermoelastic in the linear region and thermoviscoplastic in the nonlinear region, *cf.* also [104].

In the paper [105] an analytical macroscopic model was developed for analyzing the response of fiber reinforced thermoplastic matrix composites at elevated temperatures. The effect of temperature on the viscoplastic behaviour of the composites was of primary concern.

PEDERSEN [106] employed the mean field theory to discuss the problem of thermomechanical hysteresis in metal matrix composites. The experimental behaviour of a simple model system - copper with continuous tungsten fibers - serves as a guide in the relevant analysis.

In the paper [107] a simple model is formulated for a continuous fiber reinforced metal matrix composite with an interface layer when subjected to temperature change. The essential properties of an interface layer that could reduce the residual stresses in the matrix are identified. The possibility of improving the low cycle fatigue properties of the composite is also discussed.

1.11. *Plates and shells* (see also Subsections 1.4 and 1.6)

In the comprehensive paper by NOOR and BURTON [108] a review is made of recent developments in the computational modelling of high-temperature multilayered composite plates and shells. The literature concerned with eight categories of problems is reviewed: heat transfer, thermal stresses, curing, processing and residual stresses, bifurcation buckling, vibrations of heated plates and shells, large deflection and postbuckling problems, sandwich plates and shells with composite face sheets.

In the series of papers [109-113] and in the thesis [114] macroscopic models of thermoelastic perforated plates are constructed from the equations of the classical thermoelastcity by using homogenization methods. Transverse shear deformations are taken into account (Hencky - Mindlin model); only stationary problems are dealt with. The holes are distributed periodically within the thickness of the plate and may be parallel to the mid-plane.

MURAKAMI [115] proposed two shear-deformable laminate plate theories with linear and cubic variation of in-plane displacement over the thickness of the plate for predicting thermal deformations, *cf.* also Ref. [116]. The theory with a linear variation of in-plane displacements incorporates the effect of transverse shear deformations without transverse normal strain, while the theory with cubic variation of in-plane displacements incorporates both transverse shear and normal deformation.

In the paper [117] a curved nonhomogeneous and anisotropic thin layer with periodic structure is examined as a quasi-static thermoelastic problem. The averaged thermoelastic shell model is derived by using homogenization methods. The model obtained is discussed in [118].

HOROSHOUN and SHPAKOVA [119] proposed a model of layered cylindrical shells which accounts for transverse shear stresses. The shell is subjected to a stationary, non-uniformly distributed thermal field. The elastic moduli depend linearly on the temperature while the coefficients of thermal expansion are quadratic functions of the temperature.

In the paper by PISKOUNOV *et al.* [120] two-dimensional finite elements are applied to solve the problem of thermoelastic equilibrium of layered shells and plates. A piecewise linear temperature distribution along the normal to the middle surface is assumed. The discretization yields the classical system of algebraic equations

$$[\,\mathbf{K}\,]\,\{\,\mathbf{v}\,\} = \{\,\mathbf{R}\,\}$$

where $[\,\mathbf{K}\,]$ is the general stiffness matrix of the whole system, $\{\,\mathbf{v}\,\}$ is the vector of unknown nodal displacements and $\{\,\mathbf{R}\,\}$ stands for the vector of thermal loadings.

1.12. *Random media and composites* (see also Subsection 1.1)

In the paper [121] bounds for the effective conductivity of a random medium are determined.

The problem of determination of effective thermal properties of a composite made of a matrix in which randomly distributed spheres are imbedded is investigated in Refs [122, 123].

Somewhat abstract scheme is proposed for the stochastic formulation of the transition from the elastic to the inelastic response of a multi-component material by using the theory of Markov processes. Thermomechanical parameters are taken into account.

BUYEVICH [125] derived the averaged field equations for heat and mass transfer in disperse media. The method of *ensemble* averaging has been combined with the self-consistent approach. The procedure proposed was next applied in [126] for the description of the relaxation processes and the resulting dispersion effects accompanying unsteady heat transport in granular-like media.

In Ref. [127] a thermoelastic composite with stochastically inhomogeneous structure is investigated.

1.13 *Micro-heat exchangers and micro-heat pipes*

Micro-heat exchangers, for instance used in electronic devices, usually exhibit fine periodic structure. In the paper [128] a heat transfer problem is solved basing on a unit *cell*, *cf.* also [129].

2. THERMODIFFUSION IN AN ELASTIC BODY

2.1 *Nonequilibrium thermodynamics of diffusion*

The phenomenon of diffusion belongs to irreversible processes, and, if it is developing under the conditions in which the deviation from the equilbrium of the system is not too large, it obeys the laws of linear nonequilibrium thermodynamics (**LNT**), *cf.* DE GROOT and MAZUR, [130].

Essential role in **LNT** is played by the balance equation of entropy. It expresses the obvious fact that the variation of entropy (as every other quantity) is composed of two parts

$$d\,S = d_e\,S + d_i\,S \tag{2.1}$$

where d_eS is due to the entropy flow and its exchange with the surrounding, and d_iS is due to the entropy source (because of irreversibility of phenomena occurring in the system).

If all quantities describing the system are continuous as functions of space variables the exchange term variation per unit time has a divergence form

$$\frac{\partial_e\,S}{\partial t} = -\,div\,\mathbf{j}_e\,, \tag{2.2}$$

where $\mathbf{j}_e$ is the entropy flow per unit area and unit time.

If the variation of entropy in the system is due to flows of heat (*i.e.* energy) and mass, we have the following balance equation

$$T\,\dot{s} = -\,q_{i,i} + M\,j_{i,i}\,, \tag{2.3}$$

where s is the entropy of unit volume, **q** and **j** are heat and diffusion fluxes, respectively; T is the absolute temeperature and M denotes the chemical potential.

The last equation can be written as follows

$$\dot{s} = -\left(\frac{q_i - M\,j_i}{T}\right)_{,i} + \sigma \tag{2.4}$$

The first term in the r.h.s. is equal to $\partial_e S/\partial t$ and describes the entropy exchange with the surrounding while the second term, *i.e.*

$$\sigma = (q_i - M\,j_i)\left(\frac{1}{T}\right)_{,i} - j_i\,\frac{M_{,i}}{T}\,, \tag{2.5a}$$

or

$$\sigma = q_i\left(\frac{1}{T}\right)_{,i} - j_i\left(\frac{M}{T}\right)_{,i}\,, \tag{2.5b}$$

represents the entropy production and corresponds to $\partial_i S/\partial t$.

Eq. (2.5) is a specific example of the LNT law: *the entropy production is a bilinear form in the fluxes* $\dot{x}^{\alpha}$ *and thermodynamic forces* X^{α} *appearing in the phenomenological equations for which the Onsager relation is satisfied, cf.* [130,131]

$$\dot{x}^{\alpha} = \sum_{\beta} L^{(\alpha\beta)}\,X^{(\beta)}\,, \tag{2.6}$$

$$L^{(\alpha\beta)} = L^{(\beta\alpha)}\,, \tag{2.7}$$

$$\sigma = -\sum_{\alpha} X^{(\alpha)}\,\dot{x}^{\alpha}\,, \tag{2.8}$$

where the summation is caried out over all processes appearing in the system. Since one has

$$\sigma \geq 0\,,$$

the quadratic form

$$\sum_{\alpha,\beta} L^{(\alpha\beta)}\,X^{(\alpha)}\,X^{(\beta)}$$

must be *positive definite* or at least *positive*.

Comparison of (2.8) with (2.5) indicates a certain flexibility in the choice of fluxes and conjugate forces. For instance, in [130,131] the following fluxes are considered

$$\dot{\mathbf{x}}^{(1)} = \mathbf{q}' \equiv \mathbf{q} - M\,\mathbf{j}\,, \qquad \dot{\mathbf{x}}^{(2)} = \mathbf{j}\,, \tag{2.9}$$

as conjugate to the forces

$$X^{(1)} = \frac{1}{T^2}\,\nabla T\,, \qquad X^{(2)} = \frac{1}{T}\,\nabla M\,, \tag{2.10}$$

NOWACKI ([34], Ch. 4) assumes the fluxes

$$\dot{\mathbf{x}}^{(1)} = \mathbf{q} \,, \qquad \dot{\mathbf{x}}^{(2)} = \mathbf{j} \,, \tag{2.11}$$

as conjugate to the forces

$$\mathbf{X}^{(1)} = \frac{1}{T^2} \nabla T \,, \qquad \mathbf{X}^{(2)} = \nabla \frac{M}{T} \,, \tag{2.12}$$

respectively.

For the choice (2.9) and (2.10), we see that (2.10) does not involve the chemical potential M, while for the choice (2.11) and (2.12) M occurs in $(2.12)_2$. The difference between $\mathbf{q}$ and q' represents heat transferred by the diffusion and provides an example that *in diffusing mixtures the concept of heat flow can be defined in different ways* [130].

Therefore the first [130,131] *alternative* has more clear physical meaning. In the linear case both approaches coincide.

The appropriate phenomenological relations of the type (2.6), corresponding to the choice (2.9) and (2.10) are given by

$$q_i - \mathbf{M}\, \mathbf{j}_i = - \hat{L}_{ij}^{(11)} X_j^{(1)} - \hat{L}_{ij}^{(12)} X_j^{(2)} \,,$$
$$j_i = - \hat{L}_{ij}^{(21)} X_j^{(1)} - \hat{L}_{ij}^{(22)} X_j^{(2)} \,,$$

or explicitly

$$\begin{aligned} q_i - \mathbf{M}\, \mathbf{j}_i &= - \hat{L}_{ij}^{(11)} \frac{1}{T^2} T_{,j} - \hat{L}_{ij}^{(12)} \frac{1}{T} M_{,j} \,, \\ j_i &= - \hat{L}_{ij}^{(21)} \frac{1}{T^2} T_{,j} - \hat{L}_{ij}^{(22)} \frac{1}{T} M_{,j} \,, \end{aligned} \tag{2.13}$$

where

$$\hat{L}_{ij}^{(12)} = \hat{L}_{ij}^{(21)} = \hat{L}_{ji}^{(21)} \,. \tag{2.14}$$

If we put

$$L_{ij}^{(11)} = \frac{1}{T^2} \hat{L}_{ij}^{(11)} \,,$$
$$L_{ij}^{(12)} = L_{ij}^{(21)} = \frac{1}{T^2} \hat{L}_{ij}^{(12)} \,,$$
$$L_{ij}^{(22)} = \frac{1}{T} \hat{L}_{ij}^{(22)} \,,$$

the phenomenological relations take the form

$$\begin{aligned} \mathbf{q} - M\, \mathbf{j} &= - \mathbf{L}^{(11)} \nabla T - \mathbf{L}^{(12)} T \nabla M \,, \\ \mathbf{j} &= - \mathbf{L}^{(21)} \nabla T - \mathbf{L}^{(22)} T \nabla M \,. \end{aligned} \tag{2.15}$$

Here $\mathbf{L}^{(11)}$ is the heat conductivity matrix, below denoted also by $\mathbf{K}$. The matrix

$\underset{\sim}{\lambda} = K/T_0$ will also be used.

Below, the matrix $\mathbf{L}^{(22)}$ will be denoted by $\mathbf{D}$. It is related to the generally accepted definition of the diffusion matrix $\underset{\sim}{\mathcal{D}}$ according to the formula

$$\mathcal{D}_{ij} = \left(\frac{\partial M}{\partial c}\right)_{T,\underset{\sim}{\sigma}} D_{ij} = \left(\frac{\partial M}{\partial c}\right)_{T,\underset{\sim}{\sigma}} L_{ij}^{(22)} \tag{2.16}$$

where $\underset{\sim}{\sigma}$ is the stress tensor.

To corroborate this statement we calculate, after LANDAU and LIFSHITZ [131] the *gradient* of the chemical potential M considered as a function of $\underset{\sim}{\sigma}$, T and c. We have

$$\nabla M = \left(\frac{\partial M}{\partial c}\right)_{T,\underset{\sim}{\sigma}} \nabla c + \left(\frac{\partial M}{\partial T}\right)_{c,\underset{\sim}{\sigma}} \nabla T + \left(\frac{\partial M}{\partial \underset{\sim}{\sigma}}\right)_{T,c} \nabla \underset{\sim}{\sigma} \tag{2.17}$$

We note that the derivative $(\partial M/\partial\underset{\sim}{\sigma})_{T,c}$ in the last term can be replaced by $(\partial\underset{\sim}{\varepsilon}/\partial c)_{T,\underset{\sim}{\sigma}}$, since

$$d\,\varphi = -\,S\,dT - \underset{\sim}{\varepsilon}\,d\,\underset{\sim}{\sigma} + M\,dc\,, \tag{2.18}$$

where φ is the termodynamic potential of the unit volume. Hence

$$\left(\frac{\partial M}{\partial \underset{\sim}{\sigma}}\right)_{T,c} = \frac{\partial^2\varphi}{\partial\underset{\sim}{\sigma}\partial c} = -\left(\frac{\partial\underset{\sim}{\varepsilon}}{\partial c}\right)_{T,\underset{\sim}{\sigma}}. \tag{2.19}$$

Substituting (2.17) into (2.15) we get (2.16) and the matrix of thermodiffusion $[k_{Tik}\,\mathcal{D}_{kj}]$ such that

$$k_{Tik}\,\mathcal{D}_{kj} = T\left(L_{ij}^{(12)} + \left(\frac{\partial M}{\partial T}\right)_{c,\underset{\sim}{\sigma}} L_{ij}^{(22)} \right) \tag{2.20}$$

The matrix $\underset{\sim}{k}_T$ with dimensionless components in the scalar case reduces to the thermodiffusive ratio. Similarly, we get the components of barodiffusion $\underset{\sim}{k}_P$

$$k_{Pij} = -\left(\frac{\partial\varepsilon_{ij}}{\partial c}\right)_{T,\underset{\sim}{\sigma}} : \left(\frac{\partial M}{\partial c}\right)_{T,\underset{\sim}{\sigma}}$$

The coefficient $\mathbf{L}^{(12)}$ describes the phenomenon of thermal diffusion, *i.e.* flow of matter as a result of temperature gradient (*Soret-type effect*) and the reciprocal phenomenon, *i.e.* flow of heat as a result of the concentration gradient (*Dufour-effect*).

The linearization of Eqs (2.15) consists in rejection of the nonlinear term, namely **M j** and assuming small changes of temperature

$$\Theta = T - T_0, \quad \Theta \ll T_0 \tag{2.21}$$

In such a case $\mathbf{L}^{(\alpha\beta)}$ do not depend on temperature. Thus we arrive at the following relations

$$\begin{aligned} q_i &= - L_{ij}^{(11)} \Theta_{,j} - L_{ij}^{(12)} T_0 M_{,j} , \\ j_i &= - L_{ij}^{(21)} \Theta_{,j} - L_{ij}^{(22)} M_{,j} . \end{aligned} \tag{2.22}$$

For NOWACKI's [34] choice of fluxes (2.11) and forces (2.12), we obtain

$$\begin{aligned} q_i &= - \mathscr{L}_{ij}^{(11)} X_j^{(1)} - \mathscr{L}_{ij}^{(12)} X_j^{(2)} , \\ j_i &= - \mathscr{L}_{ij}^{(21)} X_j^{(1)} - \mathscr{L}_{ij}^{(22)} X_j^{(2)} , \end{aligned}$$

or

$$\begin{aligned} q_i &= - \mathscr{L}_{ij}^{(11)} \frac{1}{T^2} T_{,j} - \mathscr{L}_{ij}^{(12)} \frac{1}{T} (M_{,j} - \frac{1}{T} T_{,j}) , \\ j_i &= - \mathscr{L}_{ij}^{(21)} \frac{1}{T^2} T_{,j} - \mathscr{L}_{ij}^{(22)} \frac{1}{T} (M_{,j} - \frac{1}{T} T_{,j}) , \end{aligned} \tag{2.23}$$

where $\mathscr{L}_{ij}^{(\alpha\beta)}$ are new transport coefficients; their form readily follows from (2.22). After the linearization and appropriate definition of the material coefficients we recover the relations (2.22).

Now let us focus our considerations on *linear thermoelastic solids.*

2.2 *Basic equations of diffusion in a thermoelastic solid*

Let $\Omega \subset R^3$ be a bounded domain and $\Gamma = \partial \Omega$ its boundary. A thermo- elastic solid in which diffusion takes place will be denoted, for the sake of simplicity, by **TED**. Only physically and geometrically linear problems are investigated.

$\bar{\Omega}$ is the domain occupied by **TED** body in its natural state. Physical fields depend on $x = (x_i) \in \Omega$ (i=1,2,3) and on time t. The following notations are used: $\mathbf{u} = (u_i)$ - the displacement vector, $\mathbf{e} = (e_{ij})$ - the strain tensor, $\underset{\sim}{\sigma} = (\sigma_{ij})$ - the stress tensor, T - the absolute temperature, T_0 - the absolute temperature of a natural state, $\Theta = T - T_0$ - the relative temperature, $\mathbf{q} = (q_i)$ - the heat flux vector, s - the entropy, M - the chemical potential, $\mathbf{j} = (j_i)$ - the mass flux of a diffusing substance, $\mathbf{B} = (B_i)$ - the prescribed body forces.

The equations describing **TED** body are specified by

(i) Field equations

$$e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) , \tag{2.24}$$

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_i} + B_i, \tag{2.25}$$

$$T_0 \dot{s} = - \frac{\partial q_i}{\partial x_i} , \tag{2.26}$$

$$\dot{c} = - \frac{\partial j_i}{\partial x_i} , \tag{2.27}$$

where $\dot{u}_i = - \frac{\partial u_i}{\partial t}$, *etc.*

(ii) Constitutive equations

$$\sigma_{ij} = c_{ijkl}\, e_{kl} - \gamma_{ij}\, \Theta - \xi_{ij}\, c \tag{2.28}$$

$$s = \gamma_{ij}\, e_{ij} + b\, \Theta + d\, c \tag{2.29}$$

$$M = - \xi_{ij}\, e_{ij} - d\, \Theta + a\, c\ . \tag{2.30}$$

(iii) Flow laws

$$q_i = - K_{ij} \frac{\partial \Theta}{\partial x_j} - T_0 L_{ij} \frac{\partial M}{\partial x_j} , \tag{2.31}$$

$$j_i = - L_{ij} \frac{\partial \Theta}{\partial x_j} - D_{ij} \frac{\partial M}{\partial x_j} . \tag{2.32}$$

Here ρ denotes the density, (c_{ijkl}) is the tensor of elastic moduli satisfying usual symmetry conditions: $c_{ijkl} = c_{klij} = c_{jikl}$; (γ_{ij}) and (ξ_{ij}) are the stress-temperature tensor and the stress-diffusion tensor, respectively; moreover

$$\gamma_{ij} = c_{ijkl}\, \alpha^T_{kl} , \qquad \xi_{ij} = c_{ijkl}\, \alpha^D_{kl} ,$$

where (α^T_{kl}) and (α^D_{kl}) are the thermal expansion tensor and the diffusion expansion tensor, respectively; the last one describes the influence of diffusion on the change of dimensions of the body (swelling) ; $b = c_{e,c}/T_0$ and $c_{e,c}$ is the specific heat for fixed **e** and c; (K_{ij}) and (D_{ij}) denote the thermal conductivity and diffusion tensor, respectively, while (L_{ij}) stands for the thermodiffusion tensor. Obviously, the tensors $\underset{\sim}{\alpha}$, $\underset{\sim}{\gamma}$, **D**, **K** and **L** are symmetric.

On account of the symmetric role played by γ_{ij} and ξ_{ij} in the linear TED equations, the following notation will also be used

$$\gamma_{ij} = \gamma^{(1)}_{ij}, \qquad \xi_{ij} = \gamma^{(2)}_{ij} .$$

The material coefficients are not necessarily constant. Nevertheless, we make the following assumption :

$$\rho\ ,\ b,\ d,\ a \in L^{\infty}(\Omega), \qquad 0 < \rho_0 \le \rho(\mathbf{x}) \quad a.e.\ \mathbf{x} \in \Omega \tag{2.33}$$

$$\lambda_1(e_1^{\,2} + e_2^{\,2}) \le [e_1,e_2] \begin{bmatrix} b(\mathbf{x}) & d(\mathbf{x}) \\ d(\mathbf{x}) & a(\mathbf{x}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \le \lambda_2(e_1^{\,2} + e_2^{\,2}) \tag{234}$$

$$a.e.\ \ \mathbf{x} \in \Omega, \quad \forall\ \ e_1,\ e_2 \in \mathbb{R}$$

$$\lambda_1|\mathbf{e}|^2 \le c_{ijkl}\, e_{ij}\, e_{kl} \le \lambda_2|\mathbf{e}|^2 \qquad a.e.\ \ \mathbf{x} \in \Omega, \quad \forall\ \ \mathbf{e} \in \mathbb{E}^3_s \tag{2.35}$$

$$\lambda_1(|\mathbf{e}_1|^2 + |\mathbf{e}_2|^2) \le [\mathbf{e}_1,\mathbf{e}_2] \begin{bmatrix} \mathbf{K}(\mathbf{x}) & \mathbf{L}(\mathbf{x}) \\ (\mathbf{L})^T(\mathbf{x}) & \mathbf{D}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} \le \lambda_2(|\mathbf{e}_1|^2 + |\mathbf{e}_2|^2) \tag{2.36}$$

$$a.e.\ \ \mathbf{x} \in \Omega, \quad \forall\ \ \mathbf{e}_1,\ \mathbf{e}_2 \in \mathbb{E}^3_s$$

where $\mathbb{E}^3_s$ is the space of symmetric 3×3 matrices and $0 < \lambda_1 < \lambda_2$ are constants.

An equivalent form of the constitutive equations is the following one

$$\sigma_{ij} = \bar{c}_{ijkl}\, e_{kl} - \bar{\gamma}_{ij}\, \Theta - \bar{\xi}_{ij}\, M\ , \tag{2.37}$$

$$s = \bar{\gamma}_{ij}\, e_{ij} + \bar{b}\, \Theta + \bar{d}\, M\ , \tag{2.38}$$

$$c = \bar{\xi}_{ij}\, e_{ij} + \bar{d}\, \Theta + \bar{a}\, M\ , \tag{2.39}$$

or

$$\sigma_{ij} = \tilde{c}_{ijkl}\, e_{kl} - \tilde{\gamma}_{ij}\, s - \tilde{\xi}_{ij}\, c\ , \tag{2.40}$$

$$\Theta = -\,\tilde{\gamma}_{ij}\, e_{ij} + \tilde{b}\, s - \tilde{d}\, c\ , \tag{2.41}$$

$$M = -\,\tilde{\xi}_{ij}\, e_{ij} - \tilde{d}\, s + \tilde{a}\, c\ , \tag{2.42}$$

where

$$\bar{c}_{ijkl} = c_{ijkl} - \frac{1}{a} \xi_{ij} \xi_{kl} ,$$

$$\bar{\gamma}_{ij} = \gamma_{ij} + \frac{d}{a} \xi_{ij} := \bar{\gamma}^{(1)}_{ij}, \qquad \bar{\xi}_{ij} = \frac{1}{a} \xi_{ij} := \bar{\gamma}^{(2)}_{ij}, \tag{2.43}$$

$$\bar{b} = b + \frac{d^2}{a}, \qquad \bar{d} = \frac{d}{a} , \qquad \bar{a} = \frac{1}{a}$$

are isothermal-isopotential (*chemical potential*) material coefficients while

$$\tilde{c}_{ijkl} = c_{ijkl} - \frac{1}{b} \gamma_{ij} \gamma_{kl} := c_{ijkl} - \tilde{\gamma}_{ij} \gamma_{kl},$$

$$\tilde{\gamma}_{ij} = \frac{1}{b} \gamma_{ij} := \tilde{\gamma}^{(1)}_{ij}, \qquad \tilde{\xi}_{ij} = \xi_{ij} - \frac{d}{b} \gamma_{ij} := \tilde{\gamma}^{(2)}_{ij} \tag{2.44}$$

$$\tilde{b} = \frac{1}{b} , \quad \tilde{d} = \frac{d}{b} , \quad \tilde{a} = a + \frac{d^2}{b},$$

are adiabatic-isopotential ones.

