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ON ROUTES TO CHAOTIC MOTION IN OSCILLATORS WITH UNSYMMETRIC AND SYMMETRIC ELASTIC NONLINEARITY

1. INTRODUCTION

To a variety of nonlinear phenomena arising in physical systems governed by simple deterministic equations, such as the Duffing equation, a distincly new class of motion, called chaotic motion, has been added recently. The chaotic behaviour in nonlinear oscillators with single equilibrium position was first reported by Ueda [1-4]. More attention was paid to chaotic phenomena in systems having two or three positions of equilibrium, theoretical and computer simulation study being completed with experiments [5-10].

In the early stage of investigations it was fascinating enough to discaver the chaotic motion in simple deterministic systems, whose regular behaviour has been studied extensively for a few decades, and to record its major characteristics: Poincare map showing "strange attractor" and the averaged power spectrum. Further steps into the new branch of nonlinear mechanics bring a wide variety of unanswered questions: the question of "routes to chaos" and of any criteria that might be helpful to predict regions of system parameters where the chaotic motion is likely - belonging to the most attractive ones.

With these questions in mind some steps were made to seek criteria and physical interpretation of chaos reported in [11]. It was considered a forced nonlinear oscillator with single equlibrium position and unsymmetric elastic characteristic quadratic and cubic term . First regular motions: harmonic, and 1/2 subharmonic solutions were studied by means of the harmonic balance method. Local stability of the solutions was examined with the aid of variational Hill's type equations. This made it possible to observe the chaotic motion against a background of resonance curves of individual harmonic components and to find a link between theoretical stability limits of the 1/2 subharmonic solution and the zone of the chaotic behaviour. A combination of the approximate analytical evaluations and a computer simulation has led to observations:

- the chaotic motion is associated with the nonresonant branch of the resonance curve of harmonic solution and hence a resonant stable harmonic solution is always coexisting with the strange attractor;
- the variational Hill's type equation treated by approximate analytical methods indicates a possibility of a cascade of period doubling bifurcations. Indeed the simulation results show that a narrow zone of chaotic motion is preceded by a wide zone of frequency where a series of changes from T-periodic to 2T and then 4T-periodic solution is observed. For comparison see 12 where the second order averaging is applied to study the period doubling bifurcation problems ;
- the zone of chaotic motion occurs in the neighbouhood of the theoretical stability limit of the 1/2 subharmonic resonance i.e. close to the frequency where the resonance curves have vertical tangent.

It follows that on decreasing the frequency of external force the system shows a classical route to chaos via a cascade of period doubling bifurcations. However the other boundary of the chaotic zone is a sharp one - at certain frequency the motion, after some transients, "jumps" into the resonant harmonic solution. Thus one can talk about "jump phenomenon" referring to the rapid change from chaos to regular motion. However the jump phenomenon in the classical sense, the one which used to occur at the frequency of vertical tangents on resonance curves, has now, at certain values of system parameters, been replaced by a transition zone characterized by the chaotic motion.

While the results presented in [11] concern the oscillator with unsymmetric nonlinear restoring force, the chaotic motion was also observed in the classical Duffing equation with cubic nonlinear function [2] . It is an attempt of the present paper to trace routes to chaos in both types of systems: forced oscillators having single position of equilibrium, provided with unsymmetric and symmetric elastic characteristics.

2. PERIOD DOUBLING BIFURCATIONS IN SYSTEMS WITH UNSYMMETRIC AND SYMMETRIC ELASTIC NONLINEARITY

Consider two types of nonlinear oscillators:
I - system with unsymmetric elastic characteristic governed by an equation in the form

$$\ddot{y} + h\dot{y} + \omega_0^2 y + \varkappa y^2 + y^3 = P_1 \cos \nu t, \qquad h > 0, \quad h \ll 1/2\,\omega_0^2 \tag{1}$$

II - system with symmetric elastic function Duffing equation written as

$$\ddot{y} + h\dot{y} + \omega_0^2 y + y^3 = P_1 \cos \nu t, \tag{2}$$

and two types of qT-periodic solution represented by finite Fourier series:
an unsymmetric solution:

$$y_0^{(R)}(t) \neq -y_0^{(R)}\left(t + \frac{qT}{2}\right) = A_0 + \sum_{p=1,2,3\ldots}^{R} A_p \cos\left(p\frac{\nu}{q}t + \vartheta_p\right) \tag{3}$$

and symmetric solution:

$$y_0^{(R)}(t) = -y_0^{(R)}\left(t+\frac{qT}{2}\right) = \sum_{p=1,3,5\ldots}^{R} A_p \cos\left(p\frac{\nu}{q}t+\vartheta_p\right) \qquad (4)$$

where

$$q = 1, 2, 3 \ldots \quad , \quad T = \frac{2\pi}{\nu} \; ;$$

The solution (4) involves only odd harmonic components and is inherently associated with the symmetric system (2). The unsymmetric solution (3) containing a constant term and both even and odd harmonics is characteristic for the unsymmetric system (1), however it may also appear in the system (2).

Since a cascade of period doubling bifurcat on has been found in many physical systems as a very common rout to chaos it is esential to examine local stability of the solutions (3), (4) against a build-up of 2qT-periodic components.

Suppose that the coefficients $A_p, \vartheta_p, p=1,2,\ldots$ in equations (3,4) have been determined by the harmonic balance method and that one wishes to examine local stability of the qT-periodic solution thus evaluated. To this end one adds a small disturbance δy to the steady-state solution

$$\tilde{y}(t) = y_0(t) + \delta y \; , \qquad (5a)$$

inserts it into equation (1) and neglegts terms of higher order in δy with the result (see i.g. [13])

$$\delta\ddot{y} + h\,\delta\dot{y} + \delta y\left[\omega_0^2 + 2\varkappa y_0(t) + 3y_0^2(t)\right] = 0 \; ; \qquad (5b)$$

On substituting the unsymmetric solution (3) one obtains the variational equation as

$$\delta\ddot{y} + h\,\delta\dot{y} + \delta y\left[\omega_0^2 + \lambda_0 + \Lambda_1(t) + \Lambda_2(t)\right] = 0, \qquad (5c)$$

where $\Lambda_1(t)$ is a periodic function of time with period of the solution $y_0(t)$:

$$\Lambda_1(t)=\Lambda_1(t+qT), \tag{5d}$$

and $\Lambda_2(t)$ denotes the harmonic components of period $\frac{qT}{2}$:

$$\Lambda_2(t)=\Lambda_2(t+\tfrac{1}{2}qT); \tag{5e}$$

On applying Floquet theory (see e.g. [13, 14]) one can readily note that it is the term $\Lambda_1(t)$ which is essential to the period doubling bifurcation. Only if

$$\Lambda_1(t)\neq 0, \tag{6a}$$

one may expect a particular solution in the form

$$\delta y(t)=e^{\varepsilon t}\Phi(t), \tag{6b}$$

where $\varepsilon > 0$ and $\Phi(t)=\Phi(t+2qT)$.
When this form of instability exists, a build-up of harmonic components of the period 2qT and consequently period doubling bifurcation becomes possible.

If $\Lambda_1(t)=0$, $\Lambda_2(t)\neq 0$,

the only types of instability are those where the particular solution involve periodic function with period qT or $\frac{1}{2}qT$:

$$\delta y(t)=e^{\varepsilon t}\Phi(t),$$

$$\Phi(t)=\Phi(t+qT), \quad \text{or} \quad \Phi(t)=\Phi(t+\tfrac{1}{2}qT); \tag{7b}$$

(a) Unsymmetric system (1) and unsymmetric solution (3).

