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ON MOTION OF THE ROTOR IN FLEXIBLE NONLINEAR BEARINGS

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1. Introduction. Motion equations of the system.

The high speed rotors represent main and particularly important part of the machines of the rotor type, which have most various applications in technology. When the machines of this type operate at high speeds considerable centrifugal forces arise which can produce detrimental and frequently even dangerous vibration of the whole machine system. Besides the forced vibration discussed above, rotors can be subjected the other ones, caused by internal friction, hydrodynamic friction in bearings, aerodynamic friction of medium, and in case of a centrifuge caused by motion of liquid having free surface at the tank. Those vibrations are of the self-excited character and the frequenties of them are near to frequency of the rotor free vibration. To prevent vibration effectively it is necessary to solve a lot of problems associated with the dynamics of rotors. Because of complexity of these problems many simplifying assumptions are introduced. For instance, instead of the actual physical system, a certain mechanical model is analyzed. Research, for which a great number of publications is devoted followed two main directions : 1- the rotating system is simulated in the form of an elastic weightless shaft with a disc attached to it /see for instance, publications by Dimentberg, Grobov, Bolotin, Kushul [1 - 5] / ; 2- the system is simulated in the form of a rigid shaft rotating in two flexible bearings

/see for instance, publications by Kelson [6 - 8]/.

In the present work the subject of investigation is a rigid unbalanced vertical rotor supported elastically. The rotor carries a tank partly filled by liquid./Fig.1/. The mass M represents the mass of complete rotating system, without taking into account the liquid mass. The mass of unbalance is represented by $\bar{m}$, where $\bar{m} \ll M$. We assume that the distribution of masses in the rotating system is symmetric, so that two principal moments of inertia I are equal each other. Action of the mass $\bar{m}$ is equivalent to the action of an external centrifugal force. The rotor is supported in two flexible bearings. The upper bearing enables lateral displacements, whereas the lower bearing allows to perform the spherical motion. It is assumed, that the resultants of elasticity forces F_B^* , F_H^* act in the plane ξOZ /the ξ - axis is motionless, whereas the Z - axis as the symmetry axis of the rotor is rigidly bound to it/. Moreover, these resultants are assumed to be nonlinear, in general, functions of the spring deformations

$$F_B^* = w\, F_B(w) \quad , \qquad F_H^* = \zeta\, F_H(\zeta) \quad ,$$

where w , ζ are deformations of upper and lower spring, respectively. The assumptions concerning F_B and F_H will be given below.

We assume further, that the system is subjected to the action of damping moment $C_k \Omega$, depending linearly on angular velocity Ω of the OZ - axis, [9] .

In the case, when the centrifuge tank is partly filled by liquid with mass M_1 , moving liquid causes the force proportional to the rotor deflection velocity in the moving coordinates system. Besides of, the rotor deflections cause the different width of liquid layer in the tank walls and the different pressure on the opposite tank walls [10] , Fig. 2.

Under the assumption that the angles x , y of deflection of the rotor Z - axis from vertical position in two mutually perpendicular planes are small, the following equatios of mo-

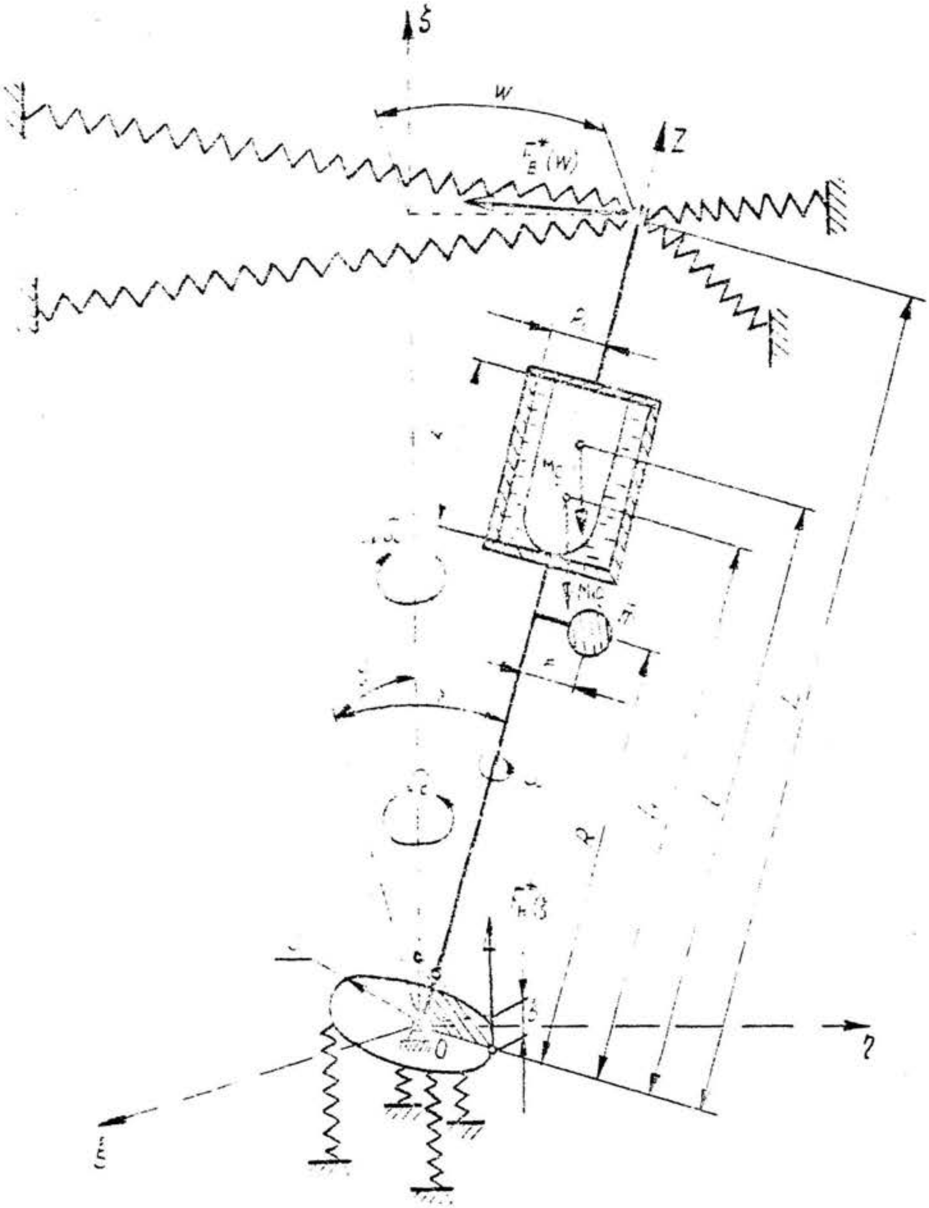

Fig. 1

tion of the rotating system have been derived

$$\ddot{x} + (a + \varkappa)\dot{x} + x\left[F(\varrho) + \eta\omega^2\right] + \tilde{\mu}\omega\dot{y} + \varkappa\omega y = -H\omega^2 \sin\omega t,$$

/1/

$$\ddot{y} + (a + \varkappa)\dot{y} + y\left[F(\varrho) + \eta\omega^2\right] - \tilde{\mu}\omega\dot{x} - \varkappa\omega x = H\omega^2 \cos\omega t,$$

where $\tilde{\mu} = \frac{I_Z}{I}$ - ratio of the axial and equatorial moment of inertia;

$H = \frac{\bar{m}\, e\, R}{I}$ - constant coefficient of the exciting force due to the action of unbalanced mass ;

ω - angular velocity of the rotor. This velocity is assumed to be constant. It means that any feedback between the system and the source of energy does not exist. Only steady states are taken into consideration ;

$a = \frac{C_\varkappa}{I}$ - constant damping coefficient of precession vibration of the rotor ;

$\varkappa = \frac{\bar{\delta}_1^3\ \varphi_1\ l_1^2}{12\pi h\, R_1^3\, I}$ - the effect of liquid with free surface coefficient ;

$\bar{\delta}_1$ - the width of liquid layer ;

φ_1 - damping coefficient, dependent from liquid viscosity;

$\eta = \frac{M_1\ l_1^2}{I}$ - the liquid mass centrifugal force moment coefficient ;

$$\varrho = \sqrt{x^2 + y^2}\ ,$$

$$F(\varrho) = \frac{1}{I}\left[L^2\ F_3(w) + s^2\ F_H(\zeta) - M\, g\, l\right]\ ,$$

$$w = L\varrho \qquad \zeta = s\varrho\ ,$$

with assumption that angles x , y are small. The remaining notations are presented in Fig. 1 and Fig. 2.