The following relations are also valid

$$\tilde{c}_{ijkl} = \bar{c}_{ijkl} + \frac{\bar{a}}{\bar{\Delta}} \bar{\gamma}_{ij} \bar{\gamma}_{kl} + \frac{\bar{b}}{\bar{\Delta}} \bar{\xi}_{ij} \bar{\xi}_{kl} - \frac{\bar{d}}{\bar{\Delta}} (\bar{\gamma}_{ij} \bar{\xi}_{kl} + \bar{\xi}_{ij} \bar{\gamma}_{kl}) ,$$

$$\tilde{d} = \bar{d} /\bar{\Delta} , \qquad \tilde{a} = \bar{b}/\bar{\Delta} , \qquad \tilde{b} = \bar{a} /\bar{\Delta} , \qquad \bar{\Delta} = \bar{a} \bar{b} - \bar{d}^2 . \tag{2.45}$$

From Eqs (2.24) - (2.32) we readily obtain the basic system of equations for the determination of three unknown fields: **u**, s and c

$$\rho \ddot{u}_i = \frac{\partial}{\partial x_i} \left[\bar{c}_{ijkl} e_{kl} (\mathbf{u}) - \tilde{\gamma}_{ij} s - \tilde{\xi}_{ij} c \right] + B_i ,$$

$$\dot{s} = \frac{\partial}{\partial x_i} \left\{ \lambda_{ij} \frac{\partial}{\partial x_j} [- \tilde{\gamma}_{kl} e_{kl}(\mathbf{u}) + \tilde{b} s - \tilde{d} c] + \right.$$

$$\left. + L_{ij} \frac{\partial}{\partial x_j} [- \tilde{\xi}_{kl} e_{kl}(\mathbf{u}) - \tilde{d} s + \tilde{a} c] \right\} , \tag{2.46}$$

$$\dot{c} = \frac{\partial}{\partial x_i} \left\{ L_{ij} \frac{\partial}{\partial x_j} [- \tilde{\gamma}_{kl} e_{kl}(\mathbf{u}) + \tilde{b} s - \tilde{d} c] + \right.$$

$$\left. + D_{ij} \frac{\partial}{\partial x_j} [- \tilde{\xi}_{kl} e_{kl}(\mathbf{u}) - \tilde{d} s + \tilde{a} c] \right\} ,$$

where

$$\lambda_{ij} = K_{ij}/T_0 \,. \tag{2.47}$$

It is worth noting that Eq. $(2.46)_2$ for the evolution of entropy is formally the same as Eq. $(2.46)_3$ for the evolution of concentration. Thus, the system (2.46) may be written in the following abbreviated form

$$\begin{aligned} \rho\, \ddot{u}_i &= \frac{\partial}{\partial x_i}\left[\tilde{c}_{ijkl}\, e_{kl}(\mathbf{u}) - \tilde{\gamma}^{\alpha}_{ij}\, s^{\alpha}\right] + B_i \,, \\ \dot{s}^{\alpha} &= \frac{\partial}{\partial x_i}\left[L^{\alpha\beta}_{ij}\, \frac{\partial}{\partial x_j}\left(-\tilde{\gamma}^{\beta}_{kl}\, e_{kl}(\mathbf{u}) + \tilde{a}^{\beta\gamma} s^{\gamma}\right)\right] , \end{aligned} \tag{2.48}$$

provided that the following notation is introduced

$$(s^{\alpha}) = (s^1, s^2) = (s, c) \,, \tag{2.49}$$

$$\left[\tilde{a}^{\alpha\beta}\right] = \begin{bmatrix} \tilde{a}^{11} & \tilde{a}^{12} \\ \tilde{a}^{21} & \tilde{a}^{22} \end{bmatrix} = \begin{bmatrix} \tilde{b} & -\tilde{d} \\ -\tilde{d} & \tilde{a} \end{bmatrix} \tag{2.50}$$

$$\left[\tilde{\gamma}^{\alpha}_{ij}\right] = \left[\tilde{\gamma}^{(1)}_{ij}\,,\ \tilde{\gamma}^{(2)}_{ij}\right] = \left[\tilde{\gamma}_{ij}\,,\ \tilde{\xi}_{ij}\right] \tag{2.51}$$

$$\left[\mathbf{L}^{\alpha\beta}\right] = \begin{bmatrix} \mathbf{L}^{11} & \mathbf{L}^{12} \\ \mathbf{L}^{21} & \mathbf{L}^{22} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\lambda}} & \mathbf{L} \\ \mathbf{L}^{T} & \mathbf{D} \end{bmatrix} . \tag{2.52}$$

The Greek superscripts take values 1 and 2, and indicate the relation between the matrix element and the relevant fluxes of heat and diffusion, respectively. The summation convention applies also to these indices. The following symmetries are evident

$$\tilde{a}^{\alpha\beta} = \tilde{a}^{\beta\alpha}, \qquad L^{\alpha\beta}_{ij} = L^{\beta\alpha}_{ij} = L^{\alpha\beta}_{ji} \tag{2.53}$$

In order to facilitate the formulation of the initial-boundary value problems, we introduce the following notations

$$(\Theta^{\alpha}) = (\Theta^1, \Theta^2) = (\Theta, M) \tag{2.54}$$

For further convenience we set

$$[\underset{\sim}{\bar{\gamma}}^{\alpha}] = [\underset{\sim}{\bar{\gamma}}^{(1)}, \underset{\sim}{\bar{\gamma}}^{(2)}] = [\ \bar{\gamma}_{ij}\ ,\ \bar{\xi}_{ij}]\ , \tag{2.55}$$

$$[\underset{\sim}{\gamma}^{\alpha}] = [\underset{\sim}{\gamma}^{(1)}, \underset{\sim}{\gamma}^{(2)}] = [\ \gamma_{ij}\ ,\ -\ \xi_{ij}]\ , \tag{2.56}$$

$$[\bar{a}^{\alpha\beta}] = \begin{bmatrix} \bar{b} & \bar{d} \\ \bar{d} & \bar{a} \end{bmatrix} \tag{2.57}$$

and

$$[a^{\alpha\beta}] = \begin{bmatrix} b & d \\ -d & a \end{bmatrix} \tag{2.58}$$

The matrix $[\bar{a}^{\alpha\beta}]$ is symmetric while $[a^{\alpha\beta}]$ a skew-symmetric one. As it is seen from the definitions (2.45) the matrices $[\bar{a}^{\alpha\beta}]$ and $[\tilde{a}^{\alpha\beta}]$ are mutually inverse

$$\tilde{a}^{\alpha\gamma}\ \bar{a}^{\gamma\beta} = \delta^{\alpha\beta} \tag{2.59}$$

Also from (2.39) we find

$$\tilde{a}^{\alpha\beta}\ \bar{\gamma}^{\beta}_{ij} = \tilde{\gamma}^{\alpha}_{ij} \tag{2.60}$$

By using (2.54) and (2.55) the constitutive equations (2.38) and (2.39) assume the following concise form

$$s^{\alpha} = \ \bar{\gamma}^{\alpha}_{ij}\ e_{ij} + \ \bar{a}^{\alpha\beta}\ \Theta^{\beta} \tag{2.61}$$

Remark 2.1

KUBIK [132] derived general equations of thermodiffusion on the basis of the mixture theory. The field equations are obtained by using the balance equations, *cf.* also KUBIK [133], KUBIK and WYRWAŁ [134].

Various physical aspects of the diffusion in solids are presented in the book by MROWEC [135], *cf.* also WERES [150].

3. EXISTENCE AND UNIQUENESS THEOREM

The initial-boundary value problem of the thermodiffusion in a nonhomogeneous anisotropic elastic body is formulated in the following form

$$\rho\ \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left(\bar{c}_{ijkl}e_{kl}(\mathbf{u}) - \bar{\gamma}_{ij}\Theta\ - \bar{\xi}_{ij}M\right) = B_i\ ,\ \text{ in } \Omega\times(0,t_o)\ , \tag{3.1}$$

$$\bar{b}\ \frac{\partial\Theta}{\partial t} + \bar{d}\ \frac{\partial M}{\partial t} - \frac{\partial}{\partial x_i}\left(\lambda_{ij}\ \frac{\partial\Theta}{\partial x_j} + L_{ij}\ \frac{\partial M}{\partial x_j}\right) + \bar{\gamma}_{ij}\ \frac{\partial \dot{u}_i}{\partial x_j} = g_1\ ,\text{in } \Omega\times(0,t_o)\ , \tag{3.2}$$

$$\bar{d}\ \frac{\partial\Theta}{\partial t} + \bar{a}\ \frac{\partial M}{\partial t} - \frac{\partial}{\partial x_i}\left(L_{ij}\ \frac{\partial\Theta}{\partial x_j} + D_{ij}\ \frac{\partial M}{\partial x_j}\right) + \bar{\xi}_{ij}\ \frac{\partial \dot{u}_i}{\partial x_j} = g_2\ ,\text{in } \Omega\times(0,t_o)\ , \tag{3.3}$$

$$\mathbf{u}(x,t) = 0, \quad \Theta(x,t) = 0, \; M(x,t) = 0, \quad \text{on } \partial\Omega\times(0,t_o) \,, \tag{3.4}$$

$$\mathbf{u}(x,0) = \mathbf{U}(x), \quad \Theta(x,0) = \Theta_o(x), \quad M(x,0) = M_o(x), \quad \dot{\mathbf{u}}(x,0) = \mathbf{V}(x) \,, \tag{3.5}$$

where $\dot{u}_i = \partial u_i/\partial t$.

We make the following, rather weak assumptions:

$$(H_1) \left| \begin{array}{l} \rho\,, \; \bar{b}, \; \bar{d}, \; \bar{a} \in L^\infty(\Omega), \quad 0 < \rho_0 \le \rho(x) \\ \\ \lambda_1(e_1^2 + e_2^2) \le [e_1,e_2] \begin{bmatrix} \bar{b}(x) & \bar{d}(x) \\ \bar{d}(x) & \bar{a}(x) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \le \lambda_2(e_1^2 + e_2^2) \end{array} \right.$$

$$\text{a.e. } x \in \Omega \quad \forall \; e_1, e_2 \in \mathbb{R}.$$

where $\lambda_2 > \lambda_1 > 0$; λ_2, λ_1 - constants;.

$$(H_2) \left| \begin{array}{l} \bar{c}_{ijkl} \in L^\infty(\Omega), \; \lambda_1|e|^2 \le \bar{c}_{ijkl}(x)e_{ij}e_{kl} \le \lambda_2|e|^2, \qquad \text{a.e. } x \in \Omega \quad \forall \; e \in \mathbb{E}^3_s, \\ \lambda_1(|e_1|^2 + |e_2|^2) \le [e_1,e_2] \begin{bmatrix} \underset{\sim}{\lambda}(x) & L(x) \\ (L)^T(x) & D(x) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \le \lambda_2(|e_1|^2 + |e_2|^2) \end{array} \right.$$

$$\text{a.e. } x \in \Omega \quad \forall \; e_1, e_2 \in \mathbb{E}^3_s.$$

Here $\mathbb{E}^3_s$ denotes the space of symmetric 3×3 matrices.

(H_3) | $\bar{\gamma}_{ij}, \bar{\xi}_{ij} \in L^\infty(\Omega)$, $|\bar{\gamma}_{ij}(x)| \le \lambda_2$, $|\bar{\xi}_{ij}(x)| \le \lambda_2$, a.e. $x \in \Omega$.

(H_4) | $B_i \in L^2(0,t_o;L^2(\Omega))$, $g_\alpha \in L^\infty(0,t_o;H^{-1}(\Omega))$, $(\alpha = 1,2)$.

(H_5) | $u_i \in H^1_o(\Omega)$; V_i, Θ_o, $M_o \in L^2(\Omega)$.

We set $H^1(\Omega)^3 = [H^1(\Omega)]^3$, *etc.*

We can now formulate

Theorem 3.1. Under the assumptions (H_1) - (H_5) there exists a unique solution $(\mathbf{u}, \Theta, M)$ of (3.1)-(3.5) and

$$\mathbf{u} \in L^\infty(0,t_o;H^1_o(\Omega)^3), \qquad \frac{\partial u_i}{\partial t} \in L^\infty(0,t_o;L^2(\Omega)),$$

$$\Theta; \; M \in L^\infty(0,t_o;L^2(\Omega)) \cap L^2(0,t_o;H^1_o(\Omega)).$$

Remark 3.1

(i) The proof of this theorem can be performed either by using the semigroup theory [136] or by the Galerkin's method [137]. In fact, existence of a solution $(\mathbf{u},\Theta,M)$

holds in the smaler class, *cf.* [39,40]

$$\mathbf{u} \in C^0(0,t_o;H^1_o(\Omega)^3), \qquad \frac{\partial \mathbf{u}}{\partial t} \in C^0(0,t_o;L^2(\Omega)^3),$$

$$\Theta;\ \mathbf{M} \in C^0(0,t_o;L^2(\Omega)) \cap L^2(0,t_o;H^1_o(\Omega)).$$

(ii) For the definitions and properties of the function spaces used throughout this paper, the reader may refer to [138,139].

(iii) The related existence and uniqueness problems are discussed in [140, 141, 142].

4. MICROPERIODIC STRUCTURE OF TED BODY AND TWO-SCALE ASYMPTOTIC EXPANSIONS

A subclass of nonhomogeneous bodies are those with microperiodic structure. Periodicity is certainly an idealization, except man-made regular composites, yet in such a case homogenization methods yield explicit formulae for the determination of overall (effective or homogenized) moduli.

In the sequel we shall apply the method of two-scale asymptotic expansion, which can likewise be used in the case of quasi-periodic (nonuniform) structures, cf. [5], [76],[143].

Let a microperiodic structure of the **TED** body considered be εY-periodic, where $\varepsilon>0$ is a small parameter and $Y = \prod_{i=1}^{3}(0,\ y_i^0)$ is the so-called basic cell. The functions:

$c_{ijkl}(y),\ \ \gamma_{ij}(y),\ \ \xi_{ij}(y),\ \ D_{ij}(y),\ \ K_{ij}(y),\ \ \lambda_{ij}(y),\ B_i(y),\ \ b(y),\ d(y),\ a(y),\ \rho(y)$

are Y-periodic and sufficiently regular. Later a weaker assumption will be discussed. For a fixed $\varepsilon > 0$ the material functions

$$\begin{aligned}
&{}^{\varepsilon}c_{ijkl}(x) = c_{ijkl}\left(\frac{x}{\varepsilon}\right),\ {}^{\varepsilon}\gamma_{ij}(x) = \gamma_{ij}\left(\frac{x}{\varepsilon}\right),\quad {}^{\varepsilon}\xi_{ij}(x) = \xi_{ij}\left(\frac{x}{\varepsilon}\right),\\
&{}^{\varepsilon}K_{ij}(x) = K_{ij}\left(\frac{x}{\varepsilon}\right),\ {}^{\varepsilon}L_{ij}(x) = L_{ij}\left(\frac{x}{\varepsilon}\right),\ {}^{\varepsilon}D_{ij}(x) = D_{ij}\left(\frac{x}{\varepsilon}\right),\\
&{}^{\varepsilon}b(x) = b\left(\frac{x}{\varepsilon}\right),\qquad {}^{\varepsilon}d(x) = d\left(\frac{x}{\varepsilon}\right),\\
&{}^{\varepsilon}a(x) = a_{ij}\left(\frac{x}{\varepsilon}\right),\quad {}^{\varepsilon}\rho(x) = \rho\left(\frac{x}{\varepsilon}\right),\qquad x \in \Omega
\end{aligned} \tag{4.1}$$

are εY-periodic. We note that in sections concerned with asymptotic expansions the notation ${}^{\varepsilon}c_{ijkl}$, *etc*. is used whereas the conventional notation c^{ε}_{ijkl} is employed in Sections 7 and 8.

According to the definitions (2.48) - (2.52) the functions $a^{\alpha\beta}_{ij}$, γ^{α}_{ij} and $L^{\alpha\beta}_{ij}$ are also εY-periodic; thus we set

$$
\begin{aligned}
&{}^{\varepsilon}c_{ijkl}(\mathbf{x}) = c_{ijkl}\left(\frac{\mathbf{x}}{\varepsilon}\right) , \ {}^{\varepsilon}\gamma^{\alpha}_{ij}(\mathbf{x}) = \gamma^{\alpha}_{ij}\left(\frac{\mathbf{x}}{\varepsilon}\right) , \\
&{}^{\varepsilon}L^{\alpha\beta}_{ij}(\mathbf{x}) = L^{\alpha\beta}_{ij}\left(\frac{\mathbf{x}}{\varepsilon}\right) , \ {}^{\varepsilon}a^{\alpha\beta}_{ij}(\mathbf{x}) = a^{\alpha\beta}_{ij}\left(\frac{\mathbf{x}}{\varepsilon}\right) , \ {}^{\varepsilon}\rho(\mathbf{x}) = \rho\left(\frac{\mathbf{x}}{\varepsilon}\right) , \ \mathbf{x} \in \Omega
\end{aligned}
\tag{4.2}
$$

We observe that for a quasi-periodic structure we would have $c_{ijkl}(\mathbf{x},\mathbf{y})$, $\gamma^{\alpha}_{ij}(\mathbf{x},\mathbf{y})$, *etc.*, where the functions $c_{ijkl}(\mathbf{x},.)$, $\gamma^{\alpha}_{ij}(\mathbf{x},.)$, *etc.* are Y-periodic and $\mathbf{x} \in \Omega$ is the macroscopic variable.

From a mathematical point of view the homogenization means a passage with ε to zero in an appropriate sense [2, 5, 10]. Strictly speaking, the method of asymptotic expansions is a formal homogenization method, nevertheless it is a powerful one.

In the periodic case the basic system of of Eqs (2.48) takes on the following form (ε > 0 and fixed) :

$$
\begin{aligned}
&{}^{\varepsilon}\rho\ {}^{\varepsilon}\ddot{u}_i = \frac{\partial}{\partial x_i}\left[{}^{\varepsilon}\tilde{c}_{ijkl}\ e_{kl}\ ({}^{\varepsilon}\mathbf{u}) - {}^{\varepsilon}\tilde{\gamma}^{\alpha}_{ij}\ {}^{\varepsilon}s^{\alpha}\right] + {}^{\varepsilon}B_i , \\
&{}^{\varepsilon}\dot{s}^{\alpha} = \frac{\partial}{\partial x_i}\left[{}^{\varepsilon}L^{\alpha\beta}_{ij}\ \frac{\partial}{\partial x_j}\left(- {}^{\varepsilon}\tilde{\gamma}^{\beta}_{kl}\ e_{kl}\ ({}^{\varepsilon}\mathbf{u}) + {}^{\varepsilon}\tilde{a}^{\beta\gamma}\ {}^{\varepsilon}s^{\gamma}\right)\right] , \text{ in } \ \Omega \times (0,\ t_0)
\end{aligned}
\tag{4.3}
$$

Obviously, the functions ${}^{\varepsilon}\mathbf{u}$ and ${}^{\varepsilon}s^{\alpha}$ depend on $x \in \Omega$ and the time $t \in (0,\ t_0)$, ($t_0 > 0$ or $t_0 = +\infty$). This system of equations has to be completed by the boundary and initial conditions. The following conditions are assumed :

(*i*) boundary conditions

$$
{}^{\varepsilon}\mathbf{u}(x,t) = 0, \quad {}^{\varepsilon}\Theta^{\alpha}(x,t) = 0 \qquad \text{on} \quad \Gamma \times (0,\ t_0) \tag{4.4}
$$

and

(*ii*) initial conditions

$$
\begin{aligned}
&{}^{\varepsilon}\mathbf{u}(x,t) = U(x), \qquad {}^{\varepsilon}\mathbf{u}(x,t) = V(x), \\
&({}^{\varepsilon}\Theta^{\alpha}(x,t)) = (\Theta^{\alpha}_0(x)) = (\Theta^1_0(x),\ \Theta^2_0(x)) = (\Theta_0(x),\ M_0(x)) , \quad x \in \Omega
\end{aligned}
\tag{4.5}
$$

where the functions U, V and Θ^{α}_0 are prescribed and sufficiently regular. It is worth noting here that various combinations of initial conditions are possible. As we shall see in Section 6, some of the non-mechanical initial conditions for the homogenized body may be different from those for the microperiodic solid.

According to the method of two-scale asymptotic expansions we make the following assumption (*ansatz*):

$$
\begin{aligned}
{}^{\varepsilon}u_i(\mathbf{x},t) &= u_i^{(0)}(\mathbf{x},\mathbf{y},t) + \varepsilon\, u_i^{(1)}(\mathbf{x},\mathbf{y},t) + \varepsilon^2 u_i^{(2)}(\mathbf{x},\mathbf{y},t) \;]+\ldots,\\
{}^{\varepsilon}s^{\alpha}(\mathbf{x},t) &= s^{(0)\alpha}(\mathbf{x},\mathbf{y},t) + \varepsilon\, s^{(1)\alpha}(\mathbf{x},\mathbf{y},t) + \varepsilon^2 s^{(2)\alpha}(\mathbf{x},\mathbf{y},t)+\ldots,
\end{aligned}
\tag{4.6}
$$

where $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$ and the functions $u^{(0)}(\mathbf{x},.,t)$, $s^{(0)\alpha}(\mathbf{x},.,t)$, $u^{(1)}(\mathbf{x},.,t)$, *etc.* are Y-periodic.

Before proceeding further we recall that for a function $f(\mathbf{x},\mathbf{y})$, where $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$, the space differentiation operator $\frac{\partial}{\partial x_i}$ should be replaced by $\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}$.

Substituting (4.5) into (4.2) we obtain

$$
\begin{aligned}
&\rho\left(\frac{\mathbf{x}}{\varepsilon}\right)(\ddot{u}_i^{(0)} + \varepsilon\, \ddot{u}_i^{(1)} + \varepsilon^2\, \ddot{u}_i^{(2)} + \ldots) = \\
&= \left(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)\Big(\tilde{c}_{ijkl}\left(\frac{\mathbf{x}}{\varepsilon}\right)\left(\frac{\partial}{\partial x_l} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_l}\right)(u_k^{(0)} + \varepsilon\, u_k^{(1)} + \varepsilon^2 u_k^{(2)} + \ldots) - \\
&\qquad - \tilde{\gamma}_{ij}^{\alpha}\left(\frac{\mathbf{x}}{\varepsilon}\right)(s^{(0)\alpha} + \varepsilon\, s^{(1)\alpha} + \varepsilon^2\, s^{(2)\alpha} + \ldots)\Big) + B_i\left(\frac{\mathbf{x}}{\varepsilon}\right),
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
&\dot{s}^{(0)\alpha} + \varepsilon\, \dot{s}^{(1)\alpha} + \varepsilon^2\, \dot{s}^{(2)\alpha} + \ldots = \\
&= \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)\Big(L_{ij}^{\alpha\beta}\left(\frac{\mathbf{x}}{\varepsilon}\right)\left(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)\cdot \\
&\qquad \cdot \Big[-\tilde{\gamma}_{kl}^{\beta}\left(\frac{\mathbf{x}}{\varepsilon}\right)\left(\frac{\partial}{\partial x_l} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_l}\right)(u_k^{(0)} + \varepsilon\, u_k^{(1)} + \varepsilon^2 u_k^{(2)} + \ldots) + \\
&\qquad\qquad + \tilde{a}^{\beta\gamma}\left(\frac{\mathbf{x}}{\varepsilon}\right)(s^{(0)\gamma} + \varepsilon\, s^{(1)\gamma} + \varepsilon^2\, s^{(2)\gamma} + \ldots)\Big]\Big)
\end{aligned}
\tag{4.8}
$$

According to the procedure of the method of asymptotic expansions we compare terms associated with the same power of ε. Consequently we obtain :

$\underline{\varepsilon^{-3}}$

$$
0 = \frac{\partial}{\partial y_i}\Big(L_{ij}^{\alpha\beta}(\mathbf{y})\, \frac{\partial}{\partial y_j}(-\, \tilde{\gamma}_{kl}^{\beta}(\mathbf{y}) e_{ykl}(\mathbf{u}^{(0)})\Big), \tag{4.9}
$$

where

$$
e_{yij}(\mathbf{v}) = \left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i} \right)/2. \tag{4.10}
$$

$\underline{\varepsilon^{-2}}$

$$0 = \frac{\partial}{\partial y_i}(\tilde{c}_{ijkl}(y)\ e_{ykl}(\mathbf{u}^{(0)})), \tag{4.11}$$

$$\begin{aligned}0 = &\frac{\partial}{\partial x_i}\Big(L^{\alpha\beta}_{ij}(y) \frac{\partial}{\partial y_j}(-\ \tilde{\gamma}^{\ \beta}_{kl}(y) e_{ykl}(\mathbf{u}^{(0)}))\Big) + \\ &+ \frac{\partial}{\partial y_i}\Big(L^{\alpha\beta}_{ij}(y) \frac{\partial}{\partial x_j}(-\ \tilde{\gamma}^{\ \beta}_{kl}(y) e_{ykl}(\mathbf{u}^{(0)}))\Big) + \\ &+ \frac{\partial}{\partial y_i}\Big(L^{\alpha\beta}_{ij}(y) \frac{\partial}{\partial y_j}(-\ \tilde{\gamma}^{\ \beta}_{kl}(y)\ e_{kl}(\mathbf{u}^{(0)}))\Big) + \\ &+ \frac{\partial}{\partial y_i}\Big(L^{\alpha\beta}_{ij}(y) \frac{\partial}{\partial y_j}(-\ \tilde{\gamma}^{\ \beta}_{kl}(y)\ e_{ykl}(\mathbf{u}^{(1)}))\Big) + \\ &+ \frac{\partial}{\partial y_i}\Big(L^{\alpha\beta}_{ij}(y) \frac{\partial}{\partial y_j}(\ \tilde{a}^{\ \gamma\beta}(y)\ s^{(0)\gamma})\Big),\end{aligned} \tag{4.12}$$

Hence it can be shown (*cf.* Appendix) that $\mathbf{u}^{(0)}$ does not depend on the local variable $\mathbf{y}$, *i.e.*

$$\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\mathbf{x},t) \tag{4.13}$$

Further, one obtains, (*cf.* Appendix)

$$s^{(0)\alpha} = \tilde{\gamma}^{\alpha}_{ij}(y)\Big(\frac{\partial u^{(1)}_i}{\partial y_j} + \frac{\partial u^{(0)}_i}{\partial x_j}\Big) + \tilde{a}^{\alpha\beta}(y)\ C^{\beta}(x,t) \tag{4.14}$$

where *a priori* unknown functions C^{β} do not depend on y.