In this case the variational equation (5b) yields

$$\delta\ddot{y}+h\delta\dot{y}+\delta y\Big[\omega_0^2+\lambda_0+\sum_{p=1,2,3}^{R}\lambda_{1p}\cos\big(p\frac{\nu}{q}t+\phi_{1p}\big)+ \tag{8a}$$

$$+\sum_{p=1,2,3\ldots}^{R}\lambda_{2p}\cos\big(2p\frac{\nu}{q}t+\phi_{2p}\big)=0,$$

where

$$\lambda_0 \equiv \lambda_0 (A_0, A_1, \dots \vartheta_1, \vartheta_2 \dots),$$

$$\lambda_{ip} \equiv \lambda_{ip} (A_0, A_1, \dots , \vartheta_1, \vartheta_2 \dots);$$

It is worth noticing that

$$\lambda_{1p} = 0 \quad \text{only if} \quad \varkappa = 0 \quad \text{and} \quad A_0 = 0; \qquad (8b)$$

It follows that period doubling bifurcation is very likely in the system with unsymmetric elastic function. It was shown in [11] that it really happens even to the first approximate solution in the form

$$y_0^{(1)}(t) = A_0 + A_1 \cos(\nu t + \vartheta_1), \qquad (9a)$$

when the variational equation (8a) becomes

$$\delta\ddot{y} + h\delta\dot{y} + \delta y\left[\omega_0^2 + \lambda_0 + \lambda_1 \cos(\nu t + \vartheta_1) + \lambda_2 \cos(2\nu t + 2\vartheta_1)\right] = 0, \qquad (9b)$$

where

$$\lambda_0 = 3/2\, A_1^2 + 2\varkappa A_0 + 3A_0^2,$$

$$\lambda_1 = 2\varkappa A_1 + 6A_0 A_1; \qquad \lambda_2 = 3/2\, A_1^2;$$

In equation (9b) the particular solution satisfying condition (6b) does exist and in the first approximation is sought as

$$\delta y(t) = e^{\varepsilon t} b_1 \cos\left(\frac{\nu}{2} t + \phi_1\right); \qquad (9c)$$

An unstable region of the type where $\varepsilon > 0$ is found in the region of frequency

$$\nu \approx 2\sqrt{\omega_0^2 + \lambda_0}; \qquad (9d)$$

It follows that the 1/2 harmonic component grows and consequently it appears that the 1/2 subharmonic solution with the period 2T is a stable steady-state solution in the region. The solution is assumed in the form

$$y_0^{(2)}(t) = A_0 + A_1 \cos(\nu t + \vartheta_1) + A_{1/2} \cos\left(\frac{\nu}{2} t + \vartheta_{1/2}\right); \quad (9e)$$

A variational equation for the solution (9e) again shows the period doubling bifurcation indicating a possibility of the build-up of harmonic components of frequencies $\frac{1}{4}\nu$ and $\frac{3}{4}\nu$ giving rise to . 4T-periodic solution. Computer simulation confirmed the approximate theoretical predictions. Indeed the computer simulation showed a series of period doubling bifurcation on decreasing the frequency ν. At least three period doublings were observed before a zone of chaotic motion. However the series of period doubling bifurçations was not necessarily followed by the chaotic behaviour (see [11]).

(b) Symmetric system (2) and symmetric solution (4).

The qT periodic term in equation (5c) proves to vanish and the variational equation equation yields

$$\delta\ddot{y} + h\delta\dot{y} + \delta y\left[\omega_0^2 + \lambda_0 + \sum_{p=1,3,5\ldots}^{R} \lambda_{2p} \cos\left(2p\frac{\nu}{q}t + \phi_{2p}\right)\right] = 0, \quad (10\text{ a})$$

where

$$\lambda_0 = \frac{3}{2} \sum_{p=1,3\ldots}^{R} A_p^2;$$

$$\lambda_{2p} \equiv \lambda_{2p}\left(A_1, A_3, \ldots A_R, \vartheta_1, \vartheta_3 \ldots \vartheta_R\right);$$

Particular solutions of eqs. (10a) are

$$\delta y(t) = e^{\varepsilon t} \Phi(t), \quad (10b)$$

where

$$\Phi(t) = \Phi\left(t + \tfrac{1}{2}qT\right), \text{ or } \Phi(t) = \Phi(t + qT);$$

Consequently in the unstable regions one may expects a build-up of harmonic components with the frequency and its higher harmonics. The analysis does not show the possiblity of the period doubling bifurcation. In spite of that a development of the symmetric solution into chaotic motion was observed and reported by Ueda (see [2] and [3]). The simulation results presented in the form of Poincare maps of system response for varying forcing parameter P_1 at constant frequency $\nu = 1.0$ did not show period doublings before a burst-out of a strange attractor. This route to chaos which appears to be a sharp one will be considered in sec. 4.

(c) Symmetric system (2) and unsymmetric solution (3).

The unsymmetric solution does not exist in the symmetric system in the first approximation i.e. at $R=1$. Indeed A_0 vanishes identically at $R=1$ and consequently the first approximate solution is

$$y_0^{(1)}(t) = A_1 \cos(\nu t + \vartheta_1); \tag{11a}$$

Unsymmetric solutions do exist, however, in higher approximations. Indeed it was shown in earlier papers (see i.g. [15]) that at low frequencies and sufficiently high forcing parameter P_1 there are regions where the harmonic solution is unstable and the higher harmonic components become dominating ones. In the neighboughood of the frequency

$$\nu \approx \frac{1}{2}\sqrt{\omega_0^2 + \lambda_0}, \tag{11b}$$

a stable solution is that which involves a constant term and the second harmonic component

$$y_0^{(2)}(t) = A_0 + A_1\cos(\nu t + \vartheta_1) + A_2\cos(2\nu t + \vartheta_2); \tag{11c}$$

It is a general rule that the constant term is associated with even order harmonic components.

It follows that in the symmetric systems we may also obtain the variational equation in the form (8a).

Note that the coefficients λ_{1p} essential to the period doubling bifurcations vanish only when both $\varkappa$ and A_0 vanish. In the case under consideration the coefficient $\varkappa$ is equal to zero, however the constant term is not:

$$\varkappa = 0 \quad , \quad A_0 \neq 0 \; ;$$

Thus we conclude that periodic solutions in symmetric systems can also undergo period doubling bifurcation, provided they are the unsymmetrical ones i.e. $A_0 \neq 0$.

3. SYSTEMS WITH UNSYMMETRIC NONLINEARITY - PERIOD DOUBLING BIFURCATIONS AND CHAOTIC MOTION

Consider the system governed by equation (1) and put

$$\omega_0^{\,2} = 3\sqrt[3]{P_0^2} \; ; \qquad \varkappa = 3\sqrt[3]{P_0} \; ; \tag{12a}$$

Then on introducing a new coordinate

$$x = y + \sqrt[3]{P_0} \, , \tag{12b}$$

the equation (1) can be transformed into a simpler form

$$\ddot{x} + h\dot{x} + x^3 = P_0 + P_1 \cos \nu t \; ; \tag{12c}$$

The system governed by eqs. (12c) was examined in [11] at the values $P_1 = 0.16$; $h = 0.05$; $P_0 = 0.020$; $P_0 = 0.045$. While at $P_0 = 0.020$ the T-periodic solution (9a) showing a sequence of period doubling bifurcation did not develop into chaotic motion, at the higher value of P_0 a strange attractor appeared in a narrow zone of the frequency (see also [2]).

Presently the behaviour of the system at the parameters

$$P_1 = 0.16 \; ; \quad h = 0.05 \; ; \quad P_0 = 0.030 \; ,$$

is analysed.

In the first approximation we seek the T-periodic solution as (see equation (9a)):

$$x_0^{(1)} = C_0 + C_1 \cos(\nu t + \vartheta) \, , \tag{13a}$$

where the unknown parameters C_0, C_1, ϑ are determined with the aid of the harmonic balance priciple i.e. they satisfy equations :

$$-\nu^2 C_1 + \tfrac{3}{4} C_1^{\,2} + 3 C_0 C_1^{\,2} = P_1 \cos \vartheta \,, \qquad (13b)$$

$$-h \nu C_1 = P_1 \sin \vartheta \,,$$

$$C_0^{\,3} + \tfrac{3}{2} C_0 C_1^{\,2} = P_0 \,;$$

On solving eqs. (13b) the resonance curves $C_1(\nu)$, $C_0(\nu)$ drawn in Fig.1 have been obtained.

The variational equation (9b) indicates, that in addition to the first order instability near the principal resonance between points of vertical tangent, another first order unstable region appears - the one where a harmonic component with the frequency $\nu/2$ can be build-up (see eqs. (9c)). This unstable region lies on the nonresonant branch of the resonance curves $C_1(\nu)$ between frequencies ν_1 and ν_2 denoted in Fig.1 and 2. The points $\nu = \nu_1$ and $\nu = \nu_2$ are critical points on the ν axis, where the T-periodic solution (13a) turns into 2T-periodic solution. The solution called 1/2 subharmonic solution is assumed in the form

$$x_0^{(2)}(t) = A_0 + A_1 \cos \nu t + A_{1/2} \cos\left(\tfrac{\nu}{2} t + \phi\right) \,, \qquad (14a)$$

and again unknown parameters $A_0, A_1, A_{1/2}, \phi$ are determined by the harmonic balance method (see [11] for details). Resonance curves $A_i \equiv A_i(\nu)$ are displayed in Fig.2 and $A_{1/2}(\nu)$ is also drawn in Fig.1. The branch of resonance curves which develop from the critical point ν_1 is unstable.