We assume, that the functions F_3 and F_H are of the class C^1. [...]s guarantees existence and uniqueness of solutions for the [d]ifferential equations /1/ at all the points of the space $\{-\infty < x\ ,\ \dot{x}\ ,\ y\ ,\ \dot{y} < \infty\ ,\ \ 0 \leqslant t < \infty\}$.

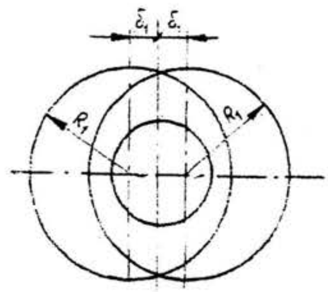

Fig. 2

We shall analyse the equations of motion /1/ for the following type of the function $F(\rho)$, representing a linear combination of elasticity forces and of gravity force. Namely, let the constants $k^* > 0$, $\rho_0 > 0$ be such, that for values of ρ not smaller than ρ_0 the following inequalities are satisfied :

$$/2/ \qquad F(\rho) - \eta\omega^2 > k^* \ , \qquad \int_0^{\rho} s\,F(s)\,ds > \rho^2 \frac{\eta\omega^2}{2} \ .$$

Those assumptions mean that there exist certain rotor deflection, such that elastic force in the bearings is greater than the gravity force of the rotor.

2. Investigation of boundedness of solutions for the equations of motion /1/.

The attempts to find the exact general solution for the system of nonlinear differential equations /1/ in a form which would be convenient for the further physical discussion have failed. In order to answer the fundamental, from the technical point of view, questions we shall apply the method of qualitative analysis.

One of the fundamental problems is the boundedness of motion, that is, the question whether the deflections of rotor from the vertical line will increase infinitely in time, or they will not.

We introduce the auxiliary function $V(x,\dot{x},y,\dot{y})$ with the following properties : for every arbitrary constant Q there exists in the space $(x,\dot{x},y,\dot{y})$ the closed surface defined by the equation

$$V\left(x, \dot{x}, y, \dot{y}\right) = Q$$

and for two arbitrary constants Q_1 , Q_2 such, that $Q_1 > Q_2$, the surface $V = Q_2$ lies inside the domain bounded by the surface $V = Q_1$. We determine the function V for every arbitrary solution $x(t)$, $\dot{x}(t)$, $y(t)$, $\dot{y}(t)$ of the system /1/. If $V\left[x(t),y(t),x(t),y(t)\right]$ is a function decreasing in time t , then the solution $x(t)$, $\dot{x}(t)$, $y(t)$, $\dot{y}(t)$ passes in the space $(x , \dot{x} , y , \dot{y})$ from the surface $V = Q_1$ in the direction of the surface $V = Q_2$ lying inside the previous one. It means that the solution $x(t)$, $\dot{x}(t)$, $y(t)$, $\dot{y}(t)$ of the system /1/ is bounded.

The function of the following form

$$V\left(x,\dot{x},y,\dot{y}\right) = \dot{x}^2 + \dot{y}^2 + 2\int_0^{\sqrt{x^2+y^2}} s\,F(s)\,ds + \varepsilon\left(x\,\dot{x} + y\,\dot{y}\right) + \left(x^2+y^2\right)\left[\frac{(a+\varkappa)\varepsilon}{2} - \eta\,\omega^2\right] \qquad /3/$$

possesses the properties mentioned above. ε is a constant, which is chosen in such a way that the following inequalities should be satisfied :

$$\dot{x}^2 + \varepsilon\,x\,\dot{x} + \frac{(a+\varkappa)\varepsilon}{2}\,x^2 > 0 \quad \text{for} \quad x^2 + \dot{x}^2 > 0\ ,$$

$$\dot{y}^2 + \varepsilon\,y\,\dot{y} + \frac{(a+\varkappa)\varepsilon}{2}\,y^2 > 0 \quad \text{for} \quad y^2 + \dot{y}^2 > 0\ .$$

It take place if

/4/ $$\varepsilon < 2(a + \varkappa) \; .$$

The function V , defined by the formula /3/ is positive definite for all x , $\dot{x}$, y , $\dot{y}$, lying outside the sphere

$$x^2 + \dot{x}^2 + y^2 + \dot{y}^2 = \varrho_0^2 + z_0^2 \; ,$$

where

$$z_0 = \frac{\varrho_0}{2} \left[\varepsilon + \sqrt{\varepsilon^2 + |(a + \varkappa)\varepsilon - 2\tilde{k}|} \right] \; .$$

The number $\tilde{k} > 0$ is defined as follows : for the domain $\varrho < \varrho_0$ there exist a number $\tilde{k} > 0$ such that $F(\varrho) > \eta\omega^2 - \tilde{k}$. If $F(\varrho) > \eta\omega^2$ for all ϱ , then V is positive definite in the whole space $(x , \dot{x} , y , \dot{y})$, provided that /4/ holds true. We take an arbitrary solution $x(t)$, $\dot{x}(t)$, $y(t)$, $\dot{y}(t)$ of the system /1/ and determine $\frac{dV}{dt}$ along this solution :

$$\frac{dV}{dt} = (\varepsilon - 2a - 2\varkappa)(\dot{x}^2 + \dot{y}^2) + \omega(\tilde{\mu}\varepsilon \qquad x y -$$

$$- x \dot{y}) - \varepsilon \left[F(\varrho) - \eta\omega^2\right](x^2 + y^2) - 2H\omega^2(\dot{x} \sin\omega t -$$

$$- \dot{y} \cos\omega t) - H\omega^2\varepsilon(x \sin\omega t - y \cos\omega t) \; .$$

We shall determine a region inside which the function $\frac{dV}{dt}$ will be negative definite.

Taking into account /2/ for $\varrho \geqslant \varrho_0$ and arbitrary $\dot{x}$, $\dot{y}$, and also for $\varrho < \varrho_0$ and

$$\sqrt{\dot{x}^2 + \dot{y}^2} > z_1 \; ,$$

where

$$z_1 = \varrho_0 \sqrt{2} \, \frac{\omega|\tilde{\mu}\varepsilon - 2\varkappa| + \sqrt{\omega^2(\tilde{\mu}\varepsilon - 2\varkappa)^2 + 2\varepsilon\tilde{k}(2a - 2\varkappa - \varepsilon)}}{2(2a + 2\varkappa - \varepsilon)}$$

the function $\frac{dV}{dt}$ can be estimated as follows :

$$\frac{dV}{dt} \leqslant (\varepsilon - 2a - 2\varkappa)(\dot{x}^2 + \dot{y}^2) + \omega(\tilde{\mu}\varepsilon - 2\varkappa)(\dot{x}y -$$

/5/ $$- x\dot{y}) + 2\sqrt{2}H\omega^2\sqrt{\dot{x}^2 + \dot{y}^2} + H\omega^2\varepsilon\sqrt{2}\sqrt{x^2 + y^2} -$$

$$- \varepsilon(x^2 + y^2)k^* .$$

We choose such constant number $\delta > 0$ that for all x, $\dot{x}$, y, $\dot{y}$ not equal simultaneously to zero, the following inequalities are satisfied :

$$(\varepsilon - 2a - 2\varkappa)\dot{x}^2 + \omega(\tilde{\mu}\varepsilon - 2\varkappa)\dot{x}y - \varepsilon y^2k^* <$$

$$< - \delta(\dot{x}^2 + \alpha y^2) ,$$

/6/

$$(\varepsilon - 2a - 2\varkappa)\dot{y}^2 - \omega(\tilde{\mu}\varepsilon - 2\varkappa)x\dot{y} - \varepsilon x^2k^* <$$

$$< - \delta(y^2 + \alpha x^2) ,$$

where α is a certain constant.