$\underline{\varepsilon^{-1}}$

By taking account of (4.13) an (4.14) we arrive at the following relations, (*cf.* Appendix)

$$\frac{\partial}{\partial y_j}\Big(\tilde{c}_{ijkl}(y)\Big(\frac{\partial u^{(1)}_k}{\partial y_l} + \frac{\partial u^{(0)}_k}{\partial x_l}\Big)\Big) = \frac{\partial \tilde{\gamma}^{\alpha}_{ij}(y)}{\partial y_j}\ C^{\alpha}(x,t)\ , \tag{4.15}$$

$$\begin{aligned}&\frac{\partial}{\partial y_i}\Big(L^{\alpha\beta}_{ij}(y) \frac{\partial}{\partial y_j}(-\ \tilde{\gamma}^{\ \beta}_{kl}(y)\Big(\frac{\partial u^{(1)}_k}{\partial x_l} + \frac{\partial u^{(2)}_k}{\partial y_l}\Big) + \tilde{a}^{\beta\gamma}(y)\ s^{(1)\gamma})\Big) = \\ &= \frac{\partial L^{\alpha\beta}_{ij}(y)}{\partial y_i}\ \frac{\partial\ C^{\beta}(x,t)}{\partial\ x_j}\end{aligned} \tag{4.16}$$

$\underline{\varepsilon^0}$

$$\rho(y)\ \ddot{u}_i^{(0)} = \frac{\partial}{\partial x_i}\left[\tilde{c}_{ijkl}(y)\ e_{kl}\ (\mathbf{u}^{(0)}) - \tilde{\gamma}_{ij}^{\alpha}(y)\ s^{(0)\alpha} + \right.$$

$$\left. + \tilde{c}_{ijkl}(y)\ e_{ykl}\ \mathbf{u}^{(1)})\right] + \frac{\partial}{\partial y_j}\{\ \ldots\ \} + B_i, \tag{4.17}$$

$$\dot{s}^{(0)\alpha} = \frac{\partial}{\partial x_i}\Big(L_{ij}^{\alpha\beta}(y)\ \frac{\partial}{\partial x_j}[-\tilde{\gamma}_{kl}^{\ \beta}(y)\ e_{kl}(\mathbf{u}^{(0)}) - \tilde{\gamma}_{ykl}^{\ \beta}(y)\ e_{kl}(\mathbf{u}^{(1)}) +$$

$$+ \tilde{a}^{\ \gamma\beta}(y)\ s^{(0)\gamma}] +$$

$$+ L_{ij}^{\alpha\beta}(y)\ \frac{\partial}{\partial y_j}[-\tilde{\gamma}_{kl}^{\ \beta}(y) e_{kl}(\mathbf{u}^{(1)}) - \tilde{\gamma}_{ykl}^{\ \beta}(y) e_{kl}(\mathbf{u}^{(2)}) +$$

$$+ \tilde{a}^{\ \gamma\beta}(y)\ s^{(1)\gamma}]\Big) + \frac{\partial}{\partial y_j}\{\ \ldots\ \}. \tag{4.18}$$

The terms in brackets {...} are unimportant for our further considerations.

5. EFFECTIVE MATERIAL COEFFICIENTS AND THE LOCAL PROBLEMS

For a function f depending on $y \in Y$ we set

$$\langle f \rangle = \frac{1}{|Y|}\int_Y f(y)\ dy\ . \tag{5.1}$$

By using our previous results, after some calculations the following field equations describing the homogenized **TED** body are obtained

$$\langle\rho\rangle\ \ddot{u}_i^{(O)} = \bar{c}_{ijkl}^{h}\ \frac{\partial^2 u_k^{(0)}}{\partial x_j \partial x_l} - \bar{\gamma}_{ij}^{h\alpha}\ \frac{\partial C^{\alpha}}{\partial x_j} + \langle B_i\rangle\ ,$$

$$\langle\dot{s}^{(O)\alpha}\rangle = L_{ij}^{h\alpha\beta}\ \frac{\partial^2 C^{\beta}}{\partial x_i \partial x_j}\ . \tag{5.2}$$

In the component form we have

$$\langle\rho\rangle\ \ddot{u}_i^{(O)} = \bar{c}_{ijkl}^{h}\ \frac{\partial^2 u_k^{(0)}}{\partial x_j \partial x_l} - \bar{\gamma}_{ij}^{h}\ \frac{\partial C_t}{\partial x_j} - \xi_{ij}^{h}\ \frac{\partial C_c}{\partial x_j} + \langle B_i\rangle\ ,$$

$$\langle\dot{s}^{(O)}\rangle = \lambda_{ij}^{h}\ \frac{\partial^2 C_t}{\partial x_i \partial x_j} + L_{ij}^{h}\ \frac{\partial^2 C_c}{\partial x_i \partial x_j}\ , \tag{5.2a}$$

$$\langle\dot{c}^{(O)}\rangle = L_{ij}^{h}\ \frac{\partial^2 C_t}{\partial x_i \partial x_j} + D_{ij}^{h}\ \frac{\partial^2 C_c}{\partial x_i \partial x_j}$$

Moreover we have

$$<s^{(O)\alpha}> = \bar{\gamma}^{h\alpha}_{ij} e_{ij}(\mathbf{u}^{(0)}) + \bar{a}^{h\alpha\beta} C^{\beta} \tag{5.3}$$

Hence, by comparing with (2.38) and (2.39) we infer that the functions C^{α}, $\alpha = 1,2$, (*i.e.* C_t and C_c) stand for the temperature and chemical potential of the homogenized solid, respectively. Consequently, it is more instructive to use the following notations, *cf.* (2.54)

$$\Theta^{h1} = \Theta^{h} = C^{1} = C_t , \qquad \Theta^{h2} = M^{h} = C^{2} = C_c \tag{5.4}$$

Then, instead of (5.3) we have

$$<s^{(O)\alpha}> = \bar{\gamma}^{h\alpha}_{ij} e_{ij}(\mathbf{u}^{(0)}) + \bar{a}^{h\alpha\beta} \Theta^{h\beta} \tag{5.3a}$$

The homogenized material constants are given by

$$\bar{c}^{h}_{ijkl} = <\bar{c}_{ijkl} + \bar{c}_{ijpq} \frac{\partial \chi^{(kl)}_{p}}{\partial y_{p}}> ,$$

$$\bar{\gamma}^{h\alpha}_{ij} = <\bar{\gamma}^{\alpha}_{ij} - \bar{c}_{ijpq} \frac{\partial \Gamma^{\alpha}_{p}}{\partial y_{q}}> , \tag{5.5}$$

$$\bar{a}^{h\alpha\beta} = <\bar{a}^{\alpha\beta} + \bar{\gamma}^{\alpha}_{ij} \frac{\partial \Gamma^{\beta}_{i}}{\partial y_{j}}> ,$$

$$L^{h\alpha\beta}_{ij} = <L^{\alpha\beta}_{ij} - L^{\alpha\gamma}_{ik} \frac{\partial \Theta^{\gamma\beta}_{j}}{\partial y_{k}}> ,$$

where

$$\bar{c}_{ijkl} = c_{ijkl} - \frac{1}{a} \xi_{ij} \xi_{kl} ,$$

$$\bar{\gamma}^{(1)}_{ij} = \bar{\gamma}_{ij} = \gamma_{ij} + \frac{d}{a} \xi_{ij} , \quad \bar{\gamma}^{(2)}_{ij} = \bar{\xi}_{ij} = \frac{1}{a} \xi_{ij} ,$$

$$\bar{a}^{\alpha\beta} = \begin{bmatrix} \bar{b} & \bar{d} \\ \bar{d} & \bar{a} \end{bmatrix}, \quad \bar{b} = b + \frac{d^2}{a}, \quad \bar{d} = \frac{d}{a}, \quad \bar{a} = \frac{1}{a} .$$

We emphasize that the superscript h always denotes a homogenized (effective) quantity. We observe that , *cf.* Eq. $(5.7)_3$ below

$$L^{h\alpha\beta}_{ij} = L^{h\alpha\beta}_{ji} \tag{5.6}$$

The functions $\underline{\chi}^{(kl)}$, $\underline{\Gamma}^{\alpha}$, $\underline{\Theta}^{\alpha\beta}$, *etc.* are Y-periodic. These local functions are solutions to the local problems posed obviously on Y :

$$\frac{\partial}{\partial y_i}\left(\bar{c}_{ijkl}(y)\, \frac{\partial \chi_k^{(pq)}}{\partial y_l} \right) = - \frac{\partial}{\partial y_j}\, \bar{c}_{ijpq}(y) \ ,$$

$$\frac{\partial}{\partial y_j}\left(\bar{c}_{ijkl}(y)\, \frac{\partial \Gamma_k^{\alpha}}{\partial y_l} \right) = \frac{\partial \bar{\gamma}_{ij}^{\alpha}(y)}{\partial y_j} \ , \tag{5.7}$$

$$\frac{\partial}{\partial y_i}\left(L_{ij}^{\alpha\gamma}(y)\, \frac{\partial \Theta_k^{\gamma\beta}}{\partial y_j} \right) = \frac{\partial L_{ik}^{\alpha\beta}(y)}{\partial y_i} \ .$$

Solutions of Eqs $(5.7)_{1,2}$ exist up to constant vectors while those of Eq. $(5.7)_3$ up to constants.

Eqs. (5.7) are strong formulation of the local problems. In such a case the periodic functions $\bar{c}_{ijkl}(y)$, $\bar{\gamma}_{ij}^{\alpha}(y)$ and $L_{ik}^{\alpha\beta}(y)$ have to be of class $C^1(y)$. This assumption may be significantly weakened provided that one passes to weak or variational formulations.

Let us set

$$H_{per}(Y) = \{ \ v \in H^1(Y) \mid v \ \text{ takes equal values at opposite sides of } \ Y \ \} \ ,$$

$$H_{per}(Y, \mathbb{R}^3) = [\ H_{per}(Y) \ ]^3 \ ,$$

where $H^1(Y)$ is the standard Sobolev space, *cf.* Refs [5], [10]. Now we assume that periodic functions $c_{ijkl}(y)$, $\gamma_{ij}^{\alpha}(y)$, $L_{ij}^{\alpha\beta}(y)$, $B_i(y)$ and $\rho(y)$ are elements of the space $L^{\infty}(Y)$.

Such an assumption comprises, for instance, layered solids with discontinuous material constants.

<u>Local problems</u>

Find $\underline{\chi}^{(pq)} = (\chi_k^{(pq)})$, $\underline{\Gamma}^{\alpha} = (\Gamma_k^{\alpha}) \in H_{per}(Y, \mathbb{R}^3)$

and $\underline{\Theta}^{\alpha\beta} = (\Theta_k^{\alpha\beta}) \in H_{per}(Y)$

$$\int_Y \bar{c}_{ijkl}(y)\, e_{ykl}(\underline{\chi}^{(pq)})\, e_{yij}(v)\, dy = - \int_Y \bar{c}_{ijpq}(y)\, e_{yij}(v)\, dy \qquad \forall \ v \in H_{per}(Y, \mathbb{R}^3)$$

$$\int_Y \bar{c}_{ijkl}(\mathbf{y})\ e_{ykl}(\underline{\Gamma}^{\alpha})\ e_{yij}(\mathbf{v})\ dy = \int_Y \bar{\gamma}^{\alpha}_{ij}(\mathbf{y})\ e_{yij}(\mathbf{v})\ dy \qquad \forall\ \mathbf{v} \in H_{per}(Y, \mathbb{R}^3) \tag{5.8}$$

$$\int_Y L^{\alpha\gamma}_{ij}(\mathbf{y})\ \frac{\partial \Theta^{\gamma\beta}_k}{\partial y_j}\ \frac{\partial w}{\partial y_i}\ dy = \int_Y L^{\alpha\beta}_{ik}(\mathbf{y})\ \frac{\partial w}{\partial y_i}\ dy \qquad \forall\ w \in H_{per}(Y)$$

Existence of solutions to the problems (5.8) results from the Lax-Milgram lemma, cf. [10].

6. INITIAL CONDITIONS FOR THE TEMPERATURE AND CHEMICAL POTENTIAL OF THE HOMOGENIZED BODY

Now we shall derive the initial conditions which should be satisfied for $t = 0$ by the pairs of the functions resolving the *homogenized* equations

$$(w^{h\alpha}) = (\ s^h,\ M^h\ )\ ,\quad (s^{h\alpha}) = (\ s^h,\ c^h) \quad \text{and} \quad (\Theta^{h\alpha}) = (\ \Theta^h,\ M^h)\ ,$$

corresponding to the pairs defined by Eqs (2.49) and (2.54), respectively, where $s^h = s^h\ (\mathbf{x},t)$, $M^h = M^h(\mathbf{x},t)$, $c^h = c^h\ (\mathbf{x},\ t)$ and $\Theta^h = \Theta^h(\mathbf{x},t)$. To this end the following result will be applied, cf. [5], [144], [145].

Lemma 6.1. If f is a Y-periodic $L^\infty(Y)$ function then $f\left(\frac{\mathbf{x}}{\varepsilon}\right)$ converges in $L^\infty(\Omega)$-weak-$*$ to $<f>$, provided that Ω is bounded. Consequently, if $g(\mathbf{x})$ is any $L^2(\Omega)$ function then $f\left(\frac{\mathbf{x}}{\varepsilon}\right)g(\mathbf{x})$ converges weakly in $L^2(\Omega)$ to $<f>g(\mathbf{x})$. ∎

For $\varepsilon > 0$ and $t = 0$, after (2.59) and (4.5), the entropy-concentration pair $(s^\alpha) = (s,c)$ in terms of the temperature-chemical potential pair $(\Theta^\alpha) = (\Theta,M)$ expresses as follows

$$^{\varepsilon}s^{\alpha}(\mathbf{x},0) = \bar{\gamma}^{\alpha}_{ij}(\tfrac{\mathbf{x}}{\varepsilon})\ e_{ij}(\mathbf{U}) + \bar{a}^{\alpha\beta}(\tfrac{\mathbf{x}}{\varepsilon})\ \underset{o}{\Theta}^{\beta}(\mathbf{x})\ , \tag{6.1}$$

where

$$\underset{o}{\Theta}^{\beta}(\mathbf{x}) = \Theta^{\beta}(\mathbf{x},0)\ .$$

By employing Lemma (6.1) we get

$$\lim_{\varepsilon \to 0}\ ^{\varepsilon}s^{\alpha}(\mathbf{x},0) = <\bar{\gamma}^{\alpha}_{ij}>\ e_{ij}(\mathbf{U}) + <\bar{a}^{\alpha\beta}>\underset{o}{\Theta}^{\beta}(\mathbf{x}) \tag{6.2}$$

On the other hand, Eq. (5.3a) yields the following relation for $t = 0$:

$$<s^{0\alpha}(\mathbf{x},\mathbf{y},0)> = \bar{\gamma}^{h\alpha}_{ij}\ e_{ij}(\mathbf{U}(\mathbf{x})) + \bar{a}^{h\alpha\beta}\ \Theta^{h\beta}(\mathbf{x},0) \tag{6.3}$$

Obviously

$$\lim_{\varepsilon\to 0} {}^{\varepsilon}s^{\alpha}(\mathbf{x},0) = \langle s^{0\alpha}(\mathbf{x},\mathbf{y},0)\rangle. \tag{6.4}$$

Hence we finally obtain the initial condition for the pair $(\Theta^{h\alpha}) = (\Theta^h, M^h)$

$$\begin{aligned}\Theta^h(\mathbf{x},0) = \Big\{&(\bar{a}^h\langle\bar{b}\rangle + \bar{d}^h\langle\bar{d}\rangle)\Theta_0(\mathbf{x}) + \\ &+(\bar{a}^h\langle\bar{d}\rangle + \bar{d}^h\langle\bar{a}\rangle)M_0(\mathbf{x}) \\ &+[\bar{a}^h(\langle\bar{\gamma}_{ij}\rangle - \bar{\gamma}^h_{ij}) + \bar{d}^h(\langle\bar{\xi}_{ij}\rangle - \bar{\xi}^h_{ij})]e_{ij}(\mathbf{U}(\mathbf{x}))\Big\}/\Delta^h,\\ M^h(\mathbf{x},0) = \Big\{&(\bar{d}^h\langle\bar{b}\rangle + \bar{b}^h\langle\bar{d}\rangle)\Theta_0(\mathbf{x}) + \\ &+(\bar{d}^h\langle\bar{d}\rangle + \bar{b}^h\langle\bar{a}\rangle)M_0(\mathbf{x}) \\ &+[\bar{d}^h(\langle\bar{\gamma}_{ij}\rangle - \bar{\gamma}^h_{ij}) + \bar{b}^h(\langle\bar{\xi}_{ij}\rangle - \bar{\xi}^h_{ij})]e_{ij}(\mathbf{U}(\mathbf{x}))\Big\}/\Delta^h.\end{aligned} \tag{6.5}$$

where Θ_0, M_0 and $\mathbf{U}$ prescribed, cf. Eq.(3.5). Moreover we have set

$$\gamma^h_{ij} = \langle\bar{\gamma}_{ij} - \frac{\bar{d}^h}{\bar{a}^h}\bar{\xi}^h_{ij}\ ,\quad \bar{d} = \frac{\bar{d}^h}{\bar{a}^h}\ ,\quad b^h = \bar{b}^h - \frac{(\bar{d}^h)^2}{\bar{a}^h}$$

and

$$\Delta^h = \bar{a}^h\,\bar{b}^h - (\bar{d}^h)^2$$

Written in a concise form Eqs (6.5) are given by

$$\begin{aligned}(\Theta^{h\alpha}(\mathbf{x},0)) = \tilde{a}^{h\alpha\beta}\ [\ &\langle\bar{a}^{\beta\gamma}\rangle\ \Theta^{\gamma}_0(\mathbf{x}) + \\ &+ (\ \langle\bar{\gamma}^{\beta}_{ij}\rangle - \bar{\gamma}^{h\beta}_{ij}\ )\ e_{ij}(\mathbf{U}(\mathbf{x}))]\end{aligned} \tag{6.5a}$$

where $\underset{\sim}{\tilde{a}}^h$ is the inverse of $\underset{\sim}{\bar{a}}^h$

$$\underset{\sim}{\tilde{a}}^h = (\ \underset{\sim}{\bar{a}}^h\ )^{-1}\quad i.e.\quad \tilde{a}^{h\alpha\gamma}\ \bar{a}^{h\gamma\alpha} = \delta^{\alpha\beta}$$

We have

$$\tilde{a}^{h11} = \tilde{b}^h = \bar{a}^h/\Delta^h,\quad \tilde{a}^{h12} = \tilde{d}^h = \bar{d}^h/\Delta^h\ ,\quad \tilde{a}^{h22} = \tilde{a}^h = \bar{b}^h/\Delta^h \tag{6.6}$$

Let us consider another pair of the initial data, for instance the pair

$$\begin{aligned}&{}^{\varepsilon}\mathbf{u}(\mathbf{x},0) = \mathbf{U}(\mathbf{x}),\qquad {}^{\varepsilon}\dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{V}(\mathbf{x}),\\ &(\ {}^{\varepsilon}Z^{\alpha}(\mathbf{x},0)) = (\ Z^{\alpha}_0(\mathbf{x})) = (\ \Theta_0(\mathbf{x}),\ G(\mathbf{x})\ )\end{aligned} \tag{6.7}$$

where $(Z_0^\alpha) = (\Theta_0, G)$ is prescribed, and

$$ {}^\varepsilon\Theta_0(\mathbf{x},0) = \Theta_0(\mathbf{x}), \quad {}^\varepsilon c(\mathbf{x},0) = G(\mathbf{x}) \tag{6.8} $$

For $\varepsilon > 0$, by using (2.29) and (6.7)-(6.8), the initial entropy expresses as follows

$$ {}^\varepsilon s(\mathbf{x},0) = \bar{\gamma}_{ij}(\tfrac{\mathbf{x}}{\varepsilon})\, e_{ij}(\mathbf{U}) + \bar{b}(\tfrac{\mathbf{x}}{\varepsilon})\, \Theta_o(\mathbf{x}) + d(\tfrac{\mathbf{x}}{\varepsilon})\, G(\mathbf{x}). \tag{6.9} $$

By employing Lemma 6.1 once again we get

$$ \lim_{\varepsilon\to 0} {}^\varepsilon s(\mathbf{x},0) = \langle\bar{\gamma}_{ij}\rangle\, e_{ij}(\mathbf{U}) + \langle\bar{b}\rangle\, \Theta_o(\mathbf{x}) + \langle d\rangle\, G(\mathbf{x}). \tag{6.10} $$

On the other hand, Eqs (5.3) and (5.4) can be written as

$$ \begin{aligned} \langle s^{(0)}\rangle &= \bar{\gamma}^h_{ij}\, e_{ij}(\mathbf{u}^{(0)}) + \bar{b}^h\, \Theta^h + \bar{d}^h\, M^h \\ \langle c^{(0)}\rangle &= \bar{\xi}^h_{ij}\, e_{ij}(\mathbf{u}^{(0)}) + \bar{d}^h\, \Theta^h + \bar{a}^h\, M^h \end{aligned} \tag{6.11} $$

After elimination of M^h and taking account of (6.8) we arrive at the following relation

$$ \langle s^{(0)}(\mathbf{x},\mathbf{y},0)\rangle = \gamma^h_{ij}\, e_{ij}(\mathbf{U}(\mathbf{x})) + b^h\, \Theta^h(\mathbf{x},0) + d^h\, G(\mathbf{x}) \;, \tag{6.12} $$

where

$$ \gamma^h_{ij} = \bar{\gamma}^h_{ij} - \frac{\bar{d}^h}{\bar{a}^h}\, \bar{\xi}^h_{ij}, \qquad b^h = \bar{b}^h - \frac{(\bar{d}^h)^2}{\bar{a}^h}, \qquad d^h = \frac{\bar{d}^h}{\bar{a}^h}\;, $$

because, *cf.* (6.8),

$$ \lim_{\varepsilon\to 0} {}^\varepsilon c(\mathbf{x},0) = \langle c^{(0)}(\mathbf{x},\mathbf{y},0)\rangle = G(\mathbf{x}) $$

As previously

$$ \lim_{\varepsilon\to 0} {}^\varepsilon s(\mathbf{x},0) = \langle s^{(0)}(\mathbf{x},\mathbf{y},0)\rangle $$

Finally, we obtain

$$ \Theta^h(\mathbf{x},0) = [\langle b\rangle\, \Theta\,(\mathbf{x}) + (\langle \gamma_{ij}\rangle - \gamma^h_{ij})\, e_{ij}(\mathbf{U}(\mathbf{x})) + (\langle d\rangle - d^h)\, G(\mathbf{x})/b^h \tag{6.13} $$

The formulae (6.5) and (6.13) generalize the corresponding result reported by FRANCFORT [61,62] in the case of homogenization of the equations of coupled thermoelasticity. From Eq.$(6.5)_1$ we conclude that the change in the initial temperature for the homogenized **TED** body is also influenced by the diffusion. This change is implied by the fact that $\langle b\rangle \neq b^h$, $\langle\gamma\rangle \neq \gamma^h$ and $\langle d\rangle \neq d^h$, in general. The initial condition for the chemical potential also changes and is given by $(6.5)_2$.