When starting with frequency $\nu > \nu_2$ and decreasing it the T-periodic solution (13a) bifurcates into a stable 2T-periodic solution (14a). Then on further decrease of ν the amplitude of the 1/2 component grows gradually until a point of vertical tangent ν_v is reached. To verify the theoretical predictions equation (12a) was simulated on a electronic computer and amplitudes of the harmonic components assumed in the solution (13a) and (14a) were detected first.

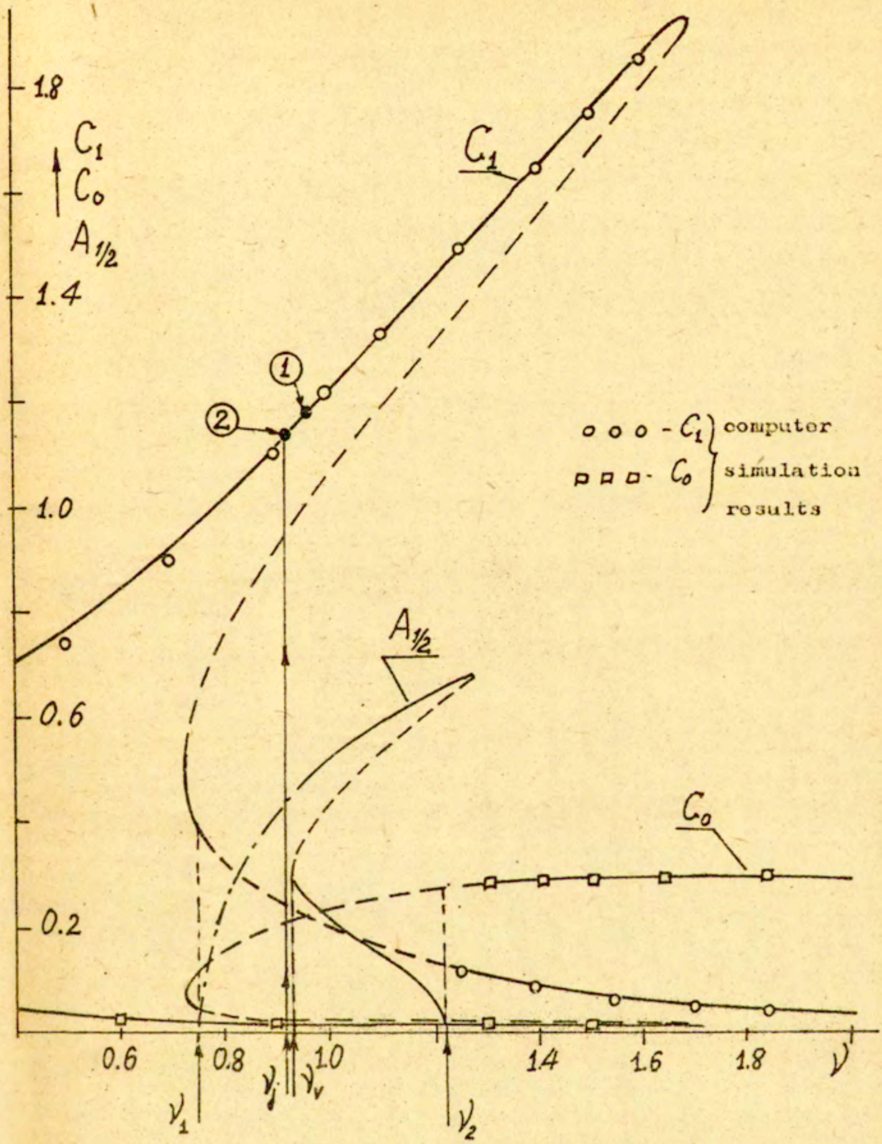


Fig.1. Resonance curves of the harmonic plus constant term solution: $P_1 = 0.16$; $P_0 = 0.030$; $h = 0.05$;

———— stable branch, – – – – – unstable branch

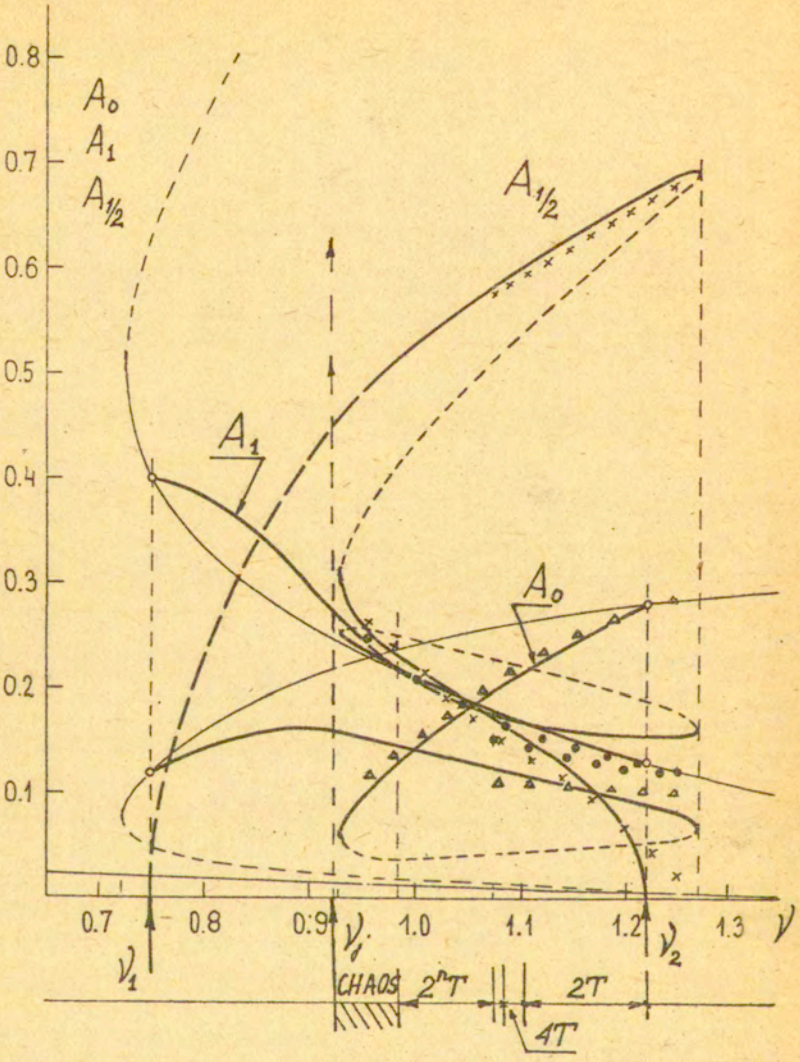


Fig.2. Resonance curves of the 1/2 subharmonic solution:

$P_1 = 0.16$; $P_0 = 0.030$; $h = 0.05$;

———————— stable branch, — — — — unstable

Computer simulation results: × × × × - $A_{1/2}$;

• • • - A_1 ; △ △ △ - A_0 .

From Fig.1 and 2 it is pretty clear that indeed the values of $C_0, C_1, A_0, A_1, A_{1/2}$ determined with the computer simulation are very close to the theoretical ones in a wide range of the frequency.

In further steps of the period doubling bifurcation analysis it was shown that the variational equation for $x_0^{(2)}$ predicts an appearance of harmonic components of the frequency $\frac{1}{4}\nu$ and $\frac{3}{4}\nu$ and hence the appearance of 4T-periodic solution. The process of development of new harmonic components in the period doubling bifurcation is illustrated in Fig.3 a-d. In Fig.3e the averaged power spectrum of the chaotic motion which develops in certain range of ν is also shown. It looks rather convincing that the continuous segment of the averaged power spectrum results from the cascade of the period doubling bifurcation: the first new components are those at the edges of the continuous segment and then those lying closer and closer to the middle value $\frac{1}{2}\nu$ appear.

To verify the predicted further period doubling bifurcation and possible chaotic motion the computer simulation results have been recorded in the form of time histories, phase portraits and Poinacre maps of the response $x(t)$ (see Figs.4-6). The Poincare map which is a stroboscopic picture of the motion in the phase plane synchronous with the forcing function gives an immedatiate answer about the period of motion - if it shows m points, the period is mT. The 2T-periodic solution corresponding to the frequency spectrum in Fig.3b is illustrated in Fig.4 a,b. While Fig.4 a shows the response which bifucates at $\nu = \nu_2$ from the T-periodic solution, Fig.4b corresponds to an isolated solution associated with the higher branch of the resonance curve $A_{1/2}(\nu)$ (see Fig.2). At lower frequency $1.075 < \nu < 1.1$ both solutions turn into the 4T-periodic periodic solutions (see Fig.5 a,b). On further decrease of the parameter ν more complex pictures of motion are obtained. The region where the motion is no more periodic but still not completely chaotic (although it depends on the definition adopted) is denoted as 2^nT on the frequency axis in Fig.2.

