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THE 1/2 SUBHARMONIC RESONANCE AND ITS TRANSITION TO CHAOTIC MOTION IN A NONLINEAR OSCILLATOR

Introduction

Steady-state vibrations of physical systems with non-symmetric elastic characteristics and single equilibrium position governed by an equation of motion:

$$\ddot{y} + h\dot{y} + \omega_0^2 y + f(y) = P_1 \cos\nu t, \qquad h > 0,$$

(1a)
$$f(0) = 0, \quad f(y) \neq 0 \quad \text{if} \quad y \neq 0,$$
$$f(y) \neq -f(-y),$$

have a wide literature. Behaviour of the system with $f(y)$ approximated by an analytical function

(1b)
$$f(y) = \mu_1 y^2 + \mu_2 y^3,$$

was studied extensively by approximate analytical methods and the phenomena of sub- and ultraresonances, jump phenomena, problems of stability of various approximate solutions and their domains of attraction were investigated and predicted with satisfactory accuracy by the first approximate solution and confirmed by an experiment or a computer simulation (see for example [1,2,3]). Systems with piece-wise nonsymmetric characteristics treated by various analytical and numerical techniques showed similar proper-ties [4,5].

However since the year 1979 when a distinctly new type of the steady-state motion - called "randomly transitional phenomena" or "chaotic motion" was reported by Ueda [6,7] it has become clear that our knowledge about the properties and behaviour of the system is far from being complete. While the chaotic behaviour in systems having three positions of equilibrium studied extensively by theoretical and experimental methods [8,9] can be intuitively explained by some physical arguments, the arguments fail in the system where $y = 0$ is the only one rest point [10].

The hitherto results on the chaotic behaviour presented in the form of Poincare maps and frequency spectrum allow to make an observation that the phenomena is associated with a transition from the subharmonic resonance to the principal one, that is, to the response of the period of excitation. The observation gives an appealing idea to try to find a link between the regular periodic solution and its stability limits studied by means of the first approximate solution and the chaotic motion described in terms of "strange attractors" to see the chaotic zone against the background of the classical resonance curves in connection with the concepts of the stability limits and jump phenomena.

1. Harmonic solution, its local stability and period doublig bifurcation.

We consider the system governed by eqs.(1a,b) where, for the sake of simplicity, it is assumed,

(1c)
$$\omega_0^2 = 3\sqrt[3]{P_0^2}\,,\qquad \mu_1 = 3\sqrt[3]{P_0}\,,\qquad \mu_2 = 1,$$

so that eqs.(1a) can be reduced into the form:

(1d) $$\ddot{x} + h\dot{x} + x^3 = P_o + P_1 \cos \nu t,$$

where

$$x = y + \sqrt[3]{P_o} \ .$$

Eqs.(1d) was examined by Ueda and the strange attractors were found at $\nu = 1.0$; $h = 0.05$; $P_1 = 0.16$; $P_o = 0.03$ and 0.045 [6].

In the first step of the theoretical analysis we consider solution with period of the exciting force, called later "T - periodic solution", in the approximate form:

(2a) $$x_0^{(1)}(t) = C_o + C_1 \cos(\nu t + \vartheta) = x_0^{(1)}(t + T), \quad T = \frac{2\pi}{\nu}.$$

Since we deal with strongly nonlinear system any simplifications inherent to the perturbation methods should be avoided and hence we will consistently make use of the harmonic balance method [1].

On substitution eqs.(2a) into eqs.(1d) and equating coefficients of $\cos(\nu t + \vartheta)$, $\sin(\nu t + \vartheta)$ and constant term separately to zero, a set of algebraic equations for the unknown C_o, C_1, ϑ is obtained:

(2b) $$\begin{aligned} -C_1 \nu^2 + \tfrac{3}{4}C_1^3 + 3C_o^2 C_1 &= P_1 \cos \vartheta, \\ -h\nu C_1 &= P_1 \sin \vartheta, \\ C_o^3 + \tfrac{3}{2} C_o C_1^2 &= P_o. \end{aligned}$$

On solving eqs.(2b) by a numerical procedure resonance curves $C_o \equiv C_o(\nu)$, $C_1 \equiv C_1(\nu)$ are obtained, (Fig.1).

To examine local stability of the solution (2a) thus determined we consider a disturbed solution,

(3a) $$\tilde{x}(t) = x_0^{(1)}(t) + \delta x^{(1)}(t),$$

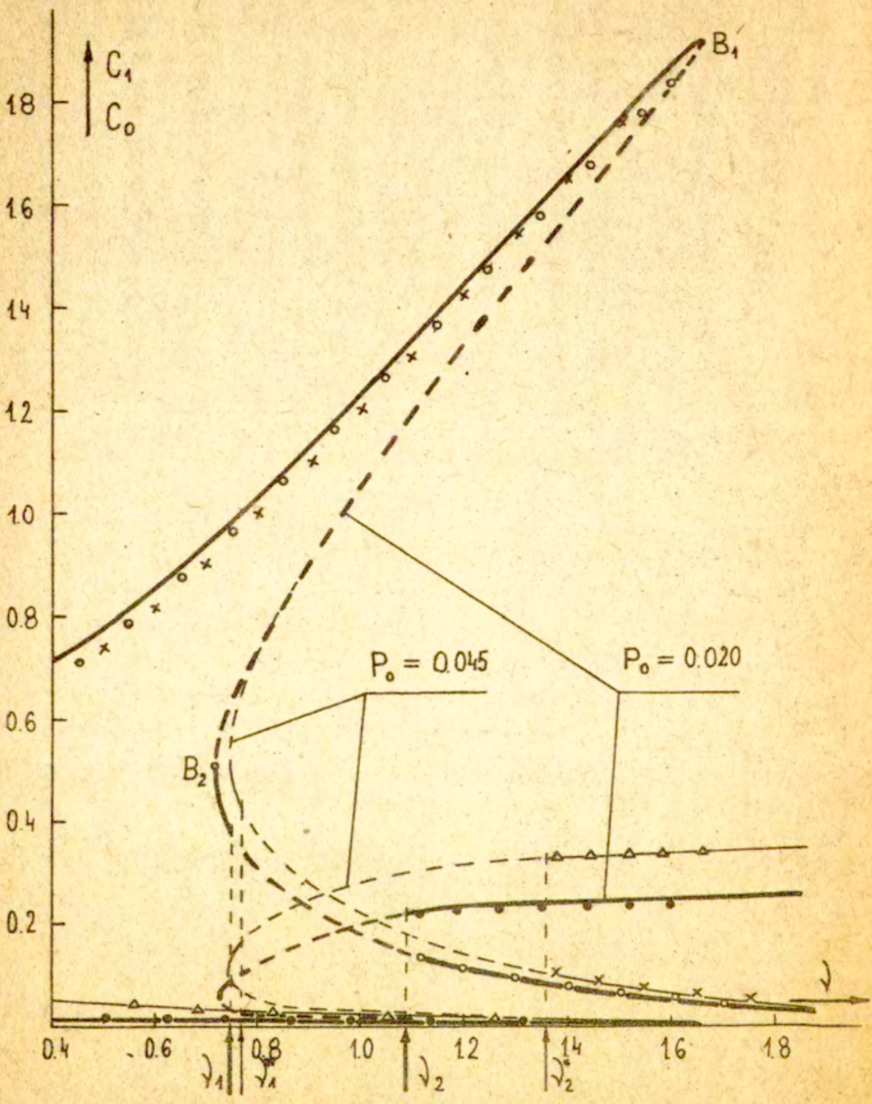


Fig.1. Resonance curves of harmonic plus constant term solution; $P_1 = 0.16$, $h = 0.05$.

and on inserting it into eqs.(1d) and taking into account eqs.(2b) we arrive at the variational equation:

$$\delta\ddot{x}^{(1)} + h\delta\dot{x}^{(1)} + 3x_0^{(1)^2}\delta x^{(1)} + 3x_0^{(1)}\delta x^{(1)^2} + \delta x^{(1)^3} = 0. \quad (3b)$$

The question of local stability is answered on ignoring higher power terms and considering the linear variational equation,

$$\delta\ddot{x}^{(1)} + h\delta\dot{x}^{(1)} + 3x_0^{(1)^2}\delta x^{(1)} = 0. \quad (3c)$$

Then inserting eqs.(2a) and expanding $x_0^{(1)^2}$ into the Fourier series yields:

$$\begin{aligned} &\delta\ddot{x}^{(1)} + h\delta\dot{x}^{(1)} + \delta x^{(1)}\left[\lambda_0 + \lambda_1\cos\theta + \lambda_2\cos 2\theta\right] = 0, \\ &\lambda_0 = 3C_0^2 + \tfrac{3}{2}C_1^2, \qquad \theta = \nu t + \vartheta, \\ &\lambda_1 = 6C_0C_1, \\ &\lambda_2 = \tfrac{3}{2}C_1^2. \end{aligned} \quad (3d)$$

Thus we arrive at the Hill's equation studied extensively in [1] by the aid of the harmonic balance method. To examine the behaviour of the disturbance $\delta x^{(1)}(t)$ and hence the stability of the harmonic solution (2a) we will make use of the method and results presented in the book. We readily see that the classical first order unstable region known in Duffing equation, which is due to the term $\lambda_2\cos 2\theta$, takes place at,

$$\nu \approx \sqrt{\lambda_0},$$

and the stability limits coincide with points of vertical tangents on the resonance curves. In Fig.1 the region is between points B_1, B_2 and is denoted by a dashed line.

However, due to the constant term C_0, a parametric term of the frequency ν – $\lambda_1\cos\theta$ appears in the variational equation. Consequently, the lowest order unstable region is that which occurs close to

$$\frac{\nu}{2} \approx \sqrt{\lambda_0}\,, \tag{4a}$$

and this type of instability needs a particular attention.