7. CONVERGENCE THEOREM IN THE GENERAL, NON-PERIODIC CASE

Suppose that a microstructure of an elastic solid in which thermodiffusion occurs is characterized by a small parameter $\varepsilon > 0$. No assumption of periodicity is imposed. Obviously, the results presented in this Section are also valid in the particular case of a periodic microstructure. Consequently the formal homogenization procedure used in Section 4 is justified by Th.7.2 below.

For a fixed $\varepsilon > 0$ the system of coupled equations of the linear thermodiffusion is assumed in the following form:

$$\rho^\varepsilon \frac{\partial^2 u_i^\varepsilon}{\partial t^2} - \frac{\partial}{\partial x_j}\left(\bar{c}^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) - \bar{\gamma}^\varepsilon_{ij}\Theta^\varepsilon - \bar{\xi}^\varepsilon_{ij} M^\varepsilon\right) = B_i\ , \quad \text{in } \Omega\times(0,t_o) \tag{7.1}$$

$$\bar{b}^\varepsilon \frac{\partial \Theta^\varepsilon}{\partial t} + \bar{d}^\varepsilon \frac{\partial M^\varepsilon}{\partial t} - \frac{\partial}{\partial x_i}\left(\lambda^\varepsilon_{ij}\frac{\partial \Theta^\varepsilon}{\partial x_j} + L^\varepsilon_{ij}\frac{\partial M^\varepsilon}{\partial x_j}\right) + \bar{\gamma}^\varepsilon_{ij}\frac{\partial \dot{u}_i^\varepsilon}{\partial x_j} = g_1, \text{in } \Omega\times(0,t_o) \tag{7.2}$$

$$\bar{d}^\varepsilon \frac{\partial \Theta^\varepsilon}{\partial t} + \bar{a}^\varepsilon \frac{\partial M^\varepsilon}{\partial t} - \frac{\partial}{\partial x_i}\left(L^\varepsilon_{ij}\frac{\partial \Theta^\varepsilon}{\partial x_j} + D^\varepsilon_{ij}\frac{\partial M^\varepsilon}{\partial x_j}\right) + \bar{\xi}^\varepsilon_{ij}\frac{\partial \dot{u}_i^\varepsilon}{\partial x_j} = g_2, \text{ in } \Omega\times(0,t_o) \tag{7.3}$$

$$\mathbf{u}^\varepsilon(\mathbf{x}.t) = 0, \quad \Theta^\varepsilon(\mathbf{x},t) = 0, \ M^\varepsilon(\mathbf{x},t) = 0, \quad \text{on } \partial\Omega\times(0,t_o)\ , \tag{7.4}$$

$$\mathbf{u}^\varepsilon(\mathbf{x}.0) = \mathbf{U}(\mathbf{x}), \quad \Theta^\varepsilon(\mathbf{x},0) = \Theta_o(\mathbf{x}), \ M^\varepsilon(\mathbf{x},0) = M_o(\mathbf{x}), \ \dot{\mathbf{u}}^\varepsilon(\mathbf{x}.0) = \mathbf{V}(\mathbf{x})\ , \tag{7.5}$$

where $\dot{u}_i = \partial u_i/\partial t$, *etc.*

We make the following assumptions:

(A_1)

$$\rho^\varepsilon,\ \bar{b}^\varepsilon,\ \bar{d}^\varepsilon,\ \bar{a}^\varepsilon \in L^\infty(\Omega), \qquad \lambda_1 \le \rho^\varepsilon(\mathbf{x}) \le \lambda_2, \quad a.e.\ \mathbf{x}\in\Omega$$

$$\lambda_1(e_1^2 + e_2^2) \le [e_1,e_2]\begin{bmatrix}\bar{b}^\varepsilon(\mathbf{x}) & \bar{d}^\varepsilon(\mathbf{x})\\ \bar{d}^\varepsilon(\mathbf{x}) & \bar{a}^\varepsilon(\mathbf{x})\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} \le \lambda_2(e_1^2 + e_2^2)$$

$$\text{a.e. } \mathbf{x}\in\Omega \quad \forall\ e_1, e_2 \in \mathbb{R}.$$

where $\lambda_2 > \lambda_1 > 0$; λ_2, λ_1 - constants;.

(A_2)

$$\bar{c}^\varepsilon_{ijkl} \in L^\infty(\Omega), \quad \lambda_1|\mathbf{e}|^2 \le \bar{c}^\varepsilon_{ijkl}(x)e_{ij}e_{kl} \le \lambda_2|\mathbf{e}|^2, \qquad \text{a.e. } \mathbf{x}\in\Omega \quad \forall\ \mathbf{e}\in\mathbb{E}^3_s,$$

$$\lambda_1(|\mathbf{e}_1|^2 + |\mathbf{e}_2|^2) \le [\mathbf{e}_1,\mathbf{e}_2]\begin{bmatrix}\underset{\sim}{\lambda}^\varepsilon(\mathbf{x}) & L^\varepsilon(\mathbf{x})\\ (L^\varepsilon)^T(\mathbf{x}) & D^\varepsilon(\mathbf{x})\end{bmatrix}\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{bmatrix} \le \lambda_2(|\mathbf{e}_1|^2 + |\mathbf{e}_2|^2)$$

$$\text{a.e. } \mathbf{x}\in\Omega \quad \forall\ \mathbf{e}_1, \mathbf{e}_2 \in \mathbb{E}^3_s.$$

We recall that $\mathbb{E}^3_s$ is the space of symmetric 3×3 matrices.

(A_3)

$$\bar{\gamma}^\varepsilon_{ij},\ \bar{\xi}^\varepsilon_{ij} \in L^\infty(\Omega), \quad |\bar{\gamma}^\varepsilon_{ij}(\mathbf{x})| \le \lambda_2, \quad |\bar{\xi}^\varepsilon_{ij}(\mathbf{x})| \le \lambda_2, \quad \text{a.e. } \mathbf{x}\in\Omega\ .$$

To obtain *a priori* estimate we multiply Eqs (7.1)-(7.3) by $\frac{\partial \mathbf{u}^\varepsilon}{\partial t}$, Θ^ε and M^ε, respectively. Next, performing integration over Ω and integration by parts one has

$$\int_\Omega \rho^\varepsilon \frac{\partial^2 u_i^\varepsilon}{\partial t^2} \frac{\partial u_i^\varepsilon}{\partial t} dx - \int_\Omega \frac{\partial u_i^\varepsilon}{\partial t} \frac{\partial}{\partial x_j} \left(\bar{c}_{ijkl}^\varepsilon e_{kl}(\mathbf{u}^\varepsilon) \right) dx + \\ + \int_\Omega \frac{\partial u_i^\varepsilon}{\partial t} \frac{\partial}{\partial x_j} \left(\bar{\gamma}_{ij}^\varepsilon \Theta^\varepsilon + \bar{\xi}_{ij}^\varepsilon M^\varepsilon \right) = \int_\Omega B_i \frac{\partial u_i^\varepsilon}{\partial t} dx \,, \tag{7.6}$$

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \bar{b}^\varepsilon |\Theta^\varepsilon|^2 dx + \int_\Omega \bar{d}^\varepsilon \Theta^\varepsilon \frac{\partial M^\varepsilon}{\partial t} dx + \\ + \int_\Omega (\lambda_{ij}^\varepsilon \Theta_{,j}^\varepsilon + L_{ij}^\varepsilon M_{,j}^\varepsilon) \Theta_{,i}^\varepsilon dx + \int_\Omega \bar{\gamma}_{ij}^\varepsilon \Theta^\varepsilon e_{ij}\left(\frac{\partial \mathbf{u}^\varepsilon}{\partial t}\right) dx = \int_\Omega g_1 \Theta^\varepsilon dx \,, \tag{7.7}$$

$$\int_\Omega \bar{d}^\varepsilon M^\varepsilon \frac{\partial \Theta^\varepsilon}{\partial t} dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \bar{a}^\varepsilon |M^\varepsilon|^2 dx + \int_\Omega (L_{ij}^\varepsilon \Theta_{,j}^\varepsilon + D_{ij}^\varepsilon M_{,j}^\varepsilon) M_{,i}^\varepsilon dx + \\ + \int_\Omega \bar{\xi}_{ij}^\varepsilon M^\varepsilon e_{ij}\left(\frac{\partial \mathbf{u}^\varepsilon}{\partial t}\right) dx = \int_\Omega g_2 M^\varepsilon dx \,, \tag{7.8}$$

since the boundary conditions (7.4) are homogeneous; here $\Theta_{,j}^\varepsilon = \frac{\partial \Theta^\varepsilon}{\partial x_j}$, etc.

Adding Eqs (7.6)-(7.8), we readily obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \rho^\varepsilon \left|\frac{\partial \mathbf{u}^\varepsilon}{\partial t}\right|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \bar{c}_{ijkl}^\varepsilon e_{kl}(\mathbf{u}^\varepsilon) e_{ij}(\mathbf{u}^\varepsilon) dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \bar{b}^\varepsilon |\Theta^\varepsilon|^2 dx + \\ + \frac{d}{dt} \int_\Omega \bar{d}^\varepsilon \Theta^\varepsilon M^\varepsilon dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \bar{a}^\varepsilon |M^\varepsilon|^2 dx + \\ + \int_\Omega \left((\lambda_{ij}^\varepsilon \Theta_{,j}^\varepsilon + L_{ij}^\varepsilon M_{,j}^\varepsilon) \Theta_{,i}^\varepsilon + (L_{ij}^\varepsilon \Theta_{,j}^\varepsilon + D_{ij}^\varepsilon M_{,j}^\varepsilon) M_{,i}^\varepsilon \right) dx = \\ = \int_\Omega \left(B_i \frac{\partial u_i^\varepsilon}{\partial t} dx + g_1 \Theta^\varepsilon + g_2 M^\varepsilon \right) dx \,.$$

On account of the initial conditions (7.5), integration in time yields

$$\frac{1}{2}\int_{\Omega}\Big(\rho^{\varepsilon}\Big|\frac{\partial \mathbf{u}^{\varepsilon}}{\partial t}\Big|^{2}+\bar{c}^{\varepsilon}_{ijkl}e_{kl}(\mathbf{u}^{\varepsilon})\,e_{ij}(\mathbf{u}^{\varepsilon})+\bar{b}^{\varepsilon}|\Theta^{\varepsilon}|^{2}+$$

$$+\,2\,\bar{d}^{\varepsilon}\Theta^{\varepsilon}M^{\varepsilon}+\bar{a}^{\varepsilon}|M^{\varepsilon}|^{2}\Big)dx+\int_{0}^{t}\int_{\Omega}\Big((\lambda^{\varepsilon}_{ij}\,\Theta^{\varepsilon}_{,j}+L^{\varepsilon}_{ij}\,M^{\varepsilon}_{,j})\,\Theta^{\varepsilon}_{,i}+$$

$$+\,(L^{\varepsilon}_{ij}\,\Theta^{\varepsilon}_{,j}+D^{\varepsilon}_{ij}\,M^{\varepsilon}_{,j})\,M^{\varepsilon}_{,i}\Big)\,dx\,ds= \tag{7.9}$$

$$\frac{1}{2}\int_{\Omega}\Big(\rho^{\varepsilon}|\mathbf{V}|^{2}+\bar{c}^{\varepsilon}_{ijkl}e_{kl}(\mathbf{U})\,e_{ij}(\mathbf{U})+\bar{b}^{\varepsilon}|\Theta_{0}|^{2}+$$

$$+\,2\,\bar{d}^{\varepsilon}\Theta_{0}M_{0}+\bar{a}^{\varepsilon}|M_{0}|^{2}\Big)dx+\int_{0}^{t}\int_{\Omega}\Big(B_{i}\,\frac{\partial u^{\varepsilon}_{i}}{\partial t}+g_{1}\,\Theta^{\varepsilon}+g_{2}\,M^{\varepsilon}\Big)dx\,ds.$$

The last relation plays an important role in mathematical developments, including existence problems. It also provides some useful hints for the study of correctors. Particularly, by using the assumptions (A_1)-(A_3), (H_4) and (H_5) combined with Gronwall lemma [146], from Eq. (7.9) we deduce that

$\{\mathbf{u}^{\varepsilon}\}_{\varepsilon>0}$ is bounded in $L^{\infty}(0,t_o;H^1_o(\Omega)^3)$,

$\left\{\frac{\partial \mathbf{u}^{\varepsilon}}{\partial t}\right\}_{\varepsilon>0}$ is bounded in $L^{\infty}(0,t_o;L^2(\Omega)^3)$,

$\{\Theta^{\varepsilon}\}_{\varepsilon>0}$ and $\{M^{\varepsilon}\}_{\varepsilon>0}$ are bounded in $L^{\infty}(0,t_o;L^2(\Omega))\cap L^2(0,t_o;H^1_o(\Omega))$.

Now we can formulate

Theorem 7.1. Under the assumptions (A_1) - (A_3), (H_4) and (H_5) there exists a unique solution $(\mathbf{u}^{\varepsilon},\Theta^{\varepsilon},M^{\varepsilon})$ of (7.1)-(7.5) and

$$\mathbf{u}^{\varepsilon}\in L^{\infty}(0,t_o;\,H^1_o(\Omega)^3),\qquad \frac{\partial u^{\varepsilon}_i}{\partial t}\in L^{\infty}(0,t_o;L^2(\Omega)),$$

$$\Theta^{\varepsilon};\ M^{\varepsilon}\in L^{\infty}(0,t_o;L^2(\Omega))\cap L^2(0,t_o;H^1_o(\Omega)).$$

Remark 7.1

Existence of the solutions $(\mathbf{u}^{\varepsilon},\Theta^{\varepsilon},M^{\varepsilon})$ holds in the smaller class:

$$\mathbf{u}^{\varepsilon}\in C^0(0,t_o;\,H^1_o(\Omega)^3),\qquad \frac{\partial u^{\varepsilon}_i}{\partial t}\in C^0(0,t_o;L^2(\Omega)),$$

$$\Theta^{\varepsilon};\ M^{\varepsilon}\in C^0(0,t_o;L^2(\Omega))\cap L^2(0,t_o;H^1_o(\Omega)).$$

Theorem 7.2. If the assumptions (A_1)-(A_3), (H_4) and (H_5) are satisfied, then there exists a subsequence $(\mathbf{u}^{\varepsilon'},\Theta^{\varepsilon'},\mathbf{M}^{\varepsilon'})$ convergent to $(\mathbf{u},\Theta,\mathbf{M})$ in the following sense

$$\mathbf{u}^{\varepsilon'} \longrightarrow \mathbf{u} \quad \text{weak-}* \ \text{ in } L^{\infty}(0,t_o;H^1_o(\Omega)^3),$$
$$\frac{\partial \mathbf{u}^{\varepsilon'}}{\partial t} \longrightarrow \frac{\partial \mathbf{u}}{\partial t} \quad \text{weak-}* \ \text{ in } L^{\infty}(0,t_o;L^2(\Omega)^3),$$
$$\Theta^{\varepsilon'} \longrightarrow \Theta \quad \text{weak-}* \ \text{ in } L^{\infty}(0,t_o;L^2(\Omega))\cap L^2(0,t_o;H^1_o(\Omega)),$$
$$M^{\varepsilon'} \longrightarrow M \quad \text{weak-}* \ \text{ in } L^{\infty}(0,t_o;L^2(\Omega))\cap L^2(0,t_o;H^1_o(\Omega)).$$

The triple $(\mathbf{u},\Theta,\mathbf{M}) = (\mathbf{u}^h,\Theta^h,\mathbf{M}^h)$ is the unique solution to the homogenized system given by

$$\overset{o}{\rho}\,\frac{\partial u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left(\bar{c}^h_{ijkl}e_{kl}(\mathbf{u}) - \bar{\gamma}^h_{ij}\Theta - \bar{\xi}^h_{ij}M\right) = B_i\,, \quad \text{in } \Omega\times(0,t_o) \tag{7.10}$$

$$(\overset{o}{b}+\kappa_1)\,\frac{\partial\Theta}{\partial t} + \overset{o}{d}\,\frac{\partial M}{\partial t} - \frac{\partial}{\partial x_i}\left(\lambda^h_{ij}\,\frac{\partial\Theta}{\partial x_j} + L^h_{ij}\,\frac{\partial M}{\partial\partial x_j}\right) + \bar{\gamma}^h_{ij}\,\frac{\partial\dot{u}_i}{\partial x_j} = g_1\,, \quad \text{in } \Omega\times(0,t_o) \tag{7.11}$$

$$\overset{o}{d}\,\frac{\partial\Theta}{\partial t} + (\overset{o}{a}+\kappa_2)\,\frac{\partial M}{\partial t} - \frac{\partial}{\partial x_i}\left(L^h_{ij}\,\frac{\partial\Theta}{\partial\partial x_j} + D^h_{ij}\,\frac{\partial M}{\partial\partial x_j}\right) + \bar{\xi}^h_{ij}\,\frac{\partial\dot{u}_i}{\partial x_j} = g_2\,, \quad \text{in } \Omega\times(0,t_o) \tag{7.12}$$

$$\mathbf{u}(\mathbf{x}.t) = 0,\quad \Theta(\mathbf{x},t) = 0,\ \mathbf{M}(\mathbf{x},t) = 0, \quad \text{on } \partial\Omega\times(0,t_o) \tag{7.13}$$

$$\mathbf{u}(\mathbf{x}.0) = \mathbf{U}(\mathbf{x}),\quad \dot{\mathbf{u}}^{\varepsilon}(\mathbf{x},0) = \mathbf{V}(\mathbf{x}) \tag{7.14}$$

$$\Theta(\mathbf{x},0) = \Theta^h_o(\mathbf{x}) \neq \Theta_o, \qquad \mathbf{M}(\mathbf{x},0) = \mathbf{M}^h_o(\mathbf{x}) \neq \mathbf{M}_o(\mathbf{x}). \tag{7.15}$$

Remark 7.2

(i) Theorem 7.2 is just a reformulation of *Th.* 3.1 for the case of a solid with a microstructure characterized by ε. The proof of *Th.* 7.2 is rather lengthy and will be given elsewhere.

(ii) As we already known from Sections 5 and 6, even in the periodic case the homogenized coefficients $\bar{a}^h = \overset{o}{a}$, $\bar{b}^h = \overset{o}{b}$ and $\bar{d}^h = \overset{o}{d}$ are not equal to their mean values.

(iii) On account of (A_1), $\rho^{\varepsilon'} \longrightarrow \overset{o}{\rho}$ weak-$*$ in $L^{\infty}(\Omega)$. Here one has a liberty in the choice of a subsequence ε'.

(iv) In the general non-periodic case the effective coefficients $\overset{o}{\rho}$, c^h_{ijkl},..etc. are not necessarily constant but may depend on $\mathbf{x} \in \Omega$.

(v) The homogenized system (7.10)-(7.12) is of the same type as the primal one or (7.1)-(7.3).

PART TWO : CORRECTORS AND EXAMPLES

In Part I of our contribution the homogenization problem was solved for linear equations of thermodiffusion in a three-dimensional solid. The formal method of two-scale asymptotic expansions was justified by Theorem 7.2. Now we will continue our considerations; particularly in Section 8 results concerning correctors will be given. Sections 9 and 10 are more specific and illustrate the general developments of Part I.

8. CORRECTORS

For $\varepsilon > 0$ the initial conditions for the enetropy and concentration are given by

$$s^{\varepsilon}(\mathbf{x},0) = \dot{\gamma}^{\varepsilon}_{ij}(\mathbf{x})\ e_{ij}(\mathbf{u}^{\varepsilon}) + \bar{b}^{\varepsilon}(\mathbf{x})\ \Theta^{\varepsilon}(\mathbf{x},0) + \bar{d}^{\varepsilon}\ M^{\varepsilon}(\mathbf{x},0), \tag{8.1}$$

$$c^{\varepsilon}(\mathbf{x},0) = \bar{\xi}^{\varepsilon}_{ij}(\mathbf{x})\ e_{ij}(\mathbf{u}^{\varepsilon}) + \bar{d}^{\varepsilon}(\mathbf{x})\ \Theta^{\varepsilon}(\mathbf{x},0) + \bar{a}^{\varepsilon}\ M^{\varepsilon}(\mathbf{x},0). \tag{8.2}$$

We recall that now no periodicity assumption on the material coefficients is *a priori* imposed. Here $(\mathbf{u}^{\varepsilon}, \Theta^{\varepsilon}, \mathbf{M}^{\varepsilon})$ is the unique solution to the system (7.1) - (7.5). We still preserve the superscript "h" for some of the homogenized coefficients; they can be calculated explicitly for a periodic microstructure studied in Part I.

The considerations which follow due much to the papers [147,148] where scalar cases are only investigated.

Suppose that $(\mathbf{v}^{\varepsilon'}, \mathbf{z}^{\varepsilon'})$ be any functions such that

$$\left.\begin{aligned} -\,div\ [\ \bar{c}^{\varepsilon'}\ e(\mathbf{v}^{\varepsilon'}) - (\underset{\sim}{\bar{\gamma}}^{\varepsilon'} - \underset{\sim}{\bar{\gamma}}^{h})\ ] = 0\ , &\qquad \text{in}\ \ \Omega \\ \mathbf{v}^{\varepsilon'} \longrightarrow 0 \text{ weakly in } &\ \ H^1_o(\Omega)^3 \end{aligned}\right\} \tag{8.3}$$

$$\left.\begin{aligned} -\,div\ [\ \bar{c}^{\varepsilon'}\ e(\mathbf{z}^{\varepsilon'}) - (\underset{\sim}{\bar{\xi}}^{\varepsilon'} - \underset{\sim}{\bar{\xi}}^{h})\ ] = 0\ , &\qquad \text{in}\ \ \Omega \\ \mathbf{z}^{\varepsilon'} \longrightarrow 0 \text{ weakly in } &\ \ H^1_o(\Omega)^3 \end{aligned}\right\} \tag{8.4}$$

Such a sequence $(\mathbf{v}^{\varepsilon'}, \mathbf{z}^{\varepsilon'})$ exists [40].

Next, we define (κ_1, κ_2) by extracting a subsequence, still denoted by ε', such that

$$\dot{\gamma}^{\varepsilon'}_{ij}(\mathbf{x})\ e_{ij}(\mathbf{v}^{\varepsilon'}) \longrightarrow \kappa_1, \text{ weakly in } L^2(\Omega)\ , \tag{8.5}$$

$$\bar{\xi}^{\varepsilon'}_{ij}(\mathbf{x})\ e_{ij}(\mathbf{z}^{\varepsilon'}) \longrightarrow \kappa_2, \text{ weakly in } L^2(\Omega)\ . \tag{8.6}$$

It can be shown that

$$\kappa_\alpha \in L^\infty(\Omega) \quad \text{and} \quad \kappa_\alpha(\mathbf{x}) \geq 0 \qquad \text{a. e.}\ \ \mathbf{x} \in \Omega\ . \tag{8.7}$$

Further, we note that one can extract a subsequence, still indexed by ε', such that

$$\rho^{\varepsilon'} \longrightarrow \overset{o}{\rho}, \ \ \bar{b}^{\varepsilon'} \longrightarrow \overset{o}{b}, \ \ \bar{d}^{\varepsilon'} \longrightarrow \overset{o}{d}, \ \ \bar{a}^{\varepsilon'} \longrightarrow \overset{o}{a} \quad \text{weak-}* \quad \text{in} \quad L^2(\Omega) \tag{8.8}$$

with $\lambda_1(e_1^{\,2} + e_2^{\,2}) \le [e_1, e_2] \begin{bmatrix} \overset{o}{b} & \overset{o}{d} \\ \overset{o}{d} & \overset{o}{a} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \le \lambda_2(e_1^{\,2} + e_2^{\,2})$

$$a.e.\ \mathbf{x} \in \Omega, \quad \forall\, e_1,\ e_2 \in \mathbb{R}\ .$$

We observe that in the periodic case $\overset{o}{\rho} = \langle \rho \rangle$, *etc.* To find the homogenized coefficients $\bar{\underset{\sim}{\gamma}}^h$ and $\bar{\underset{\sim}{\xi}}^h$ one may use the corrector tensor $\mathbf{P}^\varepsilon = (\ P^\varepsilon_{ijkl})$ associated to $\bar{\mathbf{c}}^\varepsilon$. This tensor will be defined below.

Firstly, however, let us introduce the notion of H-convergence for $\bar{\mathbf{c}}^\varepsilon$, [147,148].