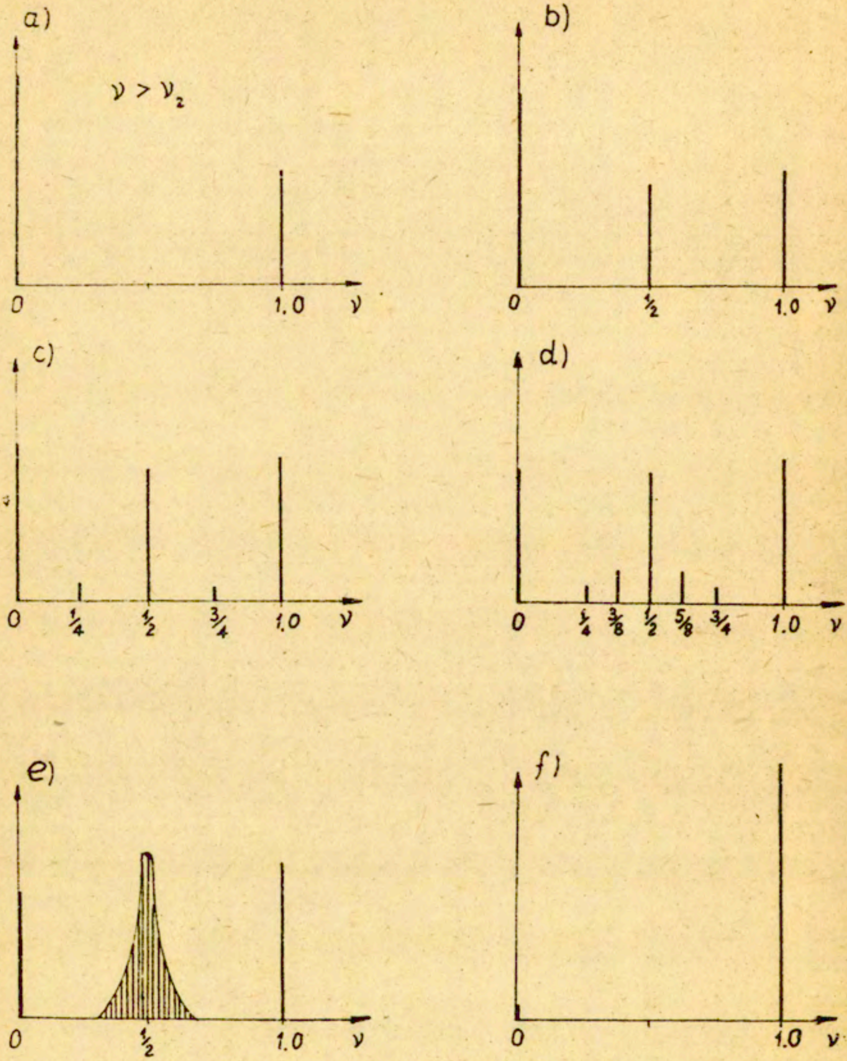


Fig.3. Development of frequency spectrum in transition to chaos of the unsymmetric solution.

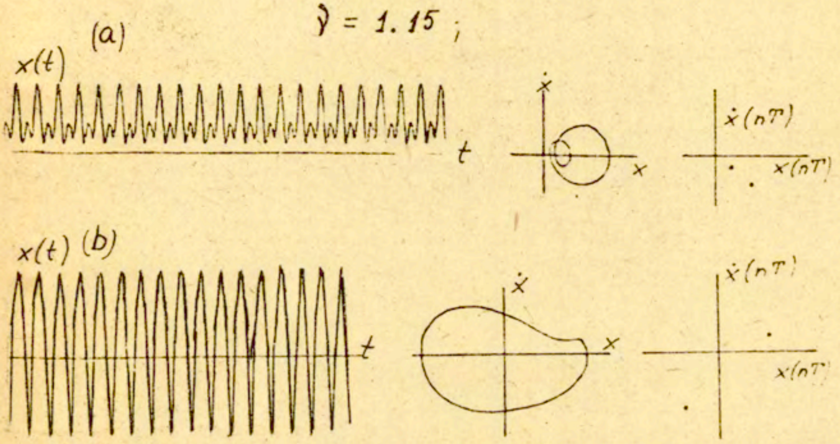


Fig.4. Time histories, phase portraits and Poincare maps of 2-Tperiodic solution: (a) – motion branching from T-periodic solution ; (b) – isolated solution.

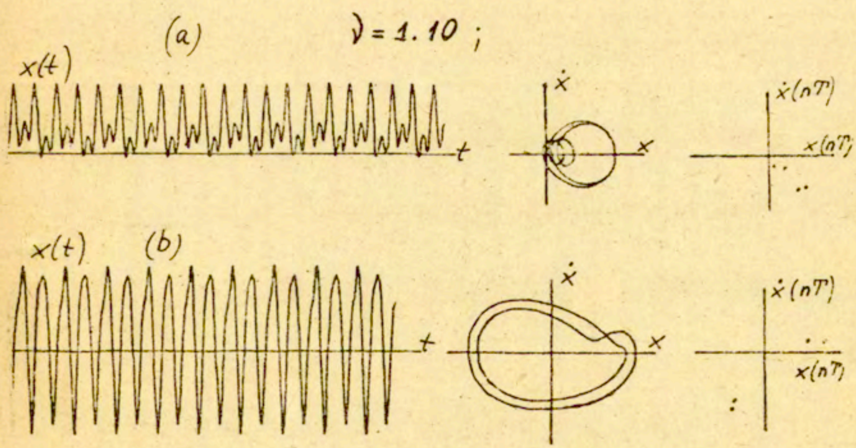


Fig.5. Time histories, phase portraits and Poincare maps of 4T-periodic solutions at ν=1.10: (a) – motion branching from T-periodic solution ; (b) isolated solution.

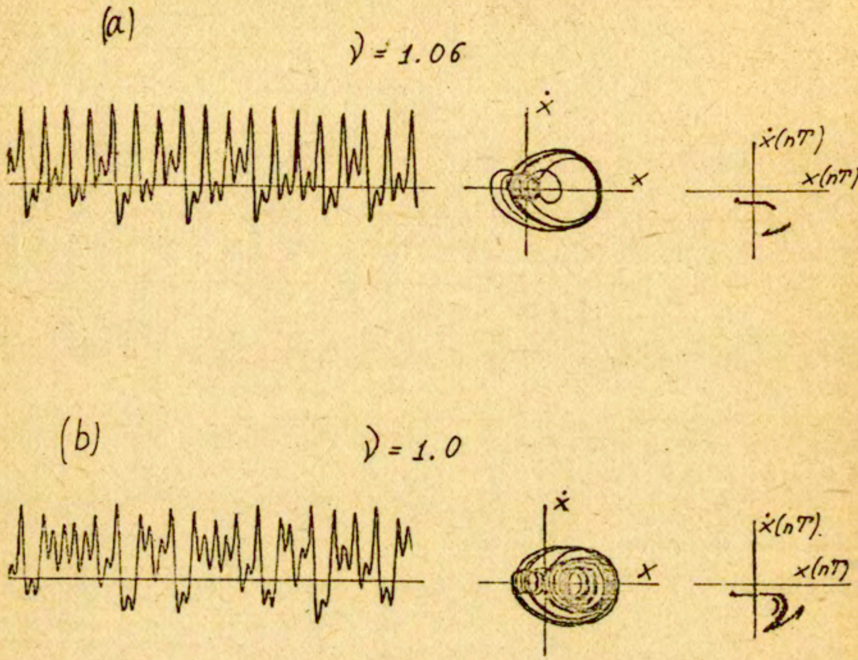


Fig.6. Phase portraits, time histories and Poncare maps in the region of 2^nT motion.

Typical pictures of the motion in the region are shown in Fig.6 a,b. It is worth noticing that even in the frequency region the three harmonic components assumed in solution (14a) - constant term fundamental and 1/2 harmonic are the dominating ones. The frequency zone

$$0.92 < \nu < 0.958 ,$$

has been denoted in Fig.2 as the zone of chaotic behaviour, although some windows of regular motion can be also observed here. The characteristic picture of the chaotic motion is presented in Fig.7 and 8. First in Fig.7 the time history of the forcing term the response $x(t)$ and the filtred one $\tilde{x}(t)$ are recorded. The filtred component of the chaotic motion is that which is intended to describe the time history of the continuous segment of the averaged power spectrum (plus constant term). This is in order to illustrate a hypothesis that the continuous segment corresponds to random-like fluctuation of the amplitude and sligh-tly of the frequency of the 1/2 ν harmonic component (see [11] for details). Then in Fig.8 the phase portrait and the characteristic picture of the strange attractor is recorded. The full orbit drawn in Fig.8b is that which describes the resonant harmonic motion coexisting with the chaotic motion (point ① in the resonant branch of $C_1(\nu)$ in Fig.1).

