TOMASZ WIERZBICKI

Viscoplastic Flow of Rotationally Symmetric Shells with Particular Application to Dynamic Loadings

1/1968

WARSZAWA

N a   p r a w a c h   r ę k o p i s u

D o   u ż y t k u   w e w n ę t r z n e g o

Zakład Mechaniki Ośrodków Ciągłych IPPT PAN

Nakład 120 egz. Ark. wyd. 0,75. Ark. druk.1,5
Oddano do drukarni w lutym 1968 r.
Wydrukowano w marcu 1968 r. Nr zam. 189/9/68

Warszawska Drukarnia Naukowa , Warszawa ,
ul. Śniadeckich 8

Viscoplastic flow of rotationally symmetric shells with particular application to dynamic loadings

Tomasz Wierzbicki /Warsaw/

1. Introduction

The Huber-Mises yield condition for rotationally symmetric shells was derived by Hodge [4]. The corresponding equations, given in the parametric form, are very complicated and therefore are not treatable mathematically. The alternative Tresca yield condition, although linear in the space of stresses, when expressed in terms of moments and stress resultants becomes again a non-linear function and so is, the appropriate flow rule, [3]. Therefore, depending upon a particular structure, various approximations to the latter yield condition were proposed mainly by means of inscribing or circumscribing much simpler geometrical figures, over the exact yield surface [5]. An extreme case is a limited interaction surface which maintains all interactions between force and force and moment and moment but neglects all interactions between force and moment [3]. Such procedures give rise to numerous objections especially when applied to the problems of dynamic loading of shells. It is well known that Tresca yield condition, although provides a fairly good approximation to the stress field it often leads to unrealistic velocity fields because the resulting strain rate tensor does not change in a continuous manner and its direction is piece-wise constant. The Tresca yield condition has proved very useful in the problems of limit analysis where only the moment distribution and values of the static load-carrying capacity were sought for [11]. In all dynamic problems for plates and shells the load already exceeds several times the limit value and one is usually concerned with the determination of the velocity field. For that purpose the Tresca hexagone is no longer a good approximation.

In the space of generalized stresses the Tresca yield condition is represented by a number of hypersurfaces, each of them is expressed analytically by a different formula. It is very unlike that the stress profile will fall within a single regime i.e. only one hypersurface will be involved. Usually two or more regimes should be taken into account to describe a given boundary value problem. In most of the dynamic problems the position of boundaries separating various regimes vary with time and should be found as a part of the solution. Therefore the problem is reduced to the solution of several systems of partial differential equations in regions with unknown and time variable boundaries. This is a formidable mathematical task. The situation is even more complicated in the case of viscoplastic material where in addition to the existing regimes new regimes are created, as shown by Prager [10] .

Finally, it should be borne in mind that the exact Tresca yield condition in the space of moments and stress resultants is seldom used if any reasonable simple solution is to be obtained. Instead reference is made to the linearized yield surface. Such an linearized yield surface would be obtained as a result of the exact transformation of the yield condition which is no longer the original Tresca hexagone. Consequently we are solving the problem for a material much different from the one we wanted to deal. This deficiency, never fully explained in the literature, has not prevented the theory from being used in practical applications. This was partly due to the fact that no competitive theory was developed neither for perfectly plastic nor for viscoplastic bodies*/. Moreover, the opinion

*/ The flow rule employed by Perrone [8] and Jones [6] are particular cases of the constitutive equations for viscoplastic shells first derived by Bykovcev et all [1]. These concepts as well as the Prager's theory [10] are all based on the Tresca yield condition.

has been widely spread out that only the yield condition was allowed to be approximated, however crudely whereas simplifying assumptions of the different nature were not permissible.

In the present paper an alternative form of the linear constitutive equations, applicable for dynamic problems are proposed. These equations, based on somewhat different arguments, describes the behaviour of a viscoplastic material in the space of generalized stresses and strain rates. All the above mentioned shortcomings of the Tresca yield condition call for an appropriate theory with the smooth yield condition. Therefore as a starting point in the present considerations the constitutive equations for rate sensitive plastic materials, based upon the Huber-Mises yield condition, are assumed. An application of the derived equations to the solution of boundary value problems for cylindrical shells at large deflections is presented.

