Elements of constitutive modelling of saturated porous materials

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The formulation of a proper constitutive theory of fluid - saturated porous media requires a strong foundation of the basic principles on which the theory should be built.

Here the continuum formulation for permeable porous solid pores of which are filled with fluid is developed where immiscibility effect and inhomogeneity of relative fluid microflow due to pore structure influence are accounted for.

By the use of the micro-macro relations of linear momentum and kinetic energy for each component ensuring their full representation at the macro level the proper motion equations and solid - fluid interphase interaction are established where pore structure is characterized by two parameters: volume porosity and structural permeability (tensor) parameter.

For such two-phase material undergoing pure mechanical deformations the non-linear and linear constitutive relations (within the elastic range with some viscous effects) are developed, where the main consequences of the immiscibility, i.e. the skeleton pore structure characteristics and mutual independence of mechanical properties of individual constituents are taken into account.

Particularly, the deformation of fluid-porous solid composition with anisotropic and isotropic pore structure is analysed and compared with description given by Biot and other authors.

Stress-strain relation satisfying the effective stress concept influenced by initial pressure conditions is discussed.

The general form of compatibility conditions matching macroscopic mechanical fields at the contact surface between fluid-saturated porous solid and adjacent bulk fluid is established. Special attention is paid to the derivation of conditions for the tangential components of the fluid phase velocities at the contact surface and to the verification of validity of the slip velocity condition postulated by Beavers and Joseph.

Key words: *constitutive modelling, micro-macro relations, compatibility conditions*

1. Pore structure. Basic balance equations for porous solid - fluid immiscible mixture

Introduction

The description of mechanical behaviour of a deformable porous medium in which a liquid is moving, is usually approached in a macroscopic way. The method most often used in the formulation of a continuum approach to analyze the mechanical macrobehaviour of saturated permeable solid is based on the Classical Mixture Theory of several components given by Truesdell and Toupin, [11] and developed by Green and Naghdi, [21] and other researchers.

In that theory, a solid-fluid composition is treated as superposition of two continua; solid (s) and fluid (f) simultaneously occupying the same region of space, i.e. the constituents are assumed to be completely miscible and, consequently, there is no pore structure characteristics evidently appearing within such a description. However, it has been observed that saturated soils, porous rocks, sintered metals, etc., consist of an identifiable solid matrix and a fluid filling its pores and therefore during a deformation process, each constituent retains its integrity. Such materials are more complicated than the classical mixtures in the sense that they have internal geometrical structure reflecting the fact of immiscibility of constituents and local architecture of pores (see Fig. 1). Features of this type strongly influence the microbehaviour of phases (increasing micro-inhomogenity of their field quantities on grain or pore scale) as well as the macrobehaviour of fluid-porous solid composition. These effects are of prime importance in understanding the bulk mechanical behaviour and the acoustic properties of porous media saturated with fluids also in elastodynamics of gels, pressure diffusion through porous permeable media, etc.

Here the macro-continuum formulation for permeable porous solids saturated with fluid is developed where immiscibility effect and inhomogeneity of relative fluid microflow (which is the most significant among other micro inhomogeneities resulting from pore architecture influence) are accounted for.

A proper way to include in the description the immiscibility effect is to introduce the constituent volume fractions or, equivalently, volume porosity parameter. In order to consider the inhomogeneity of fluid microflow in microdescription, an additional pore structure characteristic is necessary. Here, this is achieved by considering the micro-macro relations of linear momentum and kinetic energy for each component, ensuring their full representation at the macro-level.
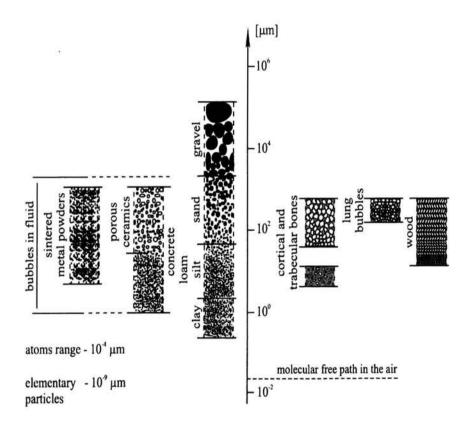
CHARACTERISTIC PORE/ GRAIN DIMENSIONS

Figure 1: Different porous materials

1.1. Macroscopic pore structure characteristics. Basic field quantities

To consider macroscopic description of saturated porous media when the immiscibility effect and the inhomogeneity of pore fluid microflow should be accounted for, it is very instructive to start with the Classical Mixture formulation (see [11]). In such case the mass continuity and motion equations for chemical inert constituents of fluid-saturated porous solid have the form

$$\frac{\partial \bar{\rho}^s}{\partial t} + div\left(\bar{\rho}^s \mathbf{v}^s\right) = 0, \quad \frac{\partial \bar{\rho}^f}{\partial t} + div\left(\bar{\rho}^f \mathbf{v}^f\right) = 0 \tag{1.1}$$

$$div\left(\mathbf{T}^\alpha\right) + \bar{\rho}^\alpha \mathbf{b} + \hat{\boldsymbol{\pi}}^\alpha = \bar{\rho}^\alpha \frac{d^\alpha}{dt}\mathbf{v}^\alpha, \quad \alpha = s - solid, \ f - fluid, \tag{1.2}$$

where $\bar{\rho}^\alpha$ stands for partial density, $\mathbf{v}^\alpha$ is the velocity of the α component, $\mathbf{T}^\alpha$ is the partial stress tensor and $\mathbf{b}$ is the body force per unit mass. For the purely mechanical process, they are complemented with constitutive relations for partial stresses $\mathbf{T}^\alpha$ and the momentum supply usually proposed as the drag force

$$\hat{\boldsymbol{\pi}}^s = -\hat{\boldsymbol{\pi}}^f \equiv \hat{\boldsymbol{\pi}} = const.\left(\mathbf{v}^f - \mathbf{v}^s\right). \tag{1.3}$$

It is clear that there is no pore structure parameter evidently appearing in the above equations. Moreover, when field quantities are considered to be locally averaged quantities (which is commonly accepted), they do not exhaustively reflect the state of the system (e.g. the kinetic energy expressed by averaged velocities is not equal to the total kinetic energy of the system).