The zone of the chaotic motion ends very sharply at $\nu \approx 0.92$ and the response, after some transients, jumps into the resonant harmonic motion - point ② in Fig.2.

The zone of the chaotic motion on the frequency axis is a narrow one and lies close to the point of vertical tangent of the resonance curves of the 1/2 subharmonic solution. This observation fits the interpretation of the chaotic motion as fluctuation of the 1/2 harmonic component - the component which is due to decay. It follows that one may view the chaotic motion as a certain early stage of instability of the subharmonic solution, when the amplitude $A_{1/2}$ is no more constant, but does not decay yet and oscillates randomly. The hypothesis appears to be useful in an attempt of interpretation of a sharp transition to chaos i.e. the transition not

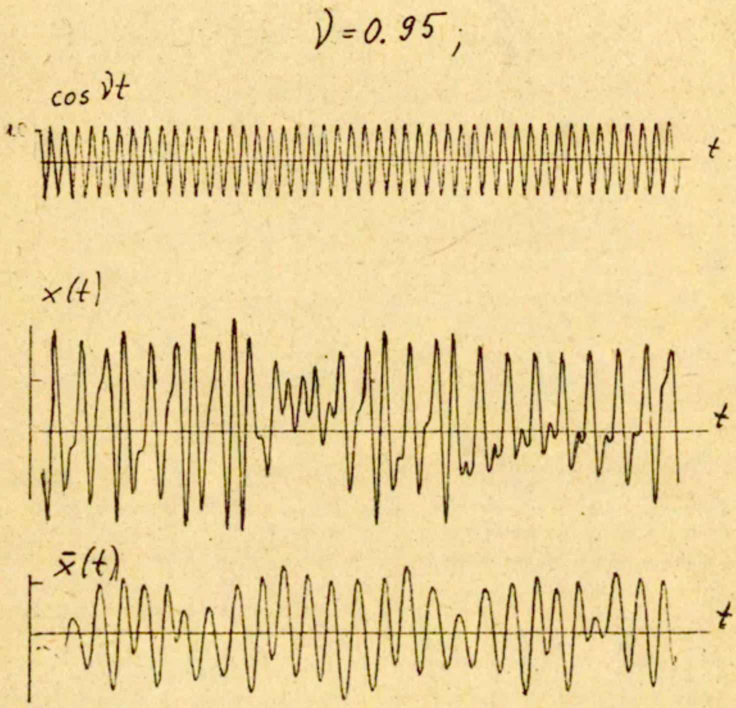


Fig.7. Forcing function, time history of chaotic motion and of the filtered component $\bar{x}(t)$ at $\nu = 0.95$;

$\nu = 0.95$

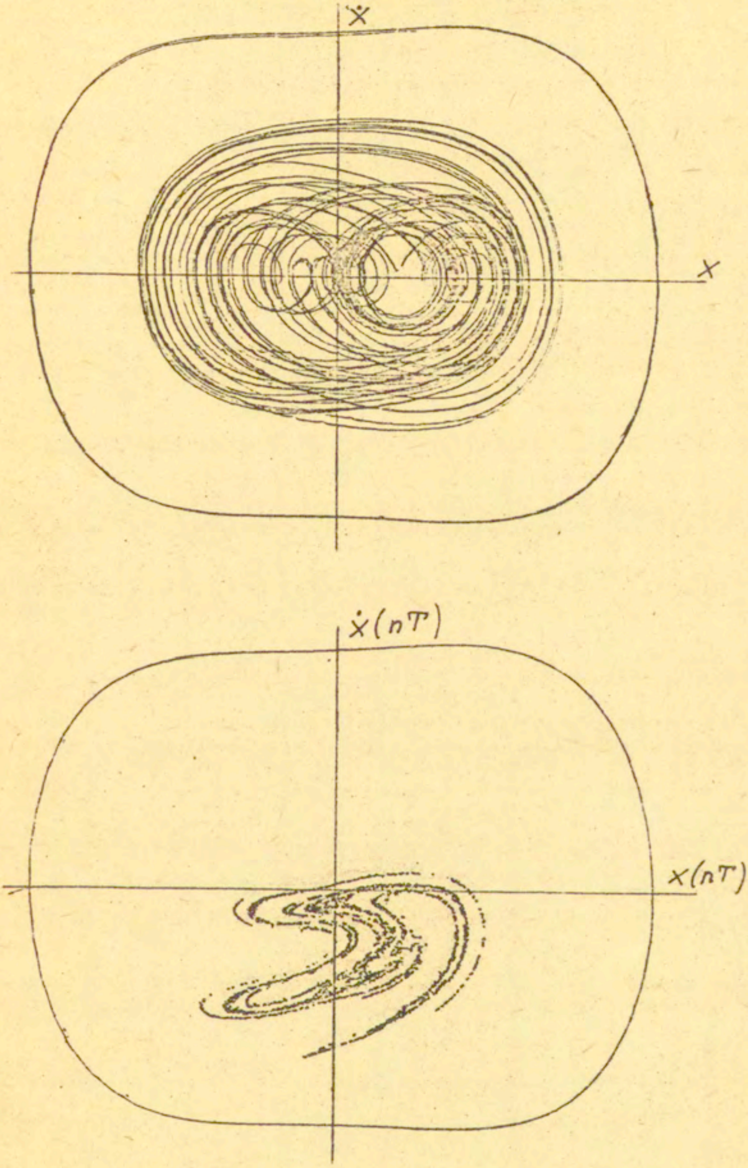


Fig.8. Phase portrait and Poincare map -strange attractor- of chaotic motion at $\nu = 0.95$;

proceded by a cascade of period doubling bifurcations.

4. A SHARP TRANSITION TO CHAOS IN SYSTEMS WITH SYMMETRIC ELASTIC NONLINEARITY

Consider the system (12b) at $P_0 = 0$

$$\ddot{x} + h\dot{x} + x^3 = P_1 \cos \nu t, \tag{15a}$$

and a transition from 3T-periodic solution to T-periodic one through a chaotic zone. The phenomenon appears at high values of the forcing parameter P_1 and was observed by Ueda [2,3] at $\nu = 1.0$ and $9 < P_1 < 13$.

Presently an analysis will be carried out at the parameters

$$P_1 = 12.0 \quad ; \quad h = 0.10 \quad ;$$

on varying the frequency ν. It is important to note that contrary to previous case the values $\nu = 1.0$ falls into a region considerably below the principal resonance and hence the chaotic motion is associated with the unique resonant branch of the resonance curve. For an illustration the resonance curve of harmonic solution at $P_1 = 12.0$ is drawn together with the one at $P_1 = 0.16$ considered in sec.3 (see Fig.9).

It is pretty clear that the harmonic solution is not adequate in the region of frequency. Even in the first approximation theory the harmonic solution shows a series of unstable regions and higher harmonic prove to be dominating ones. In higher approximations the subultraharmonic resonances appear (see i.g. [15, 3]).

In the 3T-periodic solution observed at $\nu > 1$ the harmonic component is a dominating one and the response can be approximated by a finite Fourier series as

$$x_0(t) = \sum_{p=1,3,5} A_p \cos(p\nu t + \vartheta_p) + \sum_{p=1,7} A_{p/3} \cos(p/3\, \nu t + \vartheta_{p/3}) = \tag{15b}$$

$$= -x_0\left(t + 3\frac{T}{2}\right);$$

The first sum in equation (15b) involves the T-periodic components, and the second term - the 3T-periodic harmonics. High