The first approximate solution in the unstable region of the type is,

$$\delta x^{(1)} = e^{\varepsilon t}\, b_{1/2} \cos\left(\frac{\nu}{2}t + \phi\right), \tag{4b}$$

where ε is real and positive. At the stability limit $\varepsilon = 0$ and hence,

$$\delta x^{(1)} = b_{1/2} \cos\left(\frac{\nu}{2}t + \phi\right). \tag{4c}$$

The form of solution (4b) suggest that there is a possibility of a built-up of the harmonic component of the frequency $\nu/2$, that is - bifurcation from the T-periodic solution (2a) to 2T-periodic solution. To determine boundaries of the unstable region on the resonance curves $C_0(\nu)$, $C_1(\nu)$ we insert the assumed solution (4c) into variational equation (3d) and apply the harmonic balance method. Conditions of nonzero solution for $b_{1/2}$ lead us to the following criterion to be satisfied at the stability limits:

$$\underset{\varepsilon=0}{\Delta(\nu^2)} = \left(\lambda_0 - \frac{\nu^2}{4}\right)^2 + h^2\frac{\nu^2}{4} - \frac{1}{4}\lambda_1^2 = 0. \tag{4d}$$

Inside the unstable region, that is at $\varepsilon > 0$, the determinant is negative,

$$\underset{\varepsilon>0}{\Delta(\nu^2)} < 0. \tag{4e}$$

On utilizing eqs.(4d,e) the unstable region was evaluated and denoted in Fig.1. It lies on the lower, nonresonant branch of $C_1(\nu)$ and associated upper branch of $C_0(\nu)$, between frequencies ν_1 and ν_2.

To answer the question of period doubling bifurcation [11] at the critical point $\nu = \nu_1$ on increasing the frequency and at $\nu = \nu_2$ on decreasing the frequency, we turn back to the complete nonlinear variational equation (3b) and examine existence and stability of the steady-state solution (4c) in the small neighbourhood of the two critical points. On inserting eqs.(2a) the complete variational equation takes the form:

(5a)
$$\delta\ddot{x}^{(1)} + h\delta\dot{x}^{(1)} + \delta x^{(1)}(\lambda_0 + \lambda_1\cos\theta + \lambda_2\cos 2\theta) + \delta x^{(1)2}(3C_0 + 3C_1\cos\theta) + \delta x^{(1)3} = 0,$$
$$\theta = \nu t + \vartheta \quad , \lambda_0 = 3C_0^2 + \frac{3}{2}C_1^2, \ \lambda_1 = 6C_0C_1, \ \lambda_2 = \frac{3}{2}C_1^2.$$

Values of C_0 and C_1 are assumed to be equal to those at $\nu = \nu_1$ or $\nu = \nu_2$ respectively. Since only small neighbourhood of the critical points is to be considered and we are interested in not only in the steady state solution (4c) but also in its local stability, we make use of the averaging method. To this end we set $b_{1/2}$ and ϕ as new time dependent variables and transform eqs.(5a) into standard form. Then we replace them by the averaged with time equations with the result:

(5b)
$$\frac{db_{1/2}}{dt} = \frac{1}{\pi\nu}\int_0^{2\pi} F(\psi)\sin\psi\, d\psi \equiv \frac{g_1}{\nu},$$
$$\frac{d\phi}{dt} = \frac{1}{b_{1/2}\pi\nu}\int_0^{2\pi} F(\psi)\cos\psi\, d\psi \equiv \frac{p_1}{b_{1/2}\nu}, \quad \psi = \frac{\nu}{2}t + \phi,$$

where

$$F = (\lambda_0 - \frac{\nu^2}{4})b_{1/2}\cos\psi - b_{1/2}h\frac{\nu}{2}\sin\psi + b_1\cos\psi(\lambda_1\cos\theta + \lambda_2\cos 2\theta) +$$
$$+ 3b_{1/2}^2(C_0 + C_1\cos\theta)\cos^2\psi + b_{1/2}^3\cos^3\psi =$$
$$= p_1\cos\psi + g_1\sin\psi + \text{higher harmonics},$$
$$\theta = \nu t + \vartheta = 2\psi + \vartheta - 2\phi.$$

On applying trigonometric identities we readily find coefficients p_1 and g_1 :

(5c)
$$p_1 = (\lambda_0 - \frac{\vartheta^2}{4}) b_{1/2} + b_{1/2} \frac{\lambda_1}{2} \cos(\vartheta - 2\phi) + \frac{3}{4} b_{1/2}^3 ,$$

$$g_1 = - b_{1/2} \frac{\lambda_1}{2} \sin(\vartheta - 2\phi) - h \frac{\vartheta}{2} b_{1/2} ;$$

and reduce eqs.(5b) to the form:

(5d)
$$\frac{d b_{1/2}}{dt} = -\frac{1}{\vartheta} b_{1/2} \left[\frac{\lambda_1}{2} \sin(\vartheta - 2\phi) + h \frac{\vartheta}{2} \right] = D_1 (b_{1/2}, \phi),$$

$$\frac{d\phi}{dt} = \frac{1}{\vartheta} \left[\lambda_0 - \frac{\vartheta^2}{4} + \frac{\lambda_1}{2} \cos(\vartheta - 2\phi) + \frac{3}{4} b_{1/2}^2 \right] \equiv D_2 (b_{1/2}, \phi).$$

The steady-state solution is determined by conditions,

$$\frac{d b_{1/2}}{dt} = \frac{d\phi}{dt} = 0 ,$$

which yield equations for $b_{1/2}$, ϕ :

(6a)
$$(\lambda_0 - \frac{\vartheta^2}{4}) + \frac{\lambda_1}{2} \cos(\vartheta - 2\phi) + \frac{3}{4} b_{1/2}^2 = 0 ,$$

$$\frac{\lambda_1}{2} \sin(\vartheta - 2\phi) + h \frac{\vartheta}{2} = 0 .$$

Then on eliminating the angle $(\vartheta - 2\phi)$ we obtain:

(6b)
$$\Delta(\vartheta^2, b_{1/2}) = (\lambda_0 - \frac{\vartheta^2}{4} + \frac{3}{4} b_{1/2}^2)^2 + h^2 \frac{\vartheta^2}{4} - \frac{\lambda_1^2}{4} = 0 .$$

To find bifurcation diagrams in terms of the parameter,

$\mu_1 \equiv \vartheta^2 - \vartheta_1^2$ in the vicinity of ϑ_1 , and

$\mu_2 = \vartheta_2^2 - \vartheta^2$ in the vicinity of ϑ_2 , we expand determinant (6b) into power series with the result:

$$\Delta(\nu, b_{1/2}) = \Delta(\nu^2_{1,2}, 0) + \frac{\partial\Delta}{\partial\nu^2}(\nu^2 - \nu^2_{1,2}) + \frac{\partial\Delta}{\partial b^2_{1,2}} b^2_{1,2} = 0. \tag{6c}$$

By virtue of eqs.(4d),

$$\Delta(\nu^2_{1,2}, 0) = 0,$$

and eventually the sought amplitude of the bifurcating solution (4c) is:

$$b^2_{1/2} = \frac{1}{3}\left[1 - \frac{h^2}{2(\lambda_0 - \frac{\nu_1^2}{4})}\right]\mu_1 , \quad \mu_1 \equiv \nu^2 - \nu_1^2 ,$$

(6d) or

$$b^2_{1/2} = \frac{1}{3}\left[1 + \frac{h^2}{2(\frac{\nu_2^2}{4} - \lambda_0)}\right]\mu_2 , \quad \mu_2 \equiv \nu_2^2 - \nu^2,$$

where

$$\lambda_0 - \frac{\nu_1^2}{4} > 0 \quad \text{and} \quad \frac{\nu_2^2}{4} - \lambda_0 > 0.$$

Thus we arrive at the typical bifurcation diagrams sketched in Fig.2 [11] .

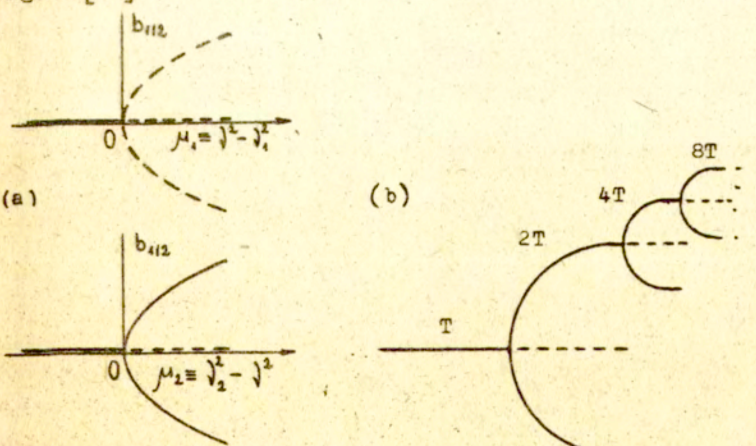


Fig.2. Period doubling bifurcation diagrams.