The inequalities /6/ are satisfied when

$$\delta < \frac{1}{2\alpha}\Big\{\alpha(2a + 2\varkappa - \varepsilon) + \varepsilon k^* -$$

/7/

$$- \sqrt{\left[\alpha(2a - 2\varkappa - \varepsilon) - \varepsilon k^*\right]^2 + \alpha\omega^2(\tilde{\mu}\varepsilon - 2\varkappa)^2}\Big\} .$$

To assure positivity of δ , the following inequalities must be satisfied :

/8/ $$\alpha(2\varkappa + 2a - \varepsilon) + \varepsilon k^* > 0 ,$$

$$/9/ \qquad 4\,\varepsilon\, k^{*}\left(2\varkappa + 2a - \varepsilon\right) - \omega^{2}\left(\tilde{\mu}\varepsilon - 2\varkappa\right)^{2} > 0\,.$$

The inequality /8/ is satisfied in virtue of /4/. From /9/ we obtain the following condition :

$$/10/ \qquad \varepsilon_{1} < \varepsilon < \varepsilon_{2}\ ,$$

where

$$\varepsilon_{1,2} = \frac{2}{\tilde{\mu}^{2}\omega^{2} + 4k^{*}}\Big[\,2k^{*}\left(\varkappa + a\right) + \omega^{2}\varkappa\tilde{\mu} \pm$$

$$\pm\, 2\sqrt{k^{*}\left[(a + \varkappa)^{2} k^{*} + (a + \varkappa)\tilde{\mu}\,\omega^{2}\varkappa - \omega^{2}\varkappa^{2}\right]}\,\Big]$$

for which, the following inequality must be satisfied :

$$/11/ \qquad (a + \varkappa)^{2} k^{*} + (a + \varkappa)\tilde{\mu}\,\varkappa\,\omega^{2} - \omega^{2}\varkappa^{2} > 0\ .$$

The condition /11/ is identical to the condition of asymptotical stability of solutions of the linearized system /1/ /i.e., for the case $F(\varrho) - \eta\omega^{2} \equiv k^{*} > 0\,,\ H \equiv 0$ / . The solutions of the linearized system have the following form :

$$x = \sum_{i=1}^{4} A_{i}\, e^{\gamma_{i} t}\ , \qquad y = \sum_{i=1}^{4} B_{i}\, e^{\gamma_{i} t}\ ,$$

where

$$\gamma_{1,2,3,4} = -\frac{a + \varkappa}{2} \pm \sqrt{W - G} \pm j\left(\frac{\tilde{\mu}\,\omega}{2} + \sqrt{W + G}\right),$$

$$W = \frac{1}{8}\sqrt{\left[4k^{*} - (a + \varkappa)^{2}\right]^{2} + \tilde{\mu}^{2}\omega^{2}\left[8k^{*} + 2(a + \varkappa)^{2} + \tilde{\mu}^{2}\omega^{2}\right] + 16\omega^{2}\varkappa\left[\varkappa - (a + \varkappa)\tilde{\mu}\right]}\ , \quad j = \sqrt{-1}\,,$$

$$G = \frac{4k^* + \tilde{\mu}^2\omega^2 - (a+\varkappa)^2}{8}$$

From the condition

$$-\frac{a+\varkappa}{2} \pm \sqrt{W - G} < 0$$

we obtain at once the inequality /11/. Thus, in the case of linearized system /1/, the solutions will be bounded and stable provided that the condition /11/ is satisfied. It results from the linear approximation that existence of the liquid with free surface in the tank of rotor $/\varkappa \neq 0/$ may cause the instability of vertical position of the device, i.e. unbounded time increase of solutions of the motion equations. Since the assumptions on nonlinear function $F(\rho)$ admits the form $F(\rho) =$ $=$ const , the condition /11/ should be satisfied to assure the boundedness of solutions for the system of nonlinear equations /1/.

If $\tilde{\mu} \geqslant 1 - \frac{k^*}{\omega^2}$ /region of subcritical speeds, high gyroscopic coupling/, then the condition /11/ is always satisfied. If $\tilde{\mu} < 1 - \frac{k^*}{\omega^2}$ /overcritical region/ then the condition /11/ is satisfied only for a sufficiently strong damping

$$/12/ \qquad a > \varkappa \frac{\omega\sqrt{\tilde{\mu}^2\omega^2 + 4k^*} - \tilde{\mu}\omega^2 - 2k^*}{2k^*} .$$

When /12/ is satisfied, the fulfilling of condition /4/ is guaranteed by /10/, because $\varepsilon_2 \leqslant 2(a+\varkappa)$, what is easy to show.

It results from the above considerations, that the constant δ should be chosen according to the expression /7/ and the constant ε according to /10/. Then the inequality /5/ can be presented in virtue of /6/ as

$$/13/ \quad \frac{dV}{dt} \leqslant -\rho\left(\alpha\delta\rho - \sqrt{2}\,H\omega^2\varepsilon\right) - z\left(\delta z - 2\sqrt{2}\,H\omega^2\right) ,$$

where

$$z = \sqrt{\dot{x}^2 + \dot{y}^2} \quad .$$

We transform /13/ in the following way :

$$\frac{dV}{dt} \leqslant -\alpha\delta\left(\rho - \frac{\varepsilon H\omega^2}{\alpha\delta\sqrt{2}}\right)^2 - \delta\left(z - \frac{\sqrt{2}\,H\omega^2}{\delta}\right)^2 + \frac{H^2\omega^4\,(\varepsilon^2 + 4\alpha)}{2\alpha\delta} \quad .$$

The function $\frac{dV}{dt}$ will be negative definite for all x , $\dot{x}$, y , $\dot{y}$ outside the region

$$/14/ \quad \alpha\delta\left(\rho - \frac{\varepsilon H\omega^2}{\alpha\delta\sqrt{2}}\right)^2 - \delta\left(z - \frac{\sqrt{2}\,H\omega^2}{\delta}\right)^2 < \frac{H^2\omega^4\,(\varepsilon^2 + 4\alpha)}{2\alpha\delta} \, .$$

Summing up : in the region

$$/15/ \quad \begin{aligned} \rho &> \max\left[\rho_0 \, , \; \frac{H\omega^2\sqrt{\varepsilon^2 + 4\alpha}}{\sqrt{2\alpha\delta}}\right] = \rho^* \, , \\ z &> \max\left[z_0 \, , \; z_1 \, , \; \frac{H\omega^2\sqrt{\varepsilon^2 + 4\alpha}}{\sqrt{2\alpha\delta}}\right] = z^* \, , \end{aligned}$$

/where ε should be chosen from the interval $\langle \varepsilon_1 \, , \; \varepsilon_2 \rangle$, the coefficient δ - according to the inequality /7/, whereas the coefficient α should be chosen so great as possible, what results from /14/ /, the function $V(x,\dot{x},y,\dot{y})$ is positive definite and its derivative $\frac{dV}{dt}$, determined with taking into account /1/ is negative definite. Hence it results, that the region /15/ containts the region of ultimate boundedness of solutions for the system of differential equations /1/. In the course of time all solutions starting inside the region /15/

at $t = 0$ do not outcome from this region, and those starting outside /15/ tend to this region.

In the particular case, when $H \equiv 0$ and $F(\rho) > \eta\omega^2$ for all values of ρ , can prove, using the same arguments, the asymptotic convergence of all solutions of equations /1/ and the asymptotic stability of the trivial solution $x = \dot{x} = y = \dot{y} = 0$.

3. Periodic solutions of differential equations of motion.

The system of differential equations of motion /1/ possesses the periodic solution

/16/
$$x = A \sin(\omega t + \gamma) \quad ,$$
$$y = - A \cos(\omega t + \gamma) \quad ,$$

describing the precession vibration of the rotor, if the following algebraic relations are fulfilled :

/17/
$$A^2 \left[\omega^2 (\mu - 1) + F(A)\right]^2 + A^2\omega^2 a^2 = H^2\omega^4 \quad ,$$

$$\gamma = \arctan \frac{a\omega}{\omega^2(1 - \mu) - F(A)} \quad ,$$

where A - constant amplitude of the periodic motion, γ - phase shift with respect to the phase of exciting force, $\mu = \tilde{\mu} - \eta$.

Observe, that periodic solution /16/ existence /i.e. steady rotor precession/ does not depend from the value of the coefficient $\varkappa$ /i.e. the same effect is obtained with empty or fullfilled tank/.

