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ON THE OPERATIONAL PERTURBATION METHOD OF SOLUTION OF THE VOLTERRA'S NONLINEAR INTEGRAL EQUATIONS

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1. Introduction

In the domain of the nonlinear mechanics of continuous media we often have to deal with nonlinear integral equations of the Volterra's type. Such equations appear for exemple, in the problems of nonlinear viscoelasticity founded on the generalized Boltzmann's principle for hereditary processes, in the theory of nonlinear vibrations in the theory of automatic control and in the problems of nonlinear random processes.

The general form of the Volterra's nonlinear integral equation of the second kind is

$$s(t) = 2\,e(t) + \int_{t_0}^{t} f\left[t,\tau,s(\tau)\right] d\tau \quad , \tag{1.1}$$

where $s(t)$ is the function to be found, $e(t)$ - a known function a-constant and $f\left[t,\tau,s(\tau)\right]$ - a known function of all its arguments $t,\tau,s(\tau)$. Let the upper limit of the integral (1.1) be the time-veriable, t_0 denoting initial instant. We assume that the known function $e(t)$ is defined and continuous in the time-interval (t_0,∞) and also the known function $f\left[t,\tau,s(\tau)\right]$ is defined and continuous with respect to all variables in the region of their veriability given as being

$$t_0 \leqslant t \leqslant \infty\,, \quad t_0 \leqslant \tau \leqslant t, \quad \bar{s} \leqslant s \leqslant \bar{\bar{s}} \quad . \tag{1.2}$$

If we denote by k_d and k_g the lower and upper bound of the function f, respectively, then

$$/1.3/ \qquad \bar{s} < k_d \leqslant k_g < \bar{\bar{s}} \quad .$$

In addition, the function $f\left[t, \tau, s(\tau)\right]$ satisfies the condition

$$/1.4/ \quad \left| f\left[t, \tau, \partial_\tau s(\tau)\right] - f\left[t, \tau, \partial_\tau^2 s(\tau)\right]\right| < \lambda \left|\partial_\tau s(\tau) - \partial_\tau^2 s(\tau)\right| ,$$

where ∂_τ denotes the derivative with respect to variable τ and λ is a positive constant.

In the problems of mechanics as mentioned above the function f can be usually presented in the form with separated variables

$$/1.5/ \qquad f\left[t, \tau, s(\tau)\right] = -\varphi\left[s(\tau)\right] K(t,\tau) \quad ,$$

where

$$/1.6/ \qquad \varphi\left[s(\tau)\right] = s(\tau)\left[F(s)\right] \quad .$$

The last representation has some physical meaning which we shall point out in considering certain application.

If we denote by Q the integral operator

$$/1.7/ \qquad Qg = \int_{t_0}^{t} g(\tau) K(t,\tau)\, d\tau \quad ,$$

where the function K is its kernel, the Eq./1.1/ can be rewritten, for the mode of representing /1.5/ and /1.6/ in the operational form

$$/1.8/ \qquad \left[1 + QF(s)\right] s = ae \quad .$$

We note that for the function F being identically equal unity

$$F(s) \equiv 1 \quad , \tag{1.9}$$

the Eq./1.8/ becomes linear

$$(1 + Q)s = ae \quad . \tag{1.10}$$

Thus, if the function F shows a small deviation from the constant value of unity then the physical process represented by means of Eq./1.8/ can be regarded as result of small nonlinear perturbation of the linear process as given by Eq./1.10/. In this way, the function F in the operator Q of Eq./1.8/ has the meaning of certain physical magnifying factor depending on the actual value of the physical variable s. On the other hand, the function F depends on the small physical constant β which is responsible for the small perturbation of the linearity

$$F = F\left[\beta\, s(\tau)\right] \quad . \tag{1.11}$$

In general, the mentioned above properties of the function F can be represented by the exponential function of the form

$$F\left[\beta\, s(\tau)\right] = \exp\left[\beta(s - s^{*})\right] \quad , \tag{1.12}$$

where s^{*} is certain fixed value of the variable s for which the Eq./1.8/ transforms into linear equation /1.10/. Thus, according to Eq./1.12/ we have for $s = s^{*}$

$$F\left[\beta\, s(\tau)\right]_{s=s^{*}} = 1 \quad , \tag{1.13}$$

and the process considered is described by linear Eq./1.10/. For $s > s^*$ the process is nonlinear in accordance with Eq./1.8/.