To examine local stability of the steady-state solution,

(7a) $$\delta x^{(1)} = b_{112}\cos\left(\frac{\vartheta}{2}t + \phi\right),$$

we turn back to eqs.(5d) and add small disturbances to determined by eqs.(6a) to obtain linear variational equations:

(7b) $$\frac{d\,\delta b_{112}}{dt} = \frac{\partial D_1}{\partial b_{112}}\delta b_{112} + \frac{\partial D_1}{\partial \phi}\delta\phi,$$
$$\frac{d\,\delta\phi}{dt} = \frac{\partial D_2}{\partial b_{112}}\delta b_{112} + \frac{\partial D_2}{\partial \phi}\delta\phi.$$

Conditions of nontrivial particular solution,

(7c) $$\delta b_{112}(t) = B_1 e^{\varepsilon t},$$
$$\delta\phi(t) = B_2 e^{\varepsilon t},$$

lead to characteristic determinant in the form,

(7d) $$\begin{vmatrix} -\varepsilon & \frac{-b_{112}\lambda_1}{\vartheta}\cos(\vartheta - 2\phi) \\ \frac{3}{2\vartheta}b_{112} & \frac{\lambda_1}{\vartheta}\sin(\vartheta - 2\phi) - \varepsilon \end{vmatrix} = 0$$

which, on virtue of eqs.(6a) is reduced to,

(7e) $$\varepsilon^2 + h\varepsilon + \frac{3}{2}\frac{b_{112}^2}{\vartheta^2}\left(\frac{\vartheta_{1,2}^2}{4} - \lambda_0\right) = 0;\ h > 0.$$

On making use of the Routh-Hurwitz criterion [1] we see that the solution (7a) is stable if,

(7f) $$\frac{3}{2}\frac{b_{112}^2}{\vartheta^2}\left(\frac{\vartheta_{1,2}^2}{4} - \lambda_0\right) > 0.$$

From eqs.(4d) we learn that,

$$\frac{\nu_1^2}{4} - \lambda_0 < 0 \quad \text{and} \quad \frac{\nu_2^2}{4} - \lambda_0 > 0.$$

Therefore the solution (7a) in the neighbourhood of ν_2 is stable and is unstable close to ν_1. The result is showed in Fig.2 where the unstable branches are denoted by broken lines.

2. The 1/2 subharmonic resonance, its local stability and further period-doubling bifurcations.

At frequencies ν far from ν_1 and ν_2 the assumption that C_0, C_1 are constant is too rough, and now we seek the $2T$ periodic solution in the whole range $\nu_1 - \nu_2$ in the form:

$$x_0^{(2)}(t) = A_0 + A_{1/2}\cos(\nu/2\, t + \phi) + A_1 \cos \nu t, \tag{8a}$$

where A_0, $A_{1/2}$, A_1, ϕ need to be determined.

On inserting eqs.(8a) into eqs.(1d) and applying the harmonic balance principle we obtain,

$$\begin{aligned}
&-\frac{\nu^2}{4} + \frac{3}{4}A_{1/2}^2 + 3A_0^2 + \frac{3}{2}A_1^2 + 3A_0A_1\cos 2\phi = 0,\\
&-A_1\nu^2 + \frac{3}{4}A_1^3 + 3A_0^2A_1 + \frac{3}{2}A_{1/2}^2A_1 + \frac{3}{2}A_0A_{1/2}^2\cos 2\phi = P_1,\\
&-\frac{1}{2}h\nu + 3A_0A_1\sin 2\phi = 0\\
&A_0^3 + \frac{3}{2}A_0A_{1/2}^2 + \frac{3}{2}A_0A_1^2 + \frac{3}{4}A_{1/2}^2A_1\cos 2\phi = P_0.
\end{aligned} \tag{8b}$$

Results of numerical evaluations for two examples, which differ only by the value of P_0 are shown in Figs.3 and 8. Character of the resonance curves $A_{1/2}(\nu)$, $A_0(\nu)$, $A_1(\nu)$ in the two cases does not show substantial difference - however at $P_0 = 0.020$ the behaviour of the system is regular, periodic or very close to periodic in the whole range of the

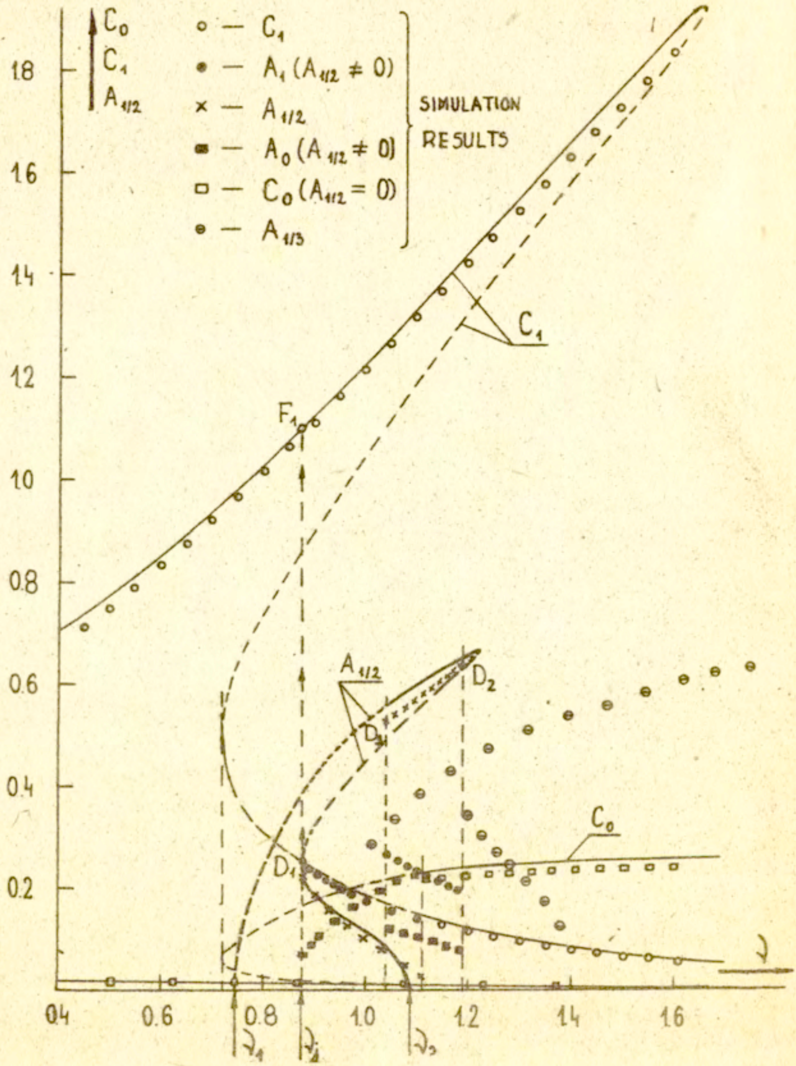


Fig.3a. Resonance curves of the 1/2 subharmonic solution; P_0 = 0.020, P_1 = 0.16, h = 0.05.

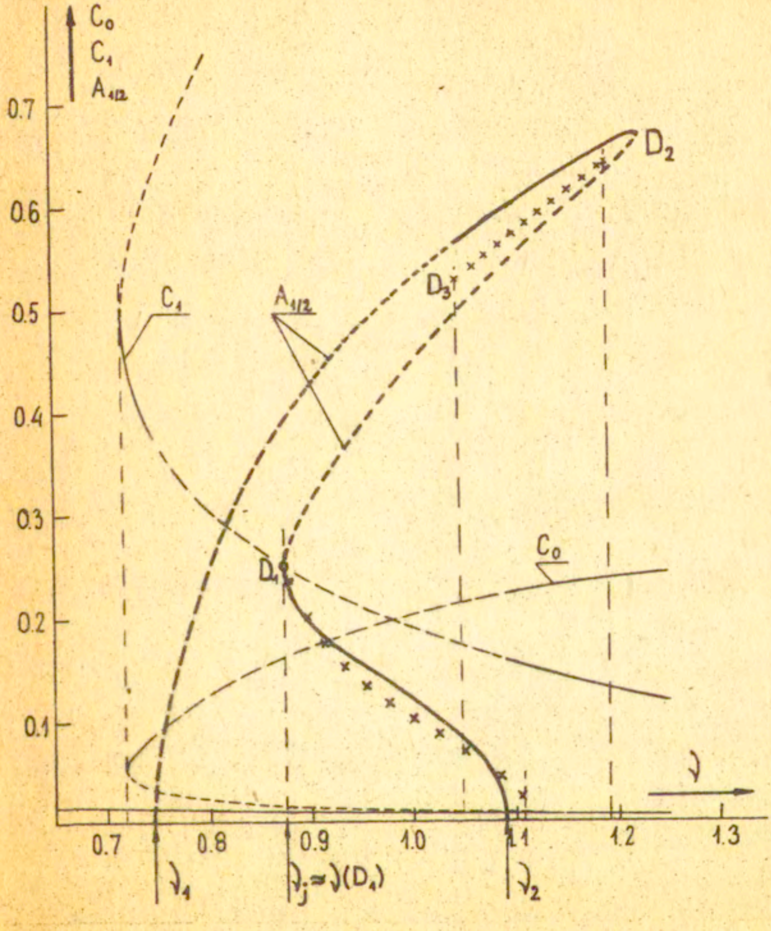


Fig.3b. Resonance curves of the 1/2 subharmonic solution; P_0 = 0.020, P_1 = 0.16, h = 0.05.

