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Fabric tensors in bone mechanics: elastic constitutive relationships and strength criteria

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The aim of this paper is to assess the usefulness of the concept of fabric tensor in bone mechanics. Elastic and inelastic properties of cancellous and cortical bone have been investigated. Some of our previous theoretical results [59–61] have been exploited and generalized. Nonlinear elastic constitutive relationships with fabric tensor have been proposed within the framework of small deformations. The theory of representation of tensor functions has been used [8, 57, 58, 126]. It has been shown that the fabric tensor plays the role of a parametric tensor. The fabric tensors used in the relevant literature have been also discussed. Taken rigorously the definition of the fabric tensor was inspired by the structure of cancellous bone, cf. [16, 19, 21, 23, 25, 26, 44, 120, 127–129]. For cortical bone the fabric tensor is to be interpreted in the sense of parametric tensor used by Boehler [8]. Particularly, orthotropic and transversely isotropic linear elasticity have been carefully examined from the point of view of interrelations of classical material constants with the proposed material parameters and eigenvalues of the fabric tensor. Hoffman's strength criterion [48] has been extended by incorporating the fabric tensor. This criterion has been also used in the elasto-plastic constitutive relationships. Anisotropic properties of some cancellous and cortical bones have been investigated by using the relations derived.

Key words: *fabric tensors, cancellous and cortical bones, elasticity, strength criterion, anisotropy.*

1. Introduction

Some materials such as woods, granular materials, bones and plastics exhibit elastic, plastic and locking behaviour under compressive stresses. The stress-deformation curves are then strongly influenced by the density of a material, cf. Figs. 10.3 and 11.5 in [36].

At the macroscopic level, there are two major forms of bone tissue: *cortical* (*compact*) bone and *trabecular* (*cancellous*, *spongious*) bone. Cortical bone is almost solid, while trabecular bone appears as a lattice of rods, plates and arches, cf. Fig. 1. These elements are collectively referred to as *trabeculae*.

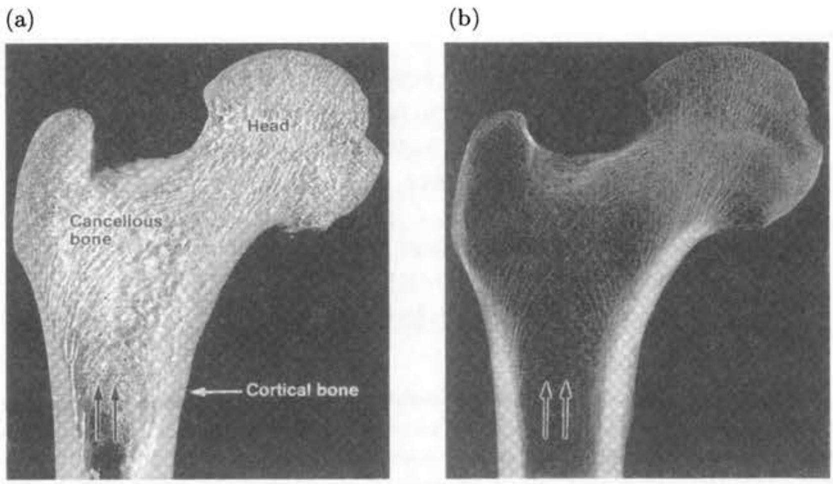

FIGURE 1. Bone section and radiograph of dried proximal end of femur. (a) Photograph of 8-mm-thick proximal femur cut in a frontal plane. The head and greater trochanter are covered by a thin shell of cortical (compact) bone, whereas the shaft is covered by a thick cylinder of cortical bone. (b) Radiograph of same bone section. The disappearance of the trabeculae is the basis of the osteoporosis, after [26].

Cancellous (spongy or trabecular) bone is quite porous; often more than half of the bone volume is associated with pore volume [19, 26, 33, 36], cf. also Figs. 1 and 2. The cellular structure of cancellous bone is made up of an interconnected network of rods or plates. A network of rods produces low-density open cells, while one of the plates gives higher-density virtually closed cells. There are some theoretical models for the elastic modulus and strength dependence upon structural density of very high porosity open cell or closed cell materials. These models help to explain the obvious trends in properties with density [26, 36]. Cancellous bone structure is anisotropic as well as porous and inhomogeneous. In the mechanics of porous materi-

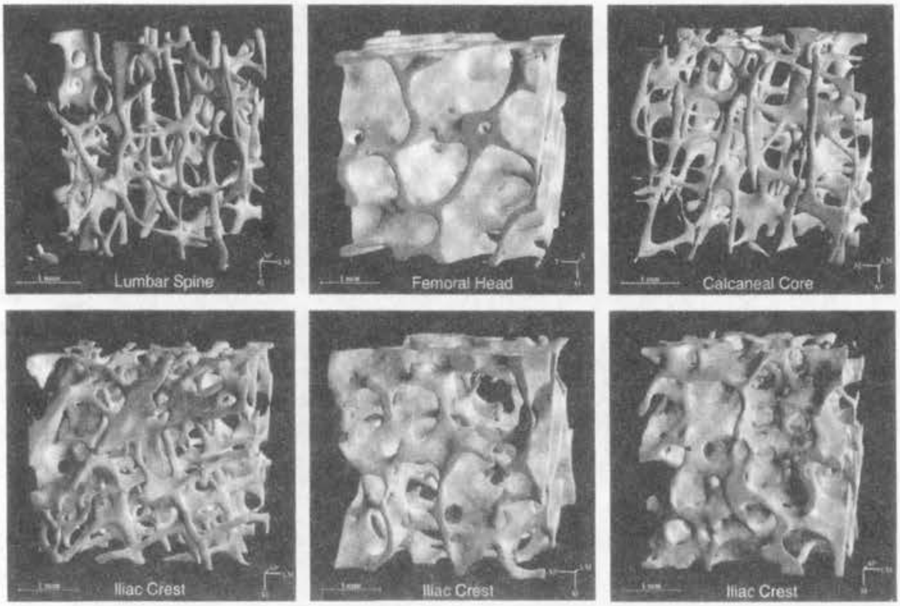

FIGURE 2. Typical 3D micro-CT images from various sites of the human skeleton (top), and the same site but different individuals (bottom), after [95].

als, it is recognized that porosity is the primary measure of local material microstructure. There appears to be general agreement that a tensor is the best second measure of local material microstructure in many porous and composite materials. Following the work of Oda [85], this tensor is generally called *fabric tensor*. The definition of the fabric tensor varies with the type of material and investigator. For example, Kanatani [64] expands the distribution density function in spherical harmonics and obtains an infinite series of even rank tensors. The first of these tensors is a second rank tensor.

In Section 2 of our paper we shall discuss structural tensors currently used in the bone mechanics. The anisotropy measure proposed by Rychlewski and Zhang [101] will be applied. Constitutive equations for geometrically linear elastic materials characterized by a positive definite structural tensor will be introduced in Section 3. Specifically, linear relationships will be discussed. In contrast to the papers [15, 16, 21, 23, 25, 26, 39, 120, 121, 127, 129], the structural tensor will be treated as a parametric tensor, and not as a variable. Such an approach is consistent with the general theory of anisotropic tensor functions [8, 57–59, 107, 109, 126]. In Section 4 we shall generalize Hoffman's [48] strength criterion, well known in the composite mechanics, in a manner suitable for defining a yield condition for the trabecular bone as well as constitutive relationships for elastic perfectly-plastic materials within the framework of small deformations. In contrast to Tsai and Wu's criterion [65, 67, 107, 112, 113, 116], which was used in the papers [14, 19, 26] as a strength criterion for bones, Hoffman's criterion requires carrying out only standard strength tests. The last criterion requires no additional hypotheses concerning determination of material parameters by performing multi-axial tests.

We observe that mechanical properties of bones have already been the subject of many books, cf. [19, 26, 33, 36] and the references cited therein. The review paper by Keaveny and Hayes [66] summarizes the state-of-the art in the trabecular bone mechanics up to 1992. For more recent developments the reader is referred to the books [1, 26, 77, 78] and the papers [9, 28, 34, 40, 42, 48, 49, 52, 54, 62, 63, 67, 71, 79, 81, 82, 83, 92, 118, 124].

2. Fabric tensor

In the mechanics of porous materials, it is recognized that porosity is the primary measure of local material microstructure. There appears to be general agreement that a tensor is the best second measure of local material microstructure in many porous and composite materials. Following the work of Oda [85], this tensor is generally called *fabric tensor*. The definition of the fabric tensor varies with the type of material and investigator. For example, Kanatani [64] expands the distribution density function in spherical harmonics and obtains an infinite series of even-rank tensors. The first of these tensors is a second rank tensor. For orthotropic and transversely isotropic materials fabric tensors are positive definite second-rank tensors.

The geometric and spatial properties of trabeculae in cancellous bone are collectively known as the *cancellous bone architecture*, cf. [86, 87]. Nowadays, X-ray micro-computed tomography (μCT) and magnetic resonance imaging (MRI) are used to study the real three-dimensional (3D) architecture of bone in a non-destructive way, cf. [75, 76, 95] and the relevant literature cited therein. Recently, a new imaging technique have been developed that can produce images of trabecular bone structures in vivo. Three-dimensional peripheral quantitative computed tomography (3D-pQCT) provides images with an isotropic resolution of $165\,\mu$m, magnetic resonance imaging can image trabecular bone in vivo at a resolution of approximately 150-300$\,\mu$m, cf. [88] and the references therein. Although not as good as that from μCT, this resolution suffices to visualize the trabecular network. We recall that μCT can provide a resolution of the order of $10\,\mu$m for a 1 cm region.

For histomorphometric studies of human trabecular bone the reader is referred to [114, 115]. There are 3D imaging tools available for standard CT and MRI systems to depict surfaces of organs from a stack of 2D images.

The resolution is about $10\,\mu$m. Odgaard [87] proposed the following classification of the methods for quantification of cancellous bone architecture: (a) basic stereological methods, (b) methods based on 3D reconstruction, (c) traditional 2D histomorphometric methods, and (d) ad hoc 2D methods, cf. Table 14.1 in [87].

The most striking, architectural property of cancellous bone is probably its anisotropy, that is, orientation of trabeculae. The mechanical properties of bone are also anisotropic, so the first requirement for a formulation between mechanics and architecture must be an ability to handle anisotropy. Different methods for quantification of anisotropy have been suggested, but it should be noted that there is no single definition of architectural anisotropy [86, 87]. Odgaard [87] has showed an example of a 2D structure that yields different results depending on the specific anisotropy measure used, cf. also [37, 38, 103, 106].

Cowin [15, 16] introduced the concept of fabric tensor in bone mechanics and defined fabric tensor as any positive definite, second-rank tensor that gives a local description of the architectural anisotropy (also called *fabric*).

In this paper we shall provide a general framework for elastic and elastic-plastic orthotropic materials, and structural anisotropy is described by a second-order tensor, called the fabric tensor, cf. [15, 16, 19, 21, 23, 25, 26, 44, 59–61, 85–87, 89, 102, 103, 105, 106, 119–121, 127–129].

Let us introduce this tensor. Firstly, however, following Whitehouse [91] we recall the notion of the mean intercept length (MIL). This author measured MIL in cancellous bone as a function of direction on polished plane sections. The basic principle of the 2D MIL method consists of placing a series of equally spaced grid lines with orientation φ onto a structure, and counting the number of intersections $n = 1, \ldots, N(\varphi)$ between the grid and the bone-marrow interfaces, see Fig. 3(a). The intersection n of a grid line with the bone structure has length I_n. The MIL is simply the total line length divided by the number of intersections:

$$\mathrm{MIL}(\varphi) = \frac{\sum\limits_{n=1}^{N} I_n}{N(\varphi)}, \quad \varphi \in [0, \pi]. \tag{2.1}$$

Whitehouse [122] observed that polar plot of Eq. (2.1) approximated an ellipse for planar sections through cancellous bone, see Fig. 3(b). A generalization of this observation into 3D space would result in an ellipsoid ($\mathrm{MIL}(\varphi, \theta)$ or $\mathrm{MIL}(\mathbf{n})$ – for 3D case). Some authors have used a modified definition of MIL which reports the mean length of intercepts through the bone phase only, cf. [86, 87, 103, 106].

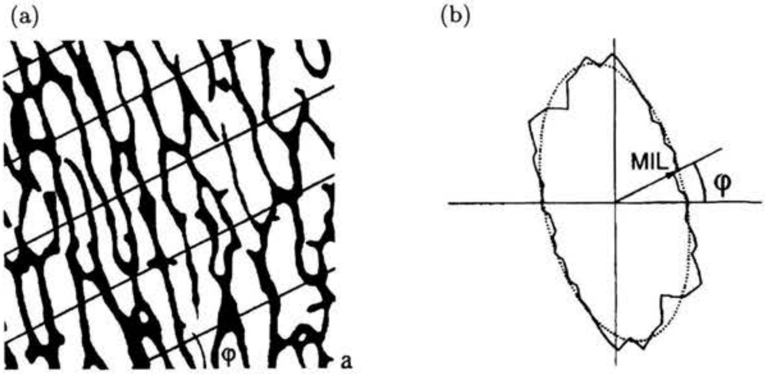

FIGURE 3. Determination of the mean intercept length (MIL). (a) A grid of test lines under an angle φ is placed on a cancellous bone specimen. The total length of the grid lines is divided by the number of intersections with the bone/marrow interfaces to find $\mathrm{MIL}(\varphi)$. (b) Polar diagram of $\mathrm{MIL}(\varphi)$. Anisotropy is characterised by a fitted ellipse, after [15, 16, 103].

We observe that Smit [103] introduced the notion of the *mean bone length* (MBL) to quantify the trabecular bone structure in all directions, cf. also [105].

Harrigan and Mann [44] extended Whitehouse's approach to the three-dimensional case and showed that $L(\mathbf{n})$, as a function of a direction $\mathbf{n}$, would be represented by ellipsoids and would therefore be equivalent to a positive definite second-order tensor.

Cowin defined the mean intercept length tensor $\mathbf{M}$ by the following formula, [21, 23]

$$\frac{1}{\mathrm{MIL}^2(\mathbf{n})} = \mathbf{n} \cdot \mathbf{M} \cdot \mathbf{n} = M_{ij} n_i n_j, \tag{2.2}$$

where $\mathbf{n}$ is the unit vector in the direction of the test line. This author defined the fabric tensor of cancellous bone to be the inverse square root of the tensor $\mathbf{M}$ [15, 16, 21, 23, 25, 26]:

$$\mathbf{H} = \frac{1}{\sqrt{\mathbf{M}}}. \tag{2.3}$$

Obviously, $\mathbf{H}$ is well defined because $\mathbf{M}$ is a positive definite and symmetric tensor, see Appendix A. The components of $\mathbf{M}$ or the mean intercept ellipsoid can be measured by using the techniques described by Harrigan and Man [44] for a cubic specimen.

Cowin [19] concluded the largest and smallest values of the set $\{H_1, H_2, H_3\}$ are associated with the largest and smallest values of Young's moduli, respectively. The fabric tensor may be normalized by the requirement [19, 119]

$$\operatorname{tr}\mathbf{H} = H_1 + H_2 + H_3 = 1. \tag{2.4}$$

Goulet et al. [39] applied the concept of the mean intercept length to investigate the relationships between the structural parameters for cancellous bone and to determine their correlation to the mechanical properties and to evaluate which parameters are important for maintaining bone strength and integrity.