2. Linearized constitutive equations for viscoplastic materials

Consider a particular case of the constitutive equations for strain rate sensitive plastic materials, derived by Perzyna [9]

$$\dot{\varepsilon}_{ij} = \gamma\left(\frac{\sqrt{J_2}}{k} - 1\right)\frac{s_{ij}}{\sqrt{J_2}} \qquad \text{for} \quad J_2 > k^2, \tag{2.1}$$

$$\dot{\varepsilon}_{ij} = 0 \qquad \text{for} \quad J_2 \le k^2,$$

where s_{ij} and $\dot{\varepsilon}_{ij}$ are components of stress and strain rate deviators respectively, J_2 denote the second invariant of the stress deviation and γ and k are material constants. Although equation /2.1/ is a far reaching idealization of the behaviour of real bodies it accounts for a simultaneous plastic and viscous effects and therefore can approximate the strain rate sensitivity characteristics of certain metals. It also constitutes a generalization of the

Saint-Venant Lévy-Mises flow rule as the latter equation can be derived from /2.1/ in the limiting case by putting $\gamma \to \infty$ and $\sqrt{J_2} \to k$, [9]

$$\dot{\varepsilon}_{ij} = \lambda s_{ij} \quad \text{and} \quad \sqrt{J_2} = k \,. \tag{2.2}$$

Define now the state of stress s°_{ij} by the relation

$$\frac{s_{ij}}{\left(\frac{1}{2} s_{ij} s_{ij}\right)^{1/2}} = \frac{s^{\circ}_{ij}}{\left(\frac{1}{2} s^{\circ}_{ij} s^{\circ}_{ij}\right)^{1/2}} \,, \tag{2.3}$$

with an additional requirement that the state s°_{ij} satisfies equation of the static yield condition

$$\frac{1}{2} s^{\circ}_{ij} s^{\circ}_{ij} = k^2 \,. \tag{2.4}$$

The state of stress s°_{ij} will be called a "state of comparison". Using /2.3/ equation /2.1/ can be rewritten in an equivalent form

$$\dot{\varepsilon}_{ij} = \gamma_0 \left(s_{ij} - s^{\circ}_{ij} \right) \quad \text{if} \quad s_{ij} > s^{\circ}_{ij} \,, \tag{2.5}$$

where $\gamma_0 = \gamma / k$ is a new material constant. Eq./2.5/ is a nonlinear relation in stresses since the term s°_{ij} , according to the definition /2.3/ is a non-linear function of s_{ij}, $s^{\circ}_{ij} = s^{\circ}_{ij}(s_{ij})$.

The components of the tensor s°_{ij} are restricted by the yield condition /2.4/ and usually by the stress boundary conditions. There is however still much freedom in the choice of the state of comparison. Therefore equations /2.5/ can not be of any use unless the state of comparison is known.

In order to determine s°_{ij} reference will be made to the specific boundary value problem since $s^{\circ}_{ij} = s^{\circ}_{ij}(x_i, t)$ vary in general both in space and time.

3. Determination of the state of comparison

Consider a rigid viscoplastic body occupying the three-dimensional region V with a regular surface S. Assume that the time-variable surface tractions $T_i(t)$ are applied to the part S_1 of S, while on the remaining part of S velocities $\dot{u}_i$ are prescribed. Assume further that the variation of surface traction is proportional

$$T_i(t) = \mu(t)\, T_i, \tag{3.1}$$

where T_i can be regarded as concentrated forces or distributed loads. At the beginning of the process the body is at rest. Let $\dot{u}_i$, $\dot{\varepsilon}_{ij}$ and σ_{ij} be a complete solution of the formulated boundary value problem satisfying the system of equations

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right), \tag{3.2}$$

$$\sigma_{ij,j} = \rho\, \ddot{u}_i, \tag{3.3}$$

supplemented by the constitutive equation /2.1/. At a given point $x_i \in V$ the solution is a function of time or any monotonically increasing parameter of the process τ. As such a parameter we can conveniently choose for example the permanent central deflection of plate or shell. For the prescribed dynamic process the stress trajectory in the s_{ij} space form a curve ACB, where the parameter τ is continuously increasing along the path ACB. At the beginning of the process $\tau = 0$ the velocities and strain are zero throughout the body, thus according to the assumed model of the material, the state of stress for a given particle of the body X_i is lying on the static yield surface, point A. As time proceeds the particle X_i is accelerated, the strain rate increases and the stress point is forming certain trajectory. The strain rate vector is always perpendicular to the subsequent yield surface. Finally, the motion of the body ceases, the particle is brought to rest at $\tau = \tau_f$ the corresponding stress point B must lay on the static yield surface. At a given instant $\tau = \tau_1$ the state of stress $s_{ij}(\tau_1)$ is represented by a point C whereas the state of stress $s^{\circ}_{ij}(\tau_1)$ by a point D on the static yield surface. According to the definition of an isotropic growth of the yield surface with the strain rate the unit normal vectors to the yield surface at the points C and D are equal. Thus the strain rate tensors considered as vectors in the nine-dimensional space are colinear

$$\dot{\varepsilon}_{ij} = \nu \dot{\varepsilon}^{\circ}_{ij} . \tag{3.4}$$

The velocity field $\dot{u}_i$ is the complete solution of the formulated boundary value problem for a viscoplastic material at the same time being the kinematically admissible velocity field for the corresponding boundary value problem but for perfectly plastic material.

