Multiscale dynamics and information: some mathematical challenges

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This talk outlines multiscale problems which combine techniques of model reduction and filtering. Multiple time scales occur in models throughout the sciences and engineering, where the rates of change of different variables differ by orders of magnitude. The basis of this work is a collection of limit theories for stochastic processes which model dynamical systems with multiple time scales. When the rates of change of different variables differ by orders of magnitude, efficient data assimilation can be accomplished by constructing nonlinear filtering equations for the coarse-grained signal. We consider the conditional law of the coarse-grained signal given the observations. In particular, we study how scaling interacts with filtering via stochastic averaging. We combine our study of stochastic dimensional reduction and nonlinear filtering to provide a rigorous framework for identifying and simulating filters which are specifically adapted to the complexities of the underlying multi-scale dynamical system.

State estimation of random dynamical systems with noisy observations has been an important problem in many areas of science and engineering. Since the true state is usually hidden and evolves according to its own dynamics, the objective is to get an optimal estimation of the true state via noisy observations. The theory of filtering attempts to give a recursive procedure for estimating an evolving signal or state from a noisy observation process. When a system has several different scales, one can seek reduced order models whose essential dynamics describe the evolution of the full system with small number of the state variables. In systems subject to both bifurcations and noise (which form one of the main components of this talk), various *singular perturbations* problems must be understood. To this end, reduced models often provide qualitatively accurate and computationally feasible descriptions. The lower-dimensional model is strictly valid only in the limit of infinitesimally small noise. Nonetheless, the stochastically averaged model should provide qualitatively correct results and be potentially helpful in developing inexpensive lower-dimensional computational models.

The first objective of this talk is concerned with certain methods of *dimensional reduction* of nonlinear systems with symmetries and small noise [1]. In the presence of a separation of scales, where the noise is asymptotically small, one exploits symmetries to use recent mathematical results concerning *stochastic averaging* to find an appropriate lower-dimensional description of the system. The unique features of the problem are interactions between bifurcations, resonances, dissipation and random perturbations. Bifurcations are where small changes in a system result in large changes in the structure of the fast orbits [2]. The subtleties of the interaction between these effects will lead to new and novel analytical techniques. Hence, we are developing techniques of stochastic dimensional reduction to find a simpler model which predicts or captures relevant dynamics of the system [3]. One of the preeminent modern frameworks for considering convergence of the laws of Markov processes is that of the *martingale problem*, which we will use in deriving the coarse-grained dynamics [4].

The second objective of this talk is to develop, with mathematical rigor, a lower - dimensional nonlinear filter by combining two ingredients, namely, stochastic dimensional reduction discussed above and nonlinear filtering. We find a reduced nonlinear filtering problem when the

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system dimension can be reduced via homogenization. We approximate the complex original nonlinear filtering equation by simpler ones with a quantifiable error. This is an extension of our previous work [5,6] to a more realistic setting where the observation depends on both slow and fast variables.

In this talk we derive a low-dimensional filtering equation, that determines conditional law of a plant, in a multi-scale environment given the observations. This talk is less concerned with specific applications and more focused on some of the theoretical aspects that deal with reduced dimensional nonlinear filters. In particular, we showed the efficient utilisation of the low-dimensional models of the signal to develop a low-dimensional filtering equation. We achieved this through the framework of homogenisation theory which enables us to average out the effects of the fast variables. Reduced models can be used in place of the original complex models, either for simulation and prediction or real-time control. To this end, reduced models often provide qualitatively accurate and computationally feasible descriptions.

In conclusion, we are interested in something of a "nongeneric" system which is not amenable to direct probabilistic asymptotic analysis, but which has been overlooked in the homogenisation literature. Another aspect of note is that our interest is specifically dimensional reduction of the plant, not homogenisation. Our analysis will hinge upon an application of the tools of stochastic averaging to a study of the Zakai equation.

The authors would like to acknowledge the support of the AFOSR under grant number FA9550-08-1-0206 and the National Science Foundation under grant numbers CNS 05-40237 and CMMI 07-58569. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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