One rather general and simple way to construct a fabric tensor for a material is from a set of N measurements of material microstructural features, each measurement characterized by a scalar m_k and a unit vector $\mathbf{n}_k$ $(\mathbf{n}_k \cdot \mathbf{n}_k = 1)$, where $k = 1, \ldots, N$. The normalized fabric tensor $\mathbf{H}$ (for the fabric tensor we use the notation $\mathbf{H}$ for brevity remembering that this is not necessarily the tensor defined by (2.3)) is defined in terms of N observations as [21]

$$\left(\sum_{k=1}^{N} m_k\right)\mathbf{H} = \sum_{k=1}^{N} m_k \mathbf{n}_k \otimes \mathbf{n}_k. \tag{2.5}$$

Due to the normalization, $\operatorname{tr}\mathbf{H} = 1$, since $\mathbf{n}_k \cdot \mathbf{n}_k = 1$ for all k.

An alternative approach to the fabric tensor has been discussed by Zysset and Curnier [127].

An elementary microstructural description is contained in a single scalar property such as relative density, while material anisotropy requires fabric tensors of higher even-rank tensors [64]. Kanatani's [64] approach can be applied to a class of materials with strictly positive morfological properties that are radially symmetric. Then we can use a scalar-valued orientation distribution function $h(\mathbf{N}) > 0$, where $\mathbf{N} = \mathbf{n} \otimes \mathbf{n}$ is the tensor product of the unit vector $\mathbf{n}$ specifying the orientation. Assuming the function to be square integrable it can be expanded in a convergent Fourier series:

$$\begin{aligned} h(\mathbf{N}) &= g(\mathbf{N})\,1 + \mathbf{G}\cdot\mathbf{F}(\mathbf{N}) + \mathsf{G} : \mathsf{F}(\mathbf{N}) + \ldots \\ &= g(\mathbf{N})\,1 + G_{ij}F_{ij}(\mathbf{N}) + \mathsf{G}_{ijkl}\mathsf{F}_{ijkl}(\mathbf{N}) + \ldots, \end{aligned} \tag{2.6}$$

where 1, $\mathbf{F}(\mathbf{N})$ and $\mathsf{F}(\mathbf{N})$ are even rank tensorial basis functions and g, $\mathbf{G}$ and $\mathsf{G}(\mathbf{N})$ the corresponding even rank fabric tensors [64]. In bone mechanics we can use an approximation based on a scalar and a symmetric, traceless second-rank fabric tensor. Then the first tensorial basis function is

$$\mathbf{F} - \frac{1}{3}\mathbf{I}, \tag{2.7}$$

while the tensorial coefficients are calculated by

$$g = \frac{1}{4\pi} \int_S h(\mathbf{N})\, dS, \qquad \mathbf{G} = \frac{15}{8\pi} \int_S h(\mathbf{N})\, \mathbf{F}(\mathbf{N})\, dS, \tag{2.8}$$

where S is the surface of the unit sphere. For the particular case of an ellipsoidal distribution function we have

$$h(\mathbf{N}) = \frac{1}{\sqrt{\mathbf{N} \cdot \mathbf{M}}}. \tag{2.9}$$

Structural tensors are not necessarily constructed according to (2.3) or (2.5). Conceptually different approach consists in measuring pore surfaces in much the same way as in continuum damage mechanics, and not just the MIL (mean intercept length). Obviously, here we do not discuss counterparts of fourth-order tensors describing damage behavior.

An alternative approach consists in using the methods of geometry of random fields, well-known in describing the geometry of porous media. For a comprehensive review the reader is referred to the paper by Adler and Thovert [2], cf. also Telega and Bielski [108]. As far as we know in abundant literature on the description of bone architecture the methods of geometry of random fields have not yet been exploited. The reason for this seems to be the fact that bone microstructure is believed to be deterministic, cf. Cowin [26]. Is it really? Has it been proven? Even if it is, one cannot preclude application of *methods* and *tools* used in description of random (and fractal) porous media, like in Adler and Thovert [2]. For example, it would be interesting to compare the approach consisting in using fabric tensors with second-order correlation functions.

Since MIL is unable to detect some forms of architectural anisotropy, volume-based measures were introduced with the volume orientation (VO) method and star volume distribution (SVD) method, see Fig. 4. Star length distribution (SLD) method provides a minor modification of the SVD measure, see Fig. 4c. All of these types of measurements of the orientation distribution function $h(\mathbf{N}) > 0$ can be used to obtained fabric tensors (MIL fabric tensor, VO fabric tensor, SVD (SLD) fabric tensor). The VO fabric tensor is a description of the typical distribution of trabecular bone volume around a typical point within a trabecula. This is in contrast to the MIL fabric tensor, which describes the orientation of interfaces between bone and marrow. For more details pertaining to VO, SVD (SLD) methods the reader is referred to papers [37, 38, 86, 87, 103, 106] and the references cited therein.

We proceed to reviewing recent papers dealing with the description of architecture of cancellous bone and measurements of $h(\mathbf{N})$. Note that those

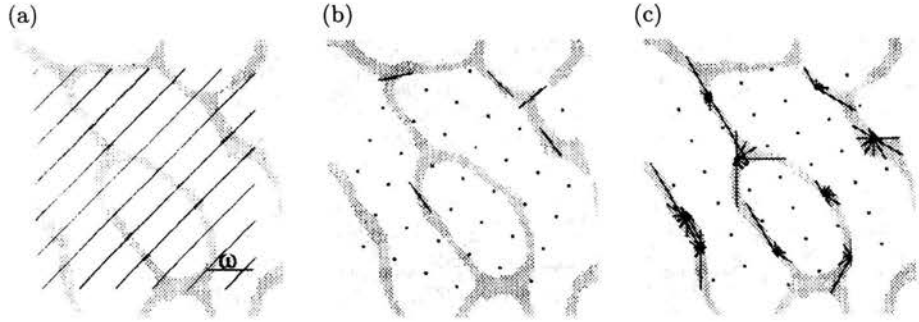

FIGURE 4. Principles for determining architectural anisotropy. (a) Mean intercept length (MIL) measurement. (b) Volume orientation (VO) measurement. A randomly translated point grid is placed on the structure. For each point hitting the phase of interest (bone) the orientation of the longest intercept through the point is determined, and this orientation is the local volume orientation for the point. In the figure, five local volume orientations are sampled and depicted by the compass needles. c) Star volume distribution (SVD) and star length distribution (SLD) measurements. A randomly translated point grid is placed on the structure. For each point hitting the phase of interest (bone) the intercept length through the point is determined for several orientations. The intercept length are used directly for SLD and cubed for the SVD measure. In the figure intercept lengths have been determined for six orientations, after [86, 87].

papers mainly deal with 2D cases. Though from the point of view of mechanics the relevant considerations are simply introductory, yet they clearly show that the manner of construction of structural tensors is important. For instance, depending upon the choice of structural tensor (and accuracy of its determination) one can obtain differing data on anisotropy of investigated structure, cf. 2D cases studied in the papers [37, 38, 86, 87, 103, 106]. In our opinion, even at the introductory stage of such investigations, it is important to precisely distinguish constructions of second-order structural tensors.

This obviously follows from the fact that, in a good approximation, cancellous bone may be treated as an orthotropic material, cf. Jemioło and Telega [60], Cowin [26]. However, it should be remembered that the orientation of the principal axes of orthotropic cancellus bone depends on location (inhomogeneity; cf. Figs. 1 and 5). Consequently, one should impose a requirement of the determination of *fields* of structural tensors. Such a requirement entails the problem of proper determination of representative samples of cancellus bone in specific regions of inhomogeneous material. Search for only "averaged quantities" (currently typical) has only a qualitative value and cannot result in good agreement of theoretical predictions with mechanical tests. Here, by theoretical predictions we mean constitutive relationships involving structural tensors.

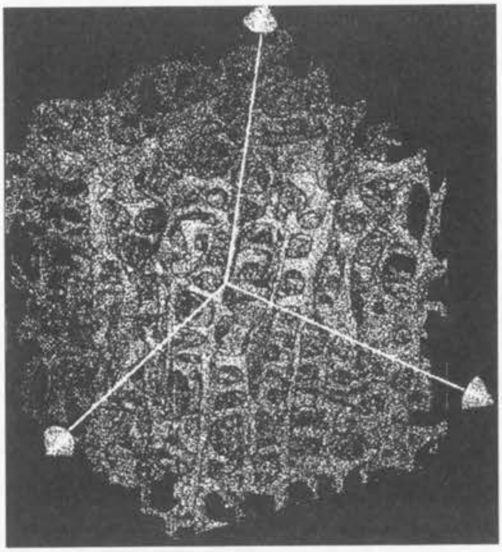

FIGURE 5. An illustration of the trabecular grain. By trabecular grain we mean a set of three ordered orthogonal directions, the first one of which lies along the local predominant trabecular direction, which is locally the stiffest direction; the second and third directions are directions orthogonal to each other in the plane perpendicular to the first direction and represent directions of extrema in stiffness in the local region of the cancellous bone. The specimen is a $7\,\mathrm{mm}^3$ cube, after [123].

Since the problem of determination of $h(\mathbf{N})$ according to the methods VO and SVD (SLD) is numerically more complex than by using MIL, therefore many authors tried to modify measurements of the function $h(\mathbf{N})$. The basic idea is a modification preserving the basic idea of MIL based on parallel grid (Fig. 4(a)), cf. [37, 38, 53, 103, 106]. Analyzing various proposed methods of measurements of the function $h(\mathbf{N})$ one should remember that the task is to determine $h(\mathbf{N}(\mathbf{x}))$. Then the manner of choice of $\mathbf{x}$ has no significant meaning. Important is what we really measure. Numerical realization, i.e. algorithms of counting of the function $h(\mathbf{N}(\mathbf{x}))$ in a representative volume can differ. This obviously influences the accuracy of determination of the function, but not the essence of measurements. Form the viewpoint of bone biomechanics important is whether the manner of measurement (and its numerical realization) does not lead to loss of data pertaining to material symmetry, preferred directions, etc.

As we have already noticed previously, in recently published papers on identification and measurement of structural anisotropy mainly 2D cases were investigated. Of interest is the paper published recently by Inglis and Pietruszczak [53] in large part containing a review, verifications and algorithms pertaining both to the measurement of the function $h(\mathbf{N})$ and to the determination of structural tensors. In the paper [53] the authors combined the concept of measuring MIL with the approach based on relations (2.6)–(2.8). MIL, SVD and other methods have been modified, cf. also [103, 106]. Note, however, that the results provided in [53] confirm the fact that the problem is still at a stage of introductory investigations.

A specific form of the fabric tensors $\mathbf{M}$, $\mathbf{H}$ or $\mathbf{J} = g\mathbf{I} + \mathbf{G}$ is not required for our subsequent developments. The only assumption is that $\mathbf{M}$, $\mathbf{H}$ and $\mathbf{J}$ be positive definite and symmetric second-order tensors. Below and in the subsequent sections we use the notation $\mathbf{H}$ for the fabric tensor, remembering that this is not necessarily the tensor defined by (2.3).

We observe that since $\mathbf{H}$ is positive definite (or defined by (2.5)) therefore it can be normalized according to (2.4). It seems, however, that a natural norm for a second-order tensor is

$$\|\mathbf{H}\| = \sqrt{\operatorname{tr}\mathbf{H}^2}. \tag{2.10}$$

Consequently, more convenient to apply is the structural tensor defined by

$$\bar{\mathbf{H}} = \frac{1}{\|\mathbf{H}\|}\mathbf{H}. \tag{2.11}$$

Rychlewski and Zhang [101] introduced the following measure of orthotropy degree of $\mathbf{H}$:

$$\delta(\mathbf{H}) = \frac{\sqrt{2}}{2}\frac{(H_1 - H_3)}{\|\mathbf{H}\|}. \tag{2.12}$$

We note that in [101] a general anisotropy measure has also be introduced for tensors, tensor functions and tensor functionals. In the specific case of symmetric second-order tensors this measure reduces to (2.12). From (2.12) we conclude that if $\mathbf{H}$ is an isotropic tensor then $\delta(\mathbf{H}) = 0$. From the results due to Turner et al. [120] it follows that in the case of human proximal tibia the average eigenvalues of $\mathbf{H}$ normalized according to (2.4) are equal to: $H_1 = 0.429$, $H_2 = 0.292$, $H_3 = 0.278$; then we have $\delta(\mathbf{H}) = 0.185$. These authors report the eigenvalues of $\mathbf{H}$ between 0.178 and 0.585. By using (2.12) we find $\delta(\mathbf{H})_{\max} = \sqrt{2}/2$. Hence we conclude that the morfological property of the bone investigated is not so strongly orthotropic. Closer inspection of the average eigenvalues of $\mathbf{H}$ given in [120] reveals that trabecular bone of the human proximal tibia behaves *approximately* as a transversely isotropic material (since $H_2 \approx H_3$). From Table 1d of [120, pp. 556] it also follows that the bone investigated is significantly inhomogeneous.

If the tensor **H** is determined by counting the function $h(\mathbf{N})$ in a manner different from that based on MIL, then the degree of anisotropy of investigated material calculated by using Eq. (2.12) will differ from the value given above. Usually, in order to determine the degree of bone anisotropy one uses the relations of principal values of the tensor **H**, cf. [10, 68, 69, 96].

3. Elasticity

3.1. Basic constitutive relationships

For small deformations both compact and cancellous bones exhibit elastic properties, cf. Fig. 6. Below we propose elastic constitutive relationships for such bones.

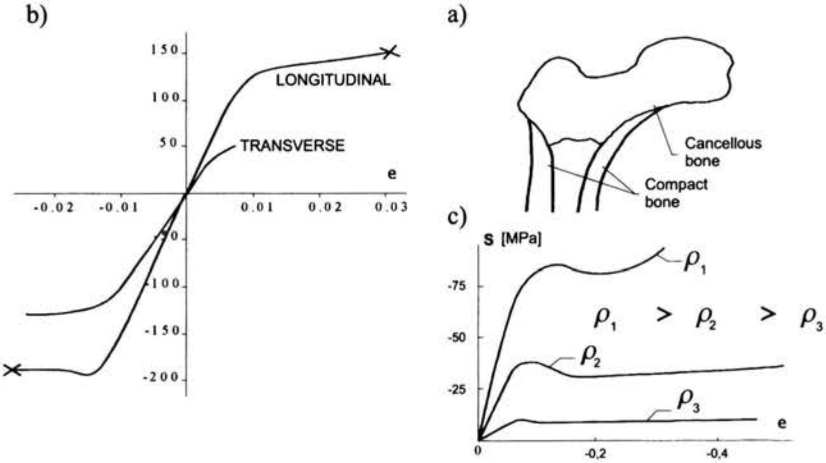

FIGURE 6. (a) A schematic drawing showing the cancellous (trabecular) bone and the compact (cortical) bone in the head of the human femur. (b) Two stress-strain curves for wet compact bone loaded in the longitudinal and transverse directions, after [28]. (c) Compressive stress-strain curves for several relative densities ρ_i $(i = 1, 2, 3)$ of wet cancellous bone, after [28].