The kinematic admissibility of $\dot{u}_i$ follows from the fact that this field satisfies all kinematically boundary conditions, is continuous and a continuous strain rate field can be obtained from Eq./3.2/. Next with each $\dot{\varepsilon}_{ij}$ or $\dot{\varepsilon}^{\circ}_{ij}$ we can uniqually associate a stress field s°_{ij} through the flow rule /2.2/. This stress field satisfies the static yield condition /2.4/. Finally the requirement of the positiveness of rate of plastic work is fulfilled

$$\int_V \sigma^{\circ}_{ij} \dot{\varepsilon}^{\circ}_{ij} \, dV \geqslant 0 . \tag{3.5}$$

Since $\dot{u}_i$ is a kinematically admissible velocity field for the perfectly plastic problem the corresponding stress field σ°_{ij} has not necessarily to be in equilibrium i.e., σ°_{ij} does not satisfy the appropriate equation of motion.

To find an approximate value for σ°_{ij} and s°_{ij} consider a stress field σ^{*}_{ij} which is in the state equilibrium

/3.6/ $$\sigma^{*}_{ij,j} = 0 .$$

Solving now the same boundary value problem using Eqs./2.4/, /3.2/, /2.2/ and /3.6/ we find a new solution $\dot{u}^{*}_{i}$, $\dot{\varepsilon}^{*}_{ij}$, σ^{*}_{ij} , s^{*}_{ij} . The velocity $\dot{u}^{*}_{i}$ differs of course from the velocity field $\dot{u}_{i}$ of the complete solution.

The stress field s^{*}_{ij} is said to be a good approximation to s°_{ij} if the velocity field for a quasi static flow of perfectly plastic material $\dot{u}^{*}_{i}$ does not deviate much from the velocity field of a dynamic problem for the viscoplastic material $\dot{u}_{i}$. In other words adding the inertia and viscous terms into the equations describing the perfectly plastic flow of a structure should not alter much the kinematics of the problem which is primarily dependent upon the boundary condition, yield condition and associate flow rule.

Introducing the assumed approximation Eq./2.5/ is replaced by

/3.9/ $$\dot{\varepsilon}_{ij} = \gamma_{o}(s_{ij} - s^{*}_{ij}) .$$

Now /3.9/ is a linear constitutive equation provided that the solution $s^{*}_{ij}(x_i , \tau)$ is known.

The new approximated constitutive equation /3.9/ is still based on the smooth yield condition, consequently the resulting velocity field is continuous.

Whether or not the assumed hypothesis is a good approximation should be investigated in each particular boundary value problem separately. However the replacement of the true velocity field by a velocity field $\dot{u}^{*}_{i}$ resulting from the same smooth yield condition seems to be a much better approximation than introduction of a discontinuous velocity profile based on the piece-wise linear yield condition. The derived constitutive equation requires consideration of a single regime thus simplifying the mathematics involved.

Note that in the above presentation of the theory nothing is said about the proportionality of the loading at a given point x_i , the hypothesis which was incorporated in the author's early papers concerning bending solutions for viscoplastic plates [7,13]. The case of proportional loading would be obtained as a particular case of the general equations if we assume that for a given x_i, $\overset{\circ}{s}_{ij}(\tau)$ is const which means that $\overset{\circ}{s}_{ij}$ is independent upon the parameter of the process. In the space of stresses the corresponding point $\overset{\circ}{s}_{ij}(x_i)$ is retaining the same position and the stress trajectory is reduced to the straight line.

4. Flow rule for axisymmetric shells

The state of stress in the thin-walled structures is essentially plane, σ_1 and σ_2 being the principal stresses, the corresponding strain rates are $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$.