The sequence $\bar{\mathbf{c}}^\varepsilon$ (strictly a subsequence $\bar{\mathbf{c}}^{\varepsilon'}$) is said to H-converge to $\bar{\mathbf{c}}^h$ if for any $\mathbf{h}^\varepsilon \longrightarrow \mathbf{h}$ strongly in $H^{-1}(\Omega)^3$, the solutions $\mathbf{v}^{\varepsilon'}$ of the elliptic system

$$-\,div\ [\ \bar{\mathbf{c}}^{\varepsilon'}\ \mathbf{e}(\mathbf{v}^{\varepsilon'})] = \mathbf{h}^{\varepsilon'} \qquad \text{in} \quad \Omega, \qquad \mathbf{v}^{\varepsilon'} \in\ H^1_o(\Omega)^3 \tag{8.9}$$

satisfy

$$\bar{\mathbf{c}}^{\varepsilon'}\ \mathbf{e}(\mathbf{v}^{\varepsilon'}) \longrightarrow\ \bar{\mathbf{c}}^h\ \mathbf{e}(\mathbf{v}) \quad \text{weakly in} \quad L^2(\Omega, \mathbb{E}^3_s)\ . \tag{8.10}$$

Here $\mathbf{v} \in H^1_o(\Omega)^3$ is the unique solution of

$$-\,div\ [\ \bar{\mathbf{c}}^h\ \mathbf{e}(\mathbf{v})] = \mathbf{h}\ , \qquad \text{in} \quad \Omega\ . \tag{8.11}$$

We note that for a general microstructure $\bar{\mathbf{c}}^h$ depends on $\mathbf{x} \in \Omega$.

The matrices $\underset{\sim}{\lambda}^h$, $\mathbf{L}^h$ and $\mathbf{D}^h$ are defined similarly. More precisely, the matrices $\underset{\sim}{\lambda}^{\varepsilon'}$, $\mathbf{L}^{\varepsilon'}$ and $\mathbf{D}^{\varepsilon'}$ are said to H-converge to $\underset{\sim}{\lambda}^h$, $\mathbf{L}^h$ and $\mathbf{D}^h$, respectively, if for any sequence $(g^\varepsilon, h^\varepsilon) \longrightarrow (g,h)$ strongly in $H^{-1}(\Omega)^2$, the solution $(v^{\varepsilon'}, w^{\varepsilon'})$ of

$$\left.\begin{aligned} &-\,div\ \ (\underset{\sim}{\lambda}^{\varepsilon'} grad\ v^{\varepsilon'} + \mathbf{L}^{\varepsilon'} grad\ w^{\varepsilon'}) = g^{\varepsilon'}, && \text{in} \quad \Omega \\ &-\,div\ [(\mathbf{L}^{\varepsilon'})^T\ grad\ v^{\varepsilon'} + \mathbf{D}^{\varepsilon'} grad\ w^{\varepsilon'}] = h^{\varepsilon'}, && \text{in} \quad \Omega \end{aligned}\right\} \tag{8.12}$$

where $(v^\varepsilon, w^\varepsilon) \in H^1_o(\Omega)^2$, satisfy

$$\underset{\sim}{\lambda}^{\varepsilon'} grad v^{\varepsilon'} \longrightarrow \underset{\sim}{\lambda}^h grad v\ ,$$

$$\mathbf{L}^{\varepsilon'} grad w^{\varepsilon'} \longrightarrow \mathbf{L}^h grad w\ ,$$

$$\mathbf{D}^{\varepsilon'} grad w^{\varepsilon'} \longrightarrow \mathbf{D}^h grad w$$

weakly in $L^2(\Omega)^3$. Here $(v,w) \in H_o(\Omega)^2$ is the unique solution of

$$\left.\begin{aligned} -\,div(\underset{\sim}{\lambda}^h grad\ v + \mathbf{L}^h grad\ w) = g\ , \qquad & \text{in}\quad \Omega \\ -\,div[(\mathbf{L}^h)^T grad\ v^{\varepsilon'} + \mathbf{D}^h grad\ w] = h\ , \qquad & \text{in}\quad \Omega \end{aligned}\right\} \tag{8.13}$$

Let $\mathbf{E} \in \mathbb{E}^3_s$; particularly one may take $\mathbf{E} = \underset{-}{\delta} = (\delta_{ij})$. Suppose that a sequence $\bar{\mathbf{c}}^{\varepsilon'}$ is H-convergent to $\bar{\mathbf{c}}^h$. Consider the function $\mathbf{w}^{\varepsilon'}_{\underline{E}} \in H^1(\Omega)^3$ such that

$$\left.\begin{aligned} \mathbf{w}^{\varepsilon'}_{\underline{E}} \longrightarrow (E_{ij}x_j) \text{ weakly in } H^1(\Omega)^3 \\ div\ (\bar{\mathbf{c}}^{\varepsilon'}\ \mathbf{e}(\mathbf{w}^{\varepsilon'}_{\underline{E}})) \longrightarrow div(\bar{\mathbf{c}}^h\ \mathbf{E}) \text{ strongly in } H^{-1}(\Omega)^3 \end{aligned}\right\} \tag{8.14}$$

It is worth noting that for a periodic microstructure the function $\mathbf{w}^{\varepsilon}_{\underline{E}}$ can be determined

explicitly by solving the local problem on the basic cell Y, cf, [5]

Now we define $\mathbf{P}^{\varepsilon'} = (P^{\varepsilon'}_{ijkl}),\ P^{\varepsilon'}_{ijkl} \in L^2(\Omega)$, by

$$\mathbf{P}^{\varepsilon'}\mathbf{E} = \mathbf{e}(\mathbf{w}^{\varepsilon'}_{\underline{E}}) \qquad \text{in } \Omega\ . \tag{8.15}$$

If $(\mathbf{v}^{\varepsilon'}, \mathbf{z}^{\varepsilon'})$ are solutions to (8.3) and (8.4) then [147,148]

$$\left.\begin{aligned} \mathbf{e}(\mathbf{v}^{\varepsilon'}) = \mathbf{P}^{\varepsilon'}\mathbf{e}(\mathbf{v}^{\varepsilon'}) + \mathbf{r}^{\varepsilon'}_1, \\ \mathbf{e}(\mathbf{z}^{\varepsilon'}) = \mathbf{P}^{\varepsilon'}\mathbf{e}(\mathbf{z}^{\varepsilon'}) + \mathbf{r}^{\varepsilon'}_2, \end{aligned}\right\} \tag{8.16}$$

with $\mathbf{r}^{\varepsilon'}_\alpha \longrightarrow 0$ strongly in $L^1(\Omega,\mathbb{E}^3_s)$, $(\alpha = 1,2)$. $\mathbf{P}^{\varepsilon'}$ is called the *corrector tensor* associated to $\bar{\mathbf{c}}^{\varepsilon'}$.

The corrector matrices associated to $\underset{\sim}{\lambda}^{\varepsilon'}$ and $\mathbf{L}^{\varepsilon'}$ are defined similarly (we recall that $(\mathbf{L}^{\varepsilon'})^T = \mathbf{L}^{\varepsilon'}$). More precisely, let $\underline{\Delta}_\alpha \in \mathbb{R}^3$ $(\alpha = 1,2)$. Consider the functions $w^{\varepsilon'}_{\underline{\Delta}_\alpha} \in H^1(\Omega)$, such that

$$\left.\begin{aligned} w^{\varepsilon'}_{\underline{\Delta}_\alpha} \longrightarrow (\underline{\Delta}_\alpha,x) = (\Lambda_{i\alpha}x_i), \qquad & \text{weakly in } H^1(\Omega) \\ div(\underset{\sim}{\lambda}^{\varepsilon'} grad\ w^{\varepsilon'}_{\underline{\Delta}_1} + \mathbf{L}^{\varepsilon'} grad\ w^{\varepsilon'}_{\underline{\Delta}_2}) \rightarrow div(\underset{\sim}{\lambda}^h\underline{\Delta}_1 + \mathbf{L}^h\underline{\Delta}_2),\ & \text{strongly in } H^{-1}(\Omega) \end{aligned}\right\} \tag{8.17}$$

We can now define the matrices $\mathbf{Q}^{\varepsilon'}, \mathbf{R}^{\varepsilon'} \in L(\Omega,\mathbb{E}^3_s)$ by

$$Q^{\varepsilon'}\ \underline{\Delta}_1 = grad\ w^{\varepsilon'}_{\underline{\Delta}_1}, \qquad \underline{R}^{\varepsilon'}\ \underline{\Delta}_2 = grad\ w^{\varepsilon'}_{\underline{\Delta}_2} \tag{8.18}$$

If $(v^{\varepsilon'}, w^{\varepsilon'})$ is a solution to (8.12) then

$$\left.\begin{aligned} grad\ v^{\varepsilon'} &= Q^{\varepsilon'}\ grad\ v + \mathbf{r}_1^{\varepsilon'}, \\ grad\ w^{\varepsilon'} &= \underline{R}^{\varepsilon'}\ grad\ w + \mathbf{r}_2^{\varepsilon'}, \quad \text{with} \quad \mathbf{r}_\alpha^{\varepsilon'} \to 0 \quad \text{strongly in} \quad L^1(\Omega)^3 \end{aligned}\right\} \tag{8.19}$$

Suppose now that the tensor $\mathbf{P}^{\varepsilon'}$ is known. Then one can define the homogenized coefficients $\bar{\underset{\sim}{\gamma}}^h$ and $\bar{\underset{\sim}{\xi}}^h$ by extracting a subsequence, still indexed by ε', such that, *cf.* [147, Section 2, Prop. 5]

$$\mathbf{P}^{\varepsilon'}\ \bar{\underset{\sim}{\gamma}}^{\varepsilon'} \longrightarrow \bar{\underset{\sim}{\gamma}}^h \quad \text{and} \quad \mathbf{P}^{\varepsilon'}\ \bar{\underset{\sim}{\xi}}^{\varepsilon'} \longrightarrow \bar{\underset{\sim}{\xi}}^h \qquad \text{weakly in} \quad L(\Omega, \mathbb{E}_s^3). \tag{8.20}$$

Denoting the r.h.s. of (8.1) and (8.2) by $(D_1^\varepsilon, D_2^\varepsilon)$, we have (for a subsequence, say ε')

$$D_\alpha^{\varepsilon'} \longrightarrow D_\alpha^0 \qquad \text{weakly in} \quad L^2(\Omega) \tag{8.21}$$

By using (8.5) and (8.6) we define $(\Theta^h(x, 0), M^h(x, 0))$ by

$$\begin{aligned} \bar{\gamma}_{ij}^h\, e_{ij}(U) + (\overset{0}{b} + \kappa_1)\, \Theta^h(x, 0) + \overset{0}{d}\, M^h(x, 0) &= D_1^0, \\ \bar{\xi}_{ij}^h\, e_{ij}(U) + \overset{0}{d}\, \Theta^h(x, 0) + (\overset{0}{a} + \kappa_2)\, M^h(x, 0) &= D_2^0 \end{aligned} \tag{8.22}$$

On the other hand we have $\bar{\gamma}_{ij}^{\varepsilon'}, \bar{\xi}_{ij}^{\varepsilon'} \in L^\infty(\Omega) \subset L^2(\Omega)$, since Ω is a bounded domain. Hence, for some subsequence - still indexed by ε' - we have

$$\bar{\gamma}_{ij}^{\varepsilon'} \longrightarrow \overset{0}{\gamma}_{ij}, \ \bar{\xi}_{ij}^{\varepsilon'} \longrightarrow \overset{0}{\xi}_{ij} \quad \text{weakly in } L^2(\Omega).$$

Thus we obtain:

$$\begin{aligned} \lim_{\varepsilon'\to 0} s^{\varepsilon'}(x,0) &= \lim_{\varepsilon'\to 0}[\bar{\gamma}_{ij}^{\varepsilon'}\, e_{ij}(U) + \bar{b}^{\varepsilon'}(x)\, \Theta_o(x) + \bar{d}^{\varepsilon'}(x)\, M_o(x)] = \\ &= \overset{0}{\gamma}_{ij}\, e_{ij}(U) + \overset{0}{b}(x)\Theta_o(x) + \overset{0}{d}(x)M_o(x) \quad , \\ \lim_{\varepsilon'\to 0} c^{\varepsilon'}(x,0) &= \lim_{\varepsilon'\to 0}[\bar{\xi}_{ij}^{\varepsilon'}\, e_{ij}(U) + \bar{d}^{\varepsilon'}(x)\, \Theta_o(x) + \bar{a}^{\varepsilon'}(x)\, M_o(x)] = \\ &= \overset{0}{\xi}_{ij}(x)\, e_{ij}(U) + \overset{0}{d}(x)\Theta_o(x) + \overset{0}{a}(x)M_o(x) \ . \end{aligned} \tag{8.23}$$

From the systems of algebraic equations (8.22) and (8.23) one derives the initial value for the temperature $\Theta_o^h(x) = \Theta^h(x,0)$ and chemical potential $M_o^h(x) = M^h(x,0)$.

Following [40] and performing energetic considerations we introduce a solution $(\tilde{u}^\varepsilon, \tilde{\Theta}^\varepsilon, \tilde{M}^\varepsilon)$ corresponding to initial conditions $(U^\varepsilon, \underline{V}, \Theta_o^\varepsilon, M_o^\varepsilon)$. The latter are defined by

$$- div\ [\bar{c}^{\varepsilon}\ \mathbf{e}(\mathbf{U}^{\varepsilon}) - \Theta_o^h(\bar{\underset{\sim}{\gamma}}^{\varepsilon} - \bar{\underset{\sim}{\gamma}}^h) - M_o^h(\bar{\underset{\sim}{\xi}}^{\varepsilon} - \bar{\underset{\sim}{\xi}}^h)] = - div\ [\bar{c}^h\ \mathbf{e}(\mathbf{U}^{\varepsilon})],\ \text{in}\ \Omega \tag{8.24}$$

$$\mathbf{U}^{\varepsilon} \in H_0^1(\Omega)^3, \quad \Theta_o^{\varepsilon} = \Theta_o^h, \quad M_o^{\varepsilon} = M_o^h .$$

After these preparations we define

$(\tilde{\mathbf{u}}^{\varepsilon},\tilde{\Theta}^{\varepsilon},\tilde{M}^{\varepsilon})$ and $(\mathbf{v}^{\varepsilon},\eta^{\varepsilon},N^{\varepsilon})$, where $(\mathbf{u}^{\varepsilon},\Theta^{\varepsilon},M^{\varepsilon}) = (\tilde{\mathbf{u}}^{\varepsilon} + \tilde{\mathbf{v}}^{\varepsilon},\tilde{\Theta}^{\varepsilon} + \eta^{\varepsilon},\tilde{M}^{\varepsilon} + N^{\varepsilon})$, as solutions to the following initial-boundary value problems:

$$\rho^{\varepsilon}\ \frac{\partial^2\tilde{\mathbf{u}}^{\varepsilon}}{\partial t^2} - div\ \left(\bar{\mathbf{c}}^{\varepsilon}\mathbf{e}(\tilde{\mathbf{u}}^{\varepsilon}) - \bar{\underset{\sim}{\gamma}}^{\varepsilon}\ \tilde{\Theta}^{\varepsilon} - \bar{\underset{\sim}{\xi}}^{\varepsilon}\ \tilde{M}^{\varepsilon}\right) = \mathbf{B} \quad , \qquad \text{in}\ \Omega\times(0,t_o)\ ,$$

$$\bar{b}^{\varepsilon}\ \frac{\partial\tilde{\Theta}^{\varepsilon}}{\partial t} + \bar{d}^{\varepsilon}\ \frac{\partial\tilde{M}^{\varepsilon}}{\partial t} - div\ \left(\underset{\sim}{\lambda}^{\varepsilon} grad\ \tilde{\Theta}^{\varepsilon} + \mathbf{L}^{\varepsilon} grad\ \tilde{M}^{\varepsilon}\right) + \bar{\underset{\sim}{\gamma}}^{\varepsilon}\ \mathbf{e}(\frac{\partial\tilde{\mathbf{u}}^{\varepsilon}}{\partial t}) = g_1\ ,$$

$$\bar{d}^{\varepsilon}\ \frac{\partial\tilde{\Theta}^{\varepsilon}}{\partial t} + \bar{a}^{\varepsilon}\ \frac{\partial\tilde{M}^{\varepsilon}}{\partial t} - div\ \left((\mathbf{L}^{\varepsilon})^T\ grad\ \tilde{\Theta}^{\varepsilon} + \mathbf{D}^{\varepsilon} grad\ \tilde{M}^{\varepsilon}\right) + \bar{\underset{\sim}{\xi}}^{\varepsilon}\ \mathbf{e}(\frac{\partial\tilde{\mathbf{u}}^{\varepsilon}}{\partial t}) = g_2\ , \qquad \text{in}\ \Omega\times(0,t_o)$$

$$\tilde{\mathbf{u}}^{\varepsilon}(\mathbf{x},t) = 0, \quad \tilde{\Theta}^{\varepsilon}(\mathbf{x},t) = 0,\ \tilde{M}^{\varepsilon}(\mathbf{x},t) = 0, \qquad \text{on}\ \partial\Omega\times(0,t_o)\ ,$$

$$\tilde{\mathbf{u}}^{\varepsilon}(\mathbf{x},0) = \mathbf{U}^{\varepsilon}(\mathbf{x}), \quad \tilde{\Theta}^{\varepsilon}(\mathbf{x},0) = \Theta_o^h(\mathbf{x})\ ,\ \tilde{M}^{\varepsilon}(\mathbf{x},0) = M_o^h(\mathbf{x})\ ,\ \frac{\partial\tilde{\mathbf{u}}^{\varepsilon}}{\partial t}\ (\mathbf{x},0) = \mathbf{V}(\mathbf{x}) \qquad \text{in}\ \Omega;$$

and

$$\rho^{\varepsilon}\ \frac{\partial^2\tilde{\mathbf{v}}^{\varepsilon}}{\partial t^2} - div\ \left(\bar{\mathbf{c}}^{\varepsilon}\mathbf{e}(\tilde{\mathbf{v}}^{\varepsilon}) - \bar{\underset{\sim}{\gamma}}^{\varepsilon}\ \eta^{\varepsilon} - \bar{\underset{\sim}{\xi}}^{\varepsilon}\ N^{\varepsilon}\right) = 0 \quad , \quad \text{in}\ \Omega\times(0,t_o)\ ,$$

$$\bar{b}^{\varepsilon}\ \frac{\partial\eta^{\varepsilon}}{\partial t} + \bar{d}^{\varepsilon}\ \frac{\partial N^{\varepsilon}}{\partial t} - div\ \left(\underset{\sim}{\lambda}^{\varepsilon} grad\ \eta^{\varepsilon} + \mathbf{L}^{\varepsilon} grad\ N^{\varepsilon}\right) + \bar{\underset{\sim}{\gamma}}^{\varepsilon}\ \mathbf{e}(\frac{\partial\tilde{\mathbf{v}}^{\varepsilon}}{\partial t}) = 0\ , \qquad \text{in}\ \Omega\times(0,t_o)$$

$$\bar{d}^{\varepsilon}\ \frac{\partial\eta^{\varepsilon}}{\partial t} + \bar{a}^{\varepsilon}\ \frac{\partial N^{\varepsilon}}{\partial t} - div\ \left((\mathbf{L}^{\varepsilon})^T\ grad\ \eta^{\varepsilon} + \mathbf{D}^{\varepsilon} grad\ N^{\varepsilon}\right) + \bar{\underset{\sim}{\xi}}^{\varepsilon}\ \mathbf{e}(\frac{\partial\tilde{\mathbf{v}}^{\varepsilon}}{\partial t}) = 0, \qquad \text{in}\ \Omega\times(0,t_o)$$

$$\tilde{\mathbf{v}}^{\varepsilon}(\mathbf{x},t) = 0, \quad \eta^{\varepsilon}(\mathbf{x},t) = 0,\ N^{\varepsilon}(\mathbf{x},t) = 0, \quad \text{on}\ \partial\Omega\times(0,t_o)$$

$$\tilde{\mathbf{v}}^{\varepsilon}(\mathbf{x},0) = \mathbf{U}(\mathbf{x}) - \mathbf{U}^{\varepsilon}(\mathbf{x}), \quad \eta^{\varepsilon}(\mathbf{x},0) = 0\ ,\ N^{\varepsilon}(\mathbf{x},0) = 0, \quad \frac{\partial\tilde{\mathbf{v}}^{\varepsilon}}{\partial t}\ (\mathbf{x},0) = 0 \qquad \text{in}\ \Omega$$

Now, we will formulate the basic result of this Section.

Theorem 8.1 For $\varepsilon \longrightarrow 0$ the following corrector result holds true

$$\frac{\partial\tilde{\mathbf{u}}^{\varepsilon}}{\partial t} \longrightarrow \frac{\partial\mathbf{u}}{\partial t} \quad \text{strongly in} \quad C^o(0,t_o;\ L^2(\Omega)^3),$$

$$\tilde{\Theta}^{\varepsilon} \longrightarrow \Theta \quad \text{strongly in} \quad C^o(0,t_o;\ L^2(\Omega)^3),$$

$$\tilde{M}^{\varepsilon} \longrightarrow M \quad \text{strongly in} \quad C^o(0,t_o;\ L^2(\Omega)^3),$$

$$\mathbf{e}(\tilde{\mathbf{u}}^{\varepsilon}) - \mathbf{P}^{\varepsilon}\ \mathbf{e}(\mathbf{u}) - \Theta\ \mathbf{e}(\mathbf{v}^{\varepsilon}) - M\ \mathbf{e}(\mathbf{z}^{\varepsilon}) \longrightarrow 0 \quad \text{strongly in}\ C^o(0,t_o;\ L^1(\Omega,\ \mathbb{E}_s^3)),$$

$$grad\ \tilde{\Theta}^{\varepsilon} - \mathbf{Q}^{\varepsilon} grad\ \Theta \longrightarrow 0 \qquad \text{strongly in} \quad L^2(0,t_o;\ L^1(\Omega)^3),$$

$$grad\ \tilde{M}^{\varepsilon} - \mathbf{R}^{\varepsilon} grad\ \mathbf{M} \longrightarrow 0 \qquad \text{strongly in} \quad L^2(0,t_o;\ L^1(\Omega)^3).$$

where $\mathbf{v}^{\varepsilon}$ and $\mathbf{z}^{\varepsilon}$ are defined by (8.3) and (8.4), respectively.

■

We recall that in Th.8.1 the fields **u**, Θ and M describe the homogenized solid. It is worth noting that the fields $\widetilde{\mathbf{v}}^\varepsilon$, η^ε and N^ε do not appear in the corrector theorem.

Deeper insight into the structure of ($\mathbf{u}^\varepsilon$, Θ^ε,M^ε), the solution to (7.1)-(7.5) is provided by the following

Theorem 8.2. For $\varepsilon \longrightarrow 0$, the solution ($\mathbf{u}^\varepsilon$, Θ^ε,M^ε) of the system (7.1)-(7.5) exibits the following structure

$$\left.\begin{aligned} \frac{\partial \mathbf{u}^\varepsilon}{\partial t} &= \frac{\partial \mathbf{u}}{\partial \partial t} + \frac{\partial \widetilde{\mathbf{v}}^\varepsilon}{\partial t} + \mathbf{r}_1^\varepsilon \\ \Theta^\varepsilon &= \Theta + \eta^\varepsilon + r_2^\varepsilon \\ M^\varepsilon &= M + N^\varepsilon + r_3^\varepsilon \end{aligned}\right\} \tag{8.25}$$

$$\left.\begin{aligned} \mathbf{e}(\mathbf{u}^\varepsilon) &= \mathbf{P}^\varepsilon \mathbf{e}(\mathbf{u}) + \Theta\, \mathbf{e}(\mathbf{v}^\varepsilon) + M\, \mathbf{e}(\mathbf{z}^\varepsilon) + \mathbf{e}(\widetilde{\mathbf{v}}^\varepsilon) + \mathbf{r}_4^\varepsilon \\ \mathit{grad}\, \Theta^\varepsilon &= \mathbf{Q}^\varepsilon \mathit{grad}\, \Theta + \mathit{grad}\, \eta^\varepsilon + \mathbf{r}_5^\varepsilon \\ \mathit{grad}\, M^\varepsilon &= \mathbf{R}^\varepsilon \mathit{grad}\, M + \mathit{grad}\, N^\varepsilon + \mathbf{r}_6^\varepsilon \end{aligned}\right\} \tag{8.26}$$

where (**u**,Θ,M) is the solution of the homogenized system (7.6)-(7.11) whereas $\mathbf{v}^\varepsilon$ and $\mathbf{z}^\varepsilon$ are defined by (8.3) and (8.4), respectively; ($\widetilde{\mathbf{v}}^\varepsilon$,$\eta^\varepsilon$,$N^\varepsilon$) satisfies

$\frac{\partial \widetilde{\mathbf{v}}^\varepsilon}{\partial t} \longrightarrow 0$ weak-* in $L^\infty(0,t_o;\ L^2(\Omega)^3)$,

η^ε, $N^\varepsilon \longrightarrow 0$ weak-* in $L^\infty(0,t_o;\ L^2(\Omega))$,

$\mathbf{e}(\widetilde{\mathbf{v}}^\varepsilon) \longrightarrow 0$ weak-* in $L^\infty(0,t_o;\ L^2(\Omega,E_s^3))$,

grad η^ε, *grad* $N^\varepsilon \longrightarrow 0$ weakly in $L^2(0,t_o;\ L^2(\Omega)^3)$.