As is well known, elastic models in Green's sense derived via energy formulation are insensitive to the loading path and the whole deformation process is reversible. Two equivalent descriptions of the constitutive relationships are possible, namely

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{E}} \qquad \text{or} \qquad \mathbf{E} = \frac{\partial W^*}{\partial \mathbf{T}}, \tag{3.1}$$

where **T** is the stress tensor and **E** is the small strain tensor. The specific elastic energy W and the specific complementary energy W^* are convex scalar-valued functions

$$W = W(\mathbf{E}), \qquad W^* = W^*(\mathbf{T}), \tag{3.2}$$

which have to satisfy the following relations

$$\begin{aligned} & W(\mathbf{0}) = 0,\ W^*(\mathbf{0}) = 0,\ \mathbf{T}\cdot\mathbf{E} = \operatorname{tr}(\mathbf{TE}) = W + W^*, \\ & \forall \mathbf{Q} \in S,\ W(\mathbf{E}) = W\left(\mathbf{QEQ}^T\right),\ W^*(\mathbf{T}) = W^*\left(\mathbf{QTQ}^T\right), \\ & \forall \tilde{\mathbf{E}} \in T^3_{\mathbf{E}},\ \tilde{\mathbf{E}}\cdot\frac{\partial^2 W}{\partial\mathbf{E}\otimes\partial\mathbf{E}}\cdot\tilde{\mathbf{E}} \geqslant 0;\ \forall \tilde{\mathbf{T}} \in T^3_{\mathbf{T}},\ \tilde{\mathbf{T}}\cdot\frac{\partial^2 W^*}{\partial\mathbf{T}\otimes\partial\mathbf{T}}\cdot\tilde{\mathbf{T}} \geqslant 0. \end{aligned} \tag{3.3}$$

Here $S \subset O(3)$ denotes material group. More precisely, once W is known, the complementary potential W^* is calculated as the Fenchel conjugate:

$$W^*(\mathbf{T}) = \sup\left\{\mathbf{T}\cdot\mathbf{E} - W(\mathbf{E}) \mid \mathbf{E} \in T^s_{\mathbf{E}}\right\}. \tag{3.4}$$

In this case W may be only piecewise regular and $(3.3)_3$ is no longer valid in the whole space. The specific energy is only once differentiable for materials with different properties in tension and compression. Biological materials like bones are of such a type. The presence of microscopic damage also influences the macroscopic response of bones. Then equation $(3.1)_1$ may have the form of the subdifferential

$$\mathbf{T} \in \partial W(\mathbf{E}), \tag{3.5}$$

where ∂ denotes the subdifferential of the convex function W. For the subdifferential calculus the reader is referred to Rockafellar [93] and Rockafellar and Wets [94].

Assumption $(3.3)_3$ can be weakened. More precisely, one can assume that W is a convex function on a convex set $A_{\mathbf{E}} \subset \mathrm{T}^3_{\mathbf{E}}$ whilst W^* is a convex function given by the Fenchel transformation

$$\begin{aligned} W^*(\mathbf{T}) &= \sup\left\{\mathbf{T}\cdot\mathbf{E} - W(\mathbf{E}) \mid \mathbf{E} \in A_{\mathbf{E}}\right\} \\ &= \sup\left\{\mathbf{T}\cdot\mathbf{E} - \tilde{W}(\mathbf{E}) \mid \mathbf{E} \in T^3_{\mathbf{E}}\right\}, \mathbf{T} \in T^3_{\mathbf{T}} \end{aligned} \tag{3.6}$$

where

$$\tilde{W}(\mathbf{E}) = \begin{cases} W(\mathbf{E}) \text{ if } \mathbf{E} \in A_{\mathbf{E}}, \\ +\infty \text{ otherwise.} \end{cases} \tag{3.7}$$

Similarly, one can prescribe the function W^* defined on a convex set $A_{\mathbf{T}}$ and then calculate $W(\mathbf{E})$ as the Fenchel conjugate

$$\begin{aligned} W(\mathbf{T}) &= \sup\left\{\mathbf{T}\cdot\mathbf{E} - W^*(\mathbf{E}) \mid \mathbf{T} \in A_{\mathbf{T}}\right\} \\ &= \sup\left\{\mathbf{T}\cdot\mathbf{E} - \tilde{W}^*(\mathbf{E}) \mid \mathbf{T} \in T^3_{\mathbf{T}}\right\}, \mathbf{E} \in T^3_{\mathbf{E}}. \end{aligned} \tag{3.8}$$

The convex sets $A_{\mathbf{E}}$ and $A_{\mathbf{T}}$ are usually given by

$$A_{\mathbf{E}} = \left\{ \mathbf{E} \in T^3_{\mathbf{E}} \mid g(\mathbf{E}) < c_1 \right\}, \; A_{\mathbf{T}} = \left\{ \mathbf{T} \in T^3_{\mathbf{T}} \mid h(\mathbf{T}) < c_2 \right\} \tag{3.9}$$

These sets contain zero elements. The surfaces

$$g(\mathbf{E}) = c_1, \quad h(\mathbf{T}) = c_2, \tag{3.10}$$

may be called strength criterions expressed in terms of strains and stress, respectively. Note that convex sets $A_{\mathbf{E}}$ and $A_{\mathbf{T}}$ may be nonsmooth. Then boundary surfaces may be determined by several functions, say $g_1, \ldots, g_k$; $h_1, \ldots, h_k$, respectively.

If W and W^* are strictly convex then the functions $g(\mathbf{E})$ and $h(\mathbf{T})$ are interrelated by relations resulting from (3.1). Consequently, the formulations of strength criterions in the space of strains and stresses are equivalent.

Proper choice of strength criterion is verified by agreement with experimental data, cf. [11, 14, 55, 56, 98, 107, 112, 113, 116].

3.2. Constitutive relationships with fabric tensor

We assume that bone is an orthotropic material (particularly transversely isotropic or isotropic), cf. Cowin [26]. Then the symmetry group of material is defined as follows

$$S \equiv \left\{ \mathbf{Q} \in O(3) \left| \mathbf{Q}\mathbf{H}\mathbf{Q}^T = \mathbf{H} \right. \right\}, \tag{3.11}$$

where the symbol „T" denotes the transpose of a tensor and $O(3)$ is the full orthogonal group of 3D Euclidian space.

If all the eigenvalues of the tensor $\mathbf{H}$ are different then the constitutive equation of type $(3.1)_1$ has the form, cf. Jemioło and Telega [60],

$$\begin{aligned} \mathbf{T} = \alpha_1 \mathbf{I} + \alpha_2 \mathbf{H} + \alpha_3 \mathbf{H}^2 + 2\alpha_4 \mathbf{E} \\ + \alpha_5 (\mathbf{EH} + \mathbf{HE}) + \alpha_6 \left(\mathbf{EH}^2 + \mathbf{H}^2\mathbf{E} \right) + 3\alpha_7 \mathbf{E}^2, \end{aligned} \tag{3.12}$$

where

$$\alpha_m = \frac{\partial f}{\partial I_m}, \quad \frac{\partial \alpha_m}{\partial I_n} = \frac{\partial \alpha_n}{\partial I_m}; \quad m, n = 1, \ldots, 7. \tag{3.13}$$

The general form of W is given by

$$\begin{aligned} W(\mathbf{E}, p) &= f(I_m(\mathbf{E}, p)) \\ &= f\left(\operatorname{tr}\mathbf{E}, \operatorname{tr}\mathbf{EH}, \operatorname{tr}\mathbf{EH}^2, \operatorname{tr}\mathbf{E}^2, \operatorname{tr}\mathbf{E}^2\mathbf{H}, \operatorname{tr}\mathbf{E}^2\mathbf{H}^2, \operatorname{tr}\mathbf{E}^3, p \right). \end{aligned} \tag{3.14}$$

In (3.14) p is an arbitrary scalar or a set of scalars. Specifically, p may represent the relative density of material, porosity or arbitrary scalar variables related to the description of bone architecture. These scalar functions do not influence on the choice of number of orthotropic invariants of the strain tensor $\mathbf{E}$. For an inhomogeneous material W and W^* depend explicitly on $\mathbf{x} \in \Omega$ through p, where $\bar{\Omega}$ denotes the closure of a domain occupied by the body considered in its undeformed configuration ($p = \hat{p}(\mathbf{x})$ and $\mathbf{H} = \hat{\mathbf{H}}(\mathbf{x})$).

Obviously, if the eigenvalues of the tensor $\mathbf{H}$ are different then the constitutive equation of the type $(3.1)_2$ has the form

$$\mathbf{E} = \beta_1\mathbf{I} + \beta_2\mathbf{H} + \beta_3\mathbf{H}^2 + 2\beta_4\mathbf{T} + \beta_5\left(\mathbf{TH} + \mathbf{HT}\right) + \beta_6\left(\mathbf{TH}^2 + \mathbf{H}^2\mathbf{T}\right) + 3\beta_7\mathbf{T}^2, \quad (3.15)$$

where

$$\beta_m = \frac{\partial g}{\partial J_m}, \quad \frac{\partial \beta_m}{\partial J_n} = \frac{\partial \beta_n}{\partial J_m}; \quad m,n = 1,...,7. \quad (3.16)$$

Now the general form of W^* is given by

$$W^*(\mathbf{T},p) = g\left(J_m(\mathbf{T}),p\right) = g\left(\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{TH}, \operatorname{tr}\mathbf{TH}^2, \operatorname{tr}\mathbf{T}^2, \operatorname{tr}\mathbf{T}^2\mathbf{H}, \operatorname{tr}\mathbf{T}^2\mathbf{H}^2, \operatorname{tr}\mathbf{T}^3, p\right). \quad (3.17)$$

We recall that our considerations are restricted to convex elastic strain energy functions, though representations (3.12), (3.14), (3.15) and (3.17) are general, cf. Jemioło and Telega [60].

If $H_1 = H_2 = H$ and $H \neq H_3$, $\mathbf{e}_3$ being a preferred direction, the bone investigated is locally transversely isotropic. Equations (3.12) becomes then much simpler since among the invariants and generators the following identities hold true, respectively,

$$\operatorname{tr}\mathbf{E}^\alpha\mathbf{H}^\beta = H^\beta \operatorname{tr}\mathbf{E}^\alpha + \left(H_3^\beta - H^\beta\right)\operatorname{tr}\mathbf{E}^\alpha\mathbf{M}, \quad (3.18)$$

where

$$\begin{aligned} \mathbf{M} &= \mathbf{e}_3 \otimes \mathbf{e}_3, \quad \alpha,\beta = 1,2, \\ \mathbf{H}^\alpha &= H^\alpha\mathbf{I} + \left(H_3^\alpha - H^\alpha\right)\mathbf{M}, \\ \mathbf{EH}^\alpha + \mathbf{H}^\alpha\mathbf{E} &= 2H^\alpha\mathbf{I} + \left(H_3^\alpha - H^\alpha\right)\left(\mathbf{EM} + \mathbf{ME}\right). \end{aligned} \quad (3.19)$$

Then Eq. (3.12) takes the form

$$\mathbf{T} = \gamma_1\mathbf{I} + \gamma_2\mathbf{M} + 2\gamma_3\mathbf{E} + \gamma_4(\mathbf{EM} + \mathbf{ME}) + 3\gamma_5\mathbf{E}^2, \quad (3.20)$$

where

$$\gamma_m = \frac{\partial f}{\partial I_m}, \quad \frac{\partial \gamma_m}{\partial I_n} = \frac{\partial \gamma_n}{\partial I_m}; \quad m,n = 1,\ldots,5. \quad (3.21)$$

Thus we have

$$W(\mathbf{E},p) = f(I_m(\mathbf{E}),p) = f\left(\operatorname{tr}\mathbf{E}, \operatorname{tr}\mathbf{EM}, \operatorname{tr}\mathbf{E}^2, \operatorname{tr}\mathbf{E}^2\mathbf{M}, \operatorname{tr}\mathbf{E}^3, p\right). \quad (3.22)$$

If $H_1 = H_2 = H_3 = H$ the description fits an isotropic material, the identities among invariants and generators are, respectively

$$\operatorname{tr}\mathbf{E}^\alpha\mathbf{H}^\beta = H^\beta \operatorname{tr}\mathbf{E}^\alpha, \quad \mathbf{H}^\alpha = H^\alpha\mathbf{I}, \quad \mathbf{EH}^\alpha + \mathbf{H}^\alpha\mathbf{E} = 2H^\alpha\mathbf{E}. \quad (3.23)$$

Equation (3.12) simply becomes

$$\mathbf{T} = \delta_1\mathbf{I} + 2\delta_2\mathbf{E} + 3\delta_3\mathbf{E}^2, \quad (3.24)$$

where

$$\delta_i = \frac{\partial f}{\partial I_i}, \quad \frac{\partial \delta_i}{\partial I_j} = \frac{\partial \delta_j}{\partial I_i}; \quad i,j = 1,\ldots,3. \quad (3.25)$$

Thus we get

$$W(\mathbf{E},p) = f(I_i(\mathbf{E}),p) = f\left(\operatorname{tr}\mathbf{E}, \operatorname{tr}\mathbf{E}^2, \operatorname{tr}\mathbf{E}^3, p\right). \quad (3.26)$$

In the case of orthotropy the choice of tensor **H** (or of a set of parametric tensors) satisfying (3.11) is not unique. This problem was discussed in our papers [57, 58], cf. also Zheng [126]. For instance, the tensor **H** may be replaced by its deviator (similarly to Kanatani's approach [64], cf. also measures (2.6)-(2.8)). More details are given in Appendix A and in Section 3.4 where the model proposed by Zysset and Curnier [127] is discussed.

The following parametric tensors are also often used, cf. Appendix A,

$$\mathbf{M}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1, \quad \mathbf{M}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2, \quad \mathbf{M}_3 = \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (3.27)$$

or

$$\mathbf{M}_1, \quad \mathbf{M}_2. \quad (3.28)$$

We recall that $\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = \mathbf{I}$. In other words $S_1 \cap S_2 \cap S_3 = S_1 \cap S_2$ where

$$S_i \equiv \left\{\mathbf{Q} \in O(3) \,\middle|\, \mathbf{Q}\mathbf{M}_i\mathbf{Q}^T = \mathbf{M}_i\right\}, \quad i = 1,2,3. \quad (3.29)$$

Note that (3.29) is consistent with considerations due to Cowin and Mehrabadi [24, 25] pertaining to Hooke's law for orthotropic materials. Relation (3.29) implies, among others, that an orthotropic, not necessarily physically linear material, possesses two planes of mirror reflections.

The invariants of the functional and polynomial bases taking into account parametric tensors (3.27) were primarily determined by Boehler [6], cf. also [45, 46] and the determination of the function W (or W^*) depending on seven invariants is of worth of being considered in bone mechanics. We think here of

$$W(\mathbf{E},p) = f(I_m(\mathbf{E}),p)$$
$$= f\left(\operatorname{tr}\mathbf{M_1}\mathbf{E}^{\alpha}, \operatorname{tr}\mathbf{M_2}\mathbf{E}^{\alpha}, \operatorname{tr}\mathbf{M_3}\mathbf{E}^{\alpha}, \operatorname{tr}\mathbf{E}^3, p\right), \quad \alpha = 1,2. \quad (3.30)$$

Such a suggestion follows, among others, from the considerations provided in Section 2. One of the most important problems pertaining to bone architecture (structure) is the determination of privileged directions of materials, i.e. the determination of principal eigenvectors of the tensor $\mathbf{H} = \hat{\mathbf{H}}(\mathbf{x})$.

From the point of view of the representation theory of orthotropic tensor functions and the elasticity theory of inhomogeneous materials there are no prerequisites pertaining to dependence of W on $p(\mathbf{x})$ (or on a set $p_k(\mathbf{x})$, $k = 1, \ldots, K$). Hence it follows that one should resort to experimental tests and to additional data on bone microstructure. The approach currently used is purely formal and polynomial dependence on $p_k(\mathbf{x})$ is proposed, see Sections 3.3-3.6.