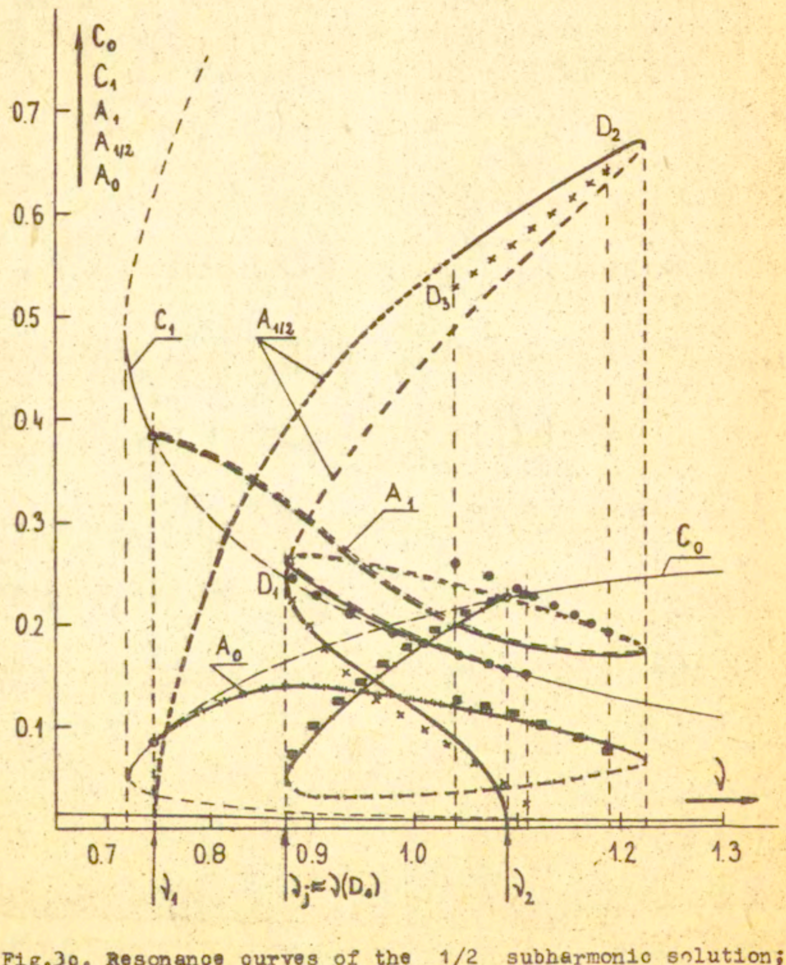


Fig.3c. Resonance curves of the 1/2 subharmonic solution; $P_0 = 0.020$, $P_1 = 0.16$, $h = 0.05$.

frequencies, while at $P_o = 0.045$ chaotic motion was observed by Ueda [6].

First we focus attention at the question of local stability of the subharmonic solution (8a). From eqs.(7)we learn that the left branch of $A_{1/2}(\nu)$ close to ν_1 is unstable. To extend validity of the criterion (7f) to larger values of $A_{1/2}$ we may replace values of C_o, C_1 at $\nu = \nu_1$ by the values of A_o, A_1 varying with ν. Such approach to the criterion (7f) allows us to state that the left branch of $A_{1/2}(\nu)$ at $P_o = 0.020$ is unstable from ν_1 up to $\nu \approx 0.89$ and at $P_o = 0.045$ up to $\nu \approx 0.93$.

To answer the question of local stability in the whole range between ν_1 and ν_2 we derive variational equations for the disturbancy $\delta x^{(2)}$:

(9a) $$\delta x^{(2)} = \tilde{x}_0^{(2)} - x_0^{(2)},$$

(9b) $$\delta \ddot{x}^{(2)} + h\,\delta\dot{x}^{(2)} + \delta x^{(2)}\Big[\lambda_0^{(2)} + \lambda_{1/2c}\cos\tfrac{\nu}{2}t + \lambda_{1/2s}\sin\tfrac{\nu}{2}t + \lambda_{3/2}\cos(\tfrac{3}{2}\nu t + \phi) + \lambda_{1c}^{(2)}\cos\nu t + \lambda_{1s}^{(2)}\sin\nu t + \lambda_2^{(2)}\cos 2\nu t\Big] = 0,$$

$$\lambda_0^{(2)} = 3(A_0^2 + \tfrac{1}{2}A_{1/2}^2 + \tfrac{1}{2}A_1^2),$$

$$\lambda_{1/2c} = 3A_{1/2}(2A_0 + A_1)\cos\phi,$$

$$\lambda_{1/2s} = 3A_{1/2}(A_1 - 2A_0)\sin\phi,$$

$$\lambda_{1c}^{(2)} = 6A_0A_1 + \tfrac{3}{2}A_{1/2}^2\cos 2\phi,$$

$$\lambda_{1s}^{(2)} = -\tfrac{3}{2}A_{1/2}^2\sin 2\phi,$$

$$\lambda_2^{(2)} = \tfrac{3}{2}A_1^2, \qquad \lambda_{3/2} = 3A_1A_{1/2}.$$

Complex form of eqs.(9b) suggests that a variety instabilities can occur. First we note the classical unstable region between the points of vertical tangents on the resonance curves, which appear at the zone of ϑ where three theoretical values of A_0, $A_{1/2}$, A_1 are found. At the boundary of unstability of the type the disturbance $\delta x^{(2)}$ in the first approximation can be assumed in the form:

$$\delta x^{(2)}(t) = b_0 + b_1 \cos(\vartheta t + \phi_1) + b_{1/2} \cos(\frac{\vartheta}{2} t + \phi_{1/2}). \tag{9c}$$

The most essential conclusion that can be drown from eqs.(9b) is that concerning the period doubling bifurcation. We see that alike in eqs.(3d) a parametric term of the period of the solution under consideration appears - now it is the term of the frequency $\vartheta/2$ and $3\vartheta/2$. It comes out that the lowest order unstable region is that associated with the terms $\lambda_{1/2}, \lambda_{3/2}$ and at the stability limit of the type we may seek approximate solution in the form:

$$\delta x^{(2)}(t) = b_{1/4} \cos(\frac{\vartheta}{4} t + \phi_{1/4}) + b_{3/4} \cos(\frac{3\vartheta}{4} t + \phi_{3/4}). \tag{9d}$$

Consequently a built-up of the $\frac{\vartheta}{4}$ and $\frac{3\vartheta}{4}$ harmonic components can be expected and the stable steady-state solution within the unstable zone may be sought in the form:

$$x_0^{(3)}(t) = A_0 + A_1 \cos\vartheta t + A_{1/2} \cos(\frac{\vartheta}{2} t + \phi_{1/2}) + A_{1/4} \cos(\frac{\vartheta}{4} t + \phi_{1/4}) + A_{3/4} \cos(\frac{3\vartheta}{4} t + \phi_{3/4}). \tag{9e}$$

Therefore we conclude that eqs.(9b) indicates a possibility of further period doubling bifurcation - from 2T periodic solution (8a) to 4T periodic solution (9e). In further step of analogous analysis we notice a possibility of a built-up of the components with frequencies $\frac{3\vartheta}{8}$ and $\frac{5\vartheta}{8}$ and then generally $\frac{n \mp 1}{2n}$, n = 2,4,8,16,... .

Summing up, variational, Hill's type equation, which involves parametric excitation component of the period of the investigated steady-state solution, suggests a possible cascade of period doubling bifurcation. The hitherto investigations indicate that such cascade may lead to chaotic behaviour [9].

3. Computer simulation results and an analysis of chaotic motion.

To verify the theoretical approximate predictions eqs.(1c) was simulated on an electronic computer and response of the system was analysed by means of various techniques. First the harmonic components dominating in regular motion were detected: constant term, fundamental and 1/2 harmonic components. The harmonic component of the frequency $\nu/3$ was also observed at certain non-zero initial conditions, becouse the 1/3 sub-harmonic resonance associated with the cubic nonlinearity is also predicted by first approximate solution. This is an isolated solution not considered here [1].

Amplitudes A_0, $A_{1/2}$, A_1 found by the simulation analysis marked in Figs.1; 3a,b,c and 8a,b.

From Figs.1 and 3a we readily see that indeed the first approximate T periodic solution (2a) describes the behaviour of the system with good accuracy in the whole range of frequency ν, where the solution is locally stable, that is on the upper resonant branch of $C_1(\nu)$ and on the lower branch expect the region $\nu_1 - \nu_2$. On approaching the region from $\nu > \nu_2$ we reach point ν_2, where the amplitude $A_{1/2}$ appears and grows gradually with further decrease of the frequency, until point close to the vertical tangent on the resonance curve. Values of the amplitude $A_{1/2}$ and of the two other components A_0 and A_1 are close to the theoretical ones. At $\nu \approx \nu_1$ a jump phenomena occurs and after a short period of transient motion the response becomes close to harmonic one with the large resonant amplitude C_1.

The response associated with the upper branch of $A_{1/2}(\nu)$ was generated on applying proper nonzero initial conditions. It comes out that the branch is stable in a relatively narrow zone of frequency between points D_2 and D_3 . It becomes evident that the whole remaining left branch of $A_{1/2}(\nu)$ and respective A_0, A_1 are unstable. At point D_3 the 1/2 subharmonic response "jumps down" onto the 1/3 subharmonic one (see also Fig.4) . Indeed in the whole range $0<\nu<\nu_j$ the only-steady-state that was generated at any initial conditions was the resonant large amplitude harmonic solution [upper branch of $C_1(\nu)$].

While the simulation results displayed in Figs.1 and 3a,b,c confirm that the harmonic components assumed in solutions (2a) and (8a) predominate over other harmonics, they do not answer the question of period doubling bifurcation. Instead of searching for possible lower period and evidently low amplitude harmonic components the response was analysed on making use the phase-plane $x - \dot{x}$, where the phase portraits and Poincare maps were recorded. The sampling time for Poincare map was $T = \frac{2\pi}{\nu}$ so that number of points m marked on $x(nT) - \dot{x}(nT)$ plane indicate the period of response $T_r = m \cdot T$.

The results presented in Figs.5,6 and illustrated in Fig.4 confirm apperance of a sequence of period doubling bifurcations. Regions of frequency at which period of response was $T_r = 2T, 4T$ are denoted in Fig.4 for the lower as well as for the upper branch of $A_{1/2}$. Figs.5 show time history, phase portraits and Poincare maps of the 2T and 4T periodic solution. Fig.6a shows more complex picture of motion. However at further decrease of ν the response again becomes a regular one, and just before point ν_j has again period 2T. Thus the jump phenomena illustrated in Fig.6d presents a jump from 2T to T periodic solution without any effects of earlier period doubling bifurcations.