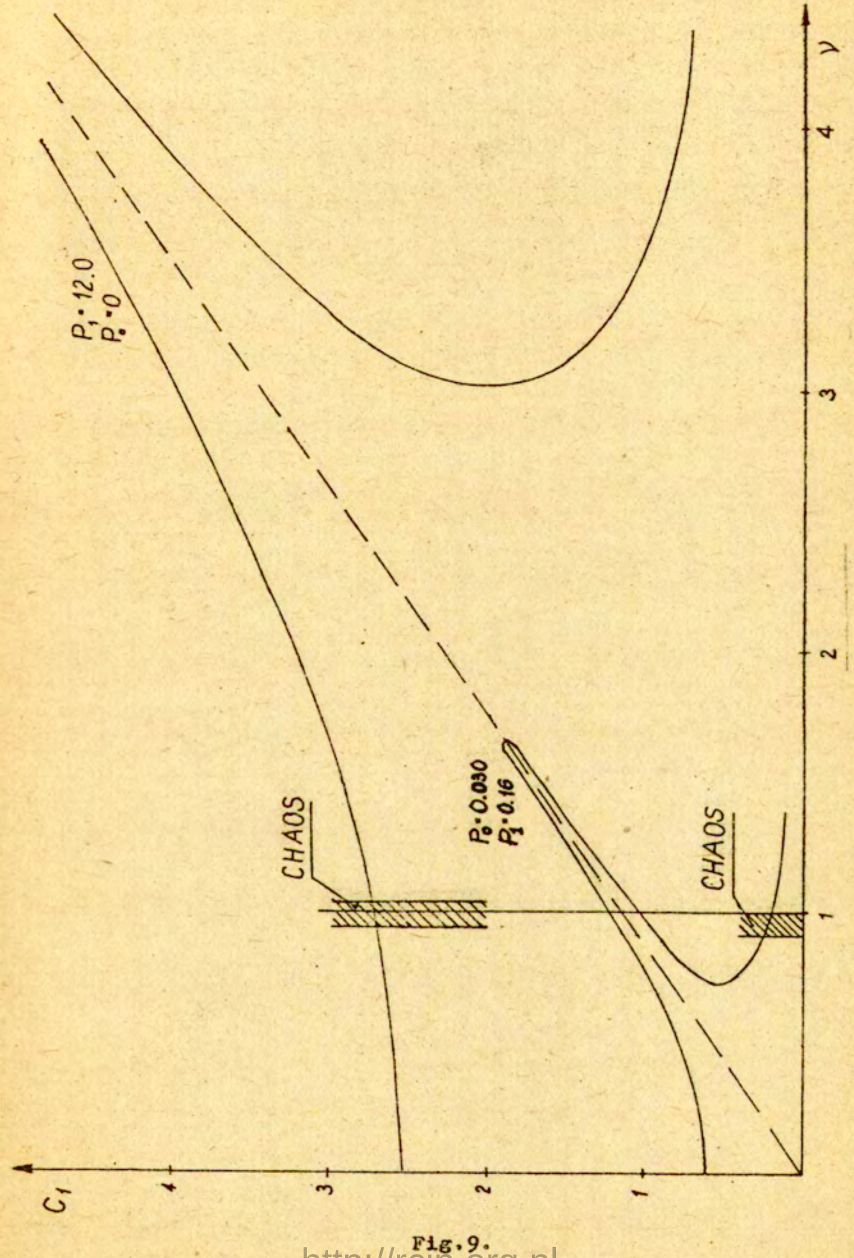


Fig.9.

number of harmonic components makes an analytical approach to an evaluation of resonance curves and stability limits unapplicable. General conclusions about the stability of the solutions presented in sec.2 are, however applicable: this is the case (b) with variational equation (10a) and consequently no period doubling bifurcation of the solution can be expected.

Yet the solution does turn into chaotic motion upon a change of a parameter and then to T-periodic solution approximated by a series

$$x_0(t) = \sum_{p=1,3,5} A_p \cos(p\nu t + \nu_p) = x(t+T) \quad , \quad T = \frac{2\pi}{\nu} ; \qquad (15c)$$

Although values of the amplitudes A_1, A_3, A_5 are certainly different than those in eqs. (15b) an essence of the difference between the two solutions lies in a disappearance of the 3T-periodic components $A_{1/3}$, $A_{7/3}$.

At the transition zone i.e. at the chaotic motion continuous segments of the averaged power spectrum appear around the two 3T-periodic components (see Fig.10). The zone of frequency where the chaotic behaviour was obserwed is marked in Fig.9.

Further computer simulation results are displayed in Fig 11a-e. First at $\nu=1.06$ the regular 3T-periodic motion is recorded. Due to large number of harmonic components the phase portrait looks very complex, however a closed orbit is clearly visible. Then at $\nu=1.04$ we note that the motion is no langer strictly periodic: the phase portrait does not show a closed orbit and the three points on the Poincare map spread into what can be described as short segments of straight lines. On further decrease of the frequency a strange attractor appears rapidly and disappears also sharply terminating into the T-periodic solution (see Fig.11 c,d,e).

It is clearly seen that the chaotic zone is not preceded by period doubling bifurcations. The only -pre-chaotic motion is that shown in Fig. 11b. It is worth noticing that results obtained by Ueda [2] at $\nu=1.0$ on changing the forcing parameter P_1 showed very similar pre-chaotic behaviour.

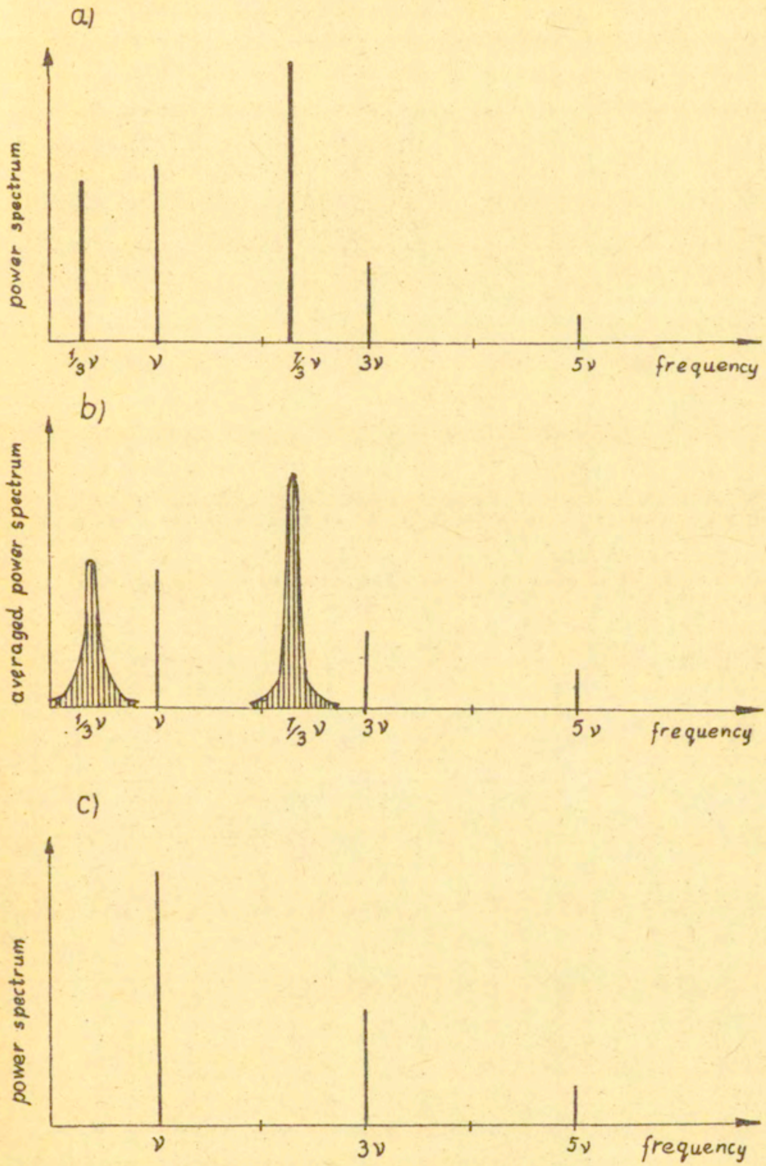


Fig.10. Character of frequency spectrum of 3T-periodic solution, chaotic motion and T-periodic solution.