In the following we shall deal with the integral equation /1.8/ where the function F is of the type /1.12/.

2. Perturbational development of the integral equation

We present the function /1.12/ in the form of power series

$$F\left[\beta\, s(\tau)\right] = \sum_{k=1}^{\infty} \beta_k s^{k-1}(\tau) \quad , \tag{2.1}$$

where the coefficients β_k are given by the equality

$$\beta_k = \frac{1}{k!}\beta^{k-1} \exp\left(-\beta\, s_o\right) = \alpha_k\, \beta^{k-1} \quad . \tag{2.2}$$

According to the development /2.1/ the equation /1.8/ may be written as follows

$$s + \sum_{k=1}^{\infty} \beta_k\, Q\, s^k = ae \quad . \tag{2.3}$$

On the other hand, we develop the function s into power series of the small parameter β

$$s(\tau) = \sum_{j=1}^{\infty} b_j \beta^j \quad , \tag{2.4}$$

where the coefficients b_j are

$$b_j(\tau) = \beta^{-1}\, s_{j-1}(\tau) \quad . \tag{2.5}$$

Introducing the series /2.4/ into Eq./2.1/ multiplied by $s(\tau)$ we have

$$/2.6/ \qquad s(\tau)F\left[\beta s(\tau)\right] = \sum_{k=1}^{\infty} \beta_k \; s^k(\tau) = \sum_{k=1}^{\infty} \beta^{k-1} \; \bar{s}_{k-1}(\tau).$$

Here, the coefficients of the series /2.6/ can be found from the formulas

$$/2.7/ \qquad \begin{aligned} \bar{s}_o &= \alpha_1 \; s_o \; , \\ \bar{s}_1 &= \alpha_1 \; s_1 + \alpha_2 \; s_o^2 \; , \\ \bar{s}_2 &= \alpha_1 \; s_2 + 2\,\alpha_2 \; s_1 \; s_o + \alpha_3 \; s_o^3 \; , \\ \bar{s}_3 &= \alpha_1 \; s_3 + \alpha_2 \; s_1^2 + 2\,\alpha_2 \; s_2 \; s_o + 3\,\alpha_3 \; s_1 \; s_o^2 + \alpha_4 s_o^4 \; , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

where the constants α_k are given by the relation /2.2/

$$/2.8/ \qquad {}_k = \frac{1}{k!} \exp\left(-\beta s^*\right) .$$

Introducing the series /2.4/ and /2.6/ into the Eq. /1.8/ we obtain finally the following condition

$$/2.9/ \qquad \sum_{j=1}^{\infty} \beta^{j-1} \left(s_{j-i} + Q\bar{s}_{j-1} \right) = ae \quad .$$

From this condition we can find, for successive powers of the parameter β , the coefficients of the development /2.4/. The condition /2.9/ corresponds to the recurrent system of linear integral equations with respect to the coefficients $s_k(\tau)$

$$/2.10/ \qquad s_k + Q\bar{s}_k = 0, \qquad /k=1,2,\ldots./ \; ,$$

the first coefficients being given by the equation

/2.11/ $$s_o + Qs_o = ae \quad .$$

3. Operational representation of the coefficients of the perturbational development

We shall evaluate the coefficients of the perturbational development s_k from the recurrent system /2.10/ and /2.11/. The last equation, when Eq./2.7/ is taken into account, can be rewritten in the form

/3.1/ $$\left(1 + \alpha_1 Q\right) s_o = ae \quad .$$

By appying the inverse operation we find from Eq./3.1/

/3.2/ $$s_o = a\left(1 + \alpha_1 Q\right)^{-1} e \quad .$$

According to the physical interpretation of the process described by the Eq./1.8/ we assume /see Eq./1.7//