Assuming the Love-Kirchhoff hypothesis the strain rates can be expressed in terms of extension rates $\dot{\lambda}_1$, $\dot{\lambda}_2$ and curvature rates $\dot{K}_1$ and $\dot{K}_2$ of the middle surface of the shell as

$$/4.1/ \qquad \dot{\varepsilon}_1 = \dot{\lambda}_1 + z\dot{K}_1 \quad , \quad \dot{\varepsilon}_2 = \dot{\lambda}_2 + z\dot{K}_2 \ .$$

The position of the shell neutral surface corresponds to such a value of z for which the component of the corresponding strain rate vector vanishes

$$/4.2/ \qquad \xi = -\frac{\dot{\lambda}_1}{\dot{K}_1} \quad , \quad \eta = -\frac{\dot{\lambda}_2}{\dot{K}_2} \ .$$

Consider a fibre distanced at $z = \zeta$ from the shell middle surface and prescribe a loading program in which ξ and η are monotonically increasing functions of the process parameter τ . A typical trajectory of the strain rate vector form on the plane $(\dot{\varepsilon}_1, \dot{\varepsilon}_2)$ a closed smooth curve passing through the origin. Possible strain rate trajectories are shown on Fig.2.

With each strain rate history we can associate the corresponding stress trajectory through the constitutive equations /2.1/. In the case of plane stress Eq./2.1/ yields two independent relations

/4.3/
$$\dot{\varepsilon}_1 = \gamma \left(\frac{\sqrt{J_2}}{k} - 1 \right) \frac{2\sigma_1 - \sigma_2}{\sqrt{J_2}} ,$$
$$\dot{\varepsilon}_2 = \gamma \left(\frac{\sqrt{J_2}}{k} - 1 \right) \frac{2\sigma_2 - \sigma_1}{\sqrt{J_2}} ,$$

where

/4.4/
$$J_2 = \sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 ,$$

represents a Huber-Mises ellipse in the plane (σ_1, σ_2). Integration of /4.3/, as mentioned in the Introduction, leads to very complicated formulas. By contrast the derived equations /3.9/ are easily integrable and their linear form is preserved after integration.

The generalized stresses corresponding to the extension rates $\dot{\lambda}_\alpha$ and curvature rates $\dot{\kappa}_\alpha$ are the membrane forces N_α and bending moments M_α , α = 1,2. These quantities are related to the stress components σ_α by

/4.5/
$$N_\alpha = \int_{-h}^{h} \sigma_\alpha \, dz , \quad M_\alpha = \int_{-h}^{h} \sigma_\alpha z \, dz .$$

The components of the stress tensor expressed in terms of strain rate vector computed from the linearized equations /3.9/ have the form

/4.6/
$$\sigma_1 - \sigma_1^* = \frac{1}{\gamma_0} \left(2\dot{\varepsilon}_1 + \dot{\varepsilon}_2 \right) ,$$
$$\sigma_2 - \sigma_2^* = \frac{1}{\gamma_0} \left(2\dot{\varepsilon}_2 + \dot{\varepsilon}_1 \right) .$$

Substituting /4.6/ and /4.1/ into /4.5/ and integrating over the shell thickness we get

$$M_1 - M_1^* = \frac{2}{3}\frac{h^3}{\gamma_0}\left(2\dot{K}_1 + \dot{K}_2\right),$$
$$M_2 - M_2^* = \frac{2}{3}\frac{h^3}{\gamma_0}\left(2\dot{K}_2 + \dot{K}_1\right), \qquad /4.7/$$
$$N_1 - N_1^* = \frac{2h}{\gamma_0}\left(2\dot{\lambda}_1 + \dot{\lambda}_2\right),$$
$$N_2 - N_2^* = \frac{2h}{\gamma_0}\left(2\dot{\lambda}_2 + \dot{\lambda}_1\right),$$

where the states of comparison $M_\alpha^*(x_\alpha,\tau)$ and $N_\alpha^*(x_\alpha,\tau)$ by hypothesis should satisfy equations of equilibrium of the corresponding quasi-static problem.

The structure of the new equations suggests that the system /4.7/ is uncoupled i.e. there is no interaction between moments and membrane forces. In fact, the interaction is incorporated in the present theory through the presence of the terms M_α^* and N_α^*. The position of the vector (M_α^*, N_α^*) on the static yield surface, derived from the exact Huber-Mises yield condition, is uniquelly determined by the generalized strain rate vector $(\dot{\lambda}_\alpha, \dot{K}_\alpha)$. This implies that the value of each components of the (M_α^*, N_α^*) vector depends upon all components of the $(\dot{\lambda}_\alpha, \dot{K}_\alpha)$ vector.