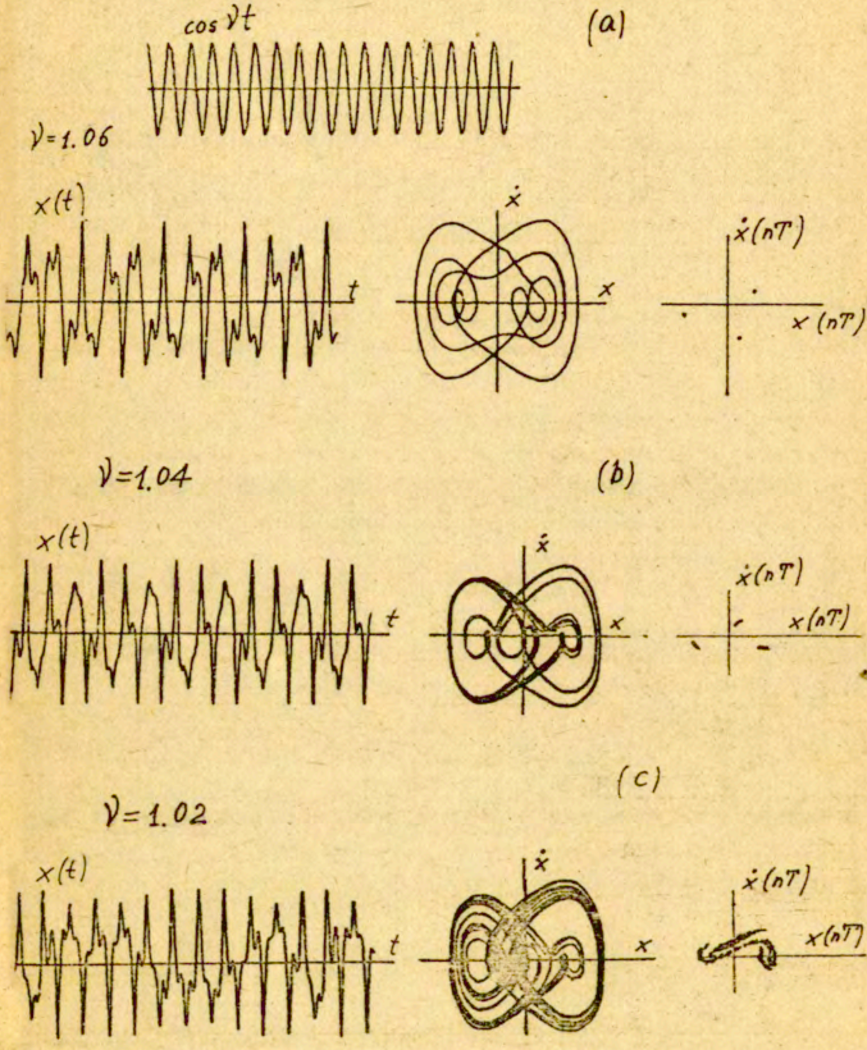
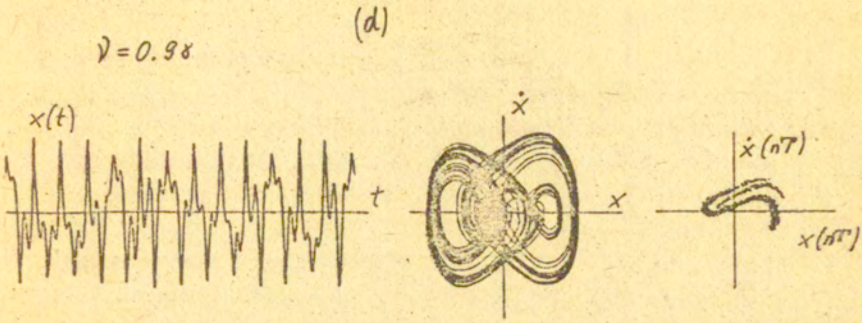


Fig. 11. Time histories, phase portraits and Poincare maps in the sharp transition to chaos: (a) - 3T-periodic solution, (b) pre-chaotic motion; (c) - chaotic motion („strange attractor").



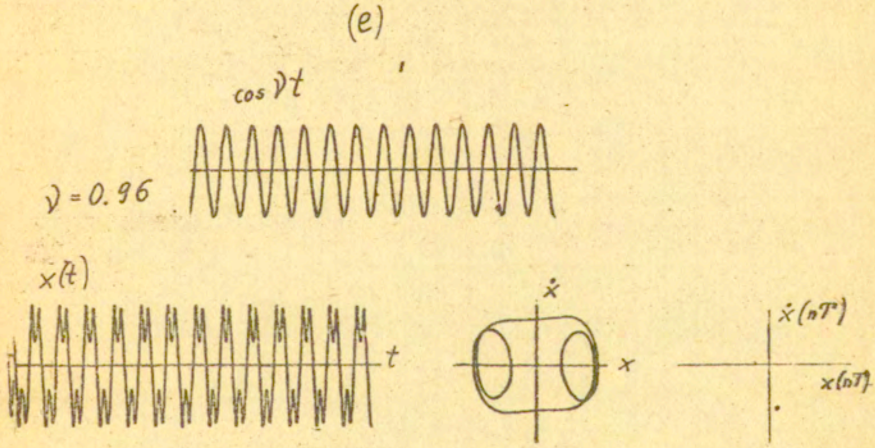


Fig.11. (continiud) (d) - chaotic motion at $\nu=0.98$;
(e) - T-periodic motion at $\nu=0.96$.

What follows is an account of an attempt to propose an approximate model of the pre-chaotic motion, the motion characterised by three short segments of straight line around the three points on the Poincare map associated with the 3T-periodic solution. A point of departure is the hypothesis presented in [11] and outlined in sec.3 of the present paper. It says that the segments of continuous power spectrum surrouding subharmonic components can be interpreted as random-like fluctuation, or random-like transients of th-ese harmonic terms. Since in the 3T-periodic solution (15a) it is the 7/3 harmonic component, which is dominating one, we can assume that in the pre-chaotic motion the aplitude $A_{7/3}$ begins to very slightly with time oscillating with very low frequency. This can be described as

$$\left[A_{7/3} + 2\,\delta A \cos \delta\nu t\right] \cos\left(\tfrac{7}{3}\nu t + \vartheta_{7/3}\right), \tag{16a}$$

or in an equivalent form as

$$A_{7/3}(t) \cos\left(\tfrac{7}{3}\nu t + \vartheta_{7/3}\right) = A_{7/3} \cos\left(\tfrac{7}{3}\nu t + \vartheta_{7/3}\right) + \delta x\,; \tag{16b}$$

$$\delta x = 2\,\delta A \cos \delta\nu t \cos\left(\tfrac{7}{3}\nu t + \vartheta_{7/3}\right) =$$

$$= \delta A \cos\left[\left(\tfrac{7}{3}\nu + \delta\nu\right)t + \vartheta_{7/3}\right] + \delta A \cos\left[\left(\tfrac{7}{3}\nu - \delta\nu\right)t + \vartheta_{7/3}\right];$$

In terms of bifurcation theory one can say that the 3T-periodic solution (15a) turns into a nonperiodic solution which contains additional harmonic components with frequencies very close to $\tfrac{7}{3}\nu$. This can be written as

$$x(t) = x_0(t) + \delta x\,, \tag{16c}$$

where $x_0(t)$ is given by eqs. (15b) and δx is definied by eqs. (16b). The power spectrum of the solution (16c) is drawn in Fig. 12.

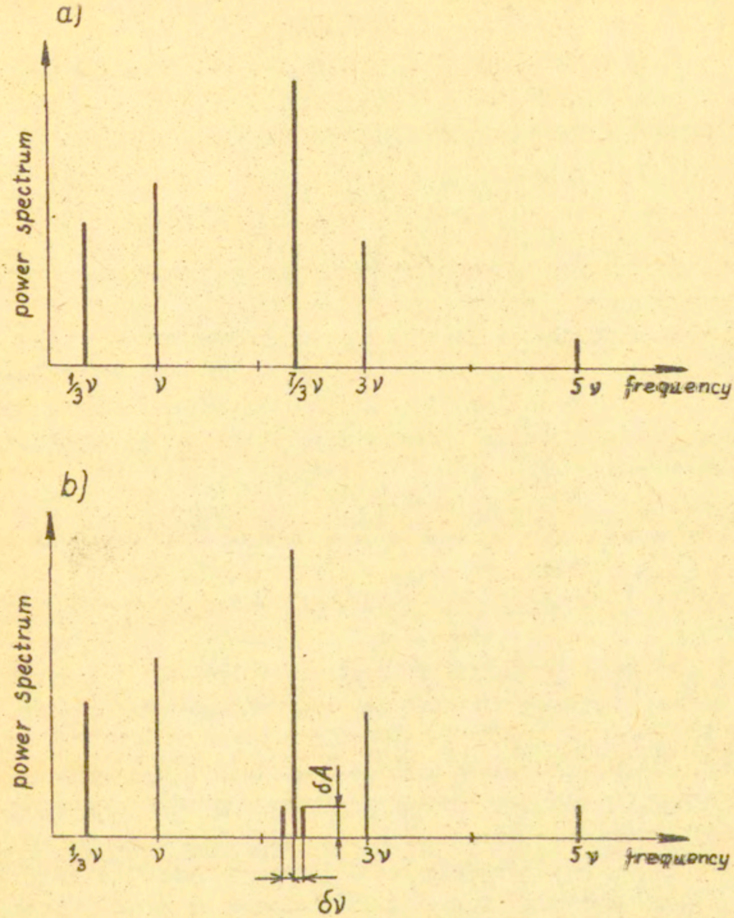


Fig.12. Character of frequency spectrum of 3T-periodic solution and the proposed model of the prechaotic-motion.