Moreover

$\mathbf{r}_1^\varepsilon \longrightarrow 0$ strongly in $C^0(0,t_o;\ L^2(\Omega)^3)$,

r_2^ε , $r_3^\varepsilon \longrightarrow 0$ strongly in $C^0(0,t_o;\ L^2(\Omega))$,

$\mathbf{r}_4^\varepsilon \longrightarrow 0$ strongly in $C^0(0,t_o;\ L^1(\Omega,E_s^3))$,

$\mathbf{r}_5^\varepsilon$, $\mathbf{r}_6^\varepsilon \longrightarrow 0$ strongly in $L^2(0,t_o;\ L^1(\Omega)^3)$.

Remark 8.1

Under additional assumptions, stronger results can be obtained, *cf.* [40,p.35].

9. MICROPERIODIC LAYERED COMPOSITES

9.1. *Specification of the general homogenization formulae*

The formulae derived in Subsection 5 will now be applied to a composite made of periodically distributed two layers. The layers are assumed to be homogenous and anisotropic. The material coefficients in both layers are denoted by

$$\overset{(1)}{c}_{ijkl}, \overset{(1)\alpha}{\gamma}_{ij}, \overset{(1)\alpha\beta}{L}_{ij}; \quad \overset{(2)}{c}_{ijkl}, \overset{(2)\alpha}{\gamma}_{ij}, \overset{(2)\alpha\beta}{L}_{ij}. \tag{9.1}$$

Suppose that the layers are orthogonal to the Ox_1-axis. For simplicity we set $y = y_1$; thus the material coefficient depend now on y only, though the dependence on macroscopic variables x_2 and x_3 is not excluded:

$$\bar{c}_{ijkl}(y) = \begin{cases} \overset{(1)}{c}_{ijkl} & \text{for } y \in (0,\xi) \\ \overset{(2)}{c}_{ijkl} & \text{for } y \in (\xi,1) \end{cases}$$

$$\gamma_{ij}(y) = \begin{cases} \overset{(1)\alpha}{\gamma}_{ij} & \text{for } y \in (0,\xi) \\ \overset{(2)\alpha}{\gamma}_{ij} & \text{for } y \in (\xi,1) \end{cases}, \quad L^{\alpha\beta}_{ij}(y) = \begin{cases} \overset{(1)\alpha\beta}{L}_{ij} & \text{for } y \in (0,\xi) \\ \overset{(2)\alpha\beta}{L}_{ij} & \text{for } y \in (\xi,1) \end{cases}$$

Now the local equations (5.7) reduce to

$$\begin{aligned} &\frac{d}{dy}\Big(\bar{c}_{i1k1}(y)\frac{d\chi^{(pq)}_k}{dy}\Big) = -\frac{d}{dy}\,\bar{c}_{i1pq}(y) \\ &\frac{d}{dy}\Big(\bar{c}_{i1k1}(y)\frac{d\Gamma^{\alpha}_k}{dy}\Big) = \frac{d}{dy}\,\bar{\gamma}^{\alpha}_{i1}(y) \\ &\frac{d}{dy}\Big(L^{\alpha\gamma}_{11}(y)\frac{d\Theta^{\gamma\beta}_k}{dy}\Big) = \frac{d}{dy}\,L^{\alpha\beta}_{1k}(y) \end{aligned} \tag{9.2}$$

According to (5.5), in order to determine the homogenized coefficients, one must find the following derivatives:

$$\frac{d\chi^{(pq)}_k}{dy}, \quad \frac{d\Gamma^{\alpha}_k}{dy}, \quad \frac{d\Theta^{\alpha\beta}_k}{dy} \tag{9.3}$$

By assuming that the matrices $[c_{ik}] = [\bar{c}_{i1k1}]$ and $[L^{\alpha\beta}] = [L^{\alpha\beta}_{11}]$ are non-singular, from (9.2) we obtain

$$\frac{d\chi_k^{(pq)}}{dy} = (c^{-1})^{ki}\left(- \bar{c}_{i1pq}(y) + \overset{\circ}{c}_{ipq}\right)$$
$$\frac{d\Gamma_k^{\alpha}}{dy} = (c^{-1})^{ki}\left(\bar{\gamma}^{\alpha}_{i1}(y) - \overset{\circ}{\gamma}{}^{\alpha}_{i}\right) \tag{9.4}$$
$$\frac{d\Theta_k^{\alpha\beta}}{dy} = (L^{-1})_{\beta\gamma}\left(L^{\alpha\gamma}_{1k}(y) - \overset{\circ}{L}{}^{\alpha\gamma}_{k}\right)$$

where $(c^{-1})^{ki}$ and $(L^{-1})_{\beta\gamma}$ are components of matrices inverse to $[c_{ik}]$ and $[L^{\alpha\beta}]$, respectively; moreover $\overset{\circ}{c}_{ipq}$, $\overset{\circ}{\gamma}{}^{\alpha}_{i}$, $\overset{\circ}{L}{}^{\alpha\beta}_{k}$ are constants of integration which can be determined by using the periodicity conditions for $\chi_k^{(\alpha\beta)}$, Γ_k^{α} ;$\Theta_k^{\alpha\beta}$ and continuity of these functions at $y = \xi$.

Solutions to the system (9.4) are sought in the following form

$$\chi_k^{(pq)}(y) = \begin{cases} \overset{(1)}{\chi}{}_k^{(pq)}(y) & \text{for } y \in (0.\xi) \\ \overset{(2)}{\chi}{}_k^{(pq)}(y) & \text{for } y \in (\xi,1) \end{cases}$$

$$\Gamma_k^{\alpha}(y) = \begin{cases} \overset{(1)}{\Gamma}{}^{k\alpha}(y) & \text{for } y \in (0.\xi) \\ \overset{(2)}{\Gamma}{}^{k\alpha}(y) & \text{for } y \in (\xi,1) \end{cases} \tag{9.5}$$

$$\Theta_k^{\alpha\beta}(y) = \begin{cases} \overset{(1)}{\Theta}{}_k^{\alpha\beta}(y) & \text{for } y \in (0.\xi) \\ \overset{(2)}{\Theta}{}_k^{\alpha\beta}(y) & \text{for } y \in (\xi,1) \end{cases}$$

From (9.4), by taking to account (9.5) one has

$$\frac{d\overset{(1)}{\chi}{}_k^{(pq)}}{dy} = (\overset{(1)}{c}{}^{-1})^{ki}\left(- \overset{(1)}{c}_{i1pq} + \overset{\circ}{c}_{ipq}\right), \qquad \frac{d\overset{(2)}{\chi}{}_k^{(pq)}}{dy} = (\overset{(2)}{c}{}^{-1})^{ki}\left(- \overset{(2)}{c}_{i1pq} + \overset{\circ}{c}_{ipq}\right)$$

$$\frac{d\overset{(1)}{\Gamma}{}_k^{\alpha}}{dy} = (\overset{(1)}{c}{}^{-1})^{ki}\left(\overset{(1)}{\gamma}{}^{\alpha}_{i1} - \overset{\circ}{\gamma}{}^{\alpha}_{i}\right), \qquad \frac{d\overset{(2)}{\Gamma}{}_k^{\alpha}}{dy} = (\overset{(2)}{c}{}^{-1})^{ki}\left(\overset{(2)}{\gamma}{}^{\alpha}_{i1} - \overset{\circ}{\gamma}{}^{\alpha}_{i}\right) \tag{9.6}$$

$$\frac{d\overset{(1)}{\Theta}{}_k^{\alpha\beta}}{dy} = (\overset{(1)}{L}{}^{-1})_{\beta\gamma}\left(\overset{(1)}{L}{}^{\alpha\gamma}_{1k} - \overset{\circ}{L}{}^{\alpha\gamma}_{k}\right), \qquad \frac{d\overset{(2)}{\Theta}{}_k^{\alpha\beta}}{dy} = (\overset{(2)}{L}{}^{-1})_{\beta\gamma}\left(\overset{(2)}{L}{}^{\alpha\gamma}_{1k} - \overset{\circ}{L}{}^{\alpha\gamma}_{k}\right)$$

where $(\overset{(1)}{c}{}^{-1})^{ki}$, $(\overset{(2)}{c}{}^{-1})^{ki}$, $(\overset{(1)}{L}{}^{-1})_{\beta\gamma}$, $(\overset{(2)}{L}{}^{-1})_{\beta\gamma}$ are components of matrices inverse to $[\overset{(1)}{c}_{i1k1}]$, $[\overset{(2)}{c}_{i1k1}]$, $[\overset{(1)}{L}{}^{\alpha\beta}_{11}]$, $[\overset{(2)}{L}{}^{\alpha\beta}_{11}]$, respectively.

Let us pass now to the determination of the constants $\overset{\circ}{c}_{ipq}$. The remaining constants, that is $\overset{\circ}{\gamma}{}^{\alpha}_{i}$ and $\overset{\circ}{L}{}^{\alpha\gamma}_{k}$ are calculated similarly.

From (9.6) we obtain

$$\overset{(1)}{\chi}{}_k^{(pq)}(y) = (\overset{(1)}{c}{}^{-1})^{ki}(-\overset{(1)}{c}_{i1pq} + \overset{\circ}{c}_{ipq})y + \overset{(1)}{A}{}^{k}_{pq}$$
$$\overset{(2)}{\chi}{}_k^{(pq)}(y) = (\overset{(2)}{c}{}^{-1})^{ki}(-\overset{(2)}{c}_{i1pq} + \overset{\circ}{c}_{ipq})y + \overset{(2)}{A}{}^{k}_{pq} \qquad (9.7)$$

Here $\overset{(1)}{A}{}^{k}_{pq}$, $\overset{(2)}{A}{}^{k}_{pq}$ are integration constants which will be eliminated.

The periodicity and continuity conditions for $\chi_k^{(pq)}$ given by ,

$$\overset{(1)}{\chi}{}_k^{(pq)}(0) = \overset{(2)}{\chi}{}_k^{(pq)}(1) \quad , \quad \overset{(1)}{\chi}{}_k^{(pq)}(\xi) = \overset{(2)}{\chi}{}_k^{(pq)}(\xi)$$

yield

$$\overset{(1)}{A}{}^{k}_{pq} = (\overset{(2)}{c}{}^{-1})^{ki}(-\overset{(2)}{c}_{i1pq} + \overset{\circ}{c}_{ipq}) + \overset{(2)}{A}{}^{k}_{pq} ,$$

and

$$(\overset{(1)}{c}{}^{-1})^{ki}(-\overset{(1)}{c}_{i1pq} + \overset{\circ}{c}_{ipq})\xi + \overset{(1)}{A}{}^{k}_{pq} = (\overset{(2)}{c}{}^{-1})^{ki}(-\overset{(2)}{c}_{i1pq} + \overset{\circ}{c}_{ipq})\xi + \overset{(2)}{A}{}^{k}_{pq}$$

respectively.

Subtracting we arrive at

$$(\overset{(1)}{c}{}^{-1})^{ki}(-\overset{(1)}{c}_{i1pq} + \overset{\circ}{c}_{ipq})\xi = (\overset{(2)}{c}{}^{-1})^{ki}(-\overset{(2)}{c}_{i1pq} + \overset{\circ}{c}_{ipq})(\xi - 1).$$

Hence

$$\overset{\circ}{c}_{ipq} = (B^{-1})_{ik}\left(\xi(\overset{(1)}{c}{}^{-1})^{kj}\overset{(1)}{c}_{j1pq} + (1 - \xi)(\overset{(2)}{c}{}^{-1})^{kj}\overset{(2)}{c}_{j1pq}\right) , \qquad (9.8)$$

where $(B^{-1})_{ik}$ are components of the matrix inverse to the matrix $B = [B^{ki}]$; here

$$B^{ki} = \xi(\overset{(1)}{c}{}^{-1})^{ki} + (1 - \xi)(\overset{(2)}{c}{}^{-1})^{ki} \qquad (9.9)$$

In a similar way one obtains the constants $\overset{\circ}{\gamma}{}^{\alpha}_{i}$, $\overset{\circ}{L}{}^{\alpha\beta}_{k}$

$$\overset{\circ}{\gamma}{}^{\alpha}_{i} = (B^{-1})_{ik}\left(\xi \, (\overset{(1)}{c}{}^{-1})^{kj} \, \overset{(1)}{\gamma}{}^{\alpha}_{j1} + (1 - \xi) \, (\overset{(2)}{c}{}^{-1})^{kj} \, \overset{(2)}{\gamma}{}^{\alpha}_{j1} \right) . \qquad (9.10)$$

$$\overset{\circ}{L}{}^{\alpha\beta}_{k} = (K^{-1})^{\alpha\gamma}\left(\xi \, (\overset{(1)}{L}{}^{-1})_{\gamma\delta} \, \overset{(1)}{L}{}^{\delta\beta}_{1k} + (1 - \xi) \, (\overset{(2)}{L}{}^{-1})_{\gamma\delta} \, \overset{(2)}{L}{}^{\delta\beta}_{1k} \right) . \qquad (9.11)$$

where $(K^{-1})^{\alpha\gamma}$ are components of the matrix $\mathbf{K}^{-1}$ inverse to $\mathbf{K}$ and

$$K_{\alpha\beta} = \xi(\overset{(1)}{L}{}^{-1})_{\alpha\beta} + (1 - \xi) \, (\overset{(2)}{L}{}^{-1})_{\alpha\beta}.$$

Substituting the integration constants $\overset{o}{c}_{ipq}$, $\overset{o}{\gamma}{}^{\alpha}_{i}$, $\overset{o}{L}{}^{\alpha\beta}_{k}$ into Eq.(9.4) and taking into account (9.5), after simple algebraic manipulations we obtain

$$\frac{d\chi_k^{(pq)}}{dy} = \begin{cases} (1-\xi)(\tilde{B}^{-1})^{kj}[\![c_{j1pq}]\!] & \text{for } y \in (0,\xi) \\ -\xi(\tilde{B}^{-1})^{kj}[\![c_{j1pq}]\!] & \text{for } y \in (\xi,1) \end{cases}$$

$$\frac{d\Gamma_k^{\alpha}}{dy} = \begin{cases} -(1-\xi)(\tilde{B}^{-1})^{kj}[\![\gamma_{j1}^{\alpha}]\!] & \text{for } y \in (0,\xi) \\ \xi(\tilde{B}^{-1})^{kj}[\![\gamma_{j1}^{\alpha}]\!] & \text{for } y \in (\xi,1) \end{cases}$$

$$\frac{d\Theta_k^{\alpha\beta}}{dy} = \begin{cases} -(1-\xi)(\tilde{K}^{-1})_{\beta\delta}[\![L_{1k}^{\delta\alpha}]\!] & \text{for } y \in (0,\xi) \\ \xi(\tilde{K}^{-1})_{\beta\delta}[\![L_{1k}^{\delta\alpha}]\!] & \text{for } y \in (\xi,1) \end{cases}$$

where $[\![.]\!]$ stands for a jump; for instance $[\![c_{ijpq}]\!] = \overset{(2)}{c}_{ijpq} - \overset{(1)}{c}_{ijpq}$; moreover $(\tilde{B}^{-1})^{kj}$ and $(\tilde{K}^{-1})_{\beta\delta}$ are components of martices inverse to $[\tilde{B}_{kj}] = [\xi\overset{(2)}{c}_{k1j1} + (1-\xi)\overset{(1)}{c}_{k1j1}]$ and $[\tilde{K}^{\alpha\beta}] = [\xi\overset{(2)}{L}{}^{\alpha\beta}_{11} + (1-\xi)\overset{(1)}{L}{}^{\alpha\beta}_{11}]$, respectively.

Thus we eventually obtain the homogenized coefficients

$$\begin{aligned} \bar{c}^{h}_{ijpq} &= \langle\bar{c}_{ijpq}\rangle - \xi(1-\xi)(\tilde{B}^{-1})^{ks}[\![c_{s1pq}]\!][\![c_{ijk1}]\!] \, , \\ \bar{\gamma}^{h\alpha}_{ij} &= \langle\bar{\gamma}^{\alpha}_{ij}\rangle - \xi(1-\xi)(\tilde{B}^{-1})^{ks}[\![c_{s1ij}]\!]\,[\![\gamma^{\alpha}_{k1}]\!] \, , \\ \bar{a}^{h\alpha\beta} &= \langle\bar{a}^{\alpha\beta}\rangle + \xi(1-\xi)(\tilde{B}^{-1})^{ks}[\![\gamma^{\alpha}_{k1}]\!][\![\gamma^{\beta}_{s1}]\!] \, , \\ L^{h\alpha\beta}_{ij} &= \langle L^{\alpha\beta}_{ij}\rangle - \xi(1-\xi)(\tilde{K}^{-1})_{\gamma\delta}\,[\![L^{\delta\beta}_{1j}]\!][\![L^{\alpha\gamma}_{i1}]\!] \, , \end{aligned} \tag{9.12}$$

where

$$\langle\bar{c}_{ijkl}\rangle = \xi\overset{(1)}{c}_{ijpq} + (1-\xi)\overset{(2)}{c}_{ijpq} \, .$$

By substituting the formulae (9.12) for the homogenized coefficients into (6.5) we obtain the following expressions for the initial temperature and chemical potential of the homogenized body

$$\Theta^h(x,0) = \Theta^o + \xi(1-\xi)\,(\tilde{B}^{-1})^{ks}(\bar{a}^h[\![\gamma_k]\!] - \bar{d}^h[\![\zeta_k]\!])(-[\![\gamma_s]\!]\,\Theta^o - [\![\zeta_s]\!]M^o + \\ + [\![\bar{c}_{slpq}]\!]e_{pq}(U))\,\frac{1}{\Delta^h} \qquad (9.13)$$

$$M^h(x,0) = M^o - \xi(1-\xi)\,(\tilde{B}^{-1})^{ks}(\bar{b}^h[\![\zeta_k]\!] - \bar{d}^h[\![\gamma_k]\!])(\,[\![\gamma_s]\!]\,\Theta^o + [\![\zeta_s]\!]M^o - \\ - [\![\bar{c}_{slpq}]\!]e_{pq}(U)) + \frac{1}{\Delta^h} \qquad (9.14)$$

where $\Delta^h = \bar{a}^h\bar{b}^h - (\bar{d}^h)^2$.

9.2. *Example : Numerical results*

By using the *formulae* (9.12) we will discuss the homogenized properties of *periodically* layered *medium* composed of two isotropic materials. For an isotropic elastic body one has

$$c_{1111} = \frac{E}{1+\nu}\left(1 + \frac{\nu}{1-2\nu}\right),$$

where E stands for the Young's *modulus* and ν is the Poisson ratio. Let $\overset{(1)}{E}$ and $\overset{(2)}{E}$ (resp. $\overset{(1)}{\nu}$ and $\overset{(2)}{\nu}$) denote Young's *moduli* (resp. Poisson ratios) of both layers. We set

$$e = \frac{\overset{(2)}{E}}{\overset{(1)}{E}}, \quad e \in (0,1)$$

Figs 1,2 and 3 show diagonal components of the homogenized elasticity tensor, particularly Fig. 1 provides $\bar{c}^h_{1111}$ and $\bar{c}^h_{2222}$ as a function of e. Poisson ratio of both components is the same: $\nu=1/4$ and the thickness of both layers of the considered medium is also the same (*i.e.* $\xi = 1/2$). The coefficient $\bar{c}^h_{3333}$ is identical with $\bar{c}^h_{2222}$, ($\bar{c}^h_{3333} = \bar{c}^h_{2222}$).
It is clear that the homogenized body is no longer an isotropic one.

The components $\bar{c}^h_{1122}$ and $\bar{c}^h_{2233}$ (resp. $\bar{c}^h_{2323}$ and $\bar{c}^h_{1212}$) are shown in Fig. 2 (resp. Fig.3). Comments given for Fig. 1 still apply.

Fig. 4 (resp. Fig.5) show $\bar{c}^h_{1111}$ (resp. $\bar{c}^h_{2222}$) as a function of the layers thickness ratio ξ, for e = 1/5, 1/2 and 3/4 while $\overset{(1)}{\nu} = \overset{(2)}{\nu} = 1/5$. Obviously, for $\xi = 1$ only the more stiff layer nr 1 exists .

Fig. 6 shows homogenized constants $\bar{c}^h_{1122}$ (black line) and $\bar{c}^h_{2233}$ (grey line) as a functions of ξ for $e = 1/2$ and $\overset{(1)}{v} = \overset{(2)}{v} = 1/4$.

Fig. 7 shows the thermoelatic coefficients $\bar{\gamma}^{1h}_{11}$ and $\bar{\gamma}^{1h}_{22}$ as functions of e, for $\overset{(1)}{v} = \overset{(2)}{v} = 1/4$. Remarks concerning homogenized coefficients $\bar{c}^h_{ijkl}$ (*cf.* Figs 1 - 4) apply again.

In Fig. 8 the dependence of $\gamma^h_{11} = \gamma^{1h}_{11}$ on the thickness ratio ξ and the ratio $g = \overset{(1)}{\gamma}{}^1_{11}/\overset{(2)}{\gamma}{}^1_{11}$ is given. The following parameters are taken into account $e = \overset{(2)}{E}/\overset{(1)}{E} = 1/2$ and $\overset{(1)}{\gamma}{}^2_{11}/\overset{(2)}{\gamma}{}^2_{11} = 1/2$.

Figs 9 and 10 exhibit the influence of the homogenization on the coefficients $\bar{a}^{\alpha\beta}$ *i.e.* on the coefficients $\bar{a}$, $\bar{b}$ and $\bar{d}$ (expressing the entropy s and concentration c in terms of the temperature Θ and chemical potential M). In these figures, performed for $\xi=1/2$, the differences between the homogenized value of the coefficients and their mean values as functions of e are shown. The parameter g is defined by $g = \overset{(1)}{\gamma}_{11}/\overset{(2)}{\gamma}_{11}$; moreover $\bar{a}^{11h} = \bar{b}^h$. We see that these correction terms, introduced by homogenization do not depend on the coefficients themselves.

In Fig. 9 the difference $\bar{a}^{11h} - \langle\bar{a}^{11}\rangle = \bar{b}^h - \langle b\rangle$ is shown, which at its turn is equal to $\bar{a}^{22h} - \langle\bar{a}^{22}\rangle = \bar{a}^h - \langle\bar{a}\rangle$. The same thickness of both layers is assumed ($\xi = 1/2$). Two ratios $\overset{(1)}{\gamma}{}^1_{11}/\overset{(2)}{\gamma}{}^1_{11}$, $(\overset{(1)}{\gamma}{}^1_{11}/\overset{(2)}{\gamma}{}^1_{11} = \overset{(1)}{\gamma}_{11}/\overset{(1)}{\gamma}_{11})$ as parameters of the plot are taken.

Fig. 10 is analogous to the previous one. As a parameter of the plot ratio ξ is taken. The lower function in Fig. 10 is identical with the upper one in Fig. 9 (note that the scale units in both figures are different).

In Fig. 11 the difference $\bar{a}^h - \langle\bar{a}\rangle = \bar{b}^h - \langle\bar{b}\rangle$ as a function of thickness ratio ξ, with e as a parameter is shown. The lower function corresponds to $e = 1/5$ and the upper one to $e = 1/2$.

Before the presentation of the results concerning homogenized transport coefficients, let us recall earlier introduced notation. K_{11} and D_{11} denote heat and diffusion coefficient and the other coefficients associated with them are :

$$L^{11}_{11} = K_{11}/T_0 = \lambda_{11} = \lambda, \qquad L^{22}_{11} = D_{11} = D$$

For isotropic layers we have, *e.g.* $\overset{11}{L}_{\alpha\beta} = T_0\, \lambda\, \delta_{\alpha\beta}$, $\overset{22}{L}_{\alpha\beta} = D\, \delta_{\alpha\beta}$.

Further we set :

$$\mu = \overset{(2)11}{L_{11}} / \overset{(1)11}{L_{11}} = \overset{(2)}{K}_{11}/\overset{(1)}{K}_{11} = \overset{(2)}{\lambda}/\overset{(1)}{\lambda} ,$$

$$\nu = \overset{(2)22}{L_{11}} / \overset{(1)22}{L_{11}} = \overset{(2)}{D}_{11}/\overset{(1)}{D}_{11} = \overset{(2)}{D}/\overset{(1)}{D} .$$

In Figs 12a and 12b the homogenized heat conductivity $\overset{11h}{L_{11}} = \lambda^h_{11}$ and diffusion $\overset{22h}{L_{11}} = D^h_{11}$ coefficients are shown as functions of ξ for $\mu = 1/2$ and $\nu = 1/5$ (Fig. 12a) and for $\mu = 1/2$ and $\nu = 1/2$ (Fig. 12b); in the last case plots of $\overset{11h}{L_{11}}$ and $\overset{22h}{L_{11}}$ coincide.