To arrive at a relevant macro-description of the mechanical behaviour of saturated porous solids, we start by defining the basic field macro-quantities for each component of the mixture in which the immiscibility effect and inhomogeneity of relative fluid micro-flow will be taken into account. This can be achieved by applying the volume averaging technique.

Under an assumption that for each component the micro-continuum description on pore or grain scale is valid, the local volume averaging technique may be used to obtain the macroscopic field quantities. Applying the averaging procedures over a Representative Elementary Volume (REV) Ω to any microscopic quantity $\psi^{\alpha\alpha}$ (scalar, vector, or tensor) of each α-component ($\alpha = s, f$), we obtain, [25, 53],

$$\bar{\psi}^\alpha = \langle\overline{\psi^{\alpha\alpha}}\rangle = \frac{1}{\Omega}\int_{\Omega^\alpha}\psi^{\alpha\alpha}d\Omega, \quad \psi^\alpha = \langle\psi^{\alpha\alpha}\rangle = \frac{1}{n^\alpha\Omega}\int_{\Omega^\alpha}\psi^{\alpha\alpha}d\Omega, \tag{1.4}$$

where $\bar{\psi}^\alpha$ and ψ^α are the *bulk* and the *effective volume average quantities*, respectively. Parameter n^α is the *volume fraction* defined by

$$n^\alpha = \frac{\Omega^\alpha}{\Omega}, \tag{1.5}$$

where Ω^α is the part of Ω occupied by the α-component. Obviously, n^α is constrained by $\sum_\alpha n^\alpha = 1$ and $0 \le n^\alpha \le 1$.

Since all pores are considered to be interconnected, then

$$n^f = \frac{\Omega^f}{\Omega} = f_v, \tag{1.6}$$

where f_v denotes *volume porosity* reflecting the immiscibility of constituents of porous solid-fluid composition.

After the use of (1.4), one can find the *effective* and *partial mass densities* for each α-phase, i.e.

$$\rho^\alpha = \langle \rho^{\alpha\alpha} \rangle, \quad \bar{\rho}^\alpha = \langle \overline{\rho^{\alpha\alpha}} \rangle = \rho^\alpha n^\alpha \tag{1.7}$$

and, next, one can define the *average velocities*

$$\mathbf{v}^\alpha = \langle \overline{\rho^{\alpha\alpha}\mathbf{v}^{\alpha\alpha}} \rangle / \langle \overline{\rho^{\alpha\alpha}} \rangle. \tag{1.8}$$

It should be pointed out here that the quantities defined above correspond to the so-called *physical components*, i.e. *the solid* and *the pore fluid*, respectively (see Fig. 2).
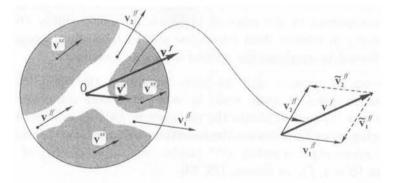

Figure 2: Description of physical components of fluid - porous solid composition

For further discussion, it is reasonable to disregard *deviations* of microscopic densities and the skeleton velocity from their averages preserving only the deviations of fluid micro-velocity from its average. Consequently, considering (1.7) and (1.8), we have

$$\rho^s = \rho^{ss}, \quad \rho^f = \rho^{ff}, \quad \mathbf{v}^s = \mathbf{v}^{ss},$$
$$\mathbf{v}^f = \left\langle \overline{\rho^{ff} \mathbf{v}^{ff}} \right\rangle / \left\langle \overline{\rho^{ff}} \right\rangle = \left\langle \mathbf{v}^{ff} \right\rangle. \tag{1.9}$$

Taking this into account, the local form of linear momentum and kinetic energy for α-phase, in relation to Ω, written as

$$\mathbf{I}^\alpha = \bar{\rho}^\alpha \mathbf{v}^\alpha, \quad 2E_k^\alpha = \bar{\rho}^\alpha \mathbf{v}^\alpha \cdot \mathbf{v}^\alpha, \quad \alpha = s, f, \tag{1.10}$$

are not satisfactory.

From (1.10) the well-known conclusion follows that the fluid kinetic energy expressed by means of the average velocity does not represent the total fluid kinetic energy, i.e.

$$\frac{1}{2}\bar{\rho}^f \mathbf{v}^f \cdot \mathbf{v}^f \neq \frac{1}{2} \left\langle \overline{\rho^{ff} \mathbf{v}^{ff} \cdot \mathbf{v}^{ff}} \right\rangle. \tag{1.11}$$

Here we propose a description ensuring full representation of the fluid kinetic energy. On introduction of decomposition of fluid micro-velocity

$$\mathbf{v}^{ff} = \mathbf{v}^s + \mathbf{u}^{ff}, \tag{1.12}$$

where $\mathbf{u}^{ff}$ is the relative fluid micro-velocity, the fluid linear momentum and kinetic energy can be written as

$$\mathbf{I}^f = \bar{\rho}^f \mathbf{v}^s + \left\langle \overline{\rho^{ff} \mathbf{u}^{ff}} \right\rangle$$
$$2E^f = \bar{\rho}^f \mathbf{v}^s \cdot \mathbf{v}^s + 2\mathbf{v}^s \cdot \left\langle \overline{\rho^{ff} \mathbf{u}^{ff}} \right\rangle + \left\langle \overline{\rho^{ff} \mathbf{u}^{ff} \cdot \mathbf{u}^{ff}} \right\rangle. \tag{1.13}$$