The period doubling bifurcations occur also on the 1/3 subharmonic resonance and this is displayed in Fig.7 in connection with Fig.4.

Then we study behaviour of the system at $P_0 = 0.045$ for which the chaotic motion was observed by Ueda [6] . The results displayed in Figs.8a,b do not show any peculiarities.

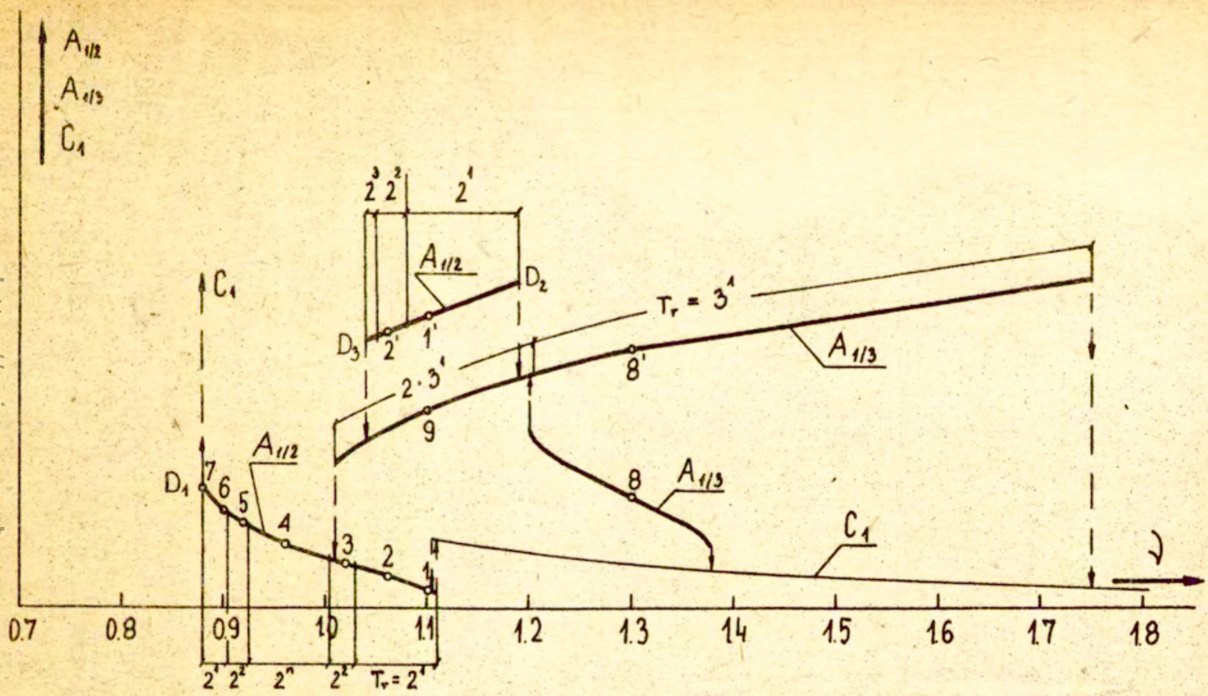


Fig.4. Schematic diagram of bifurcation of the 1/2 and 1/3 subharmonic resonances; P_o = 0.020, P_1 = 0.16, h = 0.05.

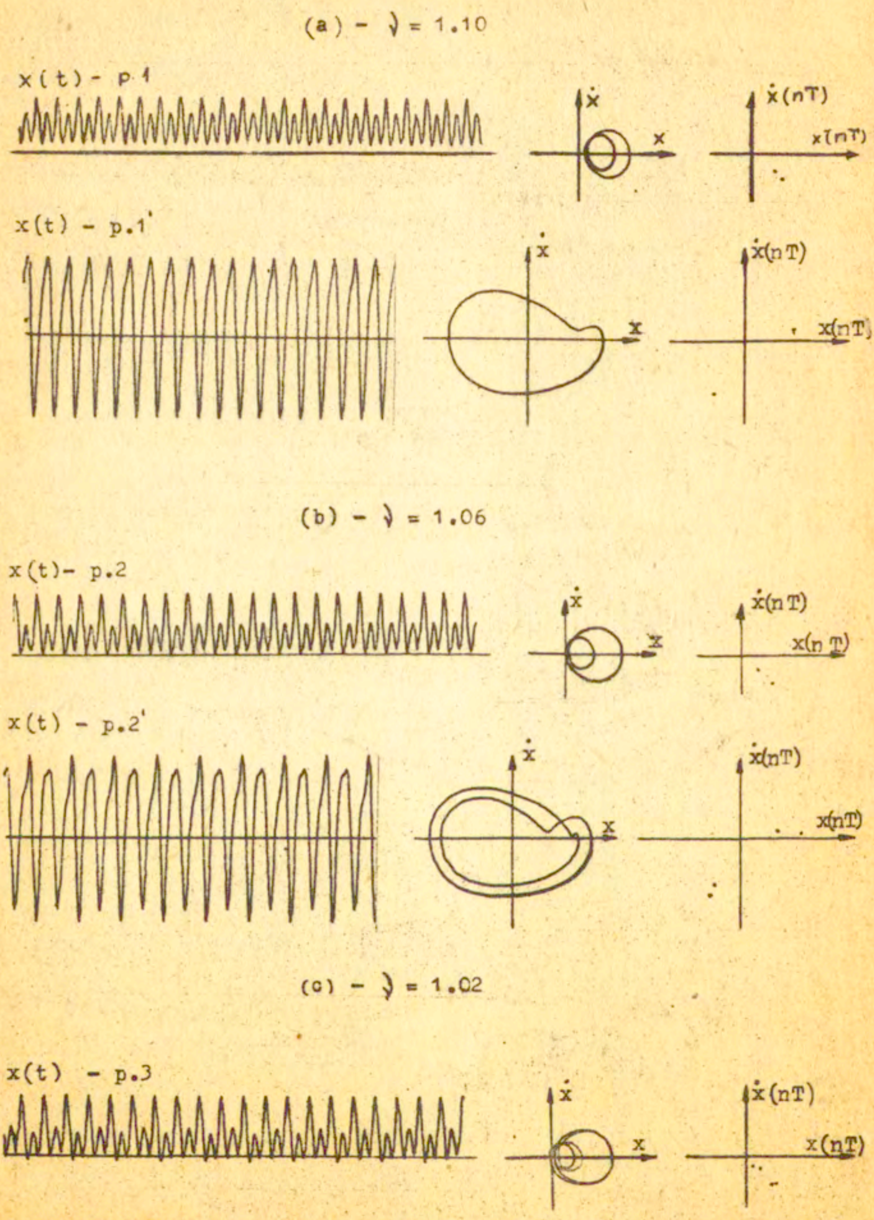
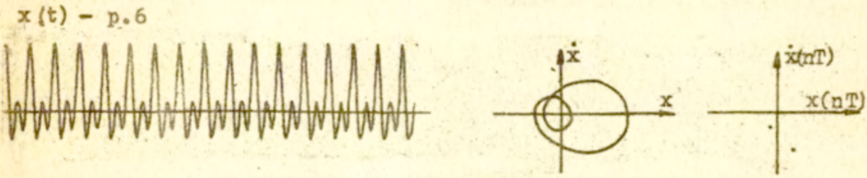


Fig.5. Time history, phase portraits and Poincare maps at points denoted in Fig.4.

(a) - $\nu = 0.9$[illegible]

(b) - $\nu = 0.92$

(c) - $\nu = 0.90$



(d) - $\nu = 0.88$

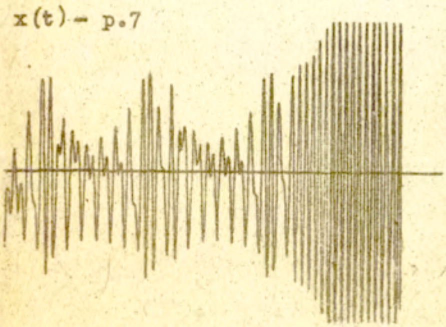


Fig.6. Time history, phase portraits and Poincare maps at points denoted in Fig.4.