The advantage of the proposed method of linearization becomes more apparent when the constitutive equations /3.9/ are combined with equation of motion of a particular structure. It can be shown that using the equation of static equilibrium all unknown terms $N_\alpha^*(x_\alpha,\tau)$ and $M_\alpha^*(x_\alpha,\tau)$ can be replaced by a single term $P(x_\alpha,\tau)$. This term represents the value of the external load required to maintain a quasi-static flow of rigid-perfectly plastic structure with Huber-Mises yield condition.

It is clear that the initial value of $P(\tau)$ /small deflections/ is equal to the load-carrying capacity of the relevent structure $P(0) = P_y$. For example for a uniformly loaded simply supported and clamped plates this value is equal respectively 6.51 M_0/R^2 and 12.5 M_0/R^2. For a cylindrical shell loaded by a ring of forces the limit load is $1.949\, M_0 \sqrt{2/hR}$ where R is the radius of the relevent structure.

For larger deflections the development of membrane forces and changes in geometry produce an increase in the value of the load carrying capacity and P is uniquelly related to the central deflection δ of the plate or shell $P = P(\delta)$.

5. Illustrative examples

Consider an infinitely long cylindrical shell of radius R subjected to a ring of forces $2Q$. Such a problem is among the simplest in the theory of thin plastic shells for which the value of the collapse load was calculated using the exact Huber-Mises yield condition [11]. To illustrate the main features of the present theory only the quasi-static flow will be considered but no difficulties arise in generalizing the solution to the case of dynamic loading.

In the abscence of axial forces the generalized stresses are the axial moment M_x and the hoop force N_φ, the corresponding generalized strain rates being $\dot{K}_x$ and $\dot{\lambda}_\varphi$. Equations of equilibrium of a cylindrical shell loaded by a ring of forces do not involve the inhomogeneous term

/5.1/ $$\frac{d^2 M_x}{dX^2} + \frac{N_\varphi}{R} = 0, \quad \frac{dM_x}{dX} = T.$$

For further convenience we introduce the following dimensionless quantities

/5.2/ $$m = \frac{M_x}{M_0}, \quad n = \frac{N_\varphi}{N_0}, \quad r = \frac{N_x}{N_0}, \quad t = \frac{T}{N_0}\sqrt{\beta},$$
$$v = \frac{1}{R}\frac{dW}{dt}, \quad x = \frac{X}{R}\sqrt{\beta}, \quad \beta = \frac{2R}{h} = \frac{N_0 R}{M_0}.$$

Since both generalized stress fields (m, n) and (m^*, n^*) satisfy Eq./5.1/ so does also their difference

/5.3/ $$\frac{d^2}{dx^2}(m - m^*) + (n - n^*) = 0.$$

The constitutive equations /4.7/ are reduced to

$$m - m^* = \frac{1}{\gamma^*}\,\frac{8}{3}\,\dot{\varkappa}_x,$$

/5.4/ $$n - n^* = \frac{1}{\gamma^*}\left(2\dot{\lambda}_\varphi + \dot{\lambda}_x\right),$$

$$\tau - \tau^* = \frac{1}{\gamma^*}\left(2\dot{\lambda}_x + \dot{\lambda}_\varphi\right) = 0,$$

where dimensionless rates of strains are expressed in terms of the velocity V as

/5.5/ $$\dot{\varkappa}_x = \frac{d^2 V}{dx^2}, \quad \dot{\lambda}_\varphi = V, \qquad \gamma^* = \sqrt{3}\,\gamma.$$

Equations /5.3/ - /5.5/ can be solved for velocity

/5.6/ $$\frac{d^4 V}{dx^4} + 4\lambda^4 V = 0, \qquad \lambda = \sqrt[4]{9/32}.$$

The latter relation is of the same form as the equation for deflection in the classical theory of elastic shells. The above mentioned analogy is easily understood in view of the linearity of the constitutive equations /4.7/ and was first noticed in [12]. The general solution of /5.6/ has the form

/5.7/ $$V(x) = e^{-\lambda x}\left(A\sin\lambda x + B\cos\lambda x\right) + e^{\lambda x}\left(C\sin\lambda x + D\cos\lambda x\right).$$

We postulate that the extent of the viscoplastic region is finite, $x = x_1$ being the boundary between the plastic and rigid zones. At $x = 0$ the shearing force T should assume the prescribed value Q and the slope of velocity is zero. At $x = x_1$, the velocity and its first derivative should vanish. Thus, the boundary conditions have the form

/5.8/ $$\left.\frac{dV}{dx}\right|_{x=0} = 0, \quad T(0) = Q, \quad V(x_1) = 0, \quad \left.\frac{dV}{dx}\right|_{x=x_1} = 0.$$

According to the assumed approximation the position of the boundary x_1 for the viscoplastic and perfectly plastic solutions is the same.