- *Isotropic pore structure*

Now, regarding isotropic pore structure, for the relative motion of the pore fluid, the following equations are to be satisfied

$$\left\langle \overline{\rho^{ff} \mathbf{u}^{ff}} \right\rangle = \overset{**}{\rho} \overset{*}{\mathbf{u}}, \quad \left\langle \overline{\rho^{ff} \mathbf{u}^{ff} \cdot \mathbf{u}^{ff}} \right\rangle = \overset{**}{\rho} \overset{*}{\mathbf{u}} \cdot \overset{*}{\mathbf{u}}. \tag{1.14}$$

They define the macro-quantities $\overset{*}{\rho}$ and $\overset{*}{\mathbf{u}}$ assuring at the macrolevel the full representation of the linear momentum $\mathbf{I}^f$ and the fluid kinetic energy E^f. Taking into account (1.9), the solution of the system (1.14) is as follows

$$\frac{\overset{*}{\rho}}{\rho^f} = \frac{\left\langle \overline{\mathbf{u}^{ff}} \right\rangle \cdot \left\langle \overline{\mathbf{u}^{ff}} \right\rangle}{\left\langle \mathbf{u}^{ff} \cdot \mathbf{u}^{ff} \right\rangle} \equiv \lambda; \quad \overset{*}{\mathbf{u}} = \frac{1}{\lambda} \left\langle \overline{\mathbf{u}^{ff}} \right\rangle; \quad \overset{*}{\rho} = \lambda \rho^f. \qquad (1.15)$$
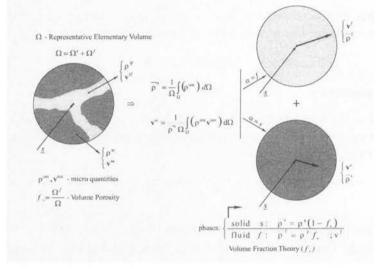

Figure 3: A REV with the introduced micro-velocity $\mathbf{v}^{ff}$ and $\mathbf{v}^{ss}$ for the fluid and the solid, respectively; $\tilde{\mathbf{v}}^{ff}$ is the fluid deviation vector from the average fluid velocity $\mathbf{v}^f$.

Introducing the parameter $\kappa = \lambda/f_v$ we obtain

$$\kappa = \frac{\lambda}{f_v} = \frac{1}{f_v} \frac{\left\langle \overline{\mathbf{u}^{ff}} \right\rangle \cdot \left\langle \overline{\mathbf{u}^{ff}} \right\rangle}{\left\langle \mathbf{u}^{ff} \cdot \mathbf{u}^{ff} \right\rangle}; \quad \overset{*}{\mathbf{u}} = \frac{1}{\kappa} \mathbf{u}^f; \quad \overset{*}{\rho} = \kappa \bar{\rho}^f. \qquad (1.16)$$

where

$$\mathbf{u}^f = \left\langle \mathbf{u}^{ff} \right\rangle = \left(\mathbf{v}^f - \mathbf{v}^s \right).$$

From (1.15) and (1.16), it is important to notice that the parameter λ (or κ) is the measure of inhomogeneity of the fluid micro-velocity in its

relative flow and is restricted by

$$0 < \kappa \leq 1 \quad \text{or} \quad 0 < \lambda \leq f_v. \tag{1.17}$$

The parameter λ (or κ) may, in general, not only reflect the fluid flow inhomogeneity due to pore structure influence but also the influence of some fluid features on that flow. However, it has been proved that when a viscous flow through the noninterpenetrating channel-like pores arbitrary oriented within the skeleton is considered, parameter λ (or κ) represents the pore architecture only and it does not depend on the features of the fluid flowing through pores, [15]. This finding can be extended to the nonviscous transient flow, particularly to the case of high frequency range.

The above discussion shows that for considerations of macroscopic mechanical behaviour of porous solid - fluid composition, when immiscibility effect and fluid flow inhomogeneity due to pore structure influence are taken into account, the pore structure has to be characterized, apart from the volume porosity, by another structural parameter.

• *Anisotropic pore structure*

In many cases when the continuum description of mechanical behaviour of saturated porous media with an anisotropic internal pore structure is considered, the assumption of isotropy of skeleton pore structure is too restrictive. In such a case instead of scalar structural parameter one should expect tensorial characteristic of pore structure.

To describe anisotropic pore space of solid skeleton and its influence on motion of saturated porous solid, one can follow the approach e.g. of Biot, [8], Bedford and Drumheller, [6] and Kubik, [35] and assume the total kinetic energy of the system reflecting the real motion of pore fluid to consist of two parts:

1. K_d - the part, responsible for the contribution of the average velocity fields of the solid and fluid components

$$K_d = \frac{1}{2}\bar{\rho}^s \mathbf{v}^s \cdot \mathbf{v}^s + \frac{1}{2}\bar{\rho}^f \mathbf{v}^f \cdot \mathbf{v}^f , \tag{1.18}$$

2. K_n - the part, responsible for the inhomogeneity of the fluid velocity field at the pore level, caused by the pore structure and expressed by means of the fluid velocity relative to the skeleton

$$K_d = \frac{1}{2}\bar{\rho}^f \left(\mathbf{v}^f - \mathbf{v}^s\right) \cdot \mathbf{A} \left(\mathbf{v}^f - \mathbf{v}^s\right) , \tag{1.19}$$

where the symmetric tensor **A** represents the influence of the geometrical structure of pores on the kinematic of the fluid flow, and assumed to be positive define.