Fig. 13 shows $\overset{11h}{L_{11}}$ and $\overset{22h}{L_{11}}$ as functions of ratio ν for $\mu = 1/2$. One sees that the ratio ν does not influence the heat conductivity $\overset{11h}{L_{11}} = \lambda^h$. The diffusion coefficient $\overset{22h}{L_{11}} = D^h$ vanishes for $\nu = 0$ (no diffusion); the result for $\nu = 1$ is obvious.

ACKNOWLEDGEMENT

This paper was supported by the State Committee for Scientific Research (KBN), Poland, under grant No 300199101.

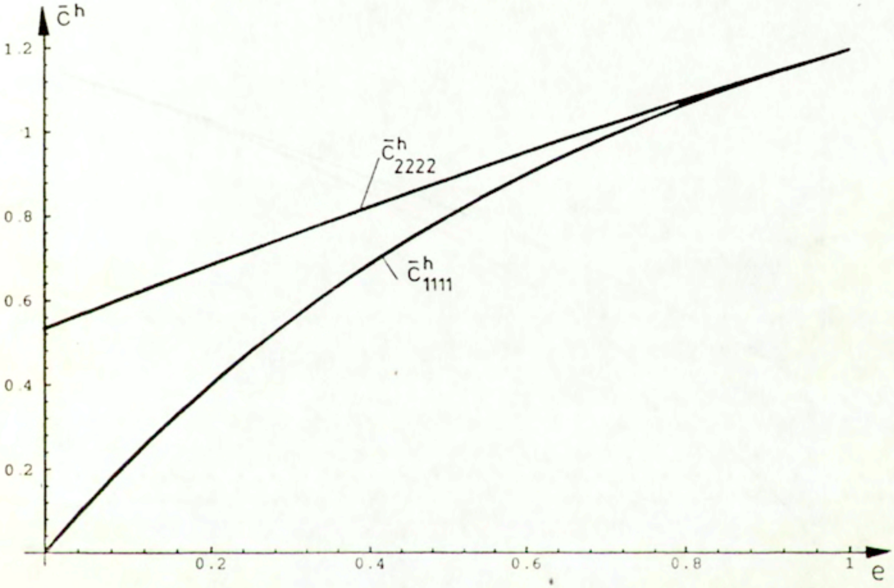

Fig.1. Diagonal components $\bar{c}^h_{1111}$ and $\bar{c}^h_{2222}$ of the homogenized elasticity tensor as functions of $e = \overset{(2)}{E}/\overset{(1)}{E}$. Poisson ratio of both composite materials is the same: $\nu = 1/4$ and the thickness of both layers is also the same (*i.e.* $\xi = 1/2$).

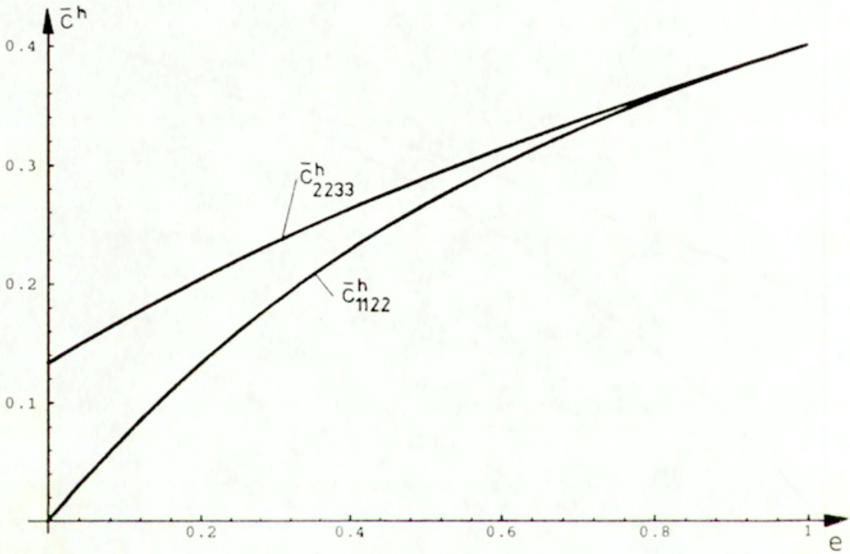

Fig.2. Components $\bar{c}^h_{1122}$ and $\bar{c}^h_{2233}$ of the homogenized elasticity tensor as functions of $e = \overset{(2)}{E}/\overset{(1)}{E}$. Poisson ratio of both composite materials is $\nu = 1/4$ and the thickness of both layers is the same (*i.e.* $\xi = 1/2$).

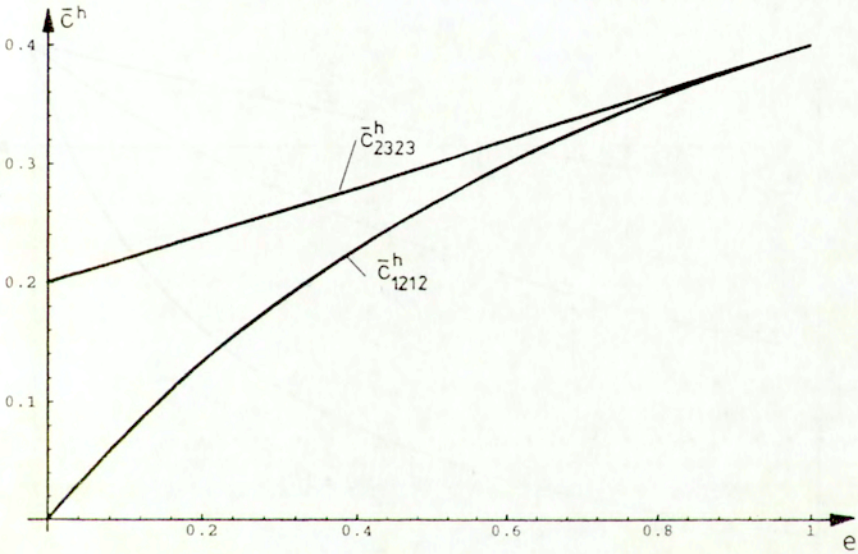

Fig.3. Components $\bar{c}^h_{1212}$ and $\bar{c}^h_{2323}$ of the homogenized elasticity tensor as functions of $e = \overset{(2)}{E}/\overset{(1)}{E}$. Poisson ratio of both composite materials is $\nu = 1/4$ and the thickness of both layers is the same (*i.e.* $\xi = 1/2$).

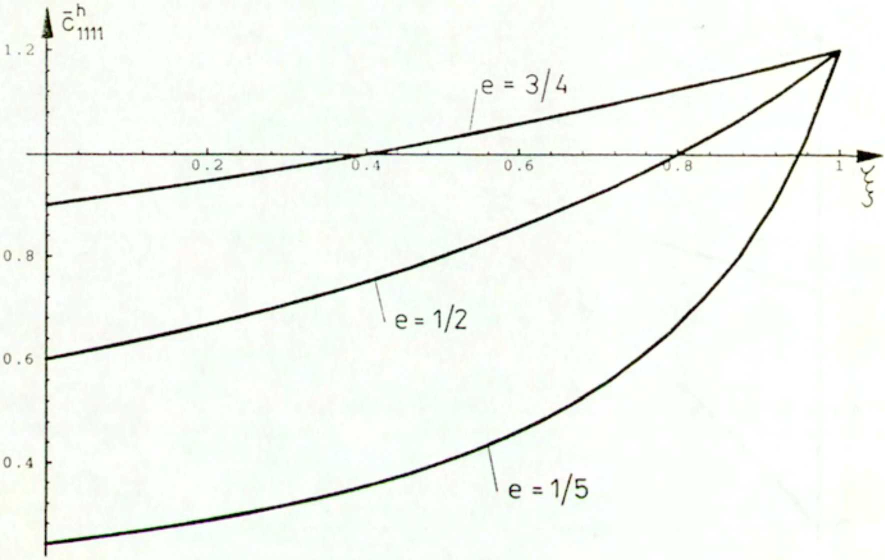

Fig.4. Component $\bar{c}^h_{1111}$ as a function of the layers thickness ratio ξ, for $e = \overset{(2)}{E}/\overset{(1)}{E} = 1/5$, 1/2 and 3/4 while $\overset{(1)}{\nu} = \overset{(2)}{\nu} = 1/5$.

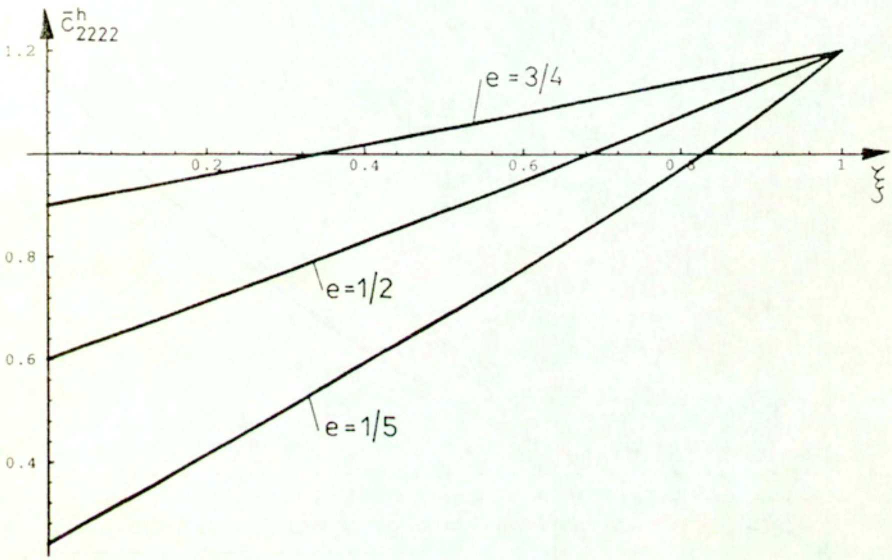

Fig.5. Component $\bar{c}^h_{2222}$ as a function of the layers thickness ratio ξ, for $e = \overset{(2)}{E}/\overset{(1)}{E} = 1/5$, 1/2 and 3/4 while $\overset{(1)}{\nu} = \overset{(2)}{\nu} = 1/5$.

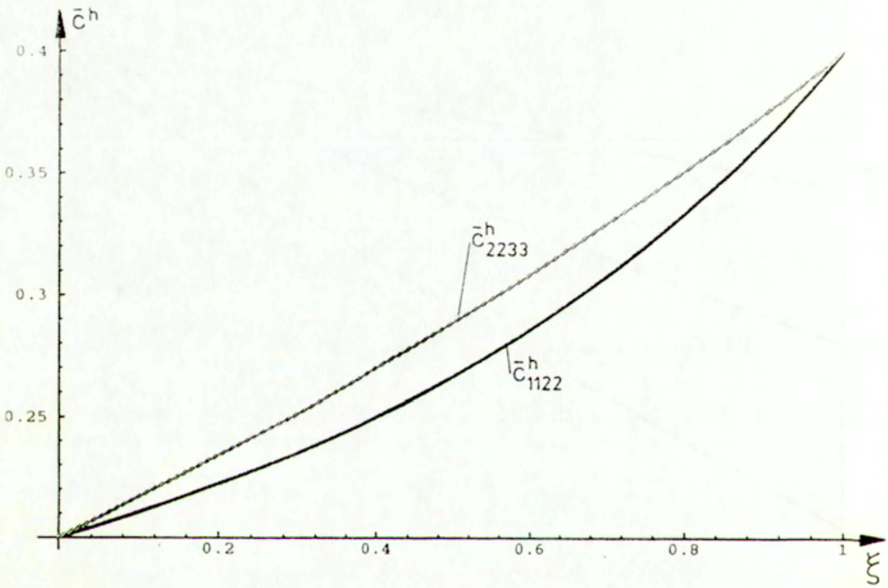

Fig.6. Homogenized constants $\bar{c}^h_{1122}$ (black line) and $\bar{c}^h_{2233}$ (grey line) as a functions of ξ for $e = \overset{(2)}{E}/\overset{(1)}{E} = 1/2$ and $\overset{(1)}{\nu} = \overset{(2)}{\nu} = 1/4$.

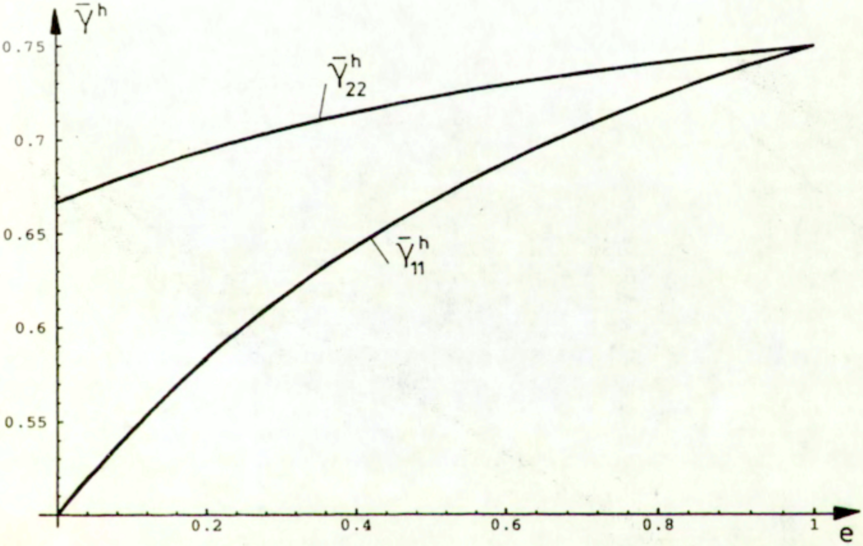

Fig.7. Homogenized thermoelastic coefficients γ^h_{11} and γ^h_{22} as functions of $e = \overset{(2)}{E} / \overset{(1)}{E}$, for $\overset{(1)}{\nu} = \overset{(2)}{\nu} = 1/4$.

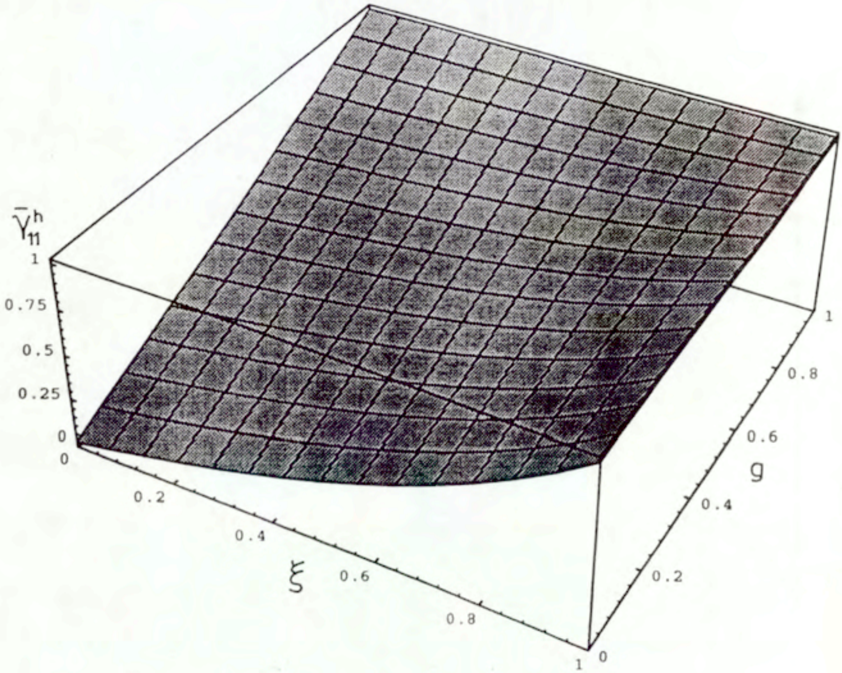

Fig.8. Dependence of γ_{11}^{h} on thickness ratio ξ and ratio $g = \overset{(1)}{\gamma}{}_{11}^{1} / \overset{(2)}{\gamma}{}_{11}^{1}$, for parameters $e = \overset{(2)}{E} / \overset{(1)}{E} = 1/2$ and $\overset{(1)}{\gamma}{}_{11}^{2} / \overset{(2)}{\gamma}{}_{11}^{2} = 1/2$.

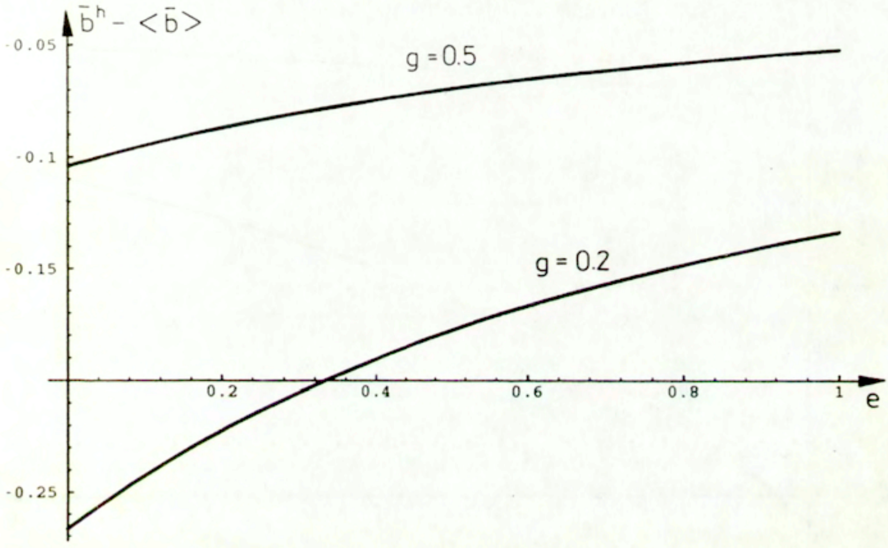

Fig.9. $\bar{b}^h$ - $<\bar{b}>$ as the function of $e = \overset{(2)}{E} / \overset{(1)}{E}$ for two parameters g = 0.2 and g = 0.5 where $g = \overset{(1)}{\gamma}_{11} / \overset{(2)}{\gamma}_{11}$.

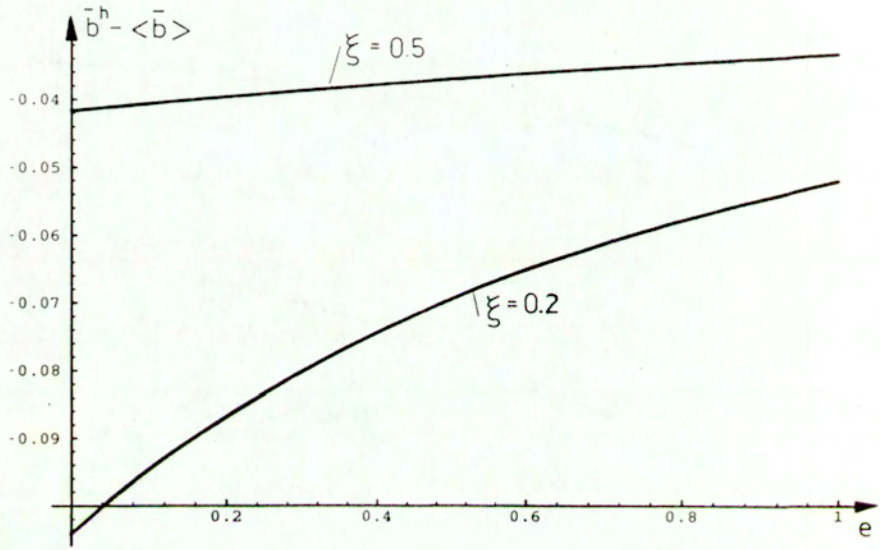

Fig.10. $\bar{b}^h$ - $\langle\bar{b}\rangle$ as the function of $e = \overset{(2)}{E}/\overset{(1)}{E}$ for two values of the parameter ξ: ξ = 1/5 and ξ = 1/2 ; ξ is the thickness ratio.

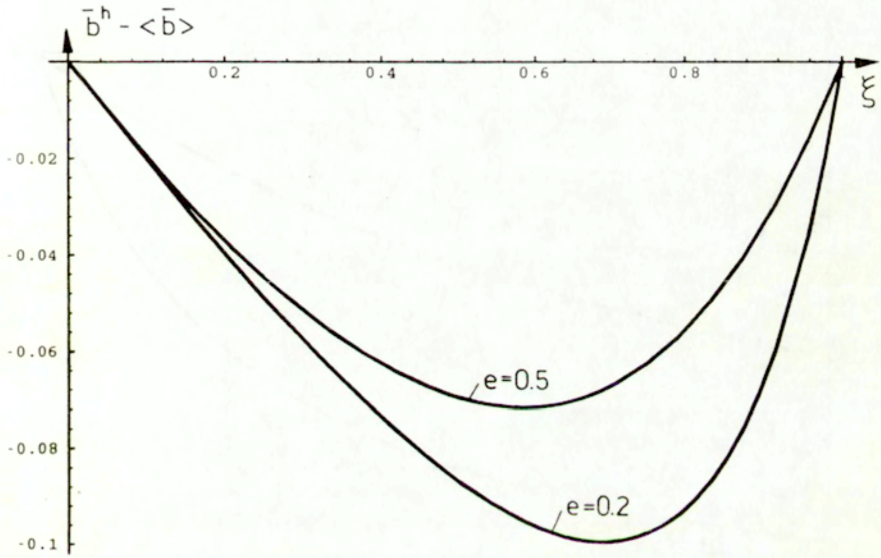

Fig.11. $\bar{b}^h$ - $\langle\bar{b}\rangle$ as the function of thickness ratio ξ for two values of the parameter $e = \overset{(2)}{E}/\overset{(1)}{E}$: e = 1/5 and e = 1/2.