We observe that the above physically nonlinear constitutive relationships are not identical with the equations proposed in the papers [15, 19, 21, 99, 120, 127]. According to our approach the structural tensor $\mathbf{H}$ is not an argument of the function (3.1) Consequently, the material functions defined by (3.13) do not depend explicitly on three invariants of $\mathbf{H}$. In Eq. (3.12) the tensor $\mathbf{H}$ describes only the microstructure of the material. Experimental data justify the assumption of small elastic deformations for bones, cf. [66]. Those deformations are of the order of 1%. The tensor $\mathbf{H}$ could be treated as an argument of the function (3.2) provided that elastic deformations would led to a significant change of this tensor. Change with time of $\mathbf{H}$ would require application of constitutive relationships completed with an evolution equation for this tensor. Such an approach would enable us to quantitatively describe the phenomenon of bone adaptation, which is out of scope of the present contribution. The reader is referred to [17, 19, 26, 33] for more details on adaptation of bones to loading. As we already know bone is an inhomogeneous material. It means that both the coefficients α_m $(m = 1, \ldots, 7)$ and the tensor $\mathbf{H}$ depend on the point in the bone. Consequently, they depend in an explicit manner on, for instance, the density of bone at this point.

3.3. Constitutive relationships with fabric tensor proposed by Cowin [15]

In Cowin's paper [15] the term fabric tensor was generalized to mean any symmetric second-rank tensor that characterizes the local arrangement of solid material or microstructure of a porous material.

From the theory of tensor function representations (cf. [58, 126] and the references cited therein) follows that if $\mathbf{T}$ is a symmetric isotropic tensor function of two symmetric tensors $\mathbf{H}$ and $\mathbf{E}$, than $\mathbf{T}$ has the representation

$$\mathbf{T} = \gamma_1\mathbf{I} + \gamma_2\mathbf{H} + \gamma_3\mathbf{H}^2 + \gamma_4\mathbf{E} + \gamma_5\mathbf{E}^2 + \gamma_6\left(\mathbf{EH} + \mathbf{HE}\right) + \gamma_7\left(\mathbf{EH}^2 + \mathbf{H}^2\mathbf{E}\right) + \gamma_8\left(\mathbf{E}^2\mathbf{H} + \mathbf{HE}^2\right) + \gamma_9\left(\mathbf{E}^2\mathbf{H}^2 + \mathbf{H}^2\mathbf{E}^2\right), \quad (3.31)$$

where γ_1 through γ_9 are functions of the following ten invariants

$$\operatorname{tr}\mathbf{H}, \operatorname{tr}\mathbf{H}^2, \operatorname{tr}\mathbf{H}^3, \operatorname{tr}\mathbf{E}, \operatorname{tr}\mathbf{E}^2, \operatorname{tr}\mathbf{E}^3, \operatorname{tr}\mathbf{EH}, \operatorname{tr}\mathbf{EH}^2, \operatorname{tr}\mathbf{E}^2\mathbf{H}, \operatorname{tr}\mathbf{E}^2\mathbf{H}^2 \quad (3.32)$$

Cowin reduced this representation by the requirement that $\mathbf{T}$ be linear in $\mathbf{E}$ and that $\mathbf{T}$ vanish when $\mathbf{E}$ vanishes, thus

$$\mathbf{T} = \gamma_1\mathbf{I} + \gamma_2\mathbf{H} + \gamma_3\mathbf{H}^2 + \gamma_4\mathbf{E} + \gamma_6\left(\mathbf{EH} + \mathbf{HE}\right) + \gamma_7\left(\mathbf{EH}^2 + \mathbf{H}^2\mathbf{E}\right), \quad (3.33)$$

where γ_1, γ_2, γ_3 must be of the form

$$\begin{aligned} \gamma_1 &= a_1 \operatorname{tr}\mathbf{E} + a_2 \operatorname{tr}\mathbf{EH} + a_3 \operatorname{tr}\mathbf{EH}^2, \\ \gamma_2 &= b_1 \operatorname{tr}\mathbf{E} + b_2 \operatorname{tr}\mathbf{EH} + b_3 \operatorname{tr}\mathbf{EH}^2, \\ \gamma_3 &= c_1 \operatorname{tr}\mathbf{E} + c_2 \operatorname{tr}\mathbf{EH} + c_3 \operatorname{tr}\mathbf{EH}^2. \end{aligned} \quad (3.34)$$

Here a_i, b_i, c_i $(i = 1,2,3)$ are functions of $\operatorname{tr}\mathbf{H}, \operatorname{tr}\mathbf{H}^2, \operatorname{tr}\mathbf{H}^3$. From $(3.1)_1$ follows that we must set

$$b_1 = a_2, \quad c_1 = a_3, \quad c_2 = b_3. \quad (3.35)$$

Cowin [15] represented the elastic material properties of the poroelastic medium, the fourth-rank tensors of elastic compliance moduli, and the second-rank tensor permeability, as functions of second-rank tensor of fabric assuming isotropy of matrix material. It seems that such an approach could be applied to cortical bone modelling.

3.4. Constitutive relationships with fabric tensor proposed by Zysset and Curnier [127]

Zysset and Curnier [127] proposed the elastic potential in the following form

$$W = W\left(\mathbf{E}, g, \mathbf{G}\right) = \hat{W}\left(\operatorname{tr}\mathbf{E}, \operatorname{tr}\mathbf{E}^2, \operatorname{tr}\mathbf{E}^3, g, \operatorname{tr}\mathbf{G}^2, \operatorname{tr}\mathbf{G}^3, \operatorname{tr}\mathbf{EG}, \operatorname{tr}\mathbf{E}^2\mathbf{G}, \operatorname{tr}\mathbf{EG}^2, \operatorname{tr}\left(\mathbf{EG}\right)^2\right), \quad (3.36)$$

where g and $\mathbf{G}$ are defined by Eqs. (2.8).

Retaining only quadratic terms in **E** to come up with linear elasticity, the general of the elastic free energy is:

$$\begin{aligned} W = \frac{d_1}{2}(\operatorname{tr}\mathbf{E})^2 + \frac{d_2}{2}\operatorname{tr}\mathbf{E}^2 + \frac{d_3}{2}(\operatorname{tr}\mathbf{EG})^2 + d_4\operatorname{tr}\mathbf{E}^2\mathbf{G} \\ + \frac{d_5}{2}\left(\operatorname{tr}\mathbf{EG}^2\right)^2 + \frac{d_6}{2}\operatorname{tr}(\mathbf{EG})^2 + d_7\operatorname{tr}\mathbf{E}\operatorname{tr}\mathbf{EG} \\ + d_8\operatorname{tr}\mathbf{EG}\operatorname{tr}\mathbf{EG}^2 + d_9\operatorname{tr}\mathbf{E}\operatorname{tr}\mathbf{EG}^2, \end{aligned} \tag{3.37}$$

where d_i $(i = 1, ..., 9)$ are functions of g and the two invariants of **G**. From $(3.1)_1$ we obtain:

$$\begin{aligned} \mathbf{T} = d_1(\operatorname{tr}\mathbf{E})\mathbf{I} + d_2\mathbf{E} + d_3(\operatorname{tr}\mathbf{EG})\mathbf{G} + d_4(\mathbf{EG} + \mathbf{GE}) \\ + d_5\left(\operatorname{tr}\mathbf{EG}^2\right)\mathbf{G}^2 + d_6\mathbf{GEG} + d_7((\operatorname{tr}\mathbf{EG})\mathbf{I} + (\operatorname{tr}\mathbf{E})\mathbf{G}) \\ + d_8\left(\left(\operatorname{tr}\mathbf{EG}^2\right)\mathbf{G} + (\operatorname{tr}\mathbf{EG})\mathbf{G}^2\right) + d_9\left(\left(\operatorname{tr}\mathbf{EG}^2\right)\mathbf{I} + (\operatorname{tr}\mathbf{E})\mathbf{G}^2\right). \end{aligned} \tag{3.38}$$

3.5. Constitutive relationships of linear elasticity with fabric tensor proposed by Jemioło and Telega [60]

Linearization of Eq. (3.12) with respect to **E** leads to the equation with the following functions α_m

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{12} & a_{22} & a_{23}\\ a_{13} & a_{23} & a_{33}\end{bmatrix}\begin{bmatrix}\operatorname{tr}\mathbf{E}\\ \operatorname{tr}\mathbf{EH}\\ \operatorname{tr}\mathbf{EH}^2\end{bmatrix}, \qquad \alpha_4 = a_{44}, \quad \alpha_5 = a_{55}, \quad \alpha_6 = a_{66}, \quad \alpha_7 = 0, \tag{3.39}$$

where $a_{ij} = a_{ji}$ $(i, j = 1, 2, 3)$, a_{44}, a_{55} and a_{66} in general depend on p.

A matrix form of Hooke's law for orthotropic materials is the following one

$$\mathbf{T}_{6\times1} = \mathbf{C}_{6\times6}\mathbf{E}_{6\times1}, \tag{3.40}$$

where

$$\mathbf{C}_{6\times6} = \begin{bmatrix}\mathbf{A}_{3\times3} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_{3\times3}\end{bmatrix} = \begin{bmatrix} e_1 & f_3 & f_2 & 0 & 0 & 0\\ f_3 & e_2 & f_1 & 0 & 0 & 0\\ f_2 & f_1 & e_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 2g_1 & 0 & 0\\ 0 & 0 & 0 & 0 & 2g_2 & 0\\ 0 & 0 & 0 & 0 & 0 & 2g_3\end{bmatrix}, \tag{3.41}$$

$$\mathbf{T}_{6\times 1} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ \sqrt{2}T_{23} \\ \sqrt{2}T_{13} \\ \sqrt{2}T_{12} \end{bmatrix}, \quad \mathbf{E}_{6\times 1} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \sqrt{2}E_{23} \\ \sqrt{2}E_{13} \\ \sqrt{2}E_{12} \end{bmatrix}. \tag{3.41 [cont.]}$$

Formulas (3.40) and (3.41) imply that classical Hooke's law: $\mathbf{T} = \mathbf{C} \cdot \mathbf{E}$ is written in the normalized basis $\mathbf{J}_K$, $K = 1, \ldots, 6$; where

$$\begin{aligned} &\mathbf{J}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1, \quad \mathbf{J}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2, \quad \mathbf{J}_3 = \mathbf{e}_3 \otimes \mathbf{e}_3, \\ &\mathbf{J}_4 = \frac{1}{\sqrt{2}} (\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2), \\ &\mathbf{J}_5 = \frac{1}{\sqrt{2}} (\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1), \\ &\mathbf{J}_6 = \frac{1}{\sqrt{2}} (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \end{aligned} \tag{3.42}$$

The matrix $\mathbf{C}_{6\times 6}$ is obviously the representation of the tensor $\mathbf{C}$ in the basis $\mathbf{J}_K \otimes \mathbf{J}_L$, $K, L = 1, \ldots, 6$.

Nine elasticity constants e_i, f_i, g_i $(i = 1, 2, 3)$ depend on the coefficients a_{ij}, a_{44}, a_{55}, a_{66} and the eigenvalues of $\mathbf{H}$ in the following manner

$$\begin{aligned} e_i =& a_{11} + 2a_{44} + 2H_i (a_{12} + a_{55}) + H_i^2 [a_{22} + 2 (a_{13} + a_{66})] \\ &+ 2a_{23} H_i^3 + a_{33} H_i^4, \\ f_i =& a_{11} + a_{12} (H_j + H_k) + \\ &+ H_j H_k [a_{22} + a_{23} (H_j + H_k) + a_{33} H_j H_k] + a_{13} \left(H_j^2 + H_k^2\right), \\ g_i =& a_{44} + \frac{1}{2} a_{55} (H_j + H_k) + \frac{1}{2} a_{66} \left(H_j^2 + H_k^2\right), \\ (i, j, k) =& (1, 2, 3); (2, 3, 1); (3, 1, 2). \end{aligned} \tag{3.43}$$

We observe that the coefficients a_{11} and a_{44} are not associated with the eigenvalues of $\mathbf{H}$. For $\mathbf{H} = \mathbf{0}$, a_{11} and a_{44} are the so-called Lamé moduli of an isotropic material. It can easily be verified that if two of the eigenvalues of $\mathbf{H}$ are equal then the matrix specified by $(3.39)_1$ contains five independent coefficients (the transverse isotropy). Further, if all of the eigenvalues coincide then only two constants are independent (the isotropy).

In the case of orthotropy six different eigenvalues of the matrix $(3.41)_1$ define six Kelvin's moduli, cf. [7, 24, 32, 80, 97-100]. The remaining three nondimensional constants, the so called stiffness distributors, determine the tensorial basis in which the matrix $(3.41)_1$ is diagonal. Kelvin's moduli are obviously the invariants of the stiffness tensor in the Hooke law. In the paper [7] it has been shown how to define the measure of the degree of orthotropy of a material provided that the spectral decomposition of the stiffness tensor is available.

If the principal axes of orthotropy are known, determination of Kelvin's moduli is easy since they have the following form:

$$\lambda_i = A_i, \quad \lambda_{i+3} = 2g_i, \quad i = 1, 2, 3. \tag{3.44}$$

Here A_i $(i = 1, 2, 3)$ are the eigenvalues of the matrix $\mathbf{A}_{3\times 3}$ and g_i denote Kirchhoff's moduli. To determine the ordered eigenvalues of this matrix, we apply formulas (A.6)-(A.8), where $\mathbf{M}$ is to be replaced by $\mathbf{A}_{3\times 3}$. The ellipticity condition $(3.3)_3$ reduces then to simple inequalities: $\lambda_K \geqslant 0$, $K = 1, \ldots, 6$. Next, to find the stiffness distributors a standard procedure of linear algebra is used. More precisely, from eigenversors of the matrix $\mathbf{A}_{3\times 3}$ an orthogonal matrix $\mathbf{R}_{3\times 3}$ is constructed. The diagonal form of $\mathbf{A}_{3\times 3}$ is then given by

$$\mathbf{R}_{3\times 3}\mathbf{A}_{3\times 3}\mathbf{R}_{3\times 3}^T = \operatorname{diag}\left[A_1, A_2, A_3\right]. \tag{3.45}$$

By using the invariant, cf. [32],

$$\cos\phi = \frac{1}{2}\left(\operatorname{tr}\mathbf{R}_{3\times 3} - \det\mathbf{R}_{3\times 3}\right), \quad \phi \in (0, \pi), \tag{3.46}$$

one can represent the elements of $\mathbf{R}_{3\times 3}$ in the following form

$$R_{ij} = \left(\det\mathbf{R}_{3\times 3}\right) r_i r_j + (\cos\phi)\left(\delta_{ij} - r_i r_j\right) - (\sin\phi)\,\epsilon_{ijk} r_k, \tag{3.47}$$

where

$$r_i = \frac{\epsilon_{ijk} R_{jk}}{2\sin\phi}. \tag{3.48}$$

Here ϵ_{ijk} are the components of Ricci's permutation symbol: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$, the remaining components being equal to zero. Note that r_i are components of the unit vector of rotation axis by angle ϕ in the basis $\mathbf{J}_i$ $(i = 1, 2, 3)$. These are not rotations or rotations with reflexions of the basis $\mathbf{e}_i$. For instance, when spherical coordinates are used, then r_i can be expressed as functions of two angular parameters, γ and η say,

$$\begin{aligned}\mathbf{r}_{3\times 1}^T = \frac{1}{2\sin\phi}\left[-R_{23} + R_{32}, R_{31} - R_{13}, R_{12} - R_{21}\right]\\ = \left[\sin\gamma\cos\eta, \sin\gamma\sin\eta, \cos\eta\right],\end{aligned} \tag{3.49}$$

where $\gamma \in (0,\pi)$, $\eta \in (0,2\pi)$. Consequently, apart from six Kelvin's moduli the angles ϕ, γ and η uniquely determine elastic moduli of an orthotropic material. The above procedure has been applied to the determination of the angles ϕ, γ and η. We observe that eigenversors are not determined in a unique manner: if $\mathbf{y}$ is an eigenversor associated with an engenvalue λ then -$\mathbf{y}$ is also an eigenversor. Spectral decomposition of the matrix $\mathbf{A}_{3\times 3}$ (and consequently of the tensor C) is, however, unique since matrix representations of eigentensors of the form:

$$\begin{aligned}\mathbf{P}^{(1)}_{3\times 3} &= \mathbf{R}^T_{3\times 3}\,\mathrm{diag}\,[1,0,0]\,\mathbf{R}_{3\times 3},\\ \mathbf{P}^{(2)}_{3\times 3} &= \mathbf{R}^T_{3\times 3}\,\mathrm{diag}\,[0,1,0]\,\mathbf{R}_{3\times 3},\\ \mathbf{P}^{(3)}_{3\times 3} &= \mathbf{R}^T_{3\times 3}\,\mathrm{diag}\,[0,0,1]\,\mathbf{R}_{3\times 3},\end{aligned} \tag{3.50}$$

associated with ordered eigenvalues are uniquely determined. The eigentensors with matrix representations (3.50) in the basis $\mathbf{J}_i \otimes \mathbf{J}_j$ $(i,j=1,2,3)$ can be determined directly from formulae (A.11) and (A.12), provided that $\mathbf{M}$ and its invariants are replaced by $A_{ij}\mathbf{J}_i \otimes \mathbf{J}_j$ and its invariants, where A_{ij} are elements of the matrix $\mathbf{A}_{3\times 3}$.