We shall point out, that Eq. /17/ can possesses several solutions A_i , $i = 1, 2, 3, \ldots$, depending on the form of function $F(A)$.

We shall rewrite Eq. /17/ as follows :

$$/18/ \qquad F(A) = \omega^2(1-\mu) \pm \sqrt{\frac{H^2\omega^4}{A^2} - a^2\omega^2}$$

and seek for solution of this equation using the graphical method. The functions of the left-hand side and the right-hand side of Eq. /18/ will be plotted in plane with the assumption that the abscissa is proportional to A^2 /Fig. 3/. As it is seen in the figure, if the function $F(A)$ is monotonic, there exist three, two or one solutions of Eq. /18/ /one solution always exists. because the left-hand side of Eq. /17/ is a continuous function of A and for $A = 0$ it is equal to zero; as $A \to \infty$, the left-hand side tends to infinity, the right-hand side of /18/ is constant positive/.

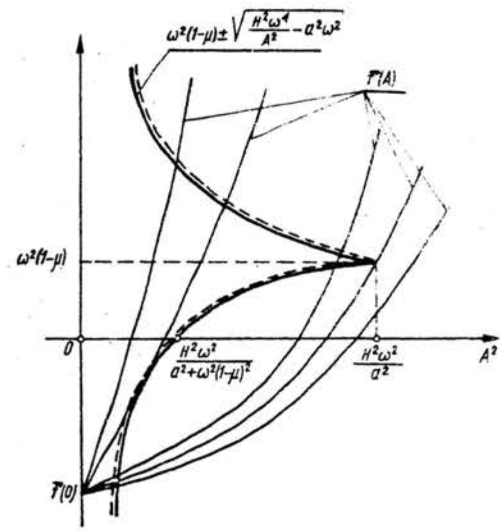

Fig. 3

If the function $F(A)$ has another form, there exist more solutions of Eq. /18/, and even, there can exist continuum set of solutions within certain interval $\langle A_1 , A_2 \rangle$ /Fig. 4 and 5/.

We shall now consider the so called resonanse curve, that is the plot of the amplitude A of periodic solution of /16/

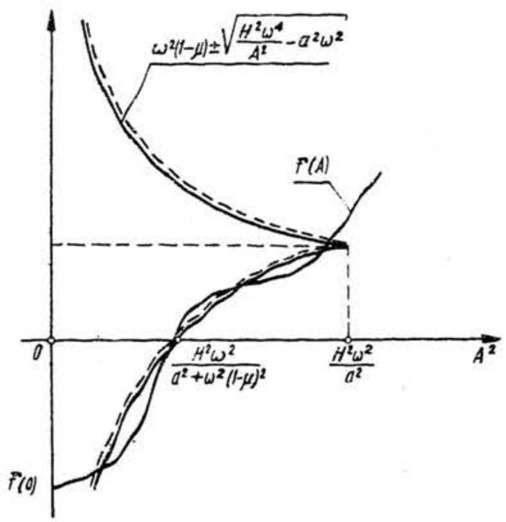

Fig. 4

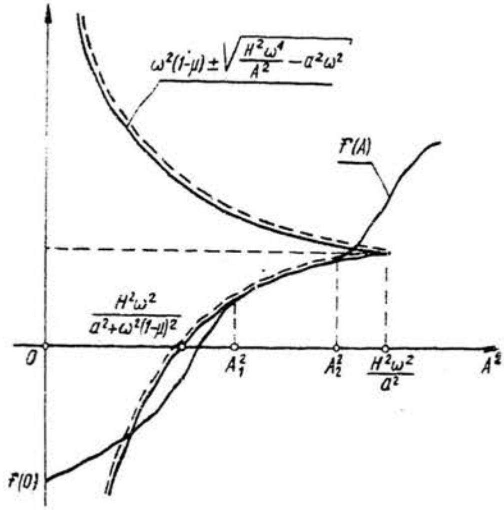

Fig. 5

versus the frequency of exciting force /which equals to the frequency of periodic solution ω /.

The function $A(\omega)$ is represented by Eq./17/. To simplify reasoning we shall consider the second powers of these variables, that is $A^2(\omega^2)$.

It follows from Eq. /17/ that if $A = 0$ then $\omega = 0$; if $\omega = 0$, then either $A = 0$, or

$$F(A) = 0 . \tag{19}$$

If Eq. /19/ possesses n solutions and if $F(0) \neq 0$, then the resonance curve $A^2(\omega^2)$ crosses the A^2 - axis at $n + 1$ points.

We shall calculate the derivative $\frac{d(A^2)}{d(\omega^2)}$. It has the following form :

$$\frac{d(A^2)}{d(\omega^2)} = \frac{2\omega^2\left[H^2 - A^2(\mu - 1)^2\right] - 2A^2 F(A)(\mu - 1) - A^2 a^2}{\left[\omega^2(\mu - 1) + F(A)\right]^2 + A\left[\omega^2(\mu - 1) + F(A)\right]\frac{dF}{dA} + \omega^2 a^2} \tag{20}$$

Hence it is obvious, that

$$\frac{d(A^2)}{d(\omega^2)} = 0 \quad \text{for} \quad \omega^2 = 0\,, \quad A = 0\,,$$

$$\frac{d(A^2)}{d(\omega^2)} = \infty \quad \text{for} \quad \omega^2 = 0\,, \quad F(A) = 0\,, \quad A \neq 0\,, \quad a \neq 0\,.$$

This means that in the origin of the coordinates $(\omega^2\,,\ A^2)$ the resonance curve is tangent to the ω^2 - axis, whereas at the points of intersection with A^2 - axis /if they exist/ it is tangent to this axis.

We shall seek for the points at which $\frac{d(A^2)}{d(\omega^2)} = 0$. Putting the numerator of expression /20/ equal to zero, after some transformations we obtain :

$$/21/ \qquad \omega^2 = \frac{A^2}{2}\,\frac{2\,F(A)\,(1-\mu) - a^2}{A^2(1-\mu)^2 - H^2} \quad .$$

We transform Eq. /17/ into the following form :

$$/22/ \quad \omega^4\left[A^2(\mu-1)^2 - H^2\right] + \omega^2 A^2\left[a^2 + 2\,F(A)(\mu-1)\right] + $$
$$+ A^2\left[F(A)\right]^2 = 0\,.$$

The expression /21/ is substituted into /22/. After transformation we have :

$$\frac{A^2}{A^2(1-\mu)^2 - H^2}\left[4\,H^2\left[F(A)\right]^2 - 4\,A^2\,F(A)\,a^2(1-\mu) + A^2\,a^4\right] = 0.$$

Hence it follows that $\frac{d(A^2)}{d(\omega^2)} = 0$ for $A = 0$, as it was shown previously, and that

$$\frac{d(A^2)}{d(\omega^2)} = 0 \quad \text{for} \quad A^2 \neq \frac{H^2}{(1-\mu)^2} \qquad \text{if}$$

$$/23/ \qquad 4\,H^2\left[F(A)\right]^2 - 4\,a^2\,A^2(1-\mu)\,F(A) + a^4\,A^2 = 0\,.$$

From another point of view the left-hand side of Eq. /22/ is a quadratic form with respect to ω^2. If the values of coefficients in Eq. /22/ and the form of function $F(A)$ are known, then for every value of A the roots of this equation can be calculated. There can exist either two or one real root, or there is no real root at all. Proceeding in such a way and changing the quantity A we can plot the exact resonance curve in the plane (ω^2, A^2).

Thus, we shall determine some general qualitative lines, which are characteristic for the resonance curves defined as "roots" of Eq. /22/.

We determine the discriminant of the expression on the left-hand side of Eq. /22/:

$$/24/ \qquad \tilde{\delta} = A^2 \left[4 H^2 \left[F(A) \right]^2 - 4 a^2 A^2 (1 - \mu) F(A) + a^4 A^2 \right] .$$

Equating each other the expressions /23/ and /24/ we obtain, that $\frac{d(A^2)}{d(\omega^2)} = 0$ if $\tilde{\delta} = 0$. Therefore, the quadratic equation with respect to ω^2, has only one root, namely the expression /21/. If $\tilde{\delta} < 0$, the real roots of Eq. /22/ do not exist.