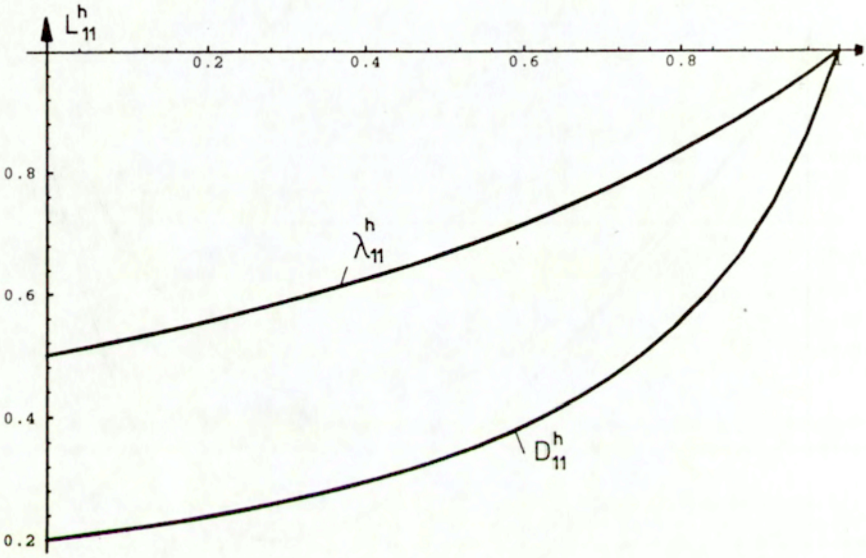

Fig.12a. Homogenized heat conductivity $L_{11}^{h(11)} = \lambda_{11}^{h}$ and diffusion $L_{11}^{h(22)} = D_{11}^{h}$ coefficients as the functions of ξ for $\omega = 1/2$ and $\upsilon = 1/5$, where

$$\omega = \overset{(2)}{L}{}_{11}^{(11)} / \overset{(1)}{L}{}_{11}^{(11)} = \overset{(2)}{K}_{11} / \overset{(1)}{K}_{11} = \overset{(2)}{\lambda} / \overset{(1)}{\lambda} \, ,$$

$$\upsilon = \overset{(2)}{L}{}_{11}^{(22)} / \overset{(1)}{L}{}_{11}^{(22)} = \overset{(2)}{D}_{11} / \overset{(1)}{D}_{11} = \overset{(2)}{D} / \overset{(1)}{D} \, .$$

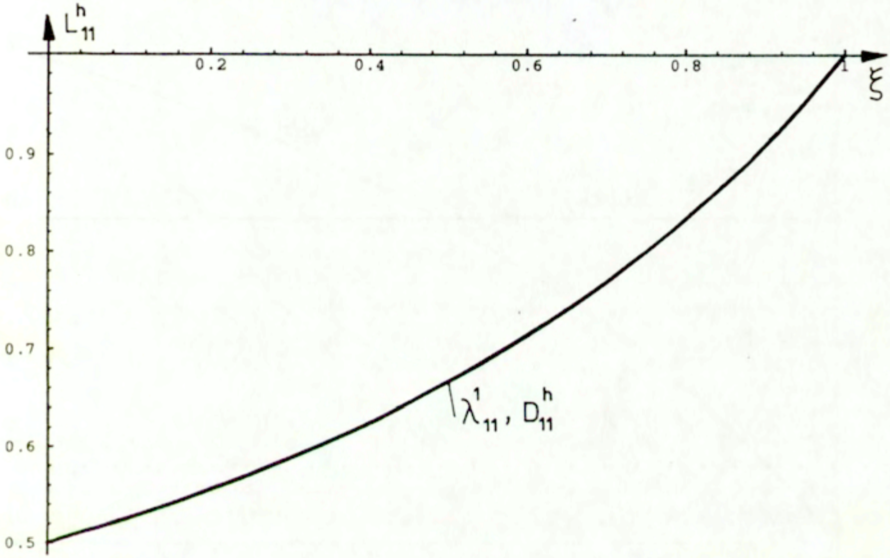

Fig.12b. Homogenized heat conductivity $L_{11}^{h(11)} = \lambda_{11}^{h}$ and diffusion $L_{11}^{h(22)} = D_{11}^{h}$ coefficients as the functions of ξ for $u = 1/2$ and $v = 1/2$; plots of $L_{11}^{h(11)}$ and $L_{11}^{h(22)}$ coincide. We have we set :

$$u = \overset{(2)}{L}{}_{11}^{(11)} / \overset{(1)}{L}{}_{11}^{(11)} = \overset{(2)}{K}_{11} / \overset{(1)}{K}_{11} = \overset{(2)}{\lambda} / \overset{(1)}{\lambda} ,$$

$$v = \overset{(2)}{L}{}_{11}^{(22)} / \overset{(1)}{L}{}_{11}^{(22)} = \overset{(2)}{D}_{11} / \overset{(1)}{D}_{11} = \overset{(2)}{D} / \overset{(1)}{D} .$$

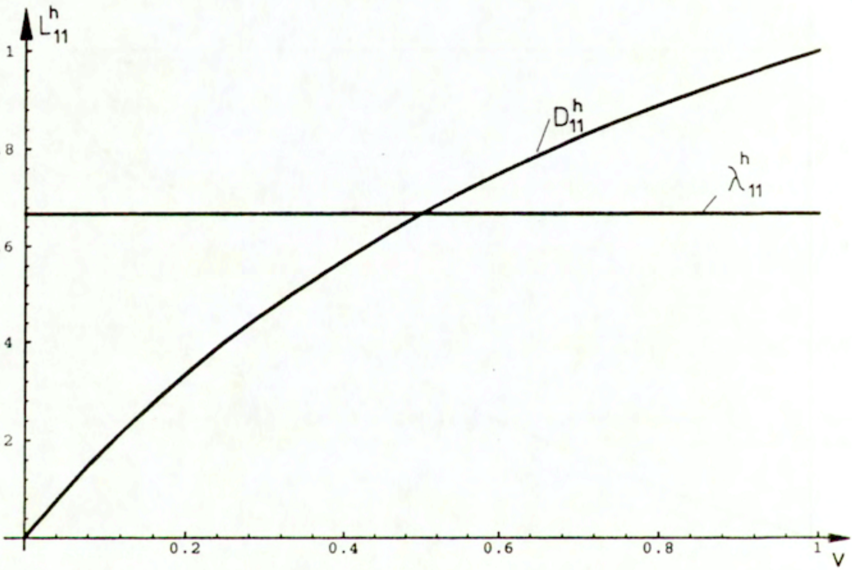

Fig.13. Homogenized heat conductivity $L_{11}^{h(11)} = \lambda_{11}^{h}$ and diffusion $L_{11}^{h(22)} = D_{11}^{h}$ coefficients as functions of ratio υ for $u = 1/2$ where

$$u = \overset{(2)}{L}{}_{11}^{(11)} / \overset{(1)}{L}{}_{11}^{(11)} = \overset{(2)}{K}_{11} / \overset{(1)}{K}_{11} = \overset{(2)}{\lambda} / \overset{(1)}{\lambda} ,$$

$$\upsilon = \overset{(2)}{L}{}_{11}^{(22)} / \overset{(1)}{L}{}_{11}^{(22)} = \overset{(2)}{D}_{11} / \overset{(1)}{D}_{11} = \overset{(2)}{D} / \overset{(1)}{D} .$$

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APPENDIX

Comments on constitutive equations and study of terms connected with ε^{-3}, ε^{-2}, ε^{-1} *and* ε^{0}

The form (2.40)-(2.42) of the constitutive relations can be obtained from, *cf.* [34, 36, 132, 133]

$$\sigma_{ij} = \frac{\partial \mathcal{U}}{\partial e_{ij}}, \qquad \Theta = \frac{\partial \mathcal{U}}{\partial s}, \qquad M = \frac{\partial \mathcal{U}}{\partial c}. \tag{A.1}$$

where $\mathcal{U}(\mathbf{x}, \mathbf{e}, s, c)$ is the internal energy per unit volume. Obviously, to obtain (2.40)-(2.42) $\mathcal{U}$ has to be assumed as a general quadratic form in $\mathbf{e}$, s and c. Then it is reasonable to make the following assumption :

$$\mathcal{U} \text{ is a strictly convex and positive function on } \mathbb{E}^3_s \times \mathbb{R} \times \mathbb{R}. \tag{A.2}$$

The last assumption implies that the elasticity matrix $[\tilde{c}_{ijkl}]$ is positive definite.

The free energy function $\mathcal{F}(\mathbf{x}, \mathbf{e}, \Theta, c)$ can be calculated as the partial concave conjugate of $\mathcal{U}$ with respect to s, *cf.* [149]

$$\mathcal{F}(\mathbf{x}, \mathbf{e}, \Theta, c) = \inf\left\{ -\Theta s + \mathcal{U}(\mathbf{x}, \mathbf{e}, s, c) \mid s \in \mathbb{R} \right\} \tag{A.3}$$

The function $\mathcal{F}$ is still strictly convex in $(\mathbf{e}, c)$, but strictly concave with respect to Θ. Consequently, the elasticity matrix $[c_{ijkl}]$ is positive definite. Eqs (2.28) - (2.30) result from

$$\sigma_{ij} = \frac{\partial \mathcal{F}}{\partial e_{ij}}, \qquad s = -\frac{\partial \mathcal{F}}{\partial \Theta}, \qquad M = \frac{\partial \mathcal{F}}{\partial c}. \tag{A.4}$$

The partial concave conjugate of $\mathcal{U}$ with respect to (s, c) is

$$\mathcal{G}(\mathbf{x}, \mathbf{e}, \Theta, M) := \inf\left\{ -\Theta s - M c + \mathcal{U}(\mathbf{x}, \mathbf{e}, s, c) \mid (s,c) \in \mathbb{R} \times \mathbb{R} \right\} \tag{A.5}$$

Under the assumption (A.2) the function $\mathcal{G}$ is strictly convex in $\mathbf{e}$ and jointly strictly concave in (Θ, M). Consequently, the elasticity matrix $[\bar{c}_{ijkl}]$ is positive definite and Eqs (2.37) - (2.39) result from

$$\sigma_{ij} = \frac{\partial \mathcal{G}}{\partial e_{ij}}, \qquad s = -\frac{\partial \mathcal{G}}{\partial \Theta}, \qquad c = -\frac{\partial \mathcal{G}}{\partial M}. \tag{A.6}$$

Let us pass now to the asymptotic analysis. From the variational form of Eq. (4.11)

$$\int_Y \tilde{c}_{ijkl}(y)\ e_{yij}(\mathbf{u}^{(0)})\ e_{ykl}(\mathbf{v})\ dy = 0 \qquad \forall\ \ \mathbf{v} \in H_{per}(Y, \mathbb{R}^3) \tag{A.7}$$

we immediately deduce that

$$e_{yij}(\mathbf{u}^{(0)}) = 0 \tag{A.8}$$

Consequently, $\mathbf{u}^{(0)}$ does not depend on $y \in Y$ and Eq. (4.9) is satisfied. Recalling that $\mathbf{u}^{(0)}$ depends on x and t only, Eq. (4.12) simplifies to

$$0 = \frac{\partial}{\partial y_i} \{ L^{\alpha\beta}_{ij} \frac{\partial}{\partial y_j} [- \tilde{\gamma}^{\beta}_{mn} (\frac{\partial u^{(0)}_m}{\partial x_n} + \frac{\partial u^{(1)}_m}{\partial y_n}) + \tilde{a}^{\beta\gamma}\ s^{(0)\gamma}] \} \tag{A.9}$$

The coercivity of $[L^{\alpha\beta}_{ij}(y)]$ and periodic behaviour with respect to y imply

$$- \tilde{\gamma}^{\beta}_{mn} (\frac{\partial u^{(0)}_m}{\partial x_n} + \frac{\partial u^{(1)}_m}{\partial y_n}) + \tilde{a}^{\beta\gamma}\ s^{(0)\gamma} = C^{\beta}(\mathbf{x},t) \tag{A.10}$$

where $C^{\beta}(\mathbf{x},t)$, $\beta=1,2$ are unknown functions of $\mathbf{x}$ and t. Hence, *cf.* (4.14)

$$s^{(0)\alpha} = \bar{\gamma}^{\alpha}_{mn}(y) (\frac{\partial u^{(0)}_m}{\partial x_n} + \frac{\partial u^{(1)}_m}{\partial y_n}) + \bar{a}^{\alpha\beta}(y)\ C^{\beta}(\mathbf{x},t) \tag{A.11}$$

By virtue of (A.11) we obtain the following identity

$$\tilde{c}_{ijkl} (\frac{\partial u^{(0)}_k}{\partial x_l} + \frac{\partial u^{(1)}_k}{\partial y_l}) - \tilde{\gamma}^{\alpha}_{ij}\ s^{(0)\alpha} =$$

$$= (c_{ijkl} + \tilde{\gamma}_{ij}\ \gamma_{kl}) (\frac{\partial u^{(0)}_k}{\partial x_l} + \frac{\partial u^{(1)}_k}{\partial y_l}) -$$

$$- \tilde{\gamma}_{ij} [\gamma_{mn} (\frac{\partial u^{(0)}_m}{\partial x_n} + \frac{\partial u^{(1)}_m}{\partial y_n}) + \bar{\gamma}^{\beta}_{mn}\ C^{\beta}(\mathbf{x},t)] =$$

$$= \bar{c}_{ijkl} (\frac{\partial u^{(0)}_k}{\partial x_l} + \frac{\partial u^{(1)}_k}{\partial y_l}) - \bar{\gamma}^{\beta}_{ij}\ C^{\beta}(\mathbf{x},t) \tag{A.12}$$

because $\tilde{\gamma}^{\alpha}_{ij}\ \bar{a}^{\alpha\beta} = \bar{\gamma}^{\beta}_{ij}$, *cf.* (2.59), (2.60) and

$$\tilde{\gamma}^{\alpha}_{ij}\ s^{(0)\alpha} = \tilde{\gamma}^{\alpha}_{ij}\ \bar{\gamma}^{\alpha}_{mn} (\frac{\partial u^{(0)}_m}{\partial x_n} + \frac{\partial u^{(1)}_m}{\partial y_n}) + \bar{\gamma}^{\beta}_{ij}\ C^{\beta}(\mathbf{x},t)$$

Terms associated with ε^{-1} in Eqs (4.7) and (4.8) yield

$$0 = \frac{\partial}{\partial x_j} [\tilde{c}_{ijkl} \frac{\partial u_k^{(0)}}{\partial y_l}] + \frac{\partial}{\partial y_j} [\tilde{c}_{ijkl} (\frac{\partial u_k^{(0)}}{\partial x_l} + \frac{\partial u_k^{(1)}}{\partial y_l}) - \tilde{\gamma}^{\alpha}_{ij} s^{(0)\alpha}]$$

$$\begin{aligned} 0 = \frac{\partial}{\partial x_i} \{ L^{\alpha\beta}_{ij} \frac{\partial}{\partial x_j} [- \tilde{\gamma}^{\beta}_{mn} \frac{\partial u_m^{(0)}}{\partial y_n})] &+ \\ + L^{\alpha\beta}_{ij} \frac{\partial}{\partial y_j} [- \tilde{\gamma}^{\beta}_{mn}(\frac{\partial u_m^{(0)}}{\partial x_n} + \frac{\partial u_m^{(1)}}{\partial y_n}) + \tilde{a}^{\beta\gamma} s^{(0)\gamma}] \} &+ \\ + \frac{\partial}{\partial y_i} \{ L^{\alpha\beta}_{ij} \frac{\partial}{\partial x_j} [- \tilde{\gamma}^{\beta}_{mn} (\frac{\partial u_m^{(0)}}{\partial x_n} + \frac{\partial u_m^{(1)}}{\partial y_n}) + \tilde{a}^{\beta\gamma} s^{(0)\gamma}] &+ \\ + L^{\alpha\beta}_{ij} \frac{\partial}{\partial y_j} [- \tilde{\gamma}^{\beta}_{mn} (\frac{\partial u_m^{(1)}}{\partial x_n} + \frac{\partial u_m^{(2)}}{\partial y_n}) + \tilde{a}^{\beta\gamma} s^{(1)\gamma}] \} & \end{aligned} \tag{A.13}$$

Taking account of (A.10) and recalling that $\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\mathbf{x}, t)$ from Eqs. (A.13) we have

$$\left| \begin{aligned} & 0 = \frac{\partial}{\partial y_j} [\tilde{c}_{ijkl} (\frac{\partial u_k^{(0)}}{\partial x_l} + \frac{\partial u_k^{(1)}}{\partial y_l}) - \tilde{\gamma}^{\alpha}_{ij} s^{(0)\alpha}] \\ & 0 = \frac{\partial}{\partial y_i} \{ L^{\alpha\beta}_{ij} \frac{\partial}{\partial x_j} C^{\beta}(\mathbf{x}) + \\ & \qquad + L^{\alpha\beta}_{ij} \frac{\partial}{\partial y_j} [- \tilde{\gamma}^{\beta}_{mn} (\frac{\partial u_m^{(1)}}{\partial x_n} + \frac{\partial u_m^{(2)}}{\partial y_n}) + \tilde{a}^{\beta\gamma} s^{(1)\gamma}] \} \end{aligned} \right. \tag{A.14}$$

By substitution of (A.12) into $(A.14)_1$ we obtain, *cf.* (4.15)-(4.16),

$$\left| \begin{aligned} & \frac{\partial}{\partial y_j} [\tilde{c}_{ijkl} (\frac{\partial u_k^{(0)}}{\partial x_l} + \frac{\partial u_k^{(1)}}{\partial y_l})] = \frac{\partial \tilde{\gamma}^{\beta}_{ij}}{\partial y_j} C^{\beta}(\mathbf{x},t) \\ & \frac{\partial}{\partial y_i} \{ L^{\alpha\beta}_{ij} \frac{\partial}{\partial y_j} [- \tilde{\gamma}^{\beta}_{mn} (\frac{\partial u_m^{(1)}}{\partial x_n} + \frac{\partial u_m^{(2)}}{\partial y_n}) + \tilde{a}^{\beta\gamma} s^{(1)\gamma}] \} = \\ & \qquad = - (\frac{\partial}{\partial y_i} L^{\alpha\beta}_{ij}) \frac{\partial}{\partial x_j} C^{\beta}(\mathbf{x},t) \end{aligned} \right. \tag{A.15}$$

The local problems yield, *cf.* $(5.7)_{1,2}$

$$\frac{\partial\ \bar{c}_{ijkl}}{\partial y_j} = -\frac{\partial}{\partial y_j}\left(\bar{c}_{ijmn}\frac{\partial\ \chi_{mkl}}{\partial y_n}\right)$$

$$\frac{\partial\ \bar{\gamma}^{\alpha}_{ij}}{\partial y_j} = \frac{\partial}{\partial y_j}\left(\bar{c}_{ijmn}\frac{\partial \Gamma^{\alpha}_{m}}{\partial y_n}\right) \tag{A.16}$$

Substituting (A.16) into $(A.14)_1$ we find

$$-\frac{\partial}{\partial y_j}\left(\bar{c}_{ijmn}\frac{\partial\ \chi_{mkl}}{\partial y_n}\right)\frac{\partial\ u^{(0)}_k}{\partial x_l} + \frac{\partial}{\partial y_j}\left(\bar{c}_{ijkl}\frac{\partial\ u^{(1)}_k}{\partial y_l}\right) = $$

$$= \frac{\partial}{\partial y_j}\left(\bar{c}_{ijmn}\frac{\partial\ \Gamma^{\beta}_{m}}{\partial y_n}\right) C^{\beta}(\mathbf{x},t)\ .$$

Hence we have

$$\frac{\partial}{\partial y_j}\left[\bar{c}_{ijmn}\frac{\partial}{\partial y_n}\left(-\chi_{mkl}\frac{\partial\ u^{(0)}_k}{\partial x_l} + u^{(1)}_m - \Gamma^{\beta}_m C^{\beta}(\mathbf{x})\right)\right] = 0 \tag{A.17}$$

and consequently

$$-\chi_{mkl}\frac{\partial\ u^{(0)}_k}{\partial x_l} + u^{(1)}_m - \Gamma^{\alpha}_m C^{\alpha}(\mathbf{x}) = w_m(\mathbf{x},t)$$

or

$$u^{(1)}_m = \chi_{mpq}(y)\frac{\partial\ u^{(0)}_p}{\partial x_q} + \Gamma_m C(\mathbf{x}) + w_m(\mathbf{x},t) \tag{A.18}$$

where $w_m(\mathbf{x},t)$ remains to be determined.

Eq. $(A.15)_2$ can be integrated by use of the functions $\Theta^{\alpha\beta}_k$, *cf.* $(5.7)_3$

$$\frac{\partial}{\partial y_i}\left(L^{\alpha\beta}_{ij}\frac{\partial\ \Theta^{\beta\gamma}_k}{\partial y_j}\right) = \frac{\partial\ L^{\alpha\gamma}_{ik}}{\partial y_i} \tag{A.19}$$

On account of (A.19), Eq. $(A.15)_2$ takes the form

$$\frac{\partial}{\partial y_i}\left\{L^{\alpha\beta}_{ij}\frac{\partial}{\partial y_j}\left[-\tilde{\gamma}^{\beta}_{mn}\left(\frac{\partial u^{(1)}_m}{\partial x_n} + \frac{\partial u^{(2)}_m}{\partial y_n}\right) + \tilde{a}^{\beta\gamma} s^{(1)\gamma} + \right.\right.$$

$$\left.\left. + \Theta^{\beta\gamma}_k\frac{\partial\ C^{\gamma}(\mathbf{x},t)}{\partial x_k}\right]\right\} = 0 \tag{A.20}$$

The coercivity of $[L^{\alpha\beta}_{ij}(y)]$ yields

$$-\tilde{\gamma}^{\beta}_{mn}\left(\frac{\partial u^{(1)}_m}{\partial x_n} + \frac{\partial u^{(2)}_m}{\partial y_n}\right) + \tilde{a}^{\beta\gamma} s^{(1)\gamma} + \Theta^{\beta\gamma}_k\frac{\partial\ C^{\gamma}(\mathbf{x},t)}{\partial x_k} = A^{\beta}(\mathbf{x},t) \tag{A.21}$$

where $A^{\beta}(\mathbf{x},t)$ is a new function. Hence

$$s^{(1)\alpha} = \bar{\gamma}^{\alpha}_{mn}\left(\frac{\partial u^{(1)}_m}{\partial x_n} + \frac{\partial u^{(2)}_m}{\partial y_n}\right) + \bar{a}^{\alpha\beta}\left[A^{\beta}(\mathbf{x},t) - \Theta^{\beta\gamma}_k\frac{\partial\ C^{\gamma}(\mathbf{x},t)}{\partial x_k}\right] \tag{A.22}$$

The terms appearing in Eqs (4.7) - (4.8) and associated with ε^0 yield, *cf.* (4.17) and (4.18)

$$\rho\ \ddot{u}_i^{(0)} = \frac{\partial}{\partial x_j}\left[\ \tilde{c}_{ijkl}\left(\frac{\partial\ u_k^{(0)}}{\partial x_l} + \frac{\partial\ u_k^{(1)}}{\partial y_l}\right) - \tilde{\gamma}_{ij}^{\alpha}\ s^{(0)\alpha}\ \right] + $$

$$+ \frac{\partial}{\partial y_j}\left[\ \tilde{c}_{ijkl}\left(\frac{\partial\ u_k^{(1)}}{\partial x_l} + \frac{\partial\ u_k^{(2)}}{\partial y_l}\right) - \tilde{\gamma}_{ij}^{\alpha}\ s^{(1)\alpha}\ \right], \qquad (A.23)$$

and

$$\dot{s}^{(0)\alpha} = \frac{\partial}{\partial x_i}\left\{ L_{ij}^{\alpha\beta}\ \frac{\partial}{\partial x_j}\left[-\tilde{\gamma}_{mn}^{\beta}\left(\frac{\partial u_m^{(0)}}{\partial x_n} + \frac{\partial u_m^{(1)}}{\partial y_n}\right) + \tilde{a}^{\beta\gamma}\ s^{(0)\gamma}\ \right] + \right.$$

$$+ L_{ij}^{\alpha\beta}\ \frac{\partial}{\partial y_j}\left[-\tilde{\gamma}_{mn}^{\beta}\left(\frac{\partial u_m^{(1)}}{\partial x_n} + \frac{\partial u_m^{(2)}}{\partial y_n}\right) + \tilde{a}^{\beta\gamma}\ s^{(1)\gamma}\ \right] \Big\} +$$

$$+ \frac{\partial}{\partial y_i}\Big\{ L_{ij}^{\alpha\beta}\ \frac{\partial}{\partial x_j}\left[-\tilde{\gamma}_{mn}^{\beta}\left(\frac{\partial u_m^{(1)}}{\partial x_n} + \frac{\partial u_m^{(2)}}{\partial y_n}\right) + \tilde{a}^{\beta\gamma}\ s^{(1)\gamma}\ \right] +$$

$$- L_{ij}^{\alpha\beta}\ \frac{\partial}{\partial y_j}\left[-\tilde{\gamma}_{mn}^{\beta}\left(\frac{\partial u_m^{(2)}}{\partial x_n} + \frac{\partial u_m^{(3)}}{\partial y_n}\right) + \tilde{a}^{\beta\gamma}\ s^{(2)\gamma}\ \right] \Big\} \qquad (A.24)$$

Taking account of (A.12), after averaging of (A.23) we obtain

$$\langle\rho\rangle\ \ddot{u}_i^{(0)} = \frac{\partial}{\partial x_j} \left\langle\ \bar{c}_{ijkl}\left(\frac{\partial\ u_k^{(0)}}{\partial x_l} + \frac{\partial\ u_k^{(1)}}{\partial y_l}\right) - \bar{\gamma}_{ij}^{\beta}\ C^{\beta}(\mathbf{x},t)\right\rangle + \langle\ B_i\rangle \qquad (A.25)$$

while taking account of (A.10) and (A.21), after averaging of (A.24) we get

$$\dot{s}^{(0)\alpha} = \frac{\partial}{\partial x_i}\left\langle L_{ij}^{\alpha\beta}\ \frac{\partial}{\partial x_j}\left[\ C^{\beta}(\mathbf{x},t)\ \right] + L_{ij}^{\alpha\beta}\ \frac{\partial}{\partial y_j}\left[-\Theta_k^{\beta\gamma}\ \frac{\partial\ C^{\gamma}(\mathbf{x},t)}{\partial x_k}\ \right]\right\rangle \qquad (A.26)$$

Hence, by substituting (A.18) into (A.25) and taking account of (5.5) and (5.7), the homogenized equations of **TED** body (5.2) are arrived at.

Finally substituting (A.18) into (A.11) we get

$$s^{(0)\alpha} = \left(\ \bar{\gamma}_{mn}^{\alpha} + \bar{\gamma}_{rs}^{\alpha}\ \frac{\partial\ \chi_{rmn}}{\partial\ y_s}\right)\frac{\partial u_m^{(0)}}{\partial x_n} + \left(\bar{a}^{\alpha\beta} + \bar{\gamma}_{rs}^{\alpha}\ \frac{\partial\ \Gamma_m^{\beta}}{\partial\ y_s}\right) C^{\beta}(\mathbf{x},t) \qquad (A.27)$$

Hence, again by use of (5.5) and (5.7), homogenized Eq.(5.3) is arrived at.