We observe that stiffness distributors are not necessarily given by the angles ϕ, γ and η. Three independent nondimensional parameters, which uniquely determine the representation of eigentensors (3.50), are likewise acceptable. For instance, Euler's angles are possible candidates for such parameters.

In the case of orthotropic materials the spectral decomposition of C is given by

$$\mathsf{C} = \lambda_1 \mathsf{P}_1 + \ldots + \lambda_6 \mathsf{P}_6, \tag{3.51}$$

where

$$\begin{gathered}\mathsf{P}_i = P^{(i)}_{kl}\mathbf{J}_k \otimes \mathbf{J}_l, \quad \mathsf{P}_{i+3} = \mathbf{J}_{i+3} \otimes \mathbf{J}_{i+3},\\ i,k,l = 1,2,3, (\text{no summation over i}).\end{gathered} \tag{3.52}$$

Here $P^{(i)}_{kl}$ are elements of the matrix $\mathbf{P}^{(i)}_{3\times 3}$ and are defined by (3.50).

As far as practical applications are concerned, an inverse of Eq. (3.40) is necessary ($\mathbf{E} = \mathsf{C}^{-1}\cdot\mathbf{T}$, $\mathsf{C}^{-1} = \lambda_1^{-1}\mathsf{P}_1 + \ldots + \lambda_6^{-1}\mathsf{P}_6$):

$$\mathbf{E}_{6\times 1} = \mathbf{C}^{-1}_{6\times 6}\mathbf{T}_{6\times 1}, \tag{3.53}$$

where

$$\mathbf{C}^{-1}_{6\times 6} = \begin{bmatrix} p_1 & r_3 & r_2 & 0 & 0 & 0\\ r_3 & p_2 & r_1 & 0 & 0 & 0\\ r_2 & r_1 & p_3 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}s_1 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2}s_2 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}s_3 \end{bmatrix}. \tag{3.54}$$

Here

$$dp_i = e_j e_k - f_i^2, \quad dr_i = f_j f_k - e_i f_i \quad \text{(no summation over } i\text{)},$$
$$s_i = \frac{1}{g_i}, \quad d = e_1 e_2 e_3 + 2 f_1 f_2 f_3 - e_1 f_1^2 - e_2 f_2^2 - e_3 f_3^2, \tag{3.55}$$
$$(i,j,k) = (1,2,3)\,;(2,3,1)\,;(3,1,2)\,.$$

The coefficients $p_i,\ r_i,\ s_i\ (i = 1,2,3)$ (in general functions of p) can be determined from standard tests performed on orthotropic material.

Obviously, in order to obtain (3.55) we can directly apply (3.15). Linearization of Eq. (3.15) under $\mathbf{T}$ leads to the equation with the following functions β_m

$$\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{12} & b_{22} & b_{23}\\ b_{13} & b_{23} & b_{33}\end{bmatrix} \begin{bmatrix}\operatorname{tr}\mathbf{T}\\ \operatorname{tr}\mathbf{TH}\\ \operatorname{tr}\mathbf{TH}^2\end{bmatrix}, \tag{3.56}$$
$$\beta_4 = b_{44}, \quad \beta_5 = b_{55}, \quad \beta_6 = b_{66}, \quad \beta_7 = 0,$$

where $b_{ij} = b_{ji}\ (i,j = 1,2,3)$, b_{44}, b_{55} and b_{66} in general depend of p. The constitutive relationship (3.15) combined with (3.56) is more convenient to experimental verification since tests are usually carried out for a given loading. Interrelationships between the coefficients (3.56) and the classical orthotropic constants p_i, r_i and s_i are analogous to (3.41). In the procedure just outlined we do not explicitly exploit the fact that (3.15) is inverse to (3.12). The coefficients p_i, r_i and s_i have a clear mechanical interpretation. We observe that a search for a simple interrelationship, for instance between Young's moduli and the eigenvalues of $\mathbf{H}$ is not justified, cf. [102]. In fact, Eqs. (3.43), (3.54)-(3.55) imply the interrelationship between the classical coefficients of an orthotropic material and the tensor $\mathbf{H}$.

Following Hayes' paper [45], cf. also Appendix B, the following relations can be established:

(i) the generalized Young moduli for an arbitrary direction $\mathbf{n}$

$$\frac{1}{E(\mathbf{n})} = p_1 n_1^4 + p_2 n_2^4 + p_3 n_3^4 + (2r_1 + s_1)\, n_2^2 n_3^2 + (2r_2 + s_2)\, n_1^2 n_3^2 + (2r_3 + s_3)\, n_1^2 n_2^2, \tag{3.57}$$

where $\mathbf{n}$ is an arbitrary versor with the components n_i,

(ii) the generalized Poisson ratios for an arbitrary plane (for a pair of orthogonal directions $\mathbf{n}$, $\mathbf{m}$)

$$-\frac{\nu(\mathbf{m},\mathbf{n})}{E(\mathbf{n})} = -\frac{\nu(\mathbf{n},\mathbf{m})}{E(\mathbf{m})} = p_1 m_1^2 n_1^2 + p_2 m_2^2 n_2^2 + p_3 m_3^2 n_3^2 + r_1\left(m_2^2 n_3^2 + m_3^2 n_2^2\right) + r_2\left(m_1^2 n_3^2 + m_3^2 n_1^2\right) + r_3\left(m_1^2 n_2^2 + m_2^2 n_1^2\right) + s_1 m_2 m_3 n_2 n_3 + s_2 m_1 m_3 n_1 n_3 + s_3 m_1 m_2 n_1 n_2, \quad (3.58)$$

where $\mathbf{m}$ is a versor with the components m_i,

(iii) the generalized Kirchhoff moduli for an arbitrary plane (for a pair of orthogonal directions $\mathbf{n}$, $\mathbf{m}$)

$$\frac{1}{G(\mathbf{m},\mathbf{n})} = \frac{1}{G(\mathbf{n},\mathbf{m})} = 4\big[p_1 n_1^2 m_1^2 + p_2 n_2^2 m_2^2 + p_3 n_3^2 m_3^2 + 2r_1 m_2 m_3 n_2 n_3 + 2r_2 m_1 m_3 n_1 n_3 + 2r_3 m_1 m_2 n_1 n_2\big] + s_1\left(n_2 m_3 + m_2 n_3\right)^2 + s_2\left(n_1 m_3 + m_1 n_3\right)^2 + s_3\left(n_1 m_2 + m_1 n_2\right)^2. \quad (3.59)$$

In Table 1 averaged experimental data of the so-called technical elastic constants are presented, which were obtained by a ultrasonic method, cf. [4, 119]. Ultrasonic methods are popular in the determination of elastic moduli. In fact, it is a subject in itself and deserves a separate overview, cf. [13, 71, 118] and the relevant papers discussed earlier. Deeper insight has been gained with the advent of acoustic microscopy [118] and MRI (magnetic resonance imaging), cf. [75].

From Eqs (3.57)-(3.59) the off-axis technical elastic constants in a plane of an orthotropic, linearly elastic material can be represented as a function of off-axis angle by the following equations, cf. Figs. 7-9,

$$\begin{aligned}
\frac{1}{E(\varphi)} &= \frac{1}{E_1}\cos^4\varphi + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)\sin^2\varphi\cos^2\varphi + \frac{1}{E_2}\sin^4\varphi, \\
\frac{1}{G(\varphi)} &= 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^2\varphi\cos^2\varphi \\
&\quad + \frac{1}{G_{12}}\left(\sin^4\varphi + \cos^4\varphi\right), \qquad (3.60) \\
\nu(\varphi) &= E(\varphi)\left[\frac{\nu_{12}}{E_1}\left(\sin^4\varphi + \cos^4\varphi\right) \right. \\
&\quad \left. - \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)\sin^2\varphi\cos^2\varphi\right],
\end{aligned}$$

where φ is referred to the 1-direction.

TABLE 1. Technical moduli (ultrasonic technique, after [4, 119]). Average technical moduli for 60 specimens of human femoral cortical bone, where the 1-direction is radial, the 2-direction is circumferential and the 3-direction is longitudinal. Average technical moduli for 9 specimens of human cancellous bone from the proximal tibia, where the 1-direction is anterior-posterior, the 2-direction is medial-lateral and the 3-direction is longitudinal. The numbers in parentheses denote standard deviations.

Technical constants (average)	Human femoral cortical bone	Human cancellous bone (proximal tibia)
E_1	11.7 (1.6) [GPa]	237 (63) [MPa]
E_2	13.2 (1.8) [GPa]	309 (93) [MPa]
E_3	19.8 (2.4) [GPa]	823 (337) [MPa]
G_{12}	4.53 (0.37) [GPa]	73 (38) [MPa]
G_{13}	5.61 (0.4) [GPa]	112 (48) [MPa]
G_{23}	6.23 (0.48) [GPa]	134 (49) [MPa]
ν_{12}	0.375 (0.095)	0.169 (0.304)
ν_{21}	0.416 (0.118)	0.209 (0.209)
ν_{23}	0.237 (0.083)	0.063 (0.217)
ν_{32}	0.346 (0.096)	0.245 (0.626)
ν_{13}	0.234 (0.088)	0.423 (0.356)
ν_{31}	0.374 (0.108)	0.145 (0.123)

Similar relationships hold true for the 1-3 plane and 2-3 plane. From Table 1 and Figs. 7, 8 and 9 it follows that the human femoral cortical bone may be approximately treated as a transversely isotropic material. On the other hand, such an approximation would not be justified for the cancellous bone. Comparing the standard deviations we conclude that the cancellous bone is

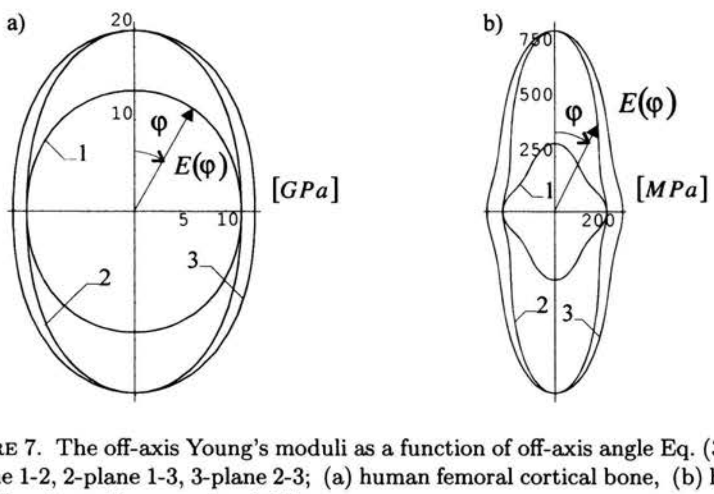

FIGURE 7. The off-axis Young's moduli as a function of off-axis angle Eq. $(3.60)_1$; 1-plane 1-2, 2-plane 1-3, 3-plane 2-3; (a) human femoral cortical bone, (b) human cancellous bone from proximal tibia.

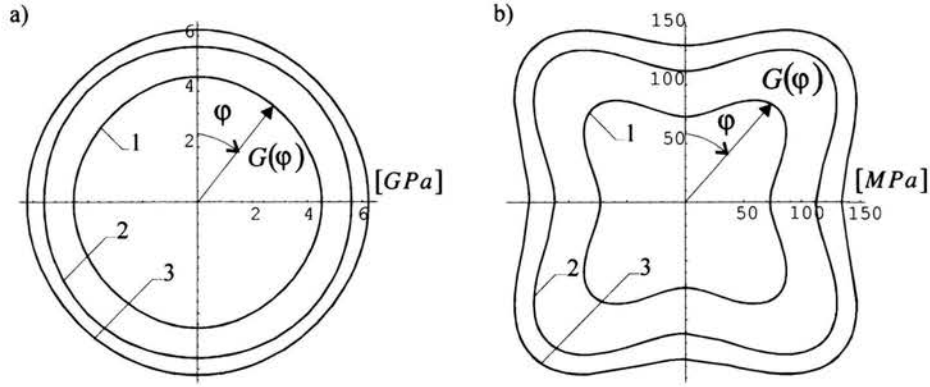

FIGURE 8. The off-axis shear moduli Eq. $(3.60)_2$; 1-plane 1-2, 2-plane 1-3, 3-plane 2-3; (a) human femoral cortical bone, (b) human cancellous bone from proximal tibia.

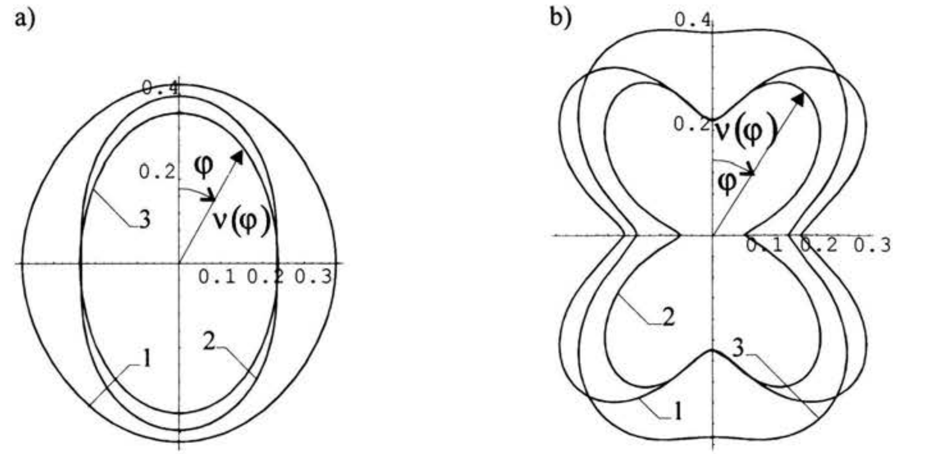

FIGURE 9. The off-axis Poisson's ratios Eq. $((3.60)_3)$; 1-plane 1-2, 2-plane 1-3, 3-plane 2-3; (a) human femoral cortical bone, (b) human cancellous bone from proximal tibia.

considerably more inhomogeneous than the cortical bone. In our opinion the tests performed by Turner et al. [89] should additionally be completed by the determination of eigenvectors of **H**. The direction of orthotropy would then be determined more precisely. We note that the data provided in Table 1 have been obtained under the assumption that the directions of orthotropy of all samples are the same.

TABLE 2.