To find out what changes are brought to the Poincare map by the small component δx we differentiate it with respect to time and neglect lower order terms with the results

$$\frac{d}{dt}(\delta x) \cong 2\,\delta A\,\tfrac{7}{3}\nu \cos \delta\nu t \,\sin\left(\tfrac{7}{3}\nu t + \vartheta_{7/3}\right); \qquad (16d)$$

By virtue of eqs. (16b) and (16d,c) the following observations can be made:

- if the 3T-periodic solution is mapped on the $x(nT)-\dot{x}(nT)$ plane by the three points with coordinates X_i, Y_i, $i=1,2,3$ (see Fig.13a), then the disturbance δx results, after sufficiently long time interval, in an appearance of large number of points within regions

$$X_i \mp \delta A \;; \qquad Y \mp \delta A\,\tfrac{7}{3}\nu \,, \qquad i=1,2,3.$$

- Becouse the ratio

$$\frac{\delta\dot{x}(nT)}{\delta x(nT)} \cong -\tfrac{7}{3}\nu\, tg\left(\tfrac{7}{3}2\pi n + \vartheta_{7/3}\right), \quad n=1,2,3,4\ldots\ldots;$$

associated with each point $i=1,2,3$ takes a constant value, the set of points due to δx form segments of straight lines (see Fig.13b).

It follows that the pre-chaotic motion can be approximately described by the solution (16b,d). Consequently the hypothesis says that in the sharp transition to chaos the continuous segments on the power spectrum develop from "inside": the first new components are those in a close neighbouhood of the middle values 7/3 ν and then 1/3 ν.

To illustrate the case (c) mentioned in sec.2 i.e the period doubling bifurcation in the oscillator with symmetric elastic nonlinearity the computer simulation was used to detect an unsymmetric periodic solution first. Indeed it was foung that in the region $\nu \approx 1.40$ the system (15a) exhibits T-periodic motion which can be approximated by eqs.(11c) i.e. the motion with a constant term, fundamental and second harmonic components playing a dominating role. Then on changing the frequency parameter and recording the Poincare maps a bifurca-

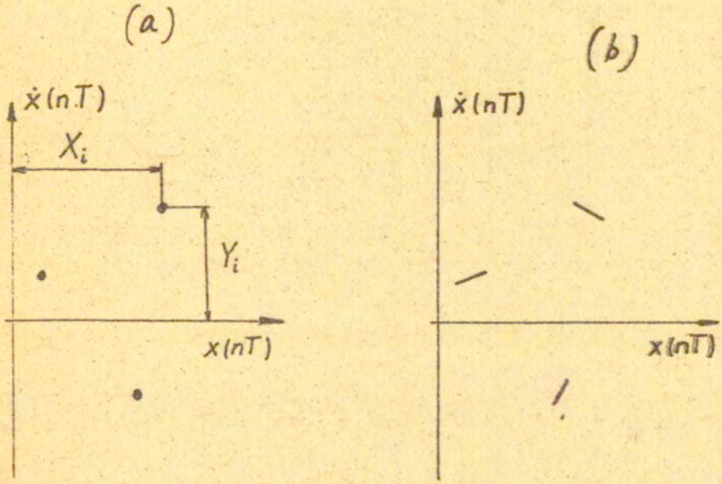


Fig.13. Poicare map of 3T-periodic solution and of the proposed model of pr-chaotic behaviour.

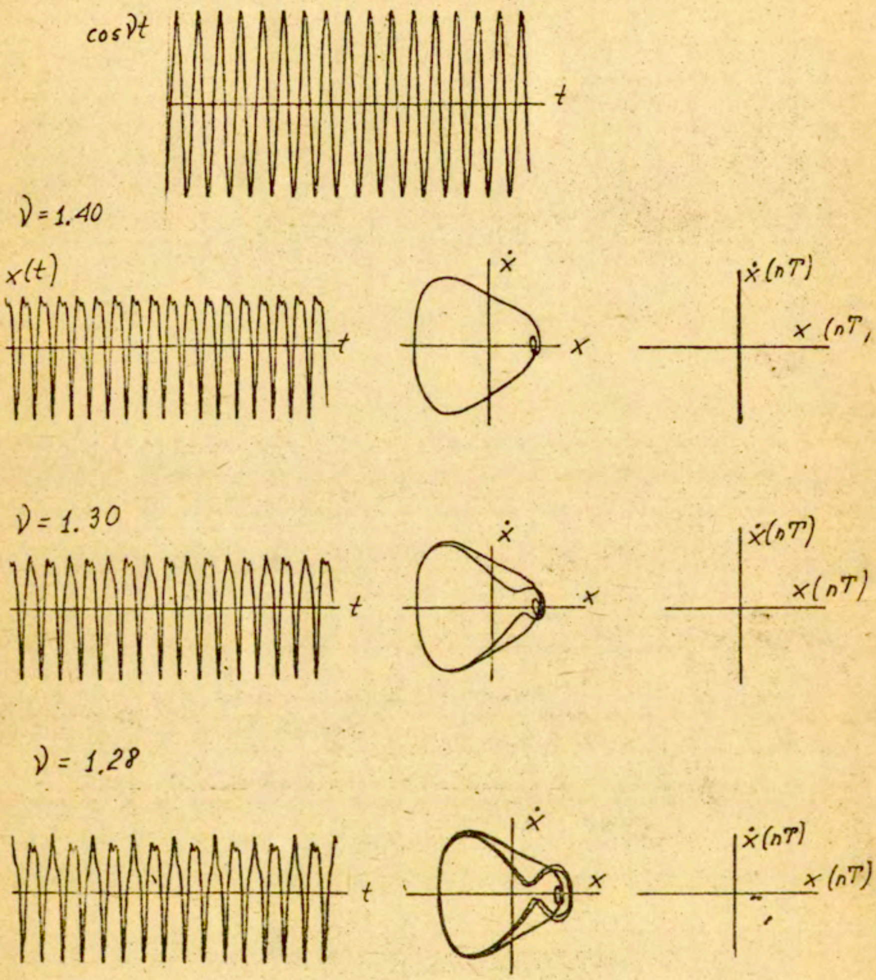


Fig.14. Period doubling bifurcations of unsymmetric solution in symmetric system.

tion into 2T-periodic and then to 4T-periodic solution was observed (see Fig.14 a-c).

5. CONCLUSIONS

Following the previous work [11] nonlinear oscillators with single equilibrium position were considered and a transition to chaotic motion on changing the frequency parameter was studied in connection with classical concepts of resonance curves stability limits of approximate subharmonic solution and estimation of period doubling bifurcation by using variational Hill's type equation.

A combination of computer simulation analysis and the theoretical evaluation of approximate periodic solutions leads to an observation that chaos is a transition zone between qT-subharmonic solution and T-periodic solution. In the theory of nonlinear oscillations the transition is due to occur at points of vertical tangent on the resonance curves of the qT-periodic solution. The presented observations indicate that when a system parameter exceeds a certain critical value the region of qT-periodic motion is separated from the region of T-periodic one by a zone of chaotic behaviour. The character of the averaged power spectrum associated with the chaotic motion gives the appealing idea to interpret the motion as random-like transients of the qT-periodic harmonic components i.e. the components which are losing stability and decay in the region of T-periodic motion.

Two routes of chaos are discussed and illustrated:

- the classical route to chaos via a cascade of period doubling bifurcations. This happens to unsymmetric solutions and hence is associated with systems having unsymmetric nonlinear characteristic. In the route to chaos the continuous segments of the averaged power spectrum are gradually build-up from "outside" i.e. components on the edges of a segment appear first.
- the "sharp" route to chaos is associated with symmetric periodic solutions and hence the system with symmetric nonlinear characteristic. On studying the pre-chaotic behaviour

an approximate mathematical model of the route to chaos is proposed. The hypothesis states that the continuous segments of the averaged power spectrum begin to develop from "inside" i.e. components in a close neighbouhood of the middle frequencies appear first.

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SUMMARY

In two types of nonlinear oscillators with single equilibrium positions the question of transition to chaos is investigated by making use of computer simulation and approximate analytical methods to study periodic solutions and their stability. The zone of chaotic motion is found as a transition zone between qT-periodic /sub- or subultraharmonic/ solution and the T-periodic solution, thus replacing the classical jump phenomenon. Two routes to chaos are studied: the gentle route via a cascade of period doubling bifurcations which appears to be characteristic in unsymmetrical oscillator, and a sharp route, which proves to accompany a stability limit of symmetrical periodic solutions. An approximate mathematical model of pre-chaotic motion in the sharp transition to chaos is proposed.