/3.3/ $$\left| \alpha_1 Q1 \right| < 1 \quad ,$$

and then the solution /3.2/ can be presented by means of the development of the inverse operator

/3.4/ $$s_o = a\sum_{i=0}^{\infty} (-1)^i \alpha_1 Q^i e \quad ,$$

Here we have

/3.5/ $$Q^o g = 1 \; ,$$
$$Q^m g = \underbrace{\int_{t_o}^{t}\int_{t_o}^{t}\dots\int_{t_o}^{t}}_{m} \underbrace{K(t,\tau_{m-2})K(\tau_{m-2},\tau_{m-3})\dots g(\tau_{m-2})}_{m} \underbrace{d\tau_{m-2}\, d\tau_{m-3}\dots}_{m} ,$$

Where we should put $\tau_{-1} = \tau$.

The first equation of the system /2.10/ can be written

$$(1 + \alpha_1 Q)\, s_1 = -\alpha_2 Q s_o^2 \quad , \tag{/3.6/}$$

where according to Eq./3.4/

$$s_o^2 = \sum_{i=o}^{\infty} (-1)^i \alpha_1^i g_i \quad . \tag{/3.7/}$$

Here denote

$$g_o = e^2, \quad g_m = \frac{1}{me} \sum_{k=1}^{m} (3k-m) Q^k e.g_{m-k}, \qquad m \geq 1 \; . \tag{/3.8/}$$

Putting the Eq./3.7/ into Eq./3.6/ and applying the inverse operation we calculate

$$s_1 = (1 + \alpha_1 Q)^{-1} \sum_{i=o}^{\infty} (-1)^{i+1} \alpha_1^i \alpha_2 a^2 Q g_i \; . \tag{/3.9/}$$

After having developed the inverse operator the Eq./3.9/ becomes

$$s_1 = \sum_{i=o}^{\infty} (-1)^i \alpha_1^i Q^i \sum_{i=o}^{\infty} (-1)^{i+1} \alpha_1^i \alpha_2 a^2 Q g_i = a^2 \alpha_2 \sum_{i=o}^{\infty} \alpha_1^i h_i \; , \tag{/3.10/}$$

where

$$h_n = \sum_{k=o}^{n} (-1)^{n+1} Q^k (Q g_{n-k}) \quad . \tag{/3.11/}$$

The second equation of the system /2.10/ gives

$$(1 + \alpha_1 Q)\, s_2 = -2\alpha_2 Q s_1 s_o - \alpha_3 Q s_o^3 \quad . \tag{/3.12/}$$

Introducing the development /3.4/ and /3.10/ into Eq./3.12/ and applying the inverse operation we obtain

$$/3.13/ \quad s_2=-(1+\alpha_1 Q)^{-1}\left\{2\alpha_2 Q\left[a^3\alpha_2\sum_{i=0}^{\infty}(-1)^i\alpha_1^i Q^i e\sum_{i=0}^{\infty}\alpha_1^i h_i\right]+ + \alpha_3 Q\left[a\sum_{i=0}^{\infty}(-1)^i\alpha_1^i Q^i e\right]^3\right\} .$$

After having carried out the indicated operations in Eq. /3.13/ we have

$$/3.14/ \quad s_2=a^3\sum_{i=0}^{\infty}(-1)^{i+1}\alpha_1^i Q^i\left[\sum_{i=0}^{\infty}\alpha_1^i Q^i\left(2\alpha_2^2 p_i+\alpha_3 q_i\right)\right] = = a^3\sum_{i=0}^{\infty}\alpha_1^i r_i .$$

Here denote

$$p_m=\sum_{k=0}^{m}(-1)^k Q^k e.h_{m-k} ,$$

$$/3.15/ \quad q_o = e^3 , \quad q_m=\frac{1}{me}\sum_{k=1}^{m}(4k-m)(-1)^k Q^k e.q_{m-k} , \quad m\geqslant 1,$$

$$r_m=\sum_{k=0}^{m}(-1)^{k+1}Q^k\left[Q\left(2\alpha_2^2 p_{m-k}+\alpha_3 q_{m-k}\right)\right] .$$

In the same manner we can evaluate the further coefficients of the development /2.4/ according to the desired accuracy of solution of the Eq./1.8/.