	Human femoral cortical bone	Human cancellous bone (proximal tibia)
λ_1	42.917 [GPa]	1190.75 [MPa]
λ_2	13.251 [GPa]	324.80 [MPa]
λ_3	8.852 [GPa]	210.43 [MPa]
λ_4	11.22 [GPa]	268 [MPa]
λ_5	12.46 [GPa]	224 [MPa]
λ_6	9.06 [GPa]	146 [MPa]
ϕ	0.6931π	0.3990π
γ	0.6927π	0.5840π
η	1.0187π	0.4842π
$\det \mathbf{R}_{3\times 3}$	1	-1

Table 2 contains Kelvin's moduli and parameters determining stiffness distributors for the analyzed cortical and cancellous bones and confirms our earlier assertions.

In the case of transverse isotropy one has four independent Kelvin's moduli and only one angular parameter defining the stiffness distributor, cf. [7, 97-100].

Closer inspection of Tables 1 and 2 with taking into account the standard deviations given in brackets in Table 1 leads to conclusion that the cortical bone may be regarded as a transversely isotropic material since the appropriately underlined values of Kelvin's moduli are practically identical while the angles ϕ and γ only insignificantly differ from each other.

The spectral decomposition of stiffness tensors of cortical and cancellous bones are obtained by applying the appropriate formulae derived earlier and the data provided in Table 2. In the calculation of spectral representation of the Hooke law it was assumed that Eq. (B.11) is satisfied. More precisely, the matrices representing stiffness and compliance tensors are symmetric. Then we get

$$\begin{aligned} S_{12} &= -\frac{1}{2}\left(\frac{\nu_{12}}{E_1} + \frac{\nu_{21}}{E_2}\right), \\ S_{13} &= -\frac{1}{2}\left(\frac{\nu_{13}}{E_1} + \frac{\nu_{31}}{E_3}\right), \\ S_{23} &= -\frac{1}{2}\left(\frac{\nu_{23}}{E_2} + \frac{\nu_{32}}{E_3}\right). \end{aligned} \tag{3.61}$$

We note that relations (B.11), resulting from assumption $(3.2)_1$, formally verify the precision of performed tests. In the case of orthotropy one determines 12 the so-called technical elasticity coefficients, (see (B.10)), and then finds 9 independent components of the elasticity tensor. The data given in

Table do not satisfy (B.11). The largest discrepancies are found for compact bone (see Table 3.1) and relations $(B.11)_2$, i.e.,

$$\frac{\nu_{13}}{E_1} = 0.001785, \quad \frac{\nu_{31}}{E_3} = 0.000176. \tag{3.62}$$

We conclude that in this case we have

$$\frac{\nu_{13}}{E_1} \approx 10^1 \frac{\nu_{31}}{E_3}. \tag{3.63}$$

Turner and Cowin [119] analysed the precision of the data given in Table 1, but from a different point of view. Among others, these authors analysed the error in the determination of Poisson's ratios, Young's and shear moduli. Unfortunately, one comment on (3.63) was provided in [119].

It is known that fourth-order tensors generate also a normed space (and also a Euclidean space, cf. [58]), for instance with the norm

$$\|\mathbf{C}\| = \sqrt{\operatorname{Tr}\mathbf{C}^2}, \quad \operatorname{Tr}\mathbf{C}^2 = \mathsf{C}_{KL}\mathsf{C}_{LK}, \tag{3.64}$$

where the representation of the tensor $\mathbf{C}$ is given in the tensor basis (3.42). Consequently we can introduce the following distance (deviation) between $\mathbf{C}$ and $\mathbf{C}_{ns}$:

$$d(\mathbf{C}, \mathbf{C}_{ns}) = \|\mathbf{C} - \mathbf{C}_{ns}\|. \tag{3.65}$$

Here the tensor $\mathbf{C}_{ns}$ may be interpreted as a tensor determined in experimental tests, not satisfying the symmetry requirements resulting from existence of (3.65), it follows that the quantity

$$\Delta(\mathbf{C}, \mathbf{C}_{ns}) = \frac{\|\mathbf{C} - \mathbf{C}_{ns}\|}{\|\mathbf{C}_{ns}\|}, \tag{3.66}$$

is a relative error enabling to estimate, simultaneously for each component of the elasticity tensor, the precision of tests performed. Relations (3.64)-(3.66) can be written both for the stiffness and compliance tensor. The data given in Table 1 and relation (3.66) imply that for compact bone the errors are relatively small:

$$\Delta(\mathbf{S}, \mathbf{S}_{ns}) = 0.438\%, \quad \Delta(\mathbf{C}, \mathbf{C}_{ns}) = 2.103\%, \tag{3.67}$$

in comparison with the errors characterizing cancellous bone:

$$\Delta(\mathbf{S}, \mathbf{S}_{ns}) = 10.696\%, \quad \Delta(\mathbf{C}, \mathbf{C}_{ns}) = 51.598\%. \tag{3.68}$$

Errors in the determination of the stiffness tensor are larger than in the case of compliance tensor is obvious. This fact follows from adequate application of formulas (3.64)-(3.66) and (B.10)-(B.14). Errors in the determination of Poisson's ratios significantly influence on values of all components of matrix $\mathbf{A}_{3\times 3}$ appearing in (3.41). For example, applying (3.61) to the determination of extremal values of Poisson's ratios in the planes given in Fig. 9b changes their values more than twice. From $(3.67)_2$ we conclude that the data given in Table 1 and pertaining to cancellous bone can only have a quantitative value is not convincing. The conclusion provided in the paper [119] are too optimistic and one can only agree with the statement that transversely isotropic approximation of cancellous bone is not justified, see Table 2.

Obviously, introducing (3.64)-(3.66) we do not propose a method of error estimation in the determination of stiffness and compliance tensors as well as of invariants of these tensors, etc. Deeper analysis requires, among others, probabilistic distribution of quantities measured.

3.6. Relationships based on fabric tensor used in the literature

The relationships developed by Cowin [15] between the elastic moduli and normalized fabric tensor eigenvalues are

$$\begin{aligned}
\mathsf{C}_{iiii} =& k_1 + 2k_6 + (k_2 + 2k_7)\,\mathrm{tr}\,\mathrm{Cof}\,\mathbf{H} + 2\,(k_3 + 2k_8)\,H_i \\
&+ (2k_4 + k_5 + 4k_9)\,H_i^2, \\
\mathsf{C}_{iijj} =& k_1 + k_2\,\mathrm{tr}\,\mathrm{Cof}\,\mathbf{H} + k_3\,(H_i + H_j) + k_4\left(H_i^2 + H_j^2\right) + k_5 H_i H_j, \\
\mathsf{C}_{ijij} =& k_6 + k_7\,\mathrm{tr}\,\mathrm{Cof}\,\mathbf{H} + k_8\,(H_i + H_j) + k_9\left(H_i^2 + H_j^2\right), \\
& i,j = 1,2,3; \quad i \neq j \text{ (no summation over i and j)}
\end{aligned} \tag{3.69}$$

with C_{ijkl} are the components of the stiffness tensor C, $\mathrm{tr}\,\mathrm{Cof}\,\mathbf{H} = H_1H_2 + H_2H_3 + H_1H_3$, and $k_1, \ldots, k_9$ are functions of the structural density or volume fraction. Similar relationships also hold between the components of the compliance tensor and the fabric tensor (C_{ijkl} is to be replaced by S_{ijkl}). The functions k_n, $(n = 1, \ldots, 9)$ are usually chosen as follows

$$k_n = \bar{a}_n + \bar{b}_n \rho^q, \quad k_n = \bar{a}_n + \bar{b}_n \rho^{-q}, \tag{3.70}$$

where $\bar{a}_n$, $\bar{b}_n$ and q are constants. Equations $(3.70)_1$ and $(3.70)_2$ are proposed for (3.69) and for relations with the compliance tensor, respectively.

An alternative model was developed by Zysset and Curnier [127]. Their relationships are as follows

$$\begin{gathered}
\mathsf{C}_{iiii} = (\lambda_c + 2\mu_c)\,H_i^{2q}, \quad \mathsf{C}_{iijj} = \lambda_c H_i^q H_j^q, \quad \mathsf{C}_{ijij} = 2\mu_c H_i^q H_j^q, \\
i,j = 1,2,3; \quad i \neq j \text{ (no summation over i and j)}
\end{gathered} \tag{3.71}$$

with λ_c and μ_c Lamé like constants. This model is more restrictive than the one developed by Cowin [15].

4. Elastic-perfectly plastic model, strength criteria

Let us denote by $\dot{\mathbf{E}}_e$, $\dot{\mathbf{E}}_p$ the elastic and plastic part of the strain rate tensor. As usual, we assume that

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}_e + \dot{\mathbf{E}}_p \tag{4.1}$$

and construct constitutive relationships for elastic perfectly-plastic materials. The elastic behaviour is described by $\dot{\mathbf{E}}_{6\times1} = \mathbf{C}^{-1}_{6\times6}\dot{\mathbf{T}}_{6\times1}$. General form of the yield function is assumed in the form

$$\begin{aligned} G(\mathbf{T}) &= F\left(J_m(\mathbf{T})\right) \\ &= F\left(\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{TH}, \operatorname{tr}\mathbf{TH}^2, \operatorname{tr}\mathbf{T}^2, \operatorname{tr}\mathbf{T}^2\mathbf{H}, \operatorname{tr}\mathbf{T}^2\mathbf{H}^2, \operatorname{tr}\mathbf{T}^3\right) \end{aligned} \tag{4.2}$$

whilst the yield surface is given by

$$G(\mathbf{T}) - 1 = 0. \tag{4.3}$$

The associated flow rule assumes the form

$$\dot{\mathbf{E}}_p = \lambda \frac{\partial G}{\partial \mathbf{T}}, \quad \lambda \geqslant 0. \tag{4.4}$$

Since cancellous bone reveals different plastic behaviour in tension and compression (cf. [11, 14, 19, 26, 65, 68]), therefore we propose to assume Hoffman's criterion [48], cf. also [55, 107-109, 112, 113]. Written in an invariant form this criterion is expressed by cf. [56],

$$\begin{aligned} c_1(K_2-K_3)^2 + c_2(K_3-K_1)^2 + c_3(K_1-K_3)^2 + 2c_4K_6 \\ + 2c_5K_5 + 2c_6K_4 + c_7K_1 + c_8K_2 + c_9K_3 - 1 = 0, \end{aligned} \tag{4.5}$$

where

$$\begin{gathered} c_1 = \frac{1}{2}\left(\frac{1}{Y_{t2}Y_{c2}} + \frac{1}{Y_{t3}Y_{c3}} - \frac{1}{Y_{t1}Y_{c1}}\right), \\ c_2 = \frac{1}{2}\left(\frac{1}{Y_{t3}Y_{c3}} + \frac{1}{Y_{t1}Y_{c1}} - \frac{1}{Y_{t2}Y_{c2}}\right), \\ c_3 = \frac{1}{2}\left(\frac{1}{Y_{t1}Y_{c1}} + \frac{1}{Y_{t2}Y_{c2}} - \frac{1}{Y_{t3}Y_{c3}}\right), \\ 2c_4 = \frac{1}{k_{23}^2}, \qquad 2c_5 = \frac{1}{k_{13}^2}, \qquad 2c_6 = \frac{1}{k_{12}^2}, \\ c_7 = \frac{Y_{c1}-Y_{t1}}{Y_{c1}Y_{t1}}, \qquad c_8 = \frac{Y_{c2}-Y_{t2}}{Y_{c2}Y_{t2}}, \qquad c_9 = \frac{Y_{c3}-Y_{t3}}{Y_{c3}Y_{t1}}. \end{gathered} \tag{4.6}$$

Here Y_{ci}, Y_{ti} and k_{ij} are the yield limit in compression and tension in the directions of orthotropy and the yield limit in shear in the principal planes of orthotropy, respectively. The invariants K_p ($p = 1, \ldots, 6$) are given by

$$\begin{aligned}
K_1 &= \operatorname{tr}\mathbf{M}_1\mathbf{T}, \quad K_2 = \operatorname{tr}\mathbf{M}_2\mathbf{T}, \quad K_3 = \operatorname{tr}\mathbf{M}_3\mathbf{T},\\
K_4 &= \frac{1}{2}\Big[(\operatorname{tr}\mathbf{M}_3\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_1\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_2\mathbf{T})^2 - \operatorname{tr}\mathbf{M}_3\mathbf{T}^2\\
&\quad + \operatorname{tr}\mathbf{M}_1\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_2\mathbf{T}^2\Big],\\
K_5 &= \frac{1}{2}\Big[(\operatorname{tr}\mathbf{M}_2\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_1\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_3\mathbf{T})^2 - \operatorname{tr}\mathbf{M}_2\mathbf{T}^2\\
&\quad + \operatorname{tr}\mathbf{M}_1\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_3\mathbf{T}^2\Big],\\
K_6 &= \frac{1}{2}\Big[(\operatorname{tr}\mathbf{M}_1\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_2\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_3\mathbf{T})^2 - \operatorname{tr}\mathbf{M}_1\mathbf{T}^2\\
&\quad + \operatorname{tr}\mathbf{M}_2\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_3\mathbf{T}^2\Big].
\end{aligned} \tag{4.7}$$

The tensors $\mathbf{M}_j = \mathbf{i}_j \otimes \mathbf{i}_j$ (no summation over j) are the eigentensors of $\mathbf{H}$, cf. Appendix A. By using the following relation

$$\begin{bmatrix}\operatorname{tr}\mathbf{T}^\alpha\\ \operatorname{tr}\mathbf{H}\mathbf{T}^\alpha\\ \operatorname{tr}\mathbf{H}^2\mathbf{T}^\alpha\end{bmatrix} = \mathbf{h}_{3\times3}\begin{bmatrix}\operatorname{tr}\mathbf{M}_1\mathbf{T}^\alpha\\ \operatorname{tr}\mathbf{M}_2\mathbf{T}^\alpha\\ \operatorname{tr}\mathbf{M}_3\mathbf{T}^\alpha\end{bmatrix}, \quad \alpha = 1, 2, \tag{4.8}$$

where

$$\mathbf{h}_{3\times3} = \begin{bmatrix}1 & 1 & 1\\ H_1 & H_2 & H_3\\ H_1^2 & H_2^2 & H_3^2\end{bmatrix}, \tag{4.9}$$

the criterion (4.5) can be written in the form (4.3).

If

$$\det(\mathbf{h}_{3\times3}) = (H_2 - H_1)(H_3 - H_1)(H_3 - H_2) \neq 0, \tag{4.10}$$

or the eigenvalues of $\mathbf{H}$ are different then, by using the inverse matrix

$$\mathbf{h}_{3\times3}^{-1} = \frac{1}{\det(\mathbf{h}_{3\times3})}\begin{bmatrix}H_2H_3^2 - H_3H_2^2 & H_2^2 - H_3^2 & H_3 - H_2\\ H_3H_1^2 - H_1H_3^2 & H_3^2 - H_1^2 & H_1 - H_3\\ H_1H_2^2 - H_2H_1^2 & H_1^2 - H_2^2 & H_2 - H_1\end{bmatrix}, \tag{4.11}$$

we establish (4.5) as claimed.

For $Y_{ci} = Y_{ti}$ the criterion (4.5) reduces to Hill's criterion [47], which was also applied in the bone mechanics, cf. [11, 89, 117].

The canonical form of Hoffman's criterion was derived in [56], where an alternative interpretation of the coefficients c_i was also provided. Moreover, on the basis of available data for the compact bone the applicability of the criterion was verified.

For an anisotropic strength criterion based on Kelvin modes the reader is referred to [3] whilst damage is examined in [6, 43, 128].