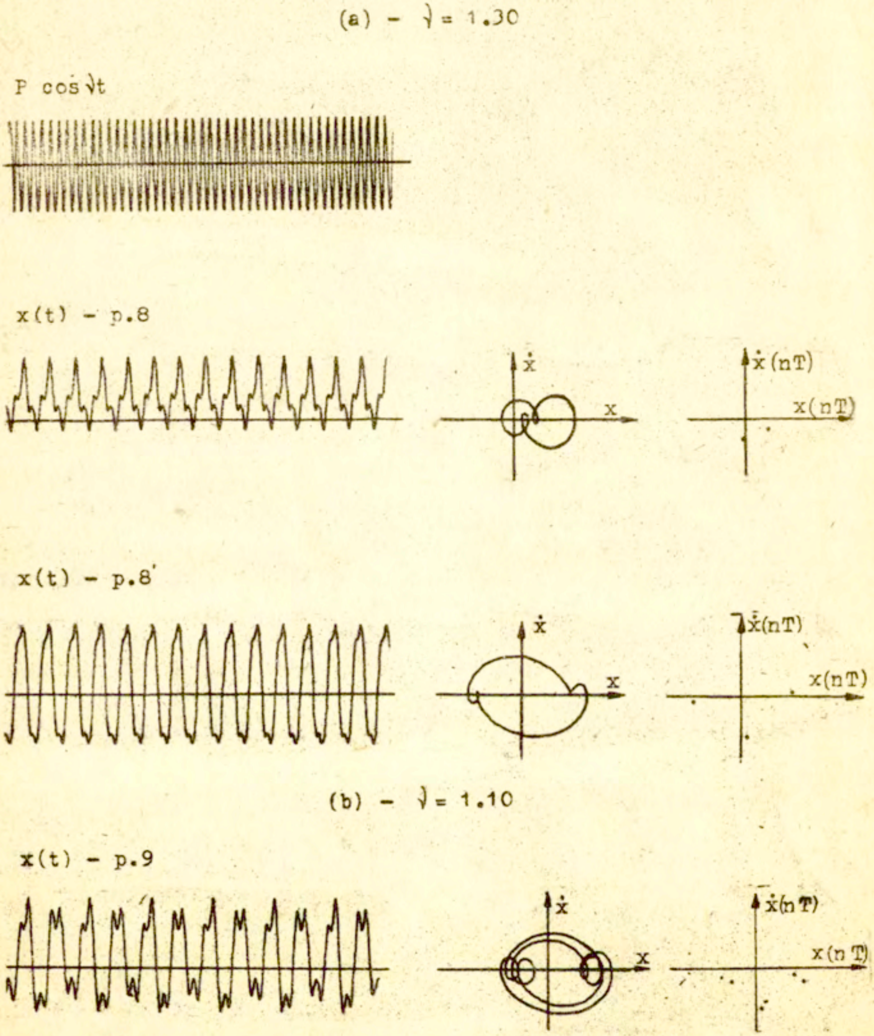


Fig.7. Time history, phase portraits and Poincare maps of the 1/3 subharmonic solution at points 8, 8' and 9 in Fig.4.

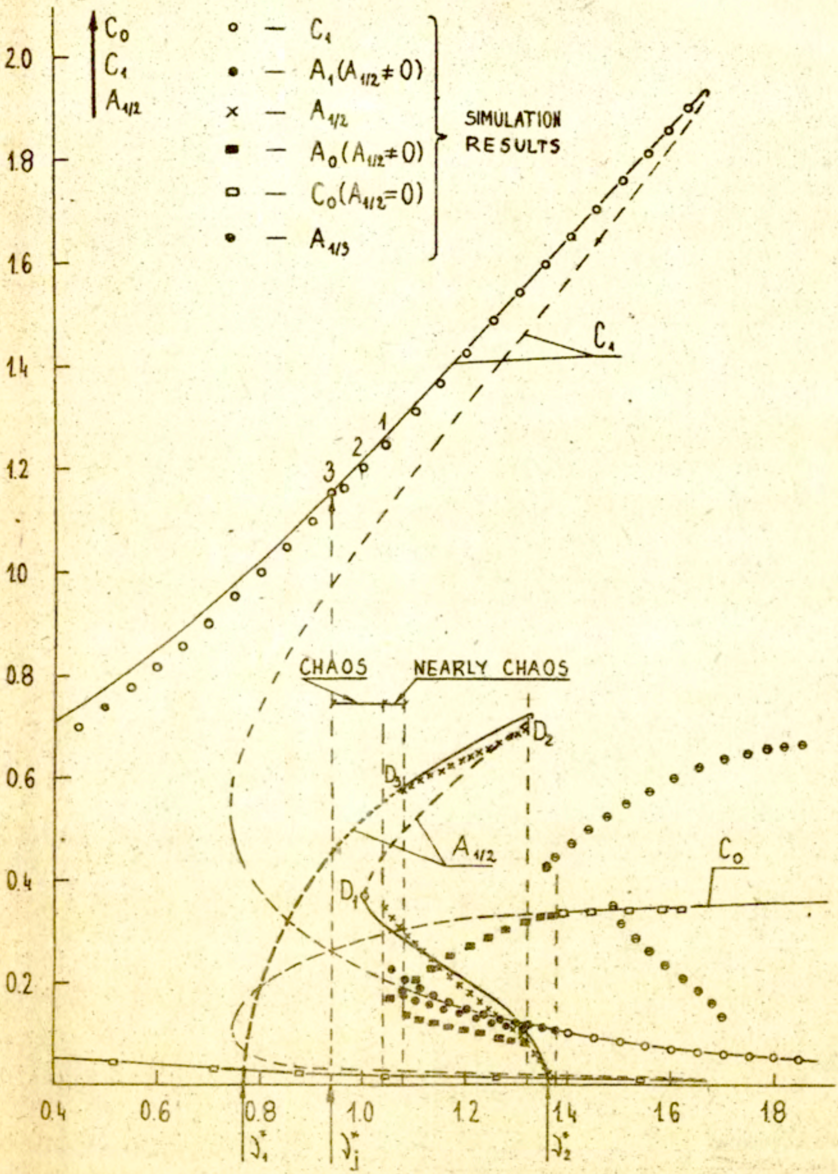


Fig.8a. Resonance curves of the 1/2 subharmonic solution; P_0 = 0.045, P_1 = 0.16, h = 0.05.

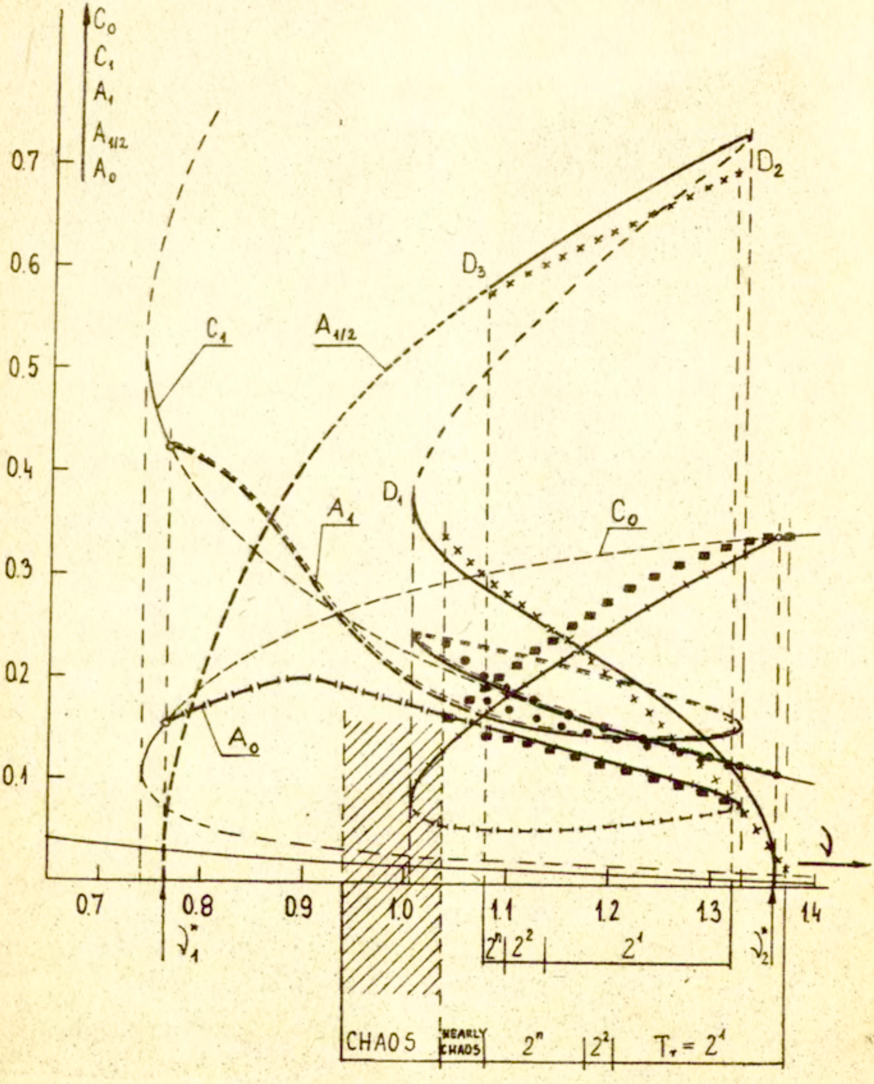


Fig.8b. Resonance curves of the 1/2 subharmonic solution, regions of periodic doubling bifurcation and chaotic behaviour.