The shearing force $t = \frac{dm}{dx}$ can be expressed in terms of the velocity

$$t - t^* = \frac{8}{3\gamma_*} \frac{d^3 v}{d x^3} . \tag{5.9}$$

Using the second boundary condition /5.8/ the latter equation yields

$$q = q^* + \frac{8}{3\gamma_*} \frac{d^3 v}{dx^3}\bigg|_{x=0} \tag{5.10}$$

where q^* = 1,949 is the value of static load-carrying capacity obtained in [11] on the basis of the exact yield condition. At the same time the corresponding value of X_1 was not computed. The necessary data are given for the sandwich shell with Huber-Mises yield condition q^* = 1,905, X_1 = 3,467.

After evaluation of the constants of integrations A, B , C, D from /5.8/ and /5.10/ the solution is found to be

$$\begin{aligned} v(x)/v(0) &= e^{-\lambda x}(0.505 \sin \lambda x + 0.495 \cos \lambda x) - e^{\lambda x}(0.00054 \sin \lambda x + \\ &\quad - 0.00054 \cos \lambda x), \\ v(0) &= \frac{3\gamma^*(q-q^*)}{8\lambda^3} . \end{aligned} \tag{5.11}$$

The plot of the velocity profile $v(x)/v(0)$ is presented in Fig.3, full line. The broken line corresponds to the exact velocity profile calculated in [11] using the Huber-Mises yield condition for sandwich shell. The general agreement of both solution indicates to the usefulness of the proposed method of linearization in computing velocity fields for rigid-viscoplastic shells with Huber-Mises yield condition.

An important feature of the solution /5.11/ is that the velocity profile of the shell $v(x)/v(0)$ is not affected by the value of the limit load q^* . This remark is of interest in view of the lack of published data on the static load-carrying capacity of structures with Huber-Mises yield condition. Usually the approximate value of q^* should be accepted but this simplification introduces no error in the computed velocity profile.

In the limiting case when $r^* \to \infty$, $q \to q^*$ the central ~~deflection~~ velocity $V(0)$ is an undetermined quantity as in the solution for perfectly plastic shell. The derived solution is valid only for small deflections.

In real facts in order to maintain the quasi-static flow of the shell the load must continuously increase since the geometry changes and development of membrane forces causes a strengthening of the structure. It was shown that this effect is significant in the early stages of the deformation process and becomes decisive on the shell response for deflections of the order of the wall thickness h, [2].

The modified equations of equilibrium which takes into account the changes in geometry have the form

$$\frac{\partial^2 m}{\partial x^2} - \beta r \frac{\partial^2 w}{\partial x^2} + n - p(x,t) = 0, \qquad /5.12/$$

$$\frac{\partial r}{\partial x} = 0,$$

where $r = \frac{N_x}{N_0}$ and $p = \frac{PR^2}{M_0}$ are respectively dimensionless membrane force and internal pressure. If the dynamic loading is considered Eqs./5.12/ should be supplemented by the proper inertia terms.

By hypothesis the components of the generalized stress vector of the auxiliary problem are in equilibrium with the value of limit load $p^*(x,\tau)$

$$\frac{\partial^2 m^*}{\partial x^2} - \beta r^* \frac{\partial^2 w}{\partial x^2} + n^* - p^*(x,\tau) = 0, \qquad /5.13/$$

$$\frac{\partial r^*}{\partial x} = 0.$$

Since the velocity and deflection profiles are in both solutions identical, substracting /5.13/ from /5.12/ we obtain

$$\frac{\partial^2}{\partial x^2}(m-m^*)-\beta(\tau-\tau^*)\frac{\partial^2 w}{\partial x^2}+(n-n^*)-(p-p^*)=0. \quad /5.14/$$

Now applying directly the flow rule /4.7/ the bending moments and membrane forces can be eliminated from /5.14/. Making use of the appropriate geometrical relation the system of governing equations can be reduced to two simultaneous partial differential equations for two components of the displacement vector (u, w) provided that the term $p^*(x,\tau)$ is known for each particular boundary value problem.
Using equations /4.7/ and /5.14/ and the appropriate load-deflection relation the variety of boundary value problems for cylindrical shells at large deflection can be formulated and solved.