4. Free vibrations of a nonlinear system with one degree of freedom

We shall point out that the problem of free nonlinear vibrations of a system with one degree of freedom reduces to the solution of the Volterra's nonlinear integral equation as given by Eq./1.8/.

The problem as mentioned above is formulated by the nonlinear differential equation of second order

$$\ddot{x}(t) + \omega^2 x(t) + \beta\omega\Phi\left[x(t), \dot{x}(t)\right] = 0, \tag{4.1}$$

or, in particular, when nonlinear term depends only on displacement $x(t)$

$$\ddot{x}(t) + \omega^2 x(t) + \beta\omega\Psi\left[x(t)\right] = 0 . \tag{4.2}$$

In the Eqs./4.1/ and /4.2/ β represents a small parameter for small perturbation of harmonic motion, ω - frequency of vibrations in linear range, Φ and Ψ - nonlinear functions depending on displacement and displacement rate and on displacement, respectively.

Both differential equations can be presented in the form of equivalent integro-differential and integral equations, respectively. Thus, Eq./4.1/ can be written as follows

$$x(t) = g(t) - \beta \int_{t_0}^{t} \Phi\left[x(\tau), \dot{x}(\tau)\right] \sin\omega(t-\tau)\, d\tau , \tag{4.3}$$

where

$$g(t) = A\cos\omega t + B\sin\omega t , \qquad A = x(t_0) , \quad B = \frac{1}{\omega}\dot{x}(t_0) . \tag{4.4}$$

The second equation /4.2/ may be presented in the form

/4.5/ $$x(t) = g(t) - \beta \int_{t_0}^{t} \Psi\left[x(\tau)\right] \sin\omega(t-\tau) d\tau \ .$$

By putting in the last equation

/4.6/ $$\Psi\left[x(\tau)\right] = x(\tau) F\left[x(\tau)\right]\frac{1}{\beta}$$

$$K\ (t-\tau) = \sin\omega(t-\tau) \ ,$$

we can wrate instead of Eq./4.5/

/4.7/ $$x(t) + \int_{t_0}^{t} x(\tau) F\left[x(\tau)\right] K(t-\tau)\, d\tau = g(t) \ ,$$

or in operational form

/4.8/ $$(1 + QF)\, x = g \ ,$$

which is formally identical with Eq./1.8/.

It is obvious that the more general integro-differential equation /4.3/ may be also reduced to the analogous operational form /4.8/ if only the function Φ can be presented as being

/4.9/ $$\Phi\left[x(\tau),\ \dot{x}(\tau)\right] = x(t)\ \bar{F}\left[x(t),\ \dot{x}(t)\right].$$

The last representation must be always allowable since its physical significance, $x(\tau)$ being the physical effect of the process considered and $\bar{F}$ - magnifying factor in nonlinear range depending on this very effect and its derivative.

Thus, we conclude that the method of solution as given in § 3 may be applied to the problem of nonlinear vibrations stated above.

5. Creep of viscoelastic material with small nonlinearity

The equation /1.8/ may be interpreted as the relation between stress s and strain e for a nonlinear viscoelastic material. If the function F is of the type /1.2/ then for sufficiently small β the process represented by Eq./1.8/ may be considered as elastic-creeping deformation under stress s(t) with small deviation from the linearity. In particular, s^* denotes the value of limiting stress at the nonlinear range and the kernel of the operator K /1.7/ is the rate of creep factor.