By using Eq. (4.5) and transformation formula of tensor components under orthogonal transformations one can readily derive the formulas for the determination of sample strength in the case of compression and tension, in the direction defined by an angle φ, in each of the principal orthotropy planes. Particularly, in the case of the orthotropy plane 1-2 this formula is given by

$$\sigma_\varphi = \frac{-B_\varphi \pm \sqrt{\Delta_\varphi}}{2A_\varphi}, \tag{4.12}$$

where

$$\begin{aligned} \Delta_\varphi &= (B_\varphi)^2 + 4A_\varphi, \quad B_\varphi = C_7 + (C_8 - C_7)\sin^2\varphi, \\ A_\varphi &= C_2 + C_3 + (-4C_3 - 2C_2 + 2C_6)\sin^2\varphi + \\ &\quad + (4C_3 + C_1 + C_2 - 2C_6)\sin^4\varphi. \end{aligned} \tag{4.13}$$

Here the sign „+" („−") refers to tension (compression).

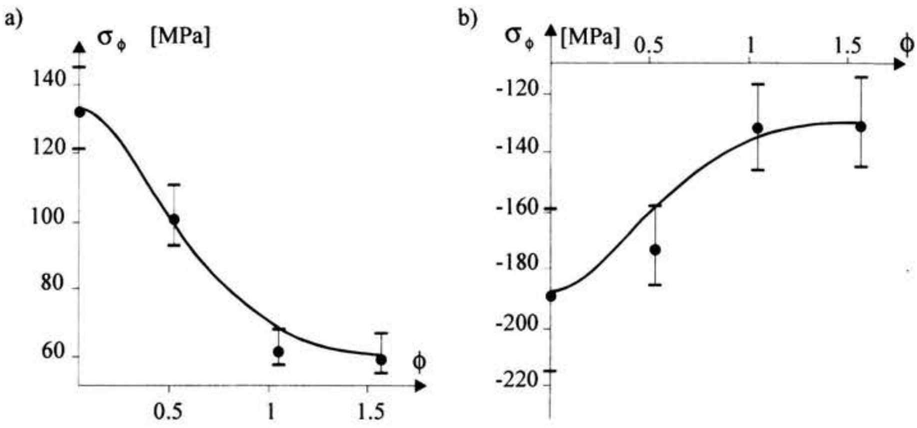

FIGURE 10. Strength limit of the human femoral cortical bone in: (a) tension, (b) compression. The angle φ is taken with respect to the long axis of the bone. • — the experimental data (jointly with error range), given in Reilly's thesis (1974), reproduced after [14].

Figure 10 depicts the relation (4.12) for a human cortical bone. As we already know, this bone may be treated as a transversely isotropic material.

The constants C_1, C_2, C_5, C_7 and C_8 have been determined by exploiting the experimental data presented in the paper [14]. Table 3 summarizes the data necessary for the calculation, which uses Eqs. (4.12) and (4.13), for this type of bone.

TABLE 3. Data for the determination of the strength in tension and compression in case of human femoral cortical bone.

[MPa]	$Y_{t1} = 132$	$Y_{c1} = 187$	$Y_t = 58$	$Y_c = 132$	$k = 67$
[MPa^{-2}]	$C_1 = 1.104 \cdot 10^{-4}$	$C_2 = C_3 = 2.026 \cdot 10^{-5}$	$C_7 = 2.23 \cdot 10^{-3}$	$C_8 = C_9 = 9.67 \cdot 10^{-3}$	$2C_5 = 2C_6 = 2.228 \cdot 10^{-4}$

5. Final remarks

The present paper confirms usefulness of the concept of fabric tensor in bone mechanics. The available experimental data reveal that bones are anisotropic materials. Their properties depend on a location, i.e. they are inhomogeneous materials. We observe that taken rigorously the definition of the fabric tensor was inspired by the structure of cancellous bone. For cortical bone the fabric tensor is to be interpreted in the sense of parametric tensor used by Boehler [8]. Our consideration have deliberately been confined to elastic and elastic perfectly-plastic modelling of bones. In fact, bone is a porous material with a very complex hierarchical structure. For instance, one can treat the bone as consisting of piezoelectric skeleton filled with a conductive biofluid, cf. Telega and Wojnar [111]. It seems, however, that a specific bone model assumed depends on a problem investigated. We believe that in contact problems of orthopaedic biomechanics the anisotropic models studied in this paper can provide reliable information on stress distribution in joints after arthroplasty. More precisely, to better model stress distribution in a human joint after arthroplasty, one has to take into account the properties of bone in a vicinity of a prosthesis. The problem is still being discussed in the relevant literature. Poroelastic parameters of cortical bone were estimated in [102]. Cowin [28] expressed elastic compliance moduli and the permeability tensor as functions of the fabric tensor.

Critical study performed in this paper shows that there is no general agreement as to geometrical description of cancellous bone. Various approaches used in the literature are not equivalent and thus we have MIL-fabric tensor, SVD-fabric tensor, etc., cf. also [73].

An enormous number of papers on experimental determination of mechanical properties of bone is available in the literature. In our opinion, however one still lacks consequent results allowing to determine in a reliable manner the components of the elasticity tensor. We have also shown that unacceptable conclusions can be drawn from available data. Such a conclusion has been illustrated on the example of Poissons ratios. We claim that in the bone mechanics we still need more precise and comprehensive experimental data on the mechanical properties of bone where geometric structure and bone inhomogeneity would be taken into account.

Bone is obviously a porous material. It seems that the methods developed for random porous media could be applied to description of geometry of cancellous bone. For an excellent review of the methods of random fields applied to random porous media the reader is referred to the comprehensive paper by Adler and Thovert [2], cf. also [108]. At this point the following problem arises: what is the relation between fabric tensor and second-order correlation functions?

We observe that in [74] μCT technique was applied to a small sample of *Perdictus potto* and *Galago sengalensis* femora to see if differences in loading environment elicit the predicted effects on trabecular structure. While the overall bone volume was approximately three times larger in the potto, there was no significant difference in the apparent volume density in the two taxa. When regional differences in the proximal femur were examined, the cancellous bone of the femoral head of *Perodicticus potto* and *Galaga senegalensis*, while not differing in volume density, showed differences in trabecular orientation, with the potto having more randomly oriented trabeculae than bushbaby.

It has been commonly assumed that trabeculae align according to what is "called Wolff's Law". In fact, it was shown by Telega and Lekszycki [110] that under multiple loads, and physiological loads are of this type, "Wolff's law" is not true. These result was achieved by exploiting modern optimisation methods combined with relaxation of compliance functional and homogenization.

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Appendix A.

The fabric tensor $\mathbf{H}$ defined by (2.3) is an isotropic tensor function of $\mathbf{M}$, $\hat{\mathbf{H}}(\mathbf{M})$ say. It means that

$$\forall \mathbf{Q} \in O(3) \quad \mathbf{Q}\hat{\mathbf{H}}(\mathbf{M})\mathbf{Q}^T = \hat{\mathbf{H}}\left(\mathbf{Q}\mathbf{M}\mathbf{Q}^T\right) = \mathbf{Q}\frac{1}{\sqrt{\mathbf{M}}}\mathbf{Q}^T. \tag{A.1}$$

Here $O(3)$ stands for the full orthogonal group:

$$O(3) \equiv \left\{\mathbf{Q} \mid \ \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}\right\}, \tag{A.2}$$

where $\mathbf{I}$ is the identity tensor; moreover $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$.

Let us pass to the determination of the function

$$\mathbf{H} = \hat{\mathbf{H}}(\mathbf{M}) = \frac{1}{\sqrt{\mathbf{M}}}. \tag{A.3}$$

Recalling that $\mathbf{M}$ is a symmetric positive-definite tensor, by applying the spectral theorem we may write

$$\mathbf{M} = M_1\mathbf{i}_1 \otimes \mathbf{i}_1 + M_2\mathbf{i}_2 \otimes \mathbf{i}_2 + M_3\mathbf{i}_3 \otimes \mathbf{i}_3, \tag{A.4}$$

where M_j $(j = 1, 2, 3)$ are eigenvalues of the tensor $\mathbf{M}$ and $\mathbf{i}_j$ its eigenvectors. It is assumed that

$$M_1 \geqslant M_2 \geqslant M_3, \tag{A.5}$$

where

$$M_i = \frac{1}{3} I_M + \frac{2}{3} \sqrt{I_M^2 - 3 II_M} \, \cos\left[\frac{2}{3}\pi\,(i-1) - \varphi\right], \quad i = 1, 2, 3 \tag{A.6}$$

and

$$\cos 3\varphi = \frac{2 I_M^3 - 9 I_M II_M + 27 III_M}{\sqrt{2\left(I_M^2 - 3 II_M\right)^3}}. \tag{A.7}$$

The basic invariants of $\mathbf{M}$ are given by

$$\begin{aligned} I_M &= \operatorname{tr}\mathbf{M}, \quad II_M = \frac{1}{2}\left(\overset{2}{\operatorname{tr}}\,\mathbf{M} - \operatorname{tr}\mathbf{M}^2\right), \\ III_M &= \det\mathbf{M} = \frac{1}{6}\left(\overset{3}{\operatorname{tr}}\,\mathbf{M} - 3\operatorname{tr}\mathbf{M}\operatorname{tr}\mathbf{M}^2 + 2\operatorname{tr}\mathbf{M}^3\right), \end{aligned} \tag{A.8}$$

where $\operatorname{tr}\mathbf{M}$ is the trace of $\mathbf{M}$. In the orthonormal basis $\{\mathbf{e}_i\}$, $(i = 1, 2, 3)$ we have: $\mathbf{M} = M_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, $\operatorname{tr}\mathbf{M} = M_{ii}$, $\left(\mathbf{M}^2\right)_{ij} = (\mathbf{MM})_{ij} = M_{ik}M_{kj}$, etc.

Note that if

$$d = 4 II_M^3 - I_M^2 II_M^2 + 4 I_M^3 III_M - 18 I_M II_M III_M + 27 III_M^2 < 0, \tag{A.9}$$

then M_i in (A.6) are different; for $d = 0$ two of the eigenvalues are equal, in other words the tensor $\mathbf{M}$ is then two-dimensional. Finally, for

$$I_M^2 = 3 II_M, \tag{A.10}$$

$\mathbf{M}$ is a spherical tensor.

In the case of three different eigenvalues, the eigentensors $\mathbf{i}_j \otimes \mathbf{i}_j$ (no summation over j) can be determined in a unique manner:

$$\mathbf{N}_j = \mathbf{i}_j \otimes \mathbf{i}_j = \frac{1}{m_j}\left[\mathbf{M}^2 - (I_M - M_j)\,\mathbf{M} + III_M M_j^{-1}\mathbf{I}\right], \quad \text{(no summation over j)}, \tag{A.11}$$

where

$$m_j = 2M_j^2 - I_M M_j + III_M M_j^{-1}. \tag{A.12}$$

Consequently, the fabric tensor (2.3) satisfying (A.1) can be represented in the following form

$$\mathbf{H} = H_1\mathbf{N}_1 + H_2\mathbf{N}_2 + H_3\mathbf{N}_3, \tag{A.13}$$

where

$$H_i = \frac{1}{\sqrt{M_i}}, \quad i = 1, 2, 3. \tag{A.14}$$

Let now $\mathbf{M}$ denote an arbitrary second-order symmetric tensor and let $\mathbf{G}$ stand for its deviator, i.e.,

$$\mathbf{G} = \mathbf{M} - \frac{1}{3}(\operatorname{tr}\mathbf{M})\,\mathbf{I}. \tag{A.15}$$

The deviator $\mathbf{G}$ is obviously an isotropic function of the tensor $\mathbf{M}$. Consequently, the eigentensors (eigenvectors) of these two tensors coincide.

Formulas (A.6) for the eigentensors of tensor $\mathbf{M}$ hold also true if $\mathbf{M}$ is not necessarily positive-definite. One can write them in an alternative form

$$M_i = \frac{1}{3} I_M + G_i, \quad i = 1, 2, 3, \tag{A.16}$$

where

$$\begin{aligned} G_1 &= \sqrt{\frac{2}{3}}\,\|\mathbf{G}\| \cos\varphi \\ G_2 &= \sqrt{\frac{2}{3}}\,\|\mathbf{G}\| \cos\left(\varphi - \frac{2}{3}\pi\right) \\ G_3 &= \sqrt{\frac{2}{3}}\,\|\mathbf{G}\| \cos\left(\varphi + \frac{2}{3}\pi\right), \end{aligned} \tag{A.17}$$

$$\|\mathbf{G}\| = \sqrt{\operatorname{tr}\mathbf{G}^2}, \quad \varphi = \frac{1}{3}\arccos\frac{3\sqrt{6}\det\mathbf{G}}{\|\mathbf{G}\|^3}, \quad \varphi \in \left[0, \frac{\pi}{3}\right]. \tag{A.18}$$

Appendix B.

The elastic properties determine mechanical behaviour of cortical and cancellous bones during normal daily activities. During such activities bone can be considered as a nonhomogeneous linear elastic anisotropic material, cf. [19,26]. Elastic properties are defined at the continuum level. At this level the bone is considered to be a continuous material with elastic properties that represent average properties of a representative bone volume. For cancellous bone a representative volume should be at least a cube approximately 3-5 mm in size.

The anisotropic form of Hooke's law is written in indicial notations as (in the normalized basis $\{\mathbf{e}_i\} \in E^3$):

$$T_{ij} = C_{ijkl} E_{kl}, \tag{B.1}$$

where C_{ijkl} are the components of the elasticity tensor $\mathbf{C}$. There are three important symmetry restrictions on the tensor $\mathbf{C}$:

$$\mathsf{C}_{ijkl} = \mathsf{C}_{ijkl} = \mathsf{C}_{ijlk} = \mathsf{C}_{klij}. \tag{B.2}$$

These restrictions follow from the symmetry of the stress tensor, the symmetry of the strain tensor, and the requirement that no work be produced by the elastic material in a closed loading cycle, respectively.

The inverse of the stress-strain relations (B.1) are the strain-stress relations

$$E_{ij} = \mathsf{S}_{ijkl} T_{kl}, \tag{B.3}$$

where S_{ijkl} are the components of the compliance tensor $\mathbf{S} = \mathbf{C}^{-1}$ ($\mathbf{S}$ is related to the elasticity tensor $\mathbf{C}$ by $\mathbf{C} : \mathbf{S} = \mathbf{S} : \mathbf{C} = \mathbf{1}$ ($\mathsf{C}_{ijmn}\mathsf{S}_{mnkl} = \mathsf{S}_{ijmn}\mathsf{C}_{mnkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$), cf. [58,100]). The stiffness and compliance tensors must be positive-definite.

Introducing the notation of Voigt, (B.1) and (B.3) can be rewritten as follows:

$$\hat{T}_p = \mathrm{C}_{pq}\hat{E}_q, \quad \hat{E}_p = \mathrm{S}_{pq}\hat{T}_q, \quad p = 1, \ldots 6, \tag{B.4}$$

where

$$\begin{aligned} \left[\hat{T}_p\right]^T &= [T_{11}, T_{22}, T_{33}, T_{23}, T_{13}, T_{12}], \\ \left[\hat{E}_p\right]^T &= [E_{11}, E_{22}, E_{33}, 2E_{23}, 2E_{13}, 2E_{12}]. \end{aligned} \tag{B.5}$$

In the general case of anisotropy the stiffness and compliance matrices have 21 coefficients. For most purposes, bone can be considered as a linear orthotropic elastic material. In the case of orthotropy, three orthogonal planes of symmetry exist, leaving 9 independent elastic coefficients to be determined from experiments. The number of independent elastic coefficients is reduced to five for the case of transverse isotropy (cortical bone). Elastic symmetries are usually determined indirectly by considering symmetries in the texture or fabric tensor of the material, cf. Sec. 2 and Fig. 5. For trabecular bone it was found that MIL fits well to an ellipsoid, which has three planes of symmetry. The nine orthotropic elastic constants in a coordinate system aligned with the fabric axes of the specimen can be determined from nine elastic constants measured by compression tests or ultrasound experiments.

