Solution of the first passage problem by asymptotic sampling

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1. Introduction

The first passage problem in random vibrations is readily written as a high-dimensional reliability problem. The the first passage probability or probability of failure P_F in an n-dimensional space of random variables $X_1, \dots X_n$ can be computed as

$$P_F = \int_{D_F} \cdots \int f_{X_1,\dots X_n}(x_1, \dots x_n) \mathrm{d}x_1 \dots \mathrm{d}x_n$$

In this equation, $f_{X_1,\dots X_n}(x_1, \dots x_n)$ denotes the joint probability function of the random variables $X_1, \dots X_n$ and D_F denotes the failure domain, i.e. the region of the n-dimensional random variable space in which failure occurs. In the context of the first passage problem, this denotes combinations of input variables such that a response variable exceeds a critical threshold value. The generalized safety index (or reliability index) β is defined by

$$\beta = \Phi^{-1}(1 - P_F)$$

Here $\Phi^{-1}(.)$ is the inverse standardized Gaussian distribution function. In [2,3], a novel method called *Asymptotic sampling* is presented which avoids some of the drawbacks associated with high-dimensional reliability analysis. The underlying concept relies on the asymptotic behavior of the failure probability in n-dimensional i.i.d Gaussian space as the standard deviation σ of the variables and hence the failure probability P_F approaches zero (see e.g. [1]). Consider a (possibly highly nonlinear) limit state function $g(\mathbf{X})$ in which $g < 0$ denotes failure. Let σ be the standard deviation of the i.i.d. Gaussian variables $X_k, k = 1 \dots n$. It is attempted to determine the functional dependence of the generalized safety index β on the standard deviation σ or its inverse $f = \frac{1}{\sigma}$ by using an appropriate sampling technique. One major advantage of this approach is its independence of the dimensionality n.

2 Numerical example

This example a single-degree-of-freedom structural model with a non-linear hysteretic restoring force according to the well-known Bouc-Wen model. This structure is subject to an earthquake-type ground excitation. The excitation model used in this example is simply an amplitude-modulated white noise (shot noise). Based on this model, the earthquake excitation $a(t)$ is generated as

$$a(t) = e(t)w(t)$$

in which $w(t)$ is white noise with intensity D_0, i.e. $R_{ww}(\tau) = D_0\delta(\tau)$, $e(t)$ is a modulating function, here chosen as

$$e(t) = 4 \cdot [\exp(-0.25t) - \exp(-0.5t)]$$

In order to apply this approach in digital simulation, the continuous time white noise excitation needs to be discretized. This is achieved by representing the white noise $w(t)$ by a sequence of i.i.d. random variables $W_k, k = 1 \dots m$ assumed to be constant values spaced at time intervals Δt. The number of random variables representing the white noise is chosen as N = 1000. The total time duration is T = 20 s, so that the time interval is $\Delta t = \frac{T}{N}$ = 0.02 s. The structural model is assumed to have one kinematic degree of freedom $x(t)$. In addition, there is an internal plastic displacement variable $z(t)$ describing the plastic behavior of the structure. The structural model has a mass m. The equation for the derivative $\dot{z}$ of the plastic variable depends on the state of the system. For the Bouc-Wen model this is defined by the differential equation

$$\dot{z} = A\dot{x} - \beta\dot{x}|z| - \gamma|\dot{x}|z$$

For the state variable x we have the equations of motion:

$$m\ddot{x} + c\dot{x} + (1-\alpha)kz + \alpha kx = -ma(t)$$

Here c is a viscous damping factor. The numerical values used in this example are k = 1 MN/m, m = 40 t, c = 5 kNs/m, α = 0.603, β = -1.8548, γ = 39.36, A = 5.868. The equations of motion are rewritten in first-order form and then numerically. Carrying out the asymptotic sampling procedure for a displacement threshold of ξ = 0.5 m yields the first passage probabilities as shown in Table 1. For reference, Monte Carlo simulation with one million samples yields the result β_{MC} = 3.75.

Table 1: Asymptotic sampling results for different number M of sample points

M	100	200	500	1000
β	3.35	3.76	3.80	3.70

References

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