We assume that at the instant t_o the stress s(t) is applied which do not exceed significantly the limiting value s^*. Then the first coefficient of the development /2.4/ is constant and can be identified with the value of limiting stress $s^* = s_o$ as bein the solution of linear equation /2.11/,

$$/5.1/ \qquad s_o = (1 + \alpha_1 Q)^{-1} e \quad .$$

Thus, the first coefficients s_o expresses the linear share of process in generally nonlinear process. In this case the linear process occurs under constant stress s_o and the nonlinearity is involved by the increase of stress over the limiting value. According to that the operator /1.7/ becomes

$$/5.2/ \qquad Q(1) = Q_1 = \int_{t_o}^{t} K(t-\tau)\, d\tau \quad ,$$

if we assume the kernel depending on the difference of its arguments. Then the Eq./5.1/ may be simpy written

$$/5.3/ \qquad s_o = \varepsilon s \left(1 + \alpha_1 Q_1\right)^{-1} \; ,$$

or after developing /see Eq./3.4//

$$s_o = ae \sum_{i=o}^{\infty} (-1)^i \alpha_1^i Q_1^i \quad . \tag{/5.4/}$$

Introducing the operator /5.2/ into Eqs./3.6/ and /3.12/ we obtain the further coefficients, respectively ,

$$s_1 = -\alpha_2 s_o^2 (1 + \alpha_1 Q_1)^{-1} Q_1 \quad , \tag{/5.5/}$$

$$s_2 = -s_o^3 (1 + \alpha_1 Q_1)^{-1} Q_1 \left[\alpha_3 - 2\alpha_2 (1 + \alpha_1 Q_1)^{-1} Q_1\right] , \tag{/5.6/}$$

where the coefficient s_o is given by the series /5.4/ and its powers can be calculated by means of the following formula

$$s_o^n = (as)^n \sum_{i=o}^{\infty} (-1)^i \alpha_1^i g_i(n) \; , \; /n=2,3,\ldots/ \; . \tag{/5.7/}$$

Here denote

$$g_o(n)=1 \; , \quad g_m(n) = \frac{1}{m} \sum_{k=1}^{m} \left[k(n+1) - m \right] Q_1^k g_{m-k}(n) \quad . \tag{/5.8/}$$

The value of the expression in brackets of Eqs./5.5/ and /5.6/ is

$$(1 + \alpha_1 Q_1)^{-1} = \sum_{i=o}^{\infty} (-1)^i \alpha_1^i Q^i \quad . \tag{/5.9/}$$

Substituting the formulas /5.7/ - /5.9/ into Eqs. /5.5/ and /5.6/ we obtain, respectively ,

$$s_1 = \alpha_2 (ae)^2 \sum_{i=o}^{\infty} \alpha_1^i h_i \; , \qquad s_2 = (ae)^3 \sum_{i=o}^{\infty} \alpha_1^i r_i \; , \tag{/5.10/}$$

$$h_m = \sum_{k=o}^{\infty} (-1)^{k+1} Q_1^{k+1} \quad ,$$

/5.11/
$$r_m = \sum_{k=o}^{m} (-1)^{k+1} Q^{k+1} \left[2\alpha_2^2 p_{m-k} + \alpha_3 g_{m-k}(3) \right] ,$$

$$p_m = \sum_{k=o}^{m} (-1)^k h_{m-k} Q_1^k \quad .$$

If we assume that the kernel of Eq./5.2/ is

/5.12/
$$K(t-\tau) = -\partial_\tau C \,(t-\tau) \; ,$$

where C denotes creep factor of the linear range, then Eq./5.2/ can be integrated giving

/5.13/
$$Q_1 = C\left(t - t_o\right) \; ,$$

since $C(0) = 0$. Thus, the solution for stress of the non-linear equation /1.8/, representing elasto-creeping deformation of viscoelastic material with small nonlinearity, is expressed by the value of limiting stress s_o and creep factor of the linear range /5.13/ which defines the stress variability in time /relaxation process/.

6. Final conclusion

On the basic of our considerations and results obtained with respect to the solution of the problem formulated above, we state what follows:

If in the integral equation /1.1/ the function e/t/ is given real function defined and continuous in the interval $t_o \leq t < \infty$ and the function $f\left[t, \tau, s(\tau)\right]$ is a given real function of the pair of points t, τ which satisfy the conditions /1.2/, continuous and satisfying Eq./1.4/ with respect to the variable s, then for the representation /1.5/ and for the inequality /3.3/ the solution of Eq./1.1/ is unique and given by the series /2.4/.

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