With the use of the above expressions and the macroscopic relative velocity

$$\mathbf{u}^f = \mathbf{v}^f - \mathbf{v}^s$$

the kinetic energy of the considered system can be written in the following form

$$E = \frac{1}{2}\bar{\rho}^s \mathbf{v}^s \cdot \mathbf{v}^s + \frac{1}{2}\bar{\rho}^f \mathbf{v}^s \cdot \mathbf{v}^s + 2\bar{\rho}^f \mathbf{v}^s \cdot \mathbf{u}^f + \mathbf{u}^f \left(1 + \mathbf{A}\right) \mathbf{u}^f , \qquad (1.20)$$

where the motion of the solid phase is singled out.

Since tensor $1 + \mathbf{A}$ is symmetric and non-singular, we can define a new second order symmetric tensor **P**, such that

$$1 + \mathbf{A} = (\mathbf{P})^{-1} f_v \quad or \quad \mathbf{P} = (1 + \mathbf{A})^{-1} f_v . \qquad (1.21)$$

Using the tensor **P** we may define a new relative velocity field for the fluid component

$$\hat{\mathbf{u}} = (\mathbf{P}^{-1})\mathbf{u}^f f_v , \qquad (1.22)$$

where macroscopic parameter **P** reflects the effect of the tortuosity of the pore structure of the skeleton on the fluid pore velocity.

In the case of an isotropic pore structure, tensor **P** is the isotropic tensor, i.e. $\mathbf{P} = \lambda \mathbf{I}$ with λ as a structural permeability parameter, the same as in relation (1.16), and relative velocity

$$\hat{\mathbf{u}} \equiv \overset{*}{\mathbf{u}} = \frac{1}{\kappa}\mathbf{u}^f . \qquad (1.23)$$

The above analysis shows that in macroscopic description when the pore structure of saturated porous media is not isotropic and the condition of full kinetic energy representation has to be satisfied, the immiscibility effect and pore space geometry require two macroscopic parameters; the volume porosity which is a scalar parameter and the second order tensor.
Thus, for further discussion, the pore structure of the considered permeable solids will be assumed to be characterized by two macro-parameters: volume porosity f_v and parameter **P** called the *structural permeability parameter* introduced by the author, [34] .

1.2. Division of solid - fluid mixture based on the kinematic criterion

• *Virtual components - isotropic pore structure*

When we use the macroscopic quantities for density and velocity fields defined by (1.7),(1.8), the assumptions (1.9), and we take into account the full representation of the kinetic energy of the solid - fluid composition including the effect of inhomogeneity of the pore fluid velocity field caused by the pore structure (1.13)-(1.16) one can write the linear momentum and kinetic energy for the porous skeleton and the fluid, respectively, as follows, [36]

$$\mathbf{I}^s = \langle \overline{\rho^{ss}\mathbf{v}^{ss}} \rangle = \bar{\rho}^s \mathbf{v}^s , \tag{1.24}$$

$$2E^s = \langle \overline{\rho^{ss}\mathbf{v}^{ss} \cdot \mathbf{v}^{ss}} \rangle = \bar{\rho}^s \mathbf{v}^s \cdot \mathbf{v}^s , \tag{1.25}$$

$$\mathbf{I}^f = \left\langle \overline{\rho^{ff}\mathbf{v}^{ff}} \right\rangle = \bar{\rho}^f \mathbf{v}^f = \bar{\rho}^f \mathbf{v}^s + \left\langle \overline{\rho^{ff}\mathbf{u}^{ff}} \right\rangle =$$
$$= \bar{\rho}^f \mathbf{v}^s + \kappa \bar{\rho}^f \overset{*}{\mathbf{u}} = \bar{\rho}^f (1 - \kappa) \overset{1}{\mathbf{v}} + \kappa \bar{\rho}^f \overset{2}{\mathbf{v}} , \tag{1.26}$$

$$2E^f = \left\langle \overline{\rho^{ff}\mathbf{v}^{ff} \cdot \mathbf{v}^{ff}} \right\rangle = \bar{\rho}^f \mathbf{v}^s \cdot \mathbf{v}^s + 2\mathbf{v}^s \cdot \left\langle \overline{\rho^{ff}\mathbf{u}^{ff}} \right\rangle + \left\langle \overline{\rho^{ff}\mathbf{u}^{ff} \cdot \mathbf{u}^{ff}} \right\rangle$$
$$= \bar{\rho}^f \mathbf{v}^s \cdot \mathbf{v}^s + 2 \overset{*}{\rho} \mathbf{v}^s \cdot \overset{*}{\mathbf{u}} + \overset{*}{\rho}\overset{*}{\mathbf{u}} \cdot \overset{*}{\mathbf{u}} = \bar{\rho}^f (1 - \kappa) \overset{1}{\mathbf{v}} \cdot \overset{1}{\mathbf{v}} + \kappa \bar{\rho}^f \overset{2}{\mathbf{v}} \cdot \overset{2}{\mathbf{v}} , \tag{1.27}$$

where the following relations

$$\overset{1}{\mathbf{v}} = \mathbf{v}^s , \quad \overset{2}{\mathbf{v}} = \mathbf{v}^s + \overset{*}{\mathbf{u}} = \mathbf{v}^s + \frac{1}{\kappa}\mathbf{u}^f , \tag{1.28}$$

have been used.