For further details on anisotropic elasticity the reader is referred to [27, 46, 84].

For materials that exhibit pure orthotropic or higher symmetries, Cowin and Mehrabadi [29, 30] developed a method of the determination of the principal orthotropic axes directly from 21 components of the stiffness matrix. If the material is not purely orthotropic, an optimization procedure has to be used to find the coordinate transformation that yields the best orthotropic representation of this matrix, cf. [63, 91, 119, 123, 127].

The matrix of stiffness coefficients and the matrix of compliance coefficients for an orthotropic material are written as follows (Nye [84], pp.141-159),

$$[C_{pq}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{B.6}$$

where

$$\begin{aligned} &C_{11} = \mathsf{C}_{1111}, \quad C_{22} = \mathsf{C}_{2222}, \quad C_{33} = \mathsf{C}_{3333}, \\ &C_{12} = \mathsf{C}_{1122} = \mathsf{C}_{2211}, \quad C_{13} = \mathsf{C}_{1133} = \mathsf{C}_{3311}, \\ &C_{23} = \mathsf{C}_{2233} = \mathsf{C}_{3322}, \\ &C_{44} = \mathsf{C}_{2323} = \mathsf{C}_{2332} = \mathsf{C}_{3223} = \mathsf{C}_{3232}, \\ &C_{55} = \mathsf{C}_{1313} = \mathsf{C}_{1331} = \mathsf{C}_{3113} = \mathsf{C}_{3131}, \\ &C_{66} = \mathsf{C}_{1212} = \mathsf{C}_{1221} = \mathsf{C}_{2112} = \mathsf{C}_{2121} \end{aligned} \tag{B.7}$$

and

$$[S_{pq}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}, \tag{B.8}$$

where

$$\begin{aligned} &S_{11} = \mathsf{S}_{1111}, \quad S_{22} = \mathsf{S}_{2222}, \quad S_{33} = \mathsf{S}_{3333}, \\ &S_{12} = \mathsf{S}_{1122} = \mathsf{S}_{2211}, \quad S_{13} = \mathsf{S}_{1133} = \mathsf{S}_{3311}, \\ &S_{23} = \mathsf{S}_{2233} = \mathsf{S}_{3322}, \\ &S_{44} = 4\mathsf{S}_{2323} = 4\mathsf{S}_{2332} = 4\mathsf{S}_{3223} = 4\mathsf{S}_{3232}, \\ &S_{55} = 4\mathsf{S}_{1313} = 4\mathsf{S}_{1331} = 4\mathsf{S}_{3113} = 4\mathsf{S}_{3131}, \\ &S_{66} = 4\mathsf{S}_{1212} = 4\mathsf{S}_{1221} = 4\mathsf{S}_{2112} = 4\mathsf{S}_{2121}. \end{aligned} \tag{B.9}$$

There are twelve non-zero components of which nine are independent. The compliance matrix for an orthotropic material expressed in terms of elastic properties such as Young's modulus, shear modulus and Poisson's ratio (technical moduli) is expressed by

$$[\mathrm{S}_{pq}] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \tag{B.10}$$

where E_i denotes the Young's modulus in the ith direction, ν_{ij} is Poisson's ratio for the strain in the j-direction with stress applied in the i-direction, and G_{ij} is the shear modulus in the i-j plane. Since the compliance matrix is symmetric, hence

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \quad \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}, \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}. \tag{B.11}$$

The matrix of elastic stiffness and the matrix of compliance coefficients are positive-definite. Hence we get

$$\begin{gathered} E_1 > 0, \quad E_2 > 0, \quad E_2 > 0, \quad G_{12} > 0, \quad G_{13} > 0, \quad G_{23} > 0, \\ 1 - \nu_{12}\nu_{21} > 0, \quad 1 - \nu_{13}\nu_{31} > 0, \quad 1 - \nu_{23}\nu_{32} > 0, \\ 1 - \nu_{12}\nu_{21} - \nu_{13}\nu_{31} - \nu_{23}\nu_{32} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32} > 0. \end{gathered} \tag{B.12}$$

From (B.11) we conclude that $\nu_{12}\nu_{23}\nu_{31} = \nu_{13}\nu_{21}\nu_{32}$. The following inequality is true: $|\nu_{ij}| < \sqrt{E_i/E_j}$, cf. also [19,26].

The stiffness and compliance matrices are mutually inverse. The components of the stiffness matrix in terms of technical constants are given by

$$\begin{gathered} \mathrm{C}_{11} = \frac{1 - \nu_{23}\nu_{32}}{E_2E_3D}, \quad \mathrm{C}_{22} = \frac{1 - \nu_{13}\nu_{31}}{E_1E_3D}, \quad \mathrm{C}_{33} = \frac{1 - \nu_{12}\nu_{21}}{E_1E_2D}, \\ \mathrm{C}_{12} = \frac{\nu_{21} - \nu_{31}\nu_{23}}{E_2E_3D} = \frac{\nu_{12} - \nu_{32}\nu_{13}}{E_1E_3D}, \\ \mathrm{C}_{13} = \frac{\nu_{31} - \nu_{21}\nu_{32}}{E_2E_3D} = \frac{\nu_{13} - \nu_{12}\nu_{23}}{E_1E_2D}, \\ \mathrm{C}_{23} = \frac{\nu_{32} - \nu_{12}\nu_{31}}{E_1E_2D} = \frac{\nu_{23} - \nu_{21}\nu_{13}}{E_1E_2D}, \\ \mathrm{C}_{44} = G_{23} = G_{32}, \quad \mathrm{C}_{55} = G_{13} = G_{31}, \quad \mathrm{C}_{66} = G_{12} = G_{21} \end{gathered} \tag{B.13}$$

where

$$D = \frac{1 - \nu_{12}\nu_{21} - \nu_{13}\nu_{31} - \nu_{23}\nu_{32} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32}}{E_1 E_2 E_2} > 0. \tag{B.14}$$

Transverse isotropy has a plane of isotropy, which is also a plane of mirror symmetry. For a transversely isotropic material the matrix of elastic coefficients and the matrix of compliance coefficients have the following forms (the vector $\mathbf{e}_3$ is normal to the plane of mirror symmetry):

$$[\mathrm{C}_{pq}] = \begin{bmatrix} \mathrm{C}_{11} & \mathrm{C}_{12} & \mathrm{C}_{13} & 0 & 0 & 0 \\ \mathrm{C}_{12} & \mathrm{C}_{11} & \mathrm{C}_{13} & 0 & 0 & 0 \\ \mathrm{C}_{13} & \mathrm{C}_{13} & \mathrm{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{C}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{C}_{11}-\mathrm{C}_{12}}{2} \end{bmatrix}, \tag{B.15}$$

$$[\mathrm{S}_{pq}] = \begin{bmatrix} \mathrm{S}_{11} & \mathrm{S}_{12} & \mathrm{S}_{13} & 0 & 0 & 0 \\ \mathrm{S}_{12} & \mathrm{S}_{11} & \mathrm{S}_{13} & 0 & 0 & 0 \\ \mathrm{S}_{13} & \mathrm{S}_{13} & \mathrm{S}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{S}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{S}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\left(\mathrm{S}_{11}-\mathrm{S}_{12}\right) \end{bmatrix} \tag{B.16}$$

where

$$\begin{gathered} \mathrm{S}_{11} = \frac{1}{E}, \quad \mathrm{S}_{12} = -\frac{\nu}{E}, \quad \mathrm{S}_{13} = -\frac{1}{2}\left(\frac{\nu_{TI}}{E_T} + \frac{\nu_{IT}}{E}\right), \\ \mathrm{S}_{33} = \frac{1}{E_T}, \quad \mathrm{S}_{44} = \frac{1}{G_T}, \\ E = E_1 = E_2, \quad E_T = E_3, \\ \nu = \nu_{12} = \nu_{21}, \quad \nu_{IT} = \nu_{13} = \nu_{23}, \quad \nu_{TI} = \nu_{31} = \nu_{32}, \\ G_T = G_{13} = G_{23}, \quad 2\left(\mathrm{S}_{11} - \mathrm{S}_{12}\right) = \frac{2(1+\nu)}{E}. \end{gathered} \tag{B.17}$$

The compliance matrix for the transversely isotropic material expressed in terms of technical moduli is given by

$$[\mathrm{S}_{pq}] = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu_{TI}}{E_T} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu_{TI}}{E_T} & 0 & 0 & 0 \\ -\frac{\nu_{IT}}{E} & -\frac{\nu_{IT}}{E} & \frac{1}{E_T} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_T} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_T} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\nu}{E} \end{bmatrix}, \quad \frac{\nu_{TI}}{E_T} = \frac{\nu_{IT}}{E}. \tag{B.18}$$

TABLE 4. Technical moduli (ultrasonic technique, after Rho [90] and Ashman et al. [4]). Average technical moduli of human tibial cortical bone, where the 1-direction is radial, the 2-direction is circumferential and the 3-direction is longitudinal. The numbers in parentheses are the standard deviations. No significant variation in elastic properties was found along the length and around the circumference of the cortical bone. There is little difference between the values of the elastic properties of the mid-diaphysis of human and canine femora [4].

Technical constants (average)	Human tibial cortical bone, [90]	Human femoral bone, [4] (60 specimens)	Canine femoral bone, [4] (120 specimens)
E_1	11.7 (1.3) [GPa]	12.0 [GPa]	12.8 [GPa]
E_2	12.2 (1.4) [GPa]	13.4 [GPa]	15.6 [GPa]
E_3	20.7 (1.9) [GPa]	20.0 [GPa]	20.1 [GPa]
G_{12}	4.1 (0.5) [GPa]	4.53 [GPa]	4.68 [GPa]
G_{13}	5.17 (0.6) [GPa]	5.61 [GPa]	5.68 [GPa]
G_{23}	5.7 (0.5) [GPa]	6.23 [GPa]	6.67 [GPa]
ν_{12}	0.420 (0.074)	0.376	0.282
ν_{21}	0.435 (0.057)	0.422	0.366
ν_{23}	0.231 (0.035)	0.235	0.265
ν_{32}	0.390 (0.021)	0.350	0.341
ν_{13}	0.237 (0.041)	0.222	0.289
ν_{31}	0.417 (0.048)	0.371	0.454

TABLE 5. Technical moduli (ultrasonic technique, after Lasaygues and Pithioux [70]).

Technical constants	Bovine bone (3)	Bovine bone (4)
E_1	20.6 [GPa]	18.7 [GPa]
E_2	23.4 [GPa]	20.0 [GPa]
E_3	30.2 [GPa]	28.0 [GPa]
G_{12}	3.0 [GPa]	2.9 [GPa]
G_{13}	3.0 [GPa]	2.8 [GPa]
G_{23}	4.6 [GPa]	3.7 [GPa]
ν_{12}	0.12	0.26
ν_{21}	0.21	0.28
ν_{23}	0.18	0.17
ν_{32}	0.24	0.25
ν_{13}	0.20	0.17
ν_{31}	0.29	0.26

Bone elastic properties vary with anatomical site and are affected by age and general health of the donor. In addition, the preparation and storage of bone specimens can affect the elastic properties of the tissue, cf. [19, 26]. Traditionally, bone elastic moduli have been determined using compression and tension tests. The determination of Poisson's ratio and shear modulus from mechanical tests is difficult to perform. As an alternative to compression tests, ultrasound measurements were used to determine all bone elastic moduli, [4, 26, 70, 72, 90]. Ultrasonic techniques (UT) offer an advantage in that they allow the use of smaller specimens of less complicated geometries than do mechanical testing methods. Another advantage of the use of UT is that more than one elastic coefficient can be measured on each specimen. A continuous wave technique for measuring the 9 independent orthotropic elastic constants was described by Ashman et al. [4]. Acoustic tests provide accurate results in small (0.5 to 5 mm) specimens. Specimens over 10 mm in length generally cannot be used, cf. [26].

Introducing the notation of tensors in the normalized basis, Eqs. (B.1) and (B.3) can be rewritten in the form:

$$T_p = \mathsf{C}_{pq} E_q, \quad E_p = \mathsf{S}_{pq} T_q, \tag{B.19}$$

where

$$\begin{aligned} [T_p]^T &= \left[T_{11}, T_{22}, T_{33}, \sqrt{2}T_{23}, \sqrt{2}T_{13}, \sqrt{2}T_{12}\right], \\ [E_p]^T &= \left[E_{11}, E_{22}, E_{33}, \sqrt{2}E_{23}, \sqrt{2}E_{13}, \sqrt{2}E_{12}\right]. \end{aligned} \tag{B.20}$$

For an orthotropic material the stiffness and compliance matrices are given by:

$$\begin{aligned} [\mathsf{C}_{pq}] &= \begin{bmatrix} \mathrm{C}_{11} & \mathrm{C}_{12} & \mathrm{C}_{13} & 0 & 0 & 0 \\ \mathrm{C}_{12} & \mathrm{C}_{22} & \mathrm{C}_{23} & 0 & 0 & 0 \\ \mathrm{C}_{13} & \mathrm{C}_{23} & \mathrm{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mathrm{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mathrm{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mathrm{C}_{66} \end{bmatrix} \\ &= \begin{bmatrix} e_1 & f_3 & f_2 & 0 & 0 & 0 \\ f_3 & e_2 & f_1 & 0 & 0 & 0 \\ f_2 & f_1 & e_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2g_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2g_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2g_3 \end{bmatrix}, \end{aligned} \tag{B.21}$$

$$[S_{pq}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{S_{44}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{S_{55}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{S_{66}}{2} \end{bmatrix} = \begin{bmatrix} p_1 & r_3 & r_2 & 0 & 0 & 0 \\ r_3 & p_2 & r_1 & 0 & 0 & 0 \\ r_2 & r_1 & p_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{s_1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{s_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{s_3}{2} \end{bmatrix}, \tag{B.22}$$

respectively.

Following the papers [45,58,97], the relations between technical moduli and compliance tensor can be established:

the generalized Young modulus in an arbitrary direction

$$\frac{1}{E(\mathbf{n})} = (\mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{S} \cdot (\mathbf{n} \otimes \mathbf{n}), \tag{B.23}$$

the generalized Poisson ratios in an arbitrary plane

$$-\frac{\nu(\mathbf{n},\mathbf{m})}{E(\mathbf{n})} = -\frac{\nu(\mathbf{m},\mathbf{n})}{E(\mathbf{m})} = (\mathbf{m} \otimes \mathbf{m}) \cdot \mathbf{S} \cdot (\mathbf{n} \otimes \mathbf{n}), \quad \mathbf{m} \cdot \mathbf{n} = 0, \tag{B.24}$$

the bulk modulus

$$\frac{1}{K} = \mathbf{I} \cdot \mathbf{S} \cdot \mathbf{I}, \tag{B.25}$$

the generalized Kirchhoff modulus in an arbitrary plane

$$\frac{1}{G(\mathbf{m},\mathbf{n})} = \frac{1}{G(\mathbf{n},\mathbf{m})} = 4(\mathbf{m} \otimes \mathbf{n}) \cdot \mathbf{S} \cdot (\mathbf{m} \otimes \mathbf{n}), \quad \mathbf{m} \cdot \mathbf{n} = 0. \tag{B.26}$$

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