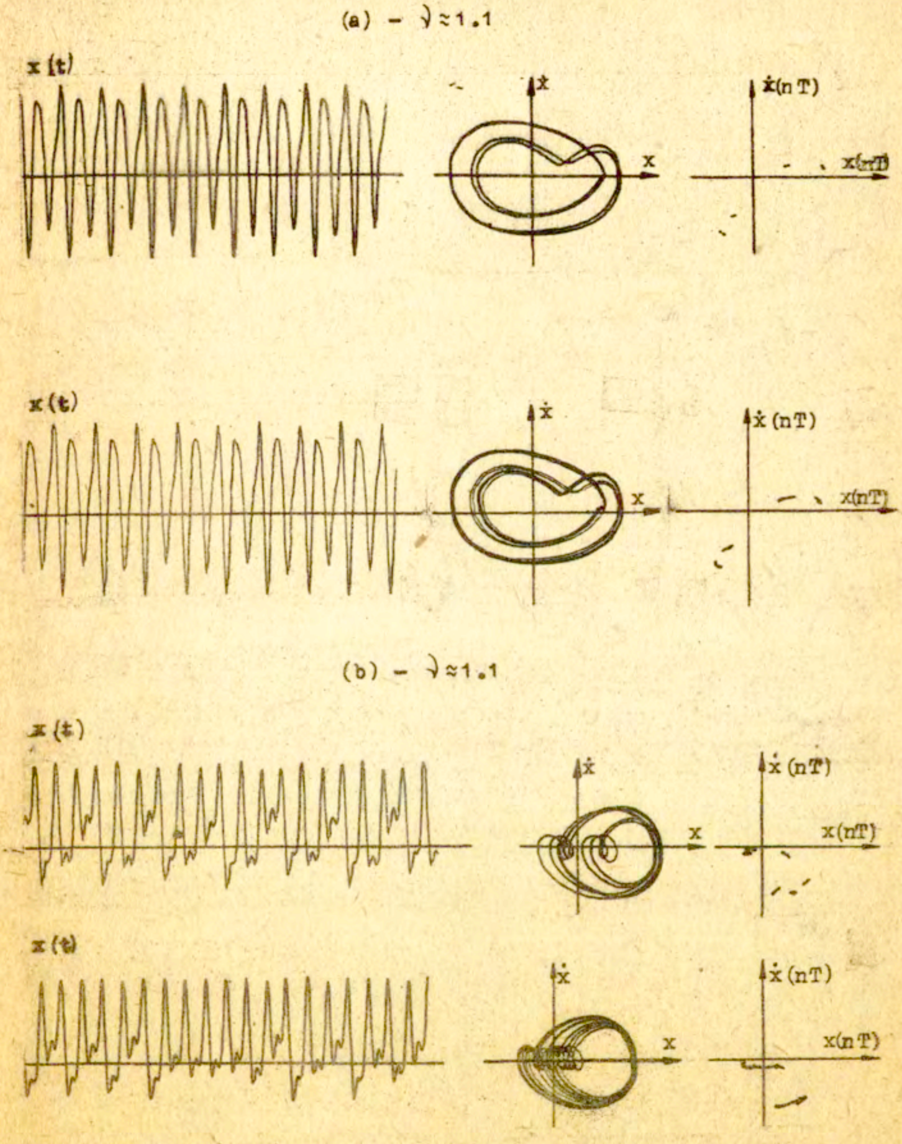


Fig.9. Time history, phase portraits and Poincare maps at $\nu \approx 1.1$, $P_0 = 0.045$, $P_1 = 0.16$, $h = 0.050$.

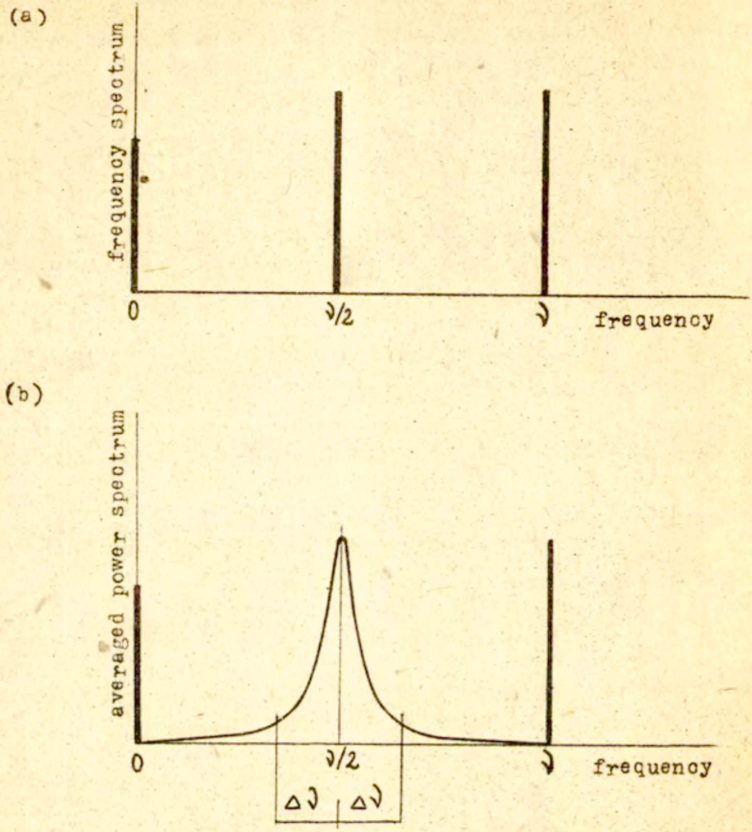


Fig.10. a - frequency spectrum of the 1/2 subharmonic resonance,
b - character of the averaged power spectrum of the chaotic motion.

It looks that again in a wide zone of ν the predicted theoretically harmonic components predominate in the response. Alike in the case $P_0 = 0.020$ we notice a sequence of period doubling bifurcations. The zones of with $T_r = 2T, 4T, \ldots$ are denoted in Fig.8b. However now, on further decrease of the frequency, the response does not return to the regular 2T periodic one and gradually the picture on Poincare map becomes more complex, Fig.9a,b. Then at $\nu \approx 1.04$ there is a "burst out" of the "strange attractor", with the results illustrated in Figs.10 and 11. An appearance of the continuous segments on the averaged power spectrum is, in addition to the characteristic picture on the Poincare map, a strong evidence of the chaotic motion.

Chaotic motion at $\nu = 1.0$ illustrated in Fig.12, is, in substance, of the same character. However the strange attractor in Fig.12c shows highly organized points arranged in what appear to be parallel lines.

The zone of frequency ν where the chaotic behaviour is observed is denoted in Fig.8b. It is readily seen that it

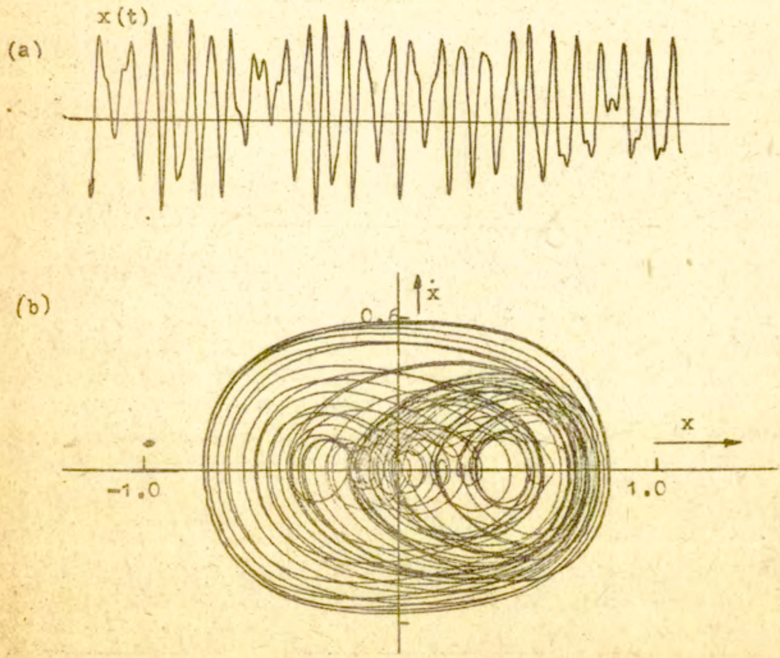


Fig.11a,b. Chaotic motion at $\nu=1.04$, $P_0=0.045$, $P_1=0.16$, $h=0.05$;
(a) - time history,
(b) - phase portrait.

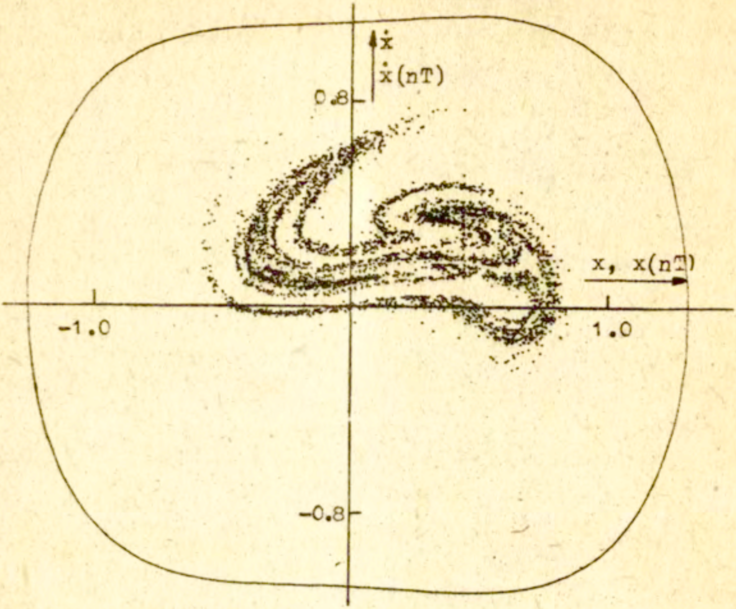


Fig.11c. Poincare map of the chaotic motion at $\nu = 1.04$, $P_0 = 0.045$, $P_1 = 0.16$, $h = 0.050$. The full orbit illustrates the large harmonic solution in point 1 in Fig.8a.

surrounds the point of the vertical tangent $\nu(D_1)$ on the theoretical resonance curves.

On further decrease of the frequency a jump phenomenon occurs - at $\nu \approx 0.94$, after some transients the chaotic motion turns into the T periodic, harmonic response, corresponding to point 3 in Fig.8a.

Therefore we note, that the chaotic motion is associated with the process of transition from regular, subharmonic resonance to the T periodic principal resonance. It appears within the zone

$$0.94 < \nu < 1.04,$$

which spreads around the theoretical, stability limit frequency, that is, the point of the vertical tangent on the resonance curve

$$\nu(D_1) \approx 1.0.$$

We may say that the regular jump phenomenon, predicted theoretically at $\nu = \nu(D_1)$ to turn the 1/2 subharmonic resonance into T periodic harmonic one, is replaced by a wide transition zone where the response is neither subharmonic nor harmonic, but highly irregular, chaotic.

While the time history, phase portrait and the Poincare map of the chaotic and the regular 1/2 subharmonic response are substantially different, the character of frequency spectrum shown in Fig.10a and b shows certain analogy. We easily make an observation that the major difference between the two types of spectra relies on an appearance of a narrow band of continuous spectrum in the neighbourhood of $1/2\,\nu$ component. This brings an appealing idea to seek an insight into the nature of chaotic behaviour by focusing an attention on the motion associated with the continuous segment of the frequency spectrum.