From the above expressions, that assure full representations of both the linear momentum and kinetic energy of particular components in relation to Ω, we conclude that from the kinematic point of view, at the macrolevel, a fluid - porous solid mixture can be considered as composed of *two virtual components*. The first one being *the skeleton and the fluid associated with it* of partial density

$$\overset{1}{\rho} = \bar{\rho}^s + \bar{\rho}^f (1 - \kappa) \tag{1.29}$$

moving at the skeleton velocity $\overset{1}{\mathbf{v}}$, and the other being *free fluid* of partial density

$$\overset{2}{\bar{\rho}} = \kappa\bar{\rho}^f \tag{1.30}$$

moving at its own velocity $\overset{2}{\mathbf{v}}$, where it is easy to show that

$$\overset{1}{\bar{\rho}} + \overset{2}{\bar{\rho}} = \bar{\rho}^s + \bar{\rho}^f .$$

Consequently, the linear momentum for the first and second virtual constituent take the form, respectively,

$$\overset{1}{\mathbf{L}} = \overset{1}{\bar{\rho}}\overset{1}{\mathbf{v}} = [\bar{\rho}^s + \bar{\rho}^f(1 - \kappa)] \overset{1}{\mathbf{v}} , \tag{1.31}$$

$$\overset{2}{\mathbf{L}} = \overset{2}{\bar{\rho}}\overset{2}{\mathbf{v}} = \kappa\bar{\rho}^f \overset{2}{\mathbf{v}} , \tag{1.32}$$

where

$$\overset{1}{\mathbf{L}} + \overset{2}{\mathbf{L}} = \mathbf{L}^s + \mathbf{L}^f = \overset{1}{\bar{\rho}}\overset{1}{\mathbf{v}} + \overset{2}{\bar{\rho}}\overset{2}{\mathbf{v}} = \bar{\rho}^s\mathbf{v}^s + \bar{\rho}^f\mathbf{v}^f ,$$

and the kinetic energy for the two component system under consideration has canonical form

$$E = \frac{1}{2} \overset{1}{\bar{\rho}}\overset{1}{\mathbf{v}} \cdot \overset{1}{\mathbf{v}} + \frac{1}{2} \overset{2}{\bar{\rho}}\overset{2}{\mathbf{v}} \cdot \overset{2}{\mathbf{v}} . \tag{1.33}$$

• *Virtual components - anisotropic pore structure*

We make use of the derived expression (1.20) for the kinetic energy of saturated porous media with an anisotropic pore structure and the relation for the respective relative fluid velocity (1.22), and after some rearranging, we obtain the canonical representation for the kinetic energy of the system, [33]

$$E = \frac{1}{2} \left\{ \overset{1}{\mathbf{v}} \cdot [\bar{\rho}^s\mathbf{I} + \bar{\rho}^f(\mathbf{I} - \frac{1}{f_v}\mathbf{P})] \overset{1}{\mathbf{v}} + \overset{2}{\mathbf{v}} \cdot \bar{\rho}^f\frac{1}{f_v}\mathbf{P} \overset{2}{\mathbf{v}} \right\} , \tag{1.34}$$

where

$$\overset{1}{\mathbf{v}} = \mathbf{v}^s , \quad \overset{2}{\mathbf{v}} = \overset{1}{\mathbf{v}} + \overset{*}{\mathbf{u}} .$$

Taking the above representation into account one can define the virtual constituents of the system for the anisotropic case. The *first virtual*

constituent is composed of the skeleton and the part of the fluid component which moves at the skeleton velocity $\overset{1}{\mathbf{v}}$, the partial mass density of which is

$$\overset{1}{\mathbf{M}} = [\bar{\rho}^s \mathbf{I} + \bar{\rho}^f (\mathbf{I} - \frac{1}{f_v} \mathbf{P})] ; \tag{1.35}$$

the *second virtual constituent* - free fluid - has the partial mass density

$$\overset{2}{\mathbf{M}} = \bar{\rho}^f \frac{1}{f_v} \mathbf{P} , \tag{1.36}$$

and it moves with the corresponding velocity $\overset{2}{\mathbf{v}}$.

Consequently, the *linear momentum* for the first virtual constituent takes the form

$$\overset{1}{\mathbf{L}} = \overset{1}{\mathbf{M}} \overset{1}{\mathbf{v}} = \left\{ \bar{\rho}^s \mathbf{I} + \bar{\rho}^f (\mathbf{I} - \frac{1}{f_v} \mathbf{P}) \right\} \overset{1}{\mathbf{v}} , \tag{1.37}$$

and for the second virtual constituent is

$$\overset{2}{\mathbf{L}} = \overset{2}{\mathbf{M}} \overset{2}{\mathbf{v}} = \bar{\rho}^f \frac{1}{f_v} \mathbf{P} \overset{2}{\mathbf{v}} . \tag{1.38}$$

At the same time, the *canonical representation of the kinetic energy* will be

$$E = \frac{1}{2} \overset{1}{\mathbf{v}} \cdot \overset{1}{\mathbf{M}} \overset{1}{\mathbf{v}} + \frac{1}{2} \overset{2}{\mathbf{v}} \cdot \overset{2}{\mathbf{M}} \overset{2}{\mathbf{v}} . \tag{1.39}$$

One can now prove that the mass densities and the linear momentum densities of the virtual constituents satisfy the following conditions:

$$\overset{1}{\mathbf{M}} + \overset{2}{\mathbf{M}} = \left\{ \bar{\rho}^s + \bar{\rho}^f \right\} \mathbf{I} ,$$
$$\overset{1}{\mathbf{M}} \overset{1}{\mathbf{v}} + \overset{2}{\mathbf{M}} \overset{2}{\mathbf{v}} = \left\{ \bar{\rho}^s \mathbf{v}^s + \bar{\rho}^f \mathbf{v}^f \right\} .$$