7. Conclusion

In the analysis of the dynamic loading of rigid visco-plastic plates and shells we are primarily interested in the determination of the velocity field and permanent deflection of the structure. Except of very simple situations the solution of thus formulated non-linear problem can be obtained through the numerical integration of governing equations. Various simplifications are commonly introduced in order to enable a analytical treatment of the problem. Usually the true velocity field of the dynamic problem is approximated by a velocity field resulting from the Tresca yield condition. It is believed that for some cases the velocity field corresponding to the static solution of the same problem for perfectly plastic material obeying the Huber-Mises yield condition would give a better approximation to the exact rate of deflection profile. Assuming this as a hypotheses a new theory has been developed in which the relation between components of generalized stress and strain rate vectors are linear. Consequently a linear form of the governing equations is obtained, formally analogous to the relevent equations in the classical theory of thin elastic shells where instead of the deflection stands the deflection rate. Therefore all elastic solution of both static and dynamic problems can be reconsidered and adapted for the viscoplastic flow of plates and shells. The presented examplary solution of the cylidrical shell under a ring of forces exhibited an excellent agreement with the exact solution of the same problem. The proposed method was shown to be also applicable in the analysis of moderately large deflection of shells.

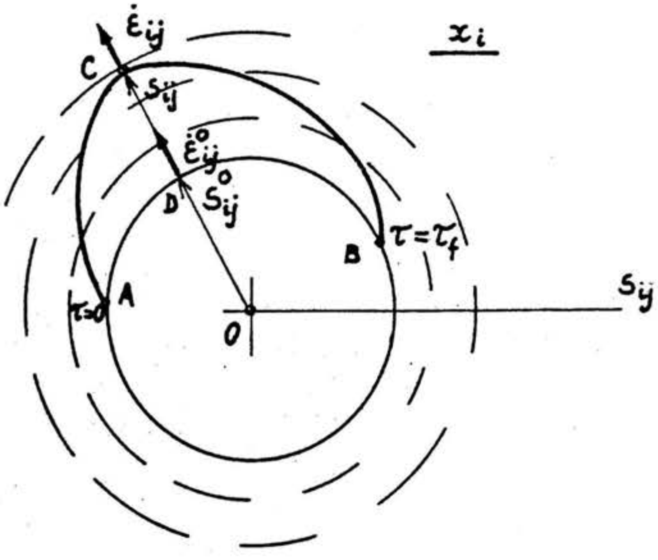

Fig.1 Stress trajectory for the prescribed dynamic process

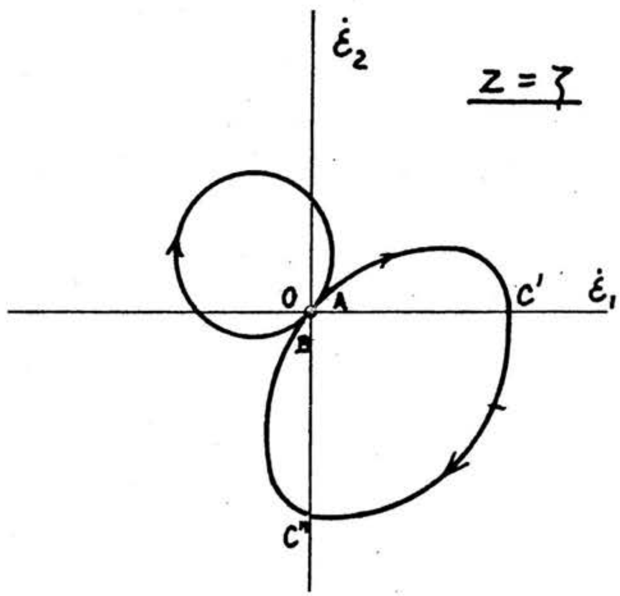

Fig.2 Possible strain rate trajectories at an arbitrary point of the uniform shell