We transform Eq. /23/ /the left-hand side of this equation is a quadratic form with respect to $F(A)$ /. We find the expression

$$/25/ \qquad \frac{2 H^2}{a^2} F(A) = \mathcal{H}_i(A^2) ,$$

where the notation is introduced

$$\mathcal{H}_i(A^2) = A^2 \left[(1 - \mu) + (-1)^i \sqrt{(1 - \mu)^2 - \frac{H^2}{A^2}} \right] ,$$

$$i = 1, 2 .$$

We shall seek for the solution of Eq. /25/ using a graphical method. These solutions are the points of intersections of the curves described separately by expressions on two different sides of /25/. The curve $\mathcal{H}_i\, A^2$ is one of the branches of the hyperbola and it possesses the following asymptotes : horizontal one

$$\mathcal{H}_{a1} = \frac{H^2}{2(1-\mu)} ,$$

and inclined one :

$$\mathcal{H}_{a2} = 2\, A^2 (1-\mu) - \frac{H^2}{2(1-\mu)} .$$

At the point $A^2 = \frac{H^2}{(1-\mu)^2}$ there is $\frac{d\mathcal{H}_i(A^2)}{d(A^2)} = \infty$.

For $i = 1$ we obtain the lower branch of hyperbola, for $i = 2$ the upper branch /Fig. 5 - provided, that $1-\mu > 0$ /.

In view of /23/ and /24/ we have /Fig. 6/ :

$\tilde{\delta} = 0$ for points lying on the curve $\mathcal{H}_i\, A^2$,

$\tilde{\delta} < 0$ for points lying inside the region bounded by curve $\mathcal{H}_i\, A^2$,

$\tilde{\delta} > 0$ for points lying outside the region bounded by curve $\mathcal{H}_i\, A^2$

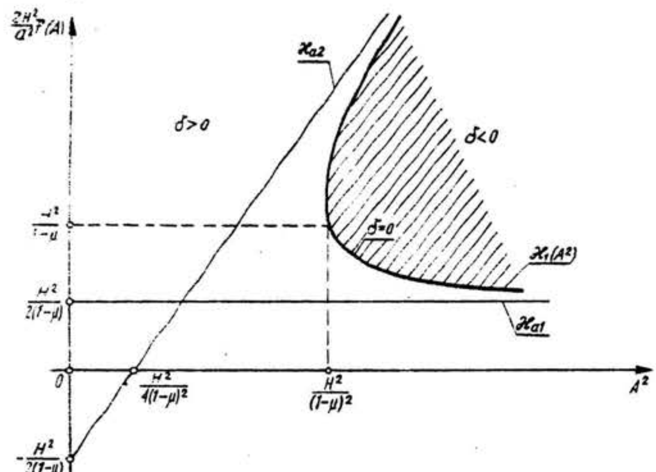

Fig. 6

Next we analyze Eq. /22/. The free term of this expression is positive everywhere. If the coefficients of ω^4 are positive, but the coefficients of ω^2 are negative as well as $\tilde{\delta} > 0$, then Eq. /22/ possesses two solutions $\omega^2(A^2)$ of the form :

$$\omega^2_{1,2} = \frac{A^2\left[2\,F(A)(1-\mu) - a^2\right] \pm \sqrt{\tilde{\delta}}}{2\left[A^2(1-\mu)^2 - H^2\right]} \qquad /26/$$

where $\tilde{\delta}$ is expressed by /24/.

If the coefficient of ω^4 is positive and the coefficient of ω^2 is non-negative, then solutions $\omega^2(A^2)$ of Eq. /22/ do not exist /because it should be $\omega^2 \geqslant 0$ /. If the coefficient of ω^4 is negative, then there exists one solution $\omega^2(A^2)$:

$$\omega^2 = \frac{A^2\left[2\,F(A)\,(1-\mu) - a^2\right] - \sqrt{\tilde{\delta}}}{2\left[A^2(1-\mu)^2 - H^2\right]} .$$

The last case takes place, if $A^2(1-\mu)^2 - H^2 < 0$, that is, if

$$A^2 < \frac{H^2}{(1-\mu)^2} . \qquad /27/$$

This is illustrated in Fig. 7.

In the region in which $A^2 \geqslant \frac{H^2}{(1-\mu)^2}$ a number of solutions depends, as it was shown above, on the sign of the coefficient of ω^2 in /22/. For

$$a^2 + 2\,F(A)(\mu - 1) < 0 \qquad /28/$$

and for $\tilde{\delta} > 0$ there exist two solutions of /26/ if $A^2 > \frac{H^2}{(1-\mu)^2}$, or one solution

$$\omega^2 = \frac{[F(A)]^2}{(1-\mu)\,F(A) - a^2}$$

if $A^2 = \frac{H^2}{(1-\mu)^2}$. For

$$a^2 + 2\,F(A)\,(\mu - 1) \geqslant 0$$

the real solutions of Eq. /22/ do not exist. The inequality /28/ can be presented in the following form :

/29/ $$\frac{2\,H^2}{a^2}\,F(A) > \frac{H^2}{1-\mu}$$

and in the plane $\left(\frac{2\,H^2}{a^2}\,F(A)\ ,\ A^2\right)$ it presents the region in which the solutions of Eq. /22/ can exist /Fig. 8/.

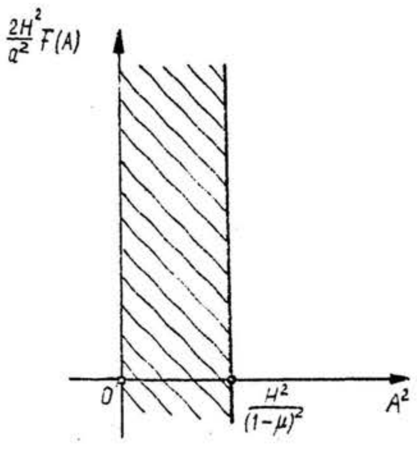

Fig. 7

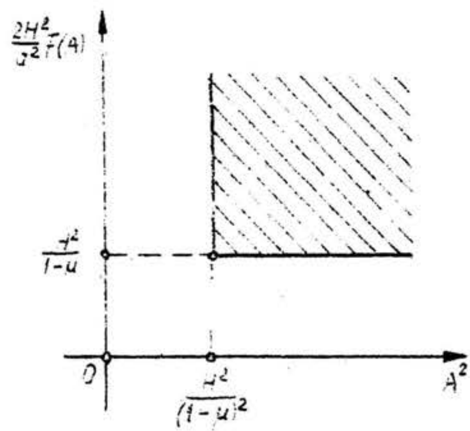

Fig. 8

Now we shall collect together in one plane the regions of existence of solutions for Eq. /22/ resulting from the inequalities /27/ and /29/ as well as from the relation /24/ /Fig. 9/.

Solving graphically Eq. /25/, changing apropriately the argument A into A^2 we are able to draw in Fig. 9 also the plot representing the function $\frac{2\,H^2}{a^2}\,F(A)$. From the analysis carried out above it follows, that for the example of the function plotted in Fig. 9 the resonance curve exists in the intervals $\left\langle 0\, ,\ \frac{H^2}{(1-\mu)^2}\right\rangle$, $\left\langle A_1^2\, ,\ A_2^2\right\rangle$ and in the first interval there exists one solution $\omega^2(A^2)$, whereas in the second interval two solutions $\omega^2(A^2)$.

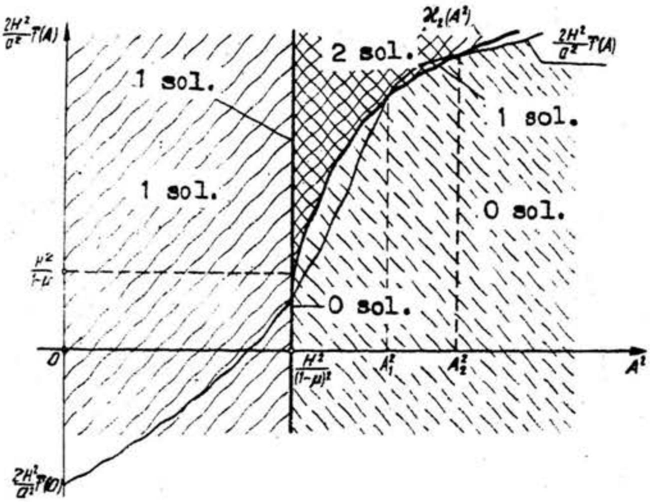

Fig. 9

In order to present the resonance curves in the plane $(\omega^2\, ,\ A^2)$ for various forms of the function $F(A)$ we shall deal, at first, with the so called "skeleton curve".