• *Continuity equations*

For the considered fluid-porous solid composition, the continuity equations for the physical components (the solid and the pore fluid) have a form

$$\frac{\partial \bar{\rho}^s}{\partial t} + div\,(\bar{\rho}^s \mathbf{v}^s) = 0, \quad \frac{\partial \bar{\rho}^f}{\partial t} + div\left(\bar{\rho}^f \mathbf{v}^f\right) = 0. \tag{1.40}$$

The mass continuity equations for the virtual components in the case of mixture with isotropic pore structure can be written as follows

$$\frac{\partial \overset{1}{\rho}}{\partial t} + div\left(\overset{1}{\rho}\overset{1}{\mathbf{v}}\right) = \overset{1}{g}, \quad \frac{\partial \overset{2}{\rho}}{\partial t} + div\left(\overset{2}{\rho}\overset{2}{\mathbf{v}}\right) = \overset{2}{g}, \tag{1.41}$$

where $\overset{1}{g}$ and $\overset{2}{g}$ are mass supply terms satisfying the condition $\overset{1}{g} + \overset{2}{g} = 0$. Combining equations (1.40) and (1.41) one can find that for a given deformation, these terms are defined by the rate of change of the effective densities and the pore structure parameters and have form

$$\overset{1}{g} = -\overset{2}{g} \equiv g = \bar{\rho}^s \frac{\overset{1}{D}}{Dt}\left[\frac{\bar{\rho}^f}{\bar{\rho}^s}(1-\kappa)\right], \quad \frac{\overset{1}{D}}{Dt} = \frac{\partial}{\partial t} + \overset{1}{v}\cdot grad \tag{1.42}$$

from which it follows that the virtual components interchange their mass and thus, the masses of these components are not conserved.

On the same way one can formulate the mass continuity equations for virtual components of fluid - porous solid composition with anisotropic pore structure. They are

$$\frac{\partial}{\partial t}\overset{1}{\mathbf{M}} + div\left(\overset{1}{\mathbf{M}}\otimes\overset{1}{\mathbf{v}}\right) = \mathbf{H}\,,$$
$$\frac{\partial}{\partial t}\overset{2}{\mathbf{M}} + div\left(\overset{2}{\mathbf{M}}\otimes\overset{2}{\mathbf{v}}\right) = -\mathbf{H}\,, \tag{1.43}$$

where $\mathbf{H}$ represents tensorial mass supply term.

1.3. Motion equations. Solid - fluid interactions

Having defined the virtual components of the fluid-porous solid composition, we associate the stress vector $\overset{k}{\mathbf{t}}$ (k = 1, 2) with each of the

components in such a way that scalar product $\overset{k}{\mathbf{t}} \cdot \overset{k}{\mathbf{v}}$ represents the rate of work of a particular component per unit area of a surface bounding the bulk material. These can be derived from the condition that the total rate of mechanical work for virtual and physical components are equal, [36],

$$\overset{1}{\mathbf{t}} \cdot \overset{1}{\mathbf{v}} + \overset{2}{\mathbf{t}} \cdot \overset{2}{\mathbf{v}} = \mathbf{t}^s \cdot \mathbf{v}^s + \mathbf{t}^f \cdot \mathbf{v}^f. \tag{1.44}$$

Thus, using (1.28), the relations between $\overset{k}{\mathbf{t}}$ and $\mathbf{t}^\alpha$ for the isotropic pore structure, are

$$\overset{1}{\mathbf{t}} = \mathbf{t}^s + (1 - \kappa)\,\mathbf{t}^f, \quad \overset{2}{\mathbf{t}} = \kappa \mathbf{t}^f. \tag{1.45}$$

Next, applying to (1.45) Cauchy's theorem valid within the Classical Mixture Theory approach, i.e. $\mathbf{t}^\alpha = \mathbf{T}^\alpha \mathbf{n}$, we obtain relations

$$\overset{1}{\mathbf{t}} = \overset{1}{\mathbf{T}}\,\mathbf{n}, \quad \overset{2}{\mathbf{t}} = \overset{2}{\mathbf{T}}\,\mathbf{n}, \tag{1.46}$$

where $\overset{1}{\mathbf{T}}$ and $\overset{2}{\mathbf{T}}$ are stress tensors for virtual components and have the form

$$\overset{1}{\mathbf{T}} = \mathbf{T}^s + (1 - \kappa)\,\mathbf{T}^f, \quad \overset{2}{\mathbf{T}} = \kappa \mathbf{T}^f. \tag{1.47}$$

It can be seen that when $\kappa = 1$ (or $\lambda = f_v$) there is no difference between the physical and virtual components.

- *Motion equations*

Motion equations for virtual components may be obtained using either the axiom of balance of linear momentum for each constituent or from the energy balance for the whole solid-fluid composition by applying invariance conditions under superposed rigid body transitions.