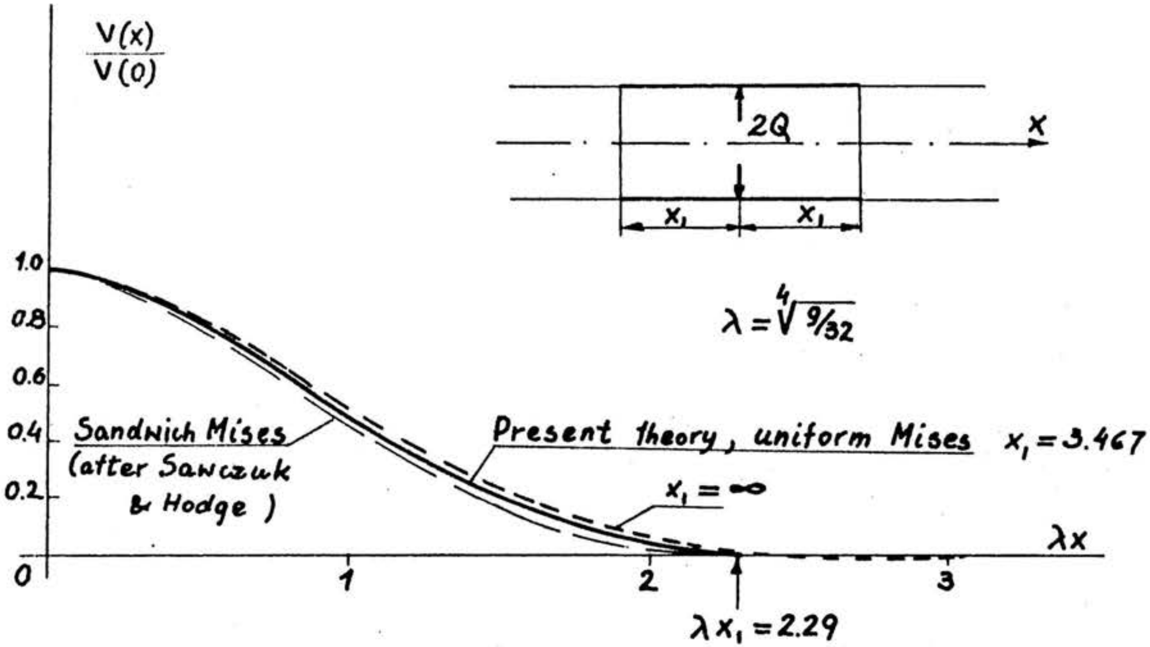

Fig.3 Comparison of velocity fields of the present theory and rigid perfectly plastic theory after Sawczuk & Hodge

References

1 G.I. Bykovcev and T.D. Semykina, On viscoplastic flow of circular plates and shells of revolution /in Russian/, Izv. Akad. Nauk, Nr 4 /1964/, 68-76.

2 M.Duszek, Plastic analysis of cylindrical shells subjected to large deflections, Arch. Mech. Stos., 18, 5 /1966/, 599-614.

3. P.G.Hodge, Yield condition for rotationally symmetric shells under axisymmetric loading, J. Appl. Mech., Trans. ASME, June 1960, 323-331.

4 P.G.Hodge, The Mises yield condition for rotationally symmetric shells, 18, 4 /1961/, 305-311.

5 D.C. Drucker and R.T. Shield, Limit analysis of symmetrically loaded thin shells of revolution, J. Appl. Mech., 29, 1 /1959/, 61-68.

6 N. Jones, Finite deflections of a rigid viscoplastic strain hardening annular plate loaded impulsively, Brown University, Report ARPA E53.

7 J.M. Kelly and T.Wierzbicki, Motion of a circular viscoplastic plate subject to projectile impact, ZAMP, 18, 2 /1967/, 236-246.

8 N. Perrone, Impulsively loaded strain rate sensitive plates, J. Appl. Mech., 33, 1 /1966/.

9 P.Perzyna, The constitutive equations for rate sensitive plastic materials, Quart. Appl. Math., 20, 4 /1963/, 321-332.

10 W.Prager, Linearization in visco-plasticity, Österr. Ing. Archiv, 15, 1-4 /1961/, 152-157.

11 A. Sawczuk and P.G.Hodge, Comparison of yield conditions for circular cylindrical shells, J. Franklin Inst., 269, 5 /1960/, 362-374.

12 T.Wierzbicki, A method of approximate solution of boundary value problems for rigid, viscoplastic structures, Acta Mechanica, 3, 1 /1967/, 56-66.

13 T. Wierzbicki, Impulsive loading of rigid viscoplastic plates, Int. J. Solids and Structures, 3 /1967/, 635-647.

Viscoplastic flow of rotationally symmetric shells with particular application to dynamic loadings

Tomasz Wierzbicki /Warsaw/

Abstract

Starting from the constitutive equations for rate sensitive plastic material obeying the Huber-Mises yield condition linear relations between components of generalized stress and strain rate vectors for rotationally symmetric shells are derived, applicable for dynamic problems. The linearization of the originally non-linear equations is achieved by defining a new stress tensor called "the state of comparison". The components of the new tensor satisfy stress boundary condition and static yield condition but not necessarily equations of equilibrium. A method is presented for the approximate determination of "the state of comparison" for a given boundary value problem. The new theory is illustrated by an example in which the deformation of a cylindrical shell under a ring of forces is considered. Extension of the present approach to the case of moderately large deflections is discussed.