We assume, that there exist two solutions $\omega^2(A^2)$ described by the expression /26/. We shall consider the "skeleton curve" that is the curve in the plane $(\omega^2\, ,\ A^2)$ with respect to which the branches of the resonance curve corresponding to the solutions /26/ are located at the identical distances at the segments which are paralell to the ω^2 - axis. For this purpose we put together the expressions /26/ /one with the sign plus of $\sqrt{\widetilde{\delta}}$, the second one with the sign minus/. Thus, we decompose the obtained expression into two parts. So the equation of the "skeleton curve" is :

$$/30/ \qquad \omega^2 = \frac{A^2\left[2\,F(A)\,(1-\mu) - a^2\right]}{2\left[A^2\,(1-\mu)^2 - H^2\right]}$$

or after some transformation

$$/31/ \qquad 2\,\omega^2\left[(1-\mu)^2 - \frac{H^2}{A^2}\right] = 2\,F(A)(1-\mu) - a^2 \; .$$

We differentiate all terms of Eq. /31/ with respect to A^2 :

$$/32/ \qquad 2\,\frac{d(\omega^2)}{d(A^2)}\left[(1-\mu)^2 - \frac{H^2}{A^2}\right] + 2\,\omega^2\,\frac{H^2}{A^4} = \frac{d\,F(A)}{d\,A}\,\frac{1-\mu}{A} \; .$$

If $\frac{d\,F(A)}{d\,A} < 0$ /this take place, for instance, for soft spring characteristics/, then the right-hand side of Eq. /32/ is negative provided that $1-\mu > 0$. Consequently, the left-hand side should be negative, too. We consider the interval $A^2 > \frac{H^2}{(1-\mu)^2}$ /since for $A^2 \leqslant \frac{H^2}{(1-\mu)^2}$ there exists only one solution and the notion of "skeleton curve" is meaningless/. In this case, the expression in brackets in /32/ is positive. To ensure the positive sign of the left-hand side of /32/ it should be $\frac{d(\omega^2)}{d(A^2)} < 0$. Hence it follows that if $\frac{d\,F(A)}{d\,A} < 0$, then the "skeleton curve" is the decreasing function of A^2 .

If $\frac{d\,F(A)}{d\,A} > 0$ /hard springs/, then for the great values of A^2 in the expression /30/ the quantity in the numerator can be neglected as compared with $2\,F(A)(1-\mu)$, as well as in the denominator the quantity H^2 in comparison with $A^2(1-\mu)^2$ The approximative equation of the "skeleton curve" has the form :

$$\omega^2 \approx \frac{F(A)}{1-\mu}$$

Hence it follows that for great values of A^2 , if $\frac{d\,F(A)}{d\,A} > 0$

the "skeleton curve" is the increasing function of A^2 .

We shall point out, that one of the solutions of /26/ posesses the vertical asymptote

$$A^2 = \frac{H^2}{(1-\mu)^2}$$

Indeed, if $A^2 \rightarrow \frac{H^2}{(1-\mu)^2}$, the numerator of the greater root of /26/ is reduced to the constant quantity

$$\left[F\left(\frac{H^2}{(1-\mu)^2}\right) - \frac{a^2}{1-\mu}\right]\frac{H^2}{1-\mu} ,$$

the denominator confines itself to zero, and therefore, for one branch of the curve $\omega^2 \rightarrow \infty$.

On the ground of the analysis presented above, we are able for various types of the function $F(A)$ /for instance, for those presented in Fig. 10/ to determine appropriately the resonance curves /Fig. 11/ .

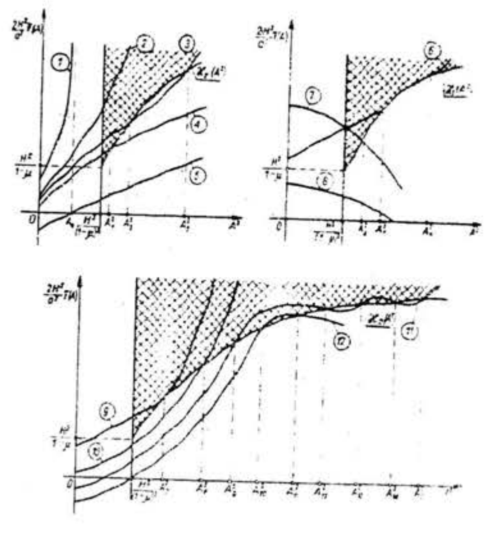

Fig. 10

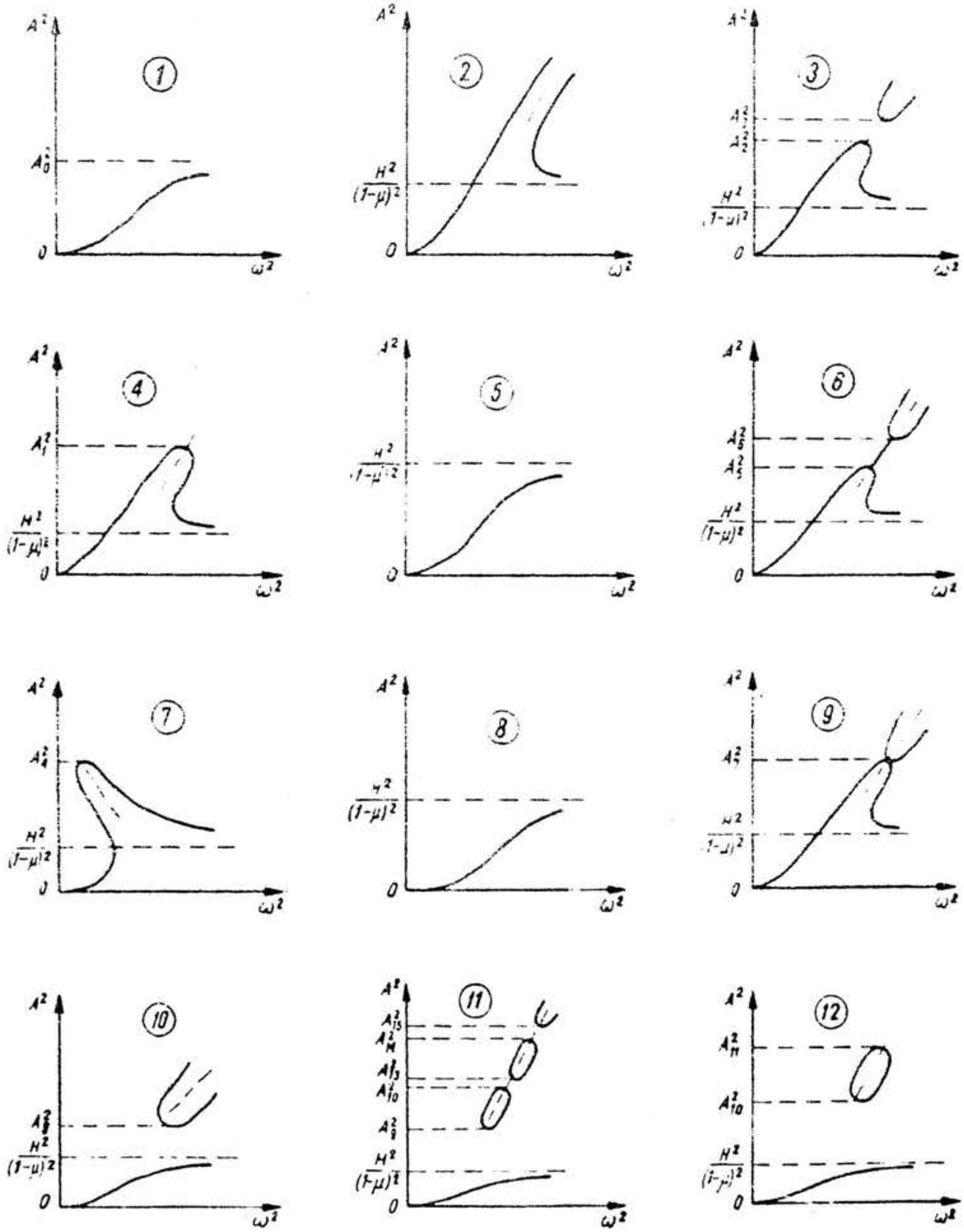

Fig. 11

The analysis can be performed analogously in the case $1 - \mu < 0$. We present here only the final results, that is, the plots of the resonance curves for some forms of the function $F(A)$ /Fig. 12 and 13/.