If multipolar stresses as well as the externally applied multipolar body forces and thermal effects from our discussion are excluded, the energy balance can be written in the following form

$$\int_V \frac{\partial}{\partial t}\left(\bar{\rho}^s e^s + \bar{\rho}^f e^f + \frac{1}{2}\left(\overset{1}{\rho}\overset{1}{\mathbf{v}} \cdot \overset{1}{\mathbf{v}} + \overset{2}{\rho}\overset{2}{\mathbf{v}} \cdot \overset{2}{\mathbf{v}}\right)\right) dV +$$

$$\int_S \left(e^s \bar{\rho}^s \overset{1}{\mathbf{v}} + e^f \left[\left(\bar{\rho}^f - \overset{2}{\rho}\right)\overset{1}{\mathbf{v}} + \overset{2}{\rho}\overset{2}{\mathbf{v}}\right] + \frac{1}{2}\left[\left(\overset{1}{\rho}\overset{1}{\mathbf{v}} \cdot \overset{1}{\mathbf{v}}\right)\overset{1}{\mathbf{v}} + \left(\overset{2}{\rho}\overset{2}{\mathbf{v}} \cdot \overset{2}{\mathbf{v}}\right)\overset{2}{\mathbf{v}}\right]\right) \cdot d\mathbf{S} =$$

$$\int_V \left(\overset{1}{\rho}\mathbf{b} \cdot \overset{1}{\mathbf{v}} + \overset{2}{\rho}\mathbf{b} \cdot \overset{2}{\mathbf{v}}\right) dV + \int_S \left(\overset{1}{\mathbf{t}} \cdot \overset{1}{\mathbf{v}} + \overset{2}{\mathbf{t}} \cdot \overset{2}{\mathbf{v}}\right) dS, \tag{1.48}$$

where e^s and e^f stand for the internal energies per unit mass of the skeleton and the fluid, respectively, and where the external body forces $\overset{1}{\rho}\,\mathbf{b}$ and $\overset{2}{\rho}\,\mathbf{b}$ are defined by their rate of work contribution, i.e. $\overset{1}{\rho}\,\mathbf{b}\cdot\overset{1}{\mathbf{v}}$ and $\overset{2}{\rho}\,\mathbf{b}\cdot\overset{2}{\mathbf{v}}$, respectively.

Using (1.41) and applying invariance conditions under the superposed rigid body translations to (1.48) we obtain the equations of motion

$$
\begin{aligned}
div\,\overset{1}{\mathbf{T}} + \overset{1}{\rho}\,\mathbf{b} + \overset{1}{\boldsymbol{\pi}} &= \overset{1}{\rho}\,\frac{\overset{1}{D}}{Dt}\,\overset{1}{\mathbf{v}} + \frac{1}{2}g\left(\overset{2}{\mathbf{v}} - \overset{1}{\mathbf{v}}\right), \\
div\,\overset{2}{\mathbf{T}} + \overset{2}{\rho}\,\mathbf{b} + \overset{2}{\boldsymbol{\pi}} &= \overset{2}{\rho}\,\frac{\overset{2}{D}}{Dt}\,\overset{2}{\mathbf{v}} + \frac{1}{2}g\left(\overset{2}{\mathbf{v}} - \overset{1}{\mathbf{v}}\right),
\end{aligned}
\tag{1.49}
$$

where $\overset{k}{\boldsymbol{\pi}}\left(\overset{1}{\boldsymbol{\pi}} = -\overset{2}{\boldsymbol{\pi}}\right)$ represents the viscous interaction force and force $\frac{1}{2}g\left(\overset{2}{\mathbf{v}} - \overset{1}{\mathbf{v}}\right)$ results from the mass exchange between virtual components. This last physical effect originates from the influence of the pore structure on the relative fluid flow.

● *Solid-fluid interaction*

Now, the formulation of the considered immiscible mixture requires a specification of the couplings between the permeable porous solid and the fluid flowing through. Therefore, we write the motion equations for the physical components taking into account a two-parameter pore structure description.
Substituting (1.47), (1.29) and (1.30) to (1.49), we obtain

$$
\begin{aligned}
div\mathbf{T}^s + \bar{\rho}^s\mathbf{b} + \mathbf{r}^s &= \bar{\rho}^s\,\frac{\overset{1}{D}}{Dt}\,\overset{1}{\mathbf{v}} \\
div\mathbf{T}^f + \bar{\rho}^f\mathbf{b} + \mathbf{r}^f &= \bar{\rho}^f\left[(1-\kappa)\frac{\overset{1}{D}}{Dt}\,\overset{1}{\mathbf{v}} + \kappa\frac{\overset{2}{D}}{Dt}\,\overset{2}{\mathbf{v}}\right] + g\left(\overset{2}{\mathbf{v}} - \overset{1}{\mathbf{v}}\right)
\end{aligned}
\tag{1.50}
$$

where $\mathbf{r}^s$ and $\mathbf{r}^f$ represent the interaction force between the solid and fluid

and take the form

$$
\mathbf{r}^s = -\mathbf{r}^s = \left[\frac{1}{\kappa} \overset{1}{\pi} + \frac{1}{\kappa} \mathbf{T}^f \, grad \, (1-\kappa) + \right.
$$
$$
\left. + \bar{\rho}^f \, (1-\kappa) \left(\frac{\overset{2}{D}}{Dt} \overset{2}{\mathbf{v}} - \frac{\overset{1}{D}}{Dt} \overset{1}{\mathbf{v}} \right) + \left(\frac{1}{2\kappa} - 1 \right) g \left(\overset{2}{\mathbf{v}} - \overset{1}{\mathbf{v}} \right) \right]. \tag{1.51}
$$

It should be noted that in (1.51), the first two terms represent the viscous interaction between the solid and the fluid and the interaction due to the inhomogeneity of the pore structure, respectively.

It can be seen that when during a deformation process parameter κ becomes constant or $\kappa = 1$, the interaction due to structure inhomogeneity vanishes. The third term in (1.51) is responsible for the dynamic coupling, whereas the fourth term represents the inertial coupling between the solid and the fluid. These couplings result from the influence of the pore structure on the relative fluid flow.