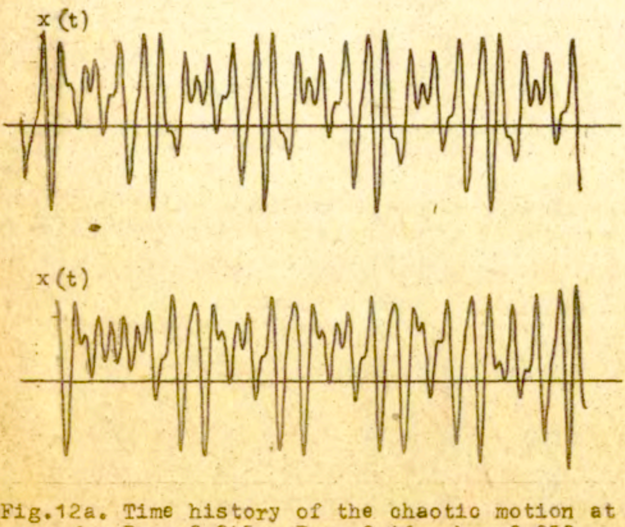


Fig.12a. Time history of the chaotic motion at $\nu = 1.0$, $P_0 = 0.045$, $P_1 = 0.16$, $h = 0.050$.

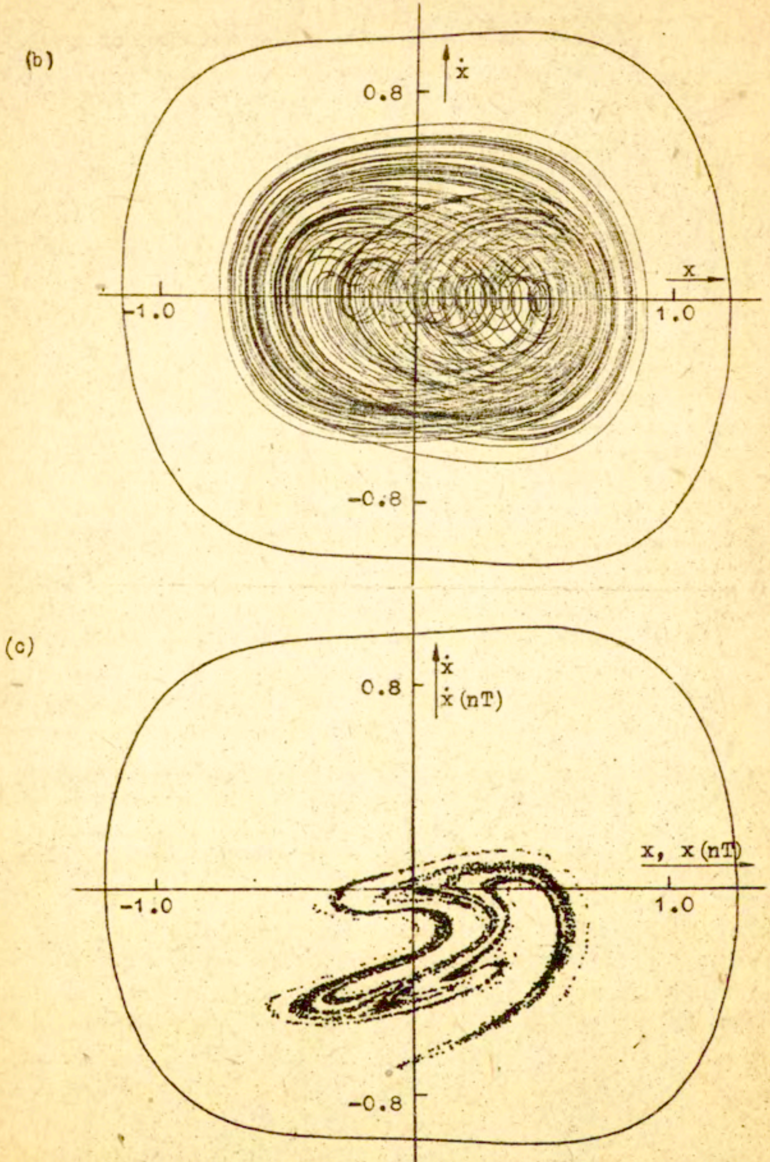


Fig.12b,c. Chaotic motion at $\nu = 1.00$,
(a) - phase portrait, (b) - Poincare map.
The full orbit illustrates large harmonic soluti
in point 2 in Fig.8a.

To this end it is useful to consider transient motion of a linear, weakly damped oscillator governed by an equation,

$$\ddot{\bar{x}} + h\dot{\bar{x}} + \left(\frac{1}{2}\nu\right)^2 \bar{x} = 0, \tag{10a}$$

and to notice that the frequency spectrum of the solution:

$$\bar{x}(t) = A_{1/2}(t) \cos\left[\left(\frac{1}{2}\nu - \Delta\omega\right)t + \phi\right], \tag{10b}$$

where

$$A_{1/2}(t) = A_{1/2}\, e^{-\frac{h}{2}t},$$

$$\frac{1}{2}\nu - \Delta\omega = \sqrt{\frac{\nu^2}{4} - \frac{h^2}{4}}, \qquad \Delta\omega \ll \frac{1}{2}\nu,$$

is very similar to that of the continuous segment of the chaotic motion [12] . Moreover it is essential to note that the character of the averaged frequency spectrum of $\bar{x}(t)$ remains unaltered if the damping coefficient h varies slightly with time.

The analogy draws an attention to the question: what is time history of the component of chaotic motion which is due to the continuous segment of the averaged power spectrum? To answer the question a filter was used which cut off the harmonic components higher than 0.8ν in the chaotic response. Results of the analysis are shown in Fig.13. The filtred response in Fig.13a corressponds to the sample of chaotic response in Fig.11a. Fig.13b illustrates another sample of the complete and filtred time history. Indeed, the time history of the filtred response $\bar{x}(t)$ appears to look close to that of harmonic oscillations described by eqs.(10b) with $A_{1/2}(t)$ and $\Delta\omega(t)$ varying randomly with time.

The result gives an suggestion that the chaotic motion in the case can be roughly described by approximate analytical solution,

$$x(t) = A_0 + A_{1/2}(t) \cos\left[\left(\frac{\nu}{2} - \Delta\omega\right)t + \phi\right] + A_1 \cos\nu t,$$

where $A_{1/2}(t)$ and $\Delta\omega \equiv \Delta\omega(t)$ are randomly varying with time.

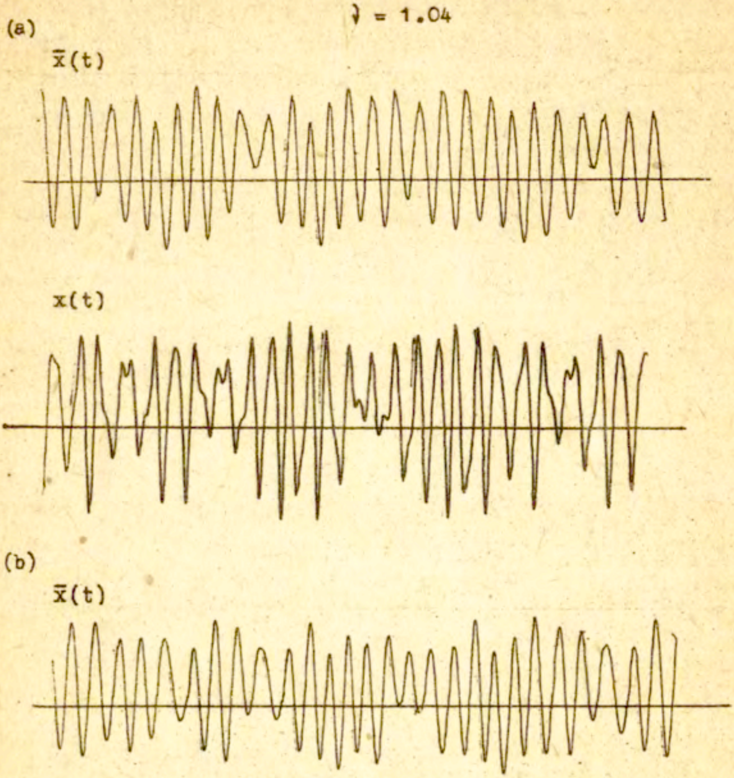


Fig.13. Time history of chaotic and filtred responses at: $\nu = 1.04$, $P_0 = 0.045$, $P_1 = 0.16$, $h = 0.050$.

Thus we can say that the essence of the chaotic behaviour relies on a sort of "instability" of the 1/2 harmonic component, which fluctuates randomly around the respective constant amplitude $1/2\,\nu$ harmonic component in the regular motion. The oscillations of the $1/2\,\nu$ harmonic components precede a strict loss of stability where the amplitude $A_{1/2}$ decays and the response turns into the T periodic, large amplitude resonant solution.

Conclusions.

The results of theoretical, first approximate solution and the computer simulation of eqs.(1c) allows us to draw the following conclusions:

- The system can exhibit chaotic motion in a narrow zone of frequency in the neighbourhood of $\nu(D_1)$, where the theoretical loss of stability of the 1/2 subharmonic resonance is predicted.
- Although the chaotic motion is preceded by a cascade of period doubling bifurcations, the three harmonic components assumed in the first approximate solution (8a) predominate in the response as long as the motion is regular, periodic one.
- The region of chaotic motion replaces a classical "jump phenomenon" and forms a transition zone between the subharmonic and resonant harmonic solution.
- The chaotic motion exhibited by the system can be roughly interpreted as random fluctuation of the $1/2\nu$ harmonic component.

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Contents