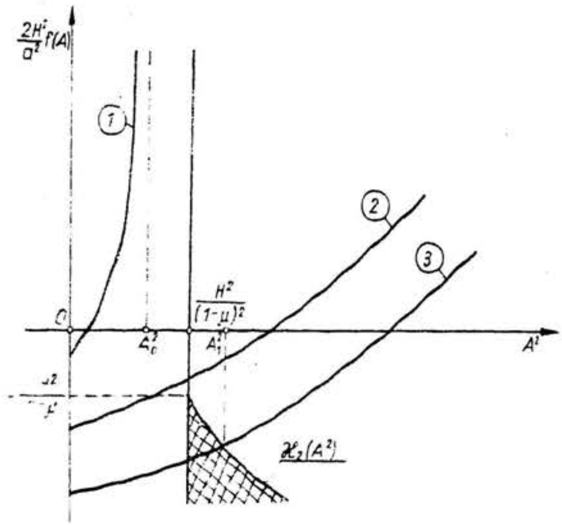

Fig. 12

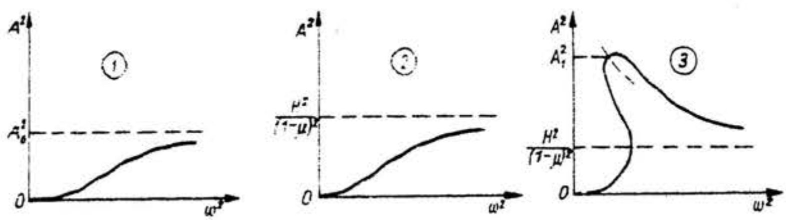

Fig. 13

At the end, in the case $1 - \mu = 0$ the following expression

$$\omega^2 = \frac{A^2 \varepsilon^2 + A\sqrt{A^2 \varepsilon^4 + 4 H^2 \left[F(A)\right]^2}}{H^2}$$

is the unique solution $\omega^2(A^2)$ of Eq. /22/.

If $F(A)$ is a monotonic function, ω^2 is a monotonically increasing function of A^2 and $\omega^2 \to \infty$ as $A^2 \to \infty$ /Fig. 14/.

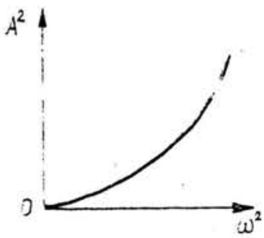

Fig. 14

The analysis given above exhaust fully the problem of harmonical resonance states for the solution of the system /1/. It should be emphasized, that the solutions /16/ are the exact solutions of the system /1/ and therefore, we obtain the resonance curves without any approximations. It is of course, very useful for practical applications to know the qualitative picture of resonance curve which is achieved with aid of simple examination of location of the curve $2H^2 a^{-2} F(A)$ on the diagram of regions of existence of separate forms of solutions for Eq. /22/. If, for instance, the function $F(A)$ /characteristics of elasticity/ is given in the form of a plot /the case frequently encountered in practice/, it is sufficient to compare the plot of this function with the graph of existence of the separate forms of solutions /Fig. 9/ and it is possible to determine directly both the qualitative picture of the resonance curve as vell as the value of greatest amplitude and frequency for which it occurs, /the formula /21/ after substitution $A = A_{max}$/. Further, it is possible to choose the spring characteristics, that is, the function $F(A)$ in such a way, that in the conditions of operation with the velocity ω, the amplitude of forced vibration does not pass above the given quantity. From plots in Fig. 10 and 12 it is seen how the damping coefficients affect the resonance process. Lowering "a" causes as if the compression of the scale of plot of the function $F(A)$, therefore the curve $2H^2 a^{-2} F(A)$ becomes less abrupt. Its point of intersection with the curve $\omega^2(A^2)$ removes itself leftwards. This means, that the greatest amplitude of forced vibration decreases. The changing of the coefficient H gives invers effects.

Let us draw attention to the case $F(\varrho) = \nu^2 \pm c\varrho^2$. Eq. /1/ possesses then one, two or three periodic solutions of the type /16/ with different amplitudes corresponding to the given value of ω /the cases 1, 2, 4, 5, 7, 8, 9 and 10 in Fig. 11/. This result is similar to those conclusions concerning the second order differential equation of the Duffing's type

$$/33/ \qquad \ddot{x} + a\dot{x} + \nu^2 x \pm c x^3 = H \sin \omega t \quad ,$$

which describes the motion of a mechanical or electrical system with one degree of freedom. It is shown, for the above mentioned equation, that the resonance curves may have the like shape as those in Fig. 11. However, it should be emphasied, that the results for Eq. /33/ were obtained by many autors in the approximative way, whereas in the present work, the results obtained for the system /1/ are exact.

If the damping in the system /1/ is small and if the function $F(\varrho)$ is increasing as well as if the principal axial moment of inertia I_z is either less than or equal to the equatorial moment I $/\mu \leqslant 1/$, then the amplitude A of forced vibration can increase infinitely as ω increases /cases 2, 3, 6, 9, 10 and 11 in Fig. 11/. Hence it follows, that if the amplitude of the exciting force is proportional to the second power of frequency ω , for high speeds the stabilizing factor, namely, the coefficient of gyroscopic coupling has the insignificant influence. If a stronger damping exists in the system, then the amplitude increases only to a certain value as ω increases.

The existence of several states of resonance was first observed by M.Z. Kolowski for the case of single equation of the second order :

$$\ddot{x} + a\,\dot{x} + x\,F(x) = H\,\omega^2 \sin \omega t \quad . \tag{34}$$

In the publication [11] the resonance curves were obtained for the approximative solution of Eq. /34/ :

$$x = A \sin(\omega t + \gamma) + B \quad ,$$

for a number of types of the function $F(x)$. The character of resonance curves obtained by Kolowski on ground of the first approximation are not different to those obtained for the system /1/ with the aid of exact method.

4. Investigation of stability of periodic solutions /16/.

In view of existence of several periodic solutions it was necessary to examine their stability. In order to investigate the Liapunov's stability of periodic solutions of the type /16/ for the system /1/ which possesses different amplitudes, we shall write the system of equations under discussion in variations. Next, we shall analyze the stability of its solutions [12] . Applying the transformation

$$x = u + A \sin(\omega t + \gamma) \quad ,$$
$$y = v - A \cos(\omega t + \gamma)$$

we obtain the linearized system of equations in variations [13, 14] :

$$\ddot{u} + (a + \varkappa)\,\dot{u} + \left[F(A) - \eta\omega^2\right]u + A \sin(\omega t + \gamma) \cdot$$
$$\cdot \frac{d\,F(A)}{d\,A}\left[u \sin(\omega t + \gamma) - v \cos(\omega t + \gamma)\right] + \tilde{\mu}\omega\dot{v} +$$
$$+ \varkappa\omega v = 0 \quad , \tag{35}$$

$$\ddot{v} + (a + \chi)\dot{v} + v\left[F(A) - \eta\omega^2\right] - A\cos(\omega t + \gamma)\cdot$$
$$\frac{d\,F(A)}{d\,A}\left[u\sin(\omega t + \gamma) - v\cos(\omega t + \gamma)\right] - \tilde{\mu}\omega\dot{u} -$$
$$- \chi\omega u = 0 . \qquad /35/$$

This is the system of homogeneous linear equatios with periodically variable coefficients. For this system we have found the solution :

$$\begin{aligned} u &= e^{mt}\left[C_1\cos(\omega t + \gamma) - C_2\sin(\omega t + \gamma)\right],\\ v &= e^{mt}\left[C_1\sin(\omega t + \gamma) + C_2\cos(\omega t + \gamma)\right], \end{aligned} \qquad /36/$$

where the constants m , C_1 , C_2 should be determined. Substituting /36/ into /35/, after some transformation we obtain the following characteristic equation to determine m :

$$m^4 + 2m^3(a + \chi) + m^2\Big[\omega^2\left(\tilde{\mu}^2 - 2\tilde{\mu} + 2 - 2\eta\right) + 2F(A) +$$
$$+ (a + \chi)^2 + A\frac{d\,F(A)}{d\,A}\Big] + m\Big\{(a + \chi)\Big[2F(A) +$$
$$+ A\frac{d\,F(A)}{d\,A}\Big] + 2a\omega^2(1 - \eta) + 2\chi\omega^2(\mu - 1)\Big\} + \qquad /37/$$
$$+ \omega^2 a^2 + \left[F(A) + \omega^2(\mu - 1)\right]^2 + A\frac{d\,F(A)}{d\,A}\Big[F(A) +$$
$$+ \omega^2(\mu - 1)\Big] = 0 .$$

Eq. /37/ has four roots m . If they are different, we obtain four independent solutions of the system /35/ in the form of /36/. Their linear combination is the general solution. However, we are interested only in the sign of the real part of the roots m determines exponential decreasing of solutions /36/ and, in consequence, the asymptotic stability of the solutions /16/ for the system /1/.

We shall apply the Hurwitz's criterium [12] . After developing the corresponding determinants we find the following conditions :

$$a + \chi > 0 \quad , \tag{38}$$

$$(a + \chi)\left[F(A) - \eta\omega^2 + (a + \chi)^2 + \frac{A}{2}\frac{d\,F(A)}{d\,A}\right] + a\omega^2(\tilde{\mu} - 1)^2 + \chi\omega^2(\tilde{\mu}^2 - 3\tilde{\mu} + 3) > 0 \quad , \tag{39}$$

$$(a + \chi)^2\left[F(A) - \eta\omega^2 + \frac{A}{2}\frac{d\,F(A)}{d\,A}\right] + (a + \chi)\tilde{\mu}\chi\omega^2 - \omega^2\chi^2 + \frac{(a + \chi)^2\,\frac{A^2}{4}\left(\frac{d\,F(A)}{d\,A}\right)^2}{(a + \chi)^2 + \omega^2(\tilde{\mu} - 2)^2} > 0 \quad , \tag{40}$$

$$\left[F(A) + \omega^2(\mu - 1)\right]^2 + a^2\omega^2 + A\,\frac{d\,F(A)}{d\,A}\left[F(A) + \omega^2(\mu - 1)\right] > 0 \quad , \tag{41}$$

which guarantees, that the real parts of m are negative. The conditions /38/, /39/ and /40/ are fulfilled if the positive dissipation exist in the system $/a > 0/$, and if, for instance, the function $F(A)$ is non-decreasing, as well as it has non-negative derivative for given A , and if the condition /11/ is fulfilled. The inequality /41/ shows directly for which amplitudes A the solutions /16/ are stable or unstable. To verify this we write Eq. /17/ in the following form :

$$\Phi(A^2) \equiv \left\{\left[F(A) + \omega^2(\mu - 1)\right]^2 + \omega^2 a^2\right\}A^2 = H^2\omega^4.$$

The function $\Phi(A^2)$ has the following properties : $\Phi(0) = 0$, $\Phi(A^2) \rightarrow \infty$, as $A^2 \rightarrow \infty$. Since $F(\varrho)$ is a continuous function, $\Phi(A^2)$ is also a continuous function of the argument

A^2. Hence it follows, that there exist such a value $A^2 = \tilde{A}^2$, for which $\Phi(\tilde{A}^2) \equiv H^2\omega^4$. If the function $\phi(A^2)$ is non-monotonic, the greater quantity of points for which $\phi(A^2) \equiv$ $= H^2\omega^4$ can exist /Fig. 15/.

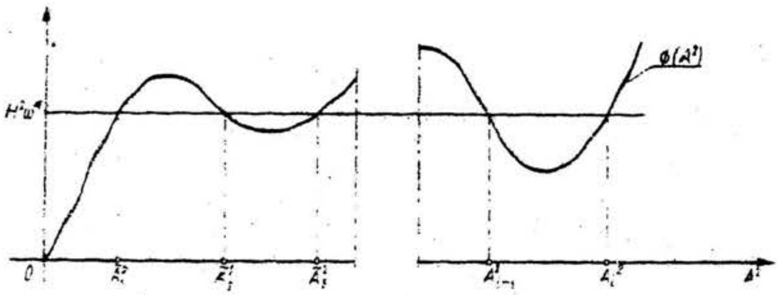

Fig. 15

We shall differentiate the function $\Phi(A^2)$:

/42/
$$\frac{d\,\Phi(A^2)}{d\,A^2} = \left[F(A) + \omega^2(\mu - 1)\right]^2 + \omega^2 s^2 + A\,\frac{d\,F(A)}{d\,A}\left[F(A) + \omega^2(\mu - 1)\right].$$

Comparing the expression /41/ and /42/ we see, that the solutions with amplitudes A_i , $i = 1, 2, 3, \ldots$, for which

$$\left.\frac{d\,\Phi(A^2)}{d\,A^2}\right|_{A = A_i} \leqslant 0$$

/that is if the function $\phi(A^2)$ at the point $A = A_i$ is non-increasing/ are unstable. For instance, in Fig. 15 the solutions with amplitudes $\tilde{A}_2$, $\tilde{A}_{i-1}$ are unstable.

If there exist three periodic solutions of the type /16/ with different amplitudes /this, as it was said above, can take place/, then the solution with mean, as to quantity, amplitude is unstable, whereas the two solutions situated extremely can be stable. This result of exact reasoning confirms the results obtained by other authors using approximative methods [11 , 15] .

5. Conclusions.

In the present work the motion of the mechanical system shown in Fig. 1 has been investigated on the ground of analysis of differential equations /1/. The boundedness of deflections of the rotor axis from the vertical position has been pointed out. It is proved, that if the solutions of homogeneous linearized differential equations system /1/ /i.e., when $F(\varrho) \equiv k^{*} > 0$, $H \equiv 0$ / are asymptotically stable, than the solutions of the nonlinear system are bounded and are coming to the certain definite domain of the ultimate boundedness. In the paper [10] it is showed, that in the linear case, the effect of liquid at the tank may cause the unlimited solutions in the overcritical domain, i.e. unlimited deflection of the centrifuge from the vertical position. One can observe this in practice. The centrifuges with tanks filled with a small quantity of liquid can be subjected the dangerous vibration, with amplitude much greater than in the case empty or fulfilled tank. In the paper it is showed, that in the case of nonlinear elastic characteristics of bearings such effect also can be obtain.

It was demonstrated the existence at least of one periodic solution of the type /16/ with the frequency of the exciting force. The existence of the solution /16/ does not depend from value of the coefficient $\varkappa$ i.e. from existence liquid with free surface in the tank. The relation between frequency and amplitude of the forced vibration has been obtained. It follows from these relations that for various frequencies of the exciting force, that is, for various angular velocities of the rotor itself, the amplitude of forced vibration changes monotonically only in certain ranges of values of ω /for fixed values of other parameters of the system/. In the defined range of values there exist two, three or more states of the steady vibration. The "jump phenomenon" is also possible, which is known for Eq. /33/ of the Duffing's type. The stability of the periodic vibration has been investigated. It is precisely determined which periodic solutions are stable and which are unstable. It should be noted that the phenomenon of existence of a number of stable steady states is only due to the action of the nonlinear elastic force of the bearings. In the case of full linearisation of motion equations, the relation amplitude - frequency has different character, namely, the moduli of amplitudes change /for the given frequency/ as well as the phenomenon of existence of several amplitude for one frequency does not appear. Designing the rotating system of such a kind, the constructor may and should choose the nonlinear elastic characteristic in such a way, that for a given angular velocity of operation the amplitude of detrimental precession vibration of the rotor should be so small, as possible.

Finally, it should be noted, that Eq. /1/ may be used to describe not only the system shown in Fig. 1, but also numerous other systems /more strictly, these are the equations of small vibration/, for instance, the motion of a massless flexible shaft rotating in rigid bearings together with the disc attached eccentrically [2, 15], the motion of some gyroscopic devices, gyroscopic stabilizers and others.

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