1.4. Internal energy balance

The local form of the internal energy balance equation can be formulated either for the individual constituents or for the whole porous solid-fluid aggregate. In our case we apply the second formulation that allows us to avoid the specification of terms describing the interchange of energy between the constituents.

When the immiscibility of the physical constituents is taken into account the internal energy of the porous solid-fluid aggregate is the sum of internal energies of the porous solid and of the fluid filling its pores. Therefore, disregarding thermal effects, the internal energy balance equation for the whole aggregate can be written in the following form

$$
\bar{\rho}^s \frac{D^s e^s}{Dt} + \bar{\rho}^f \frac{D^f e^f}{Dt} = \pi \cdot \left(\mathbf{v}^f - \mathbf{v}^s \right) + tr \left(\mathbf{T}^s \mathbf{L}^s \right) + tr \left(\mathbf{T}^f \mathbf{L}^f \right), \tag{1.52}
$$

where e^f and e^s are the internal energies per unit mass for the fluid and porous skeleton, respectively, and tensors

$$
\mathbf{L}^s = grad \, (\mathbf{v}^s), \quad \mathbf{L}^f = grad \left(\mathbf{v}^f \right) \tag{1.53}
$$

are the velocity gradients of constituents.

Equation (1.52) will be used in the further part of this work to derive necessary constitutive relations.

1.5. Relations to other descriptions

The obtained results can be reduced to two particular but very important cases; namely to the *Volume Fraction Theory* (VFT), [6], and to *Biot's linear dynamic equations*, (see [7]).

• *Relation to Volume Fraction Theory*

If we assume that the pore structure of the skeleton is characterized by the volume porosity parameter only, i.e. if $\kappa = 1$ (or $\lambda = f_v$) Equations (1.50) take the form

$$
\begin{aligned}
div\mathbf{T}^s + \bar{\rho}^s\mathbf{b} + \mathbf{r}^s &= \bar{\rho}^s\frac{D^s}{Dt}\mathbf{v}^s, \\
div\mathbf{T}^f + \bar{\rho}^f\mathbf{b} + \mathbf{r}^f &= \bar{\rho}^f\frac{D^f}{Dt}\mathbf{v}^f,
\end{aligned}
\tag{1.54}
$$

which are identical to those of the VFT. Moreover, the interaction force (1.51) is reduced to viscous interaction only, i.e.

$$
\mathbf{r}^s = \overset{1}{\boldsymbol{\pi}} ; \quad \mathbf{r}^f = \overset{2}{\boldsymbol{\pi}}
\tag{1.55}
$$

and the dynamic and inertial coupling vanishes. The above results prove that dynamic and inertial couplings are not evidently included in VFT and they may be taken into account only by postulating such effects when the form of forces (1.55) is formulated. This statement is the basic difference between the proposed formulation of immiscible mixture and the VFT.

• *Relation to Biot's Theory*

Linear dynamic equations for a fluid saturated porous solid, widely used for analyzing such a material, were proposed by Biot, [7]. They are

$$
\begin{aligned}
div\mathbf{T}^s + \bar{\rho}^s\mathbf{b} + \mathbf{R}^s &= \bar{\rho}^s\frac{\partial}{\partial t}\mathbf{v}^s + \rho_{12}\frac{\partial}{\partial t}\left(\mathbf{v}^f - \mathbf{v}^s\right), \\
div\mathbf{T}^f + \bar{\rho}^f\mathbf{b} + \mathbf{R}^f &= \bar{\rho}^f\frac{\partial}{\partial t}\mathbf{v}^f - \rho_{12}\frac{\partial}{\partial t}\left(\mathbf{v}^f - \mathbf{v}^s\right),
\end{aligned}
\tag{1.56}
$$

where ρ_{12} is Biot's mass coupling parameter and $\mathbf{R}^s$ and $\mathbf{R}^f$ are viscous interaction forces of the form:

$$\mathbf{R}^s = -\mathbf{R}^f = b\left(\mathbf{v}^f - \mathbf{v}^s\right), \quad b = const. \tag{1.57}$$

It should be pointed out that Equation (1.56) cannot be derived directly from the integral form of the balance of linear momentum for physical components.

Moreover, despite the structural origin, coefficient ρ_{12}, being responsible for the dynamic coupling, has not been related to any pore-structure parameter.

It will be shown that Biot's dynamic equations can be obtained from general equations (1.49) and that the coupling parameter ρ_{12} is related with the volume porosity and the structural permeability quantity.

The linearized form of Equations (1.49) is

$$\begin{aligned}
div\,\overset{1}{\mathbf{T}} + \overset{1}{\rho}\mathbf{b} + \overset{1}{\pi} &= \overset{1}{\rho}\frac{\partial}{\partial t}\overset{1}{\mathbf{v}}, \\
div\,\overset{2}{\mathbf{T}} + \overset{2}{\rho}\mathbf{b} + \overset{2}{\pi} &= \overset{2}{\rho}\frac{\partial}{\partial t}\overset{2}{\mathbf{v}},
\end{aligned} \tag{1.58}$$

in which parameters f_v and κ are constant. Applying relations (1.47) to (1.58) and accounting for (1.28), we obtain

$$\begin{aligned}
div\,\mathbf{T}^s + \bar{\rho}^s\mathbf{b} + \mathbf{r}^s &= \bar{\rho}^s\frac{\partial}{\partial t}\mathbf{v}^s, \\
div\,\mathbf{T}^f + \bar{\rho}^f\mathbf{b} + \mathbf{r}^f &= \bar{\rho}^f\frac{\partial}{\partial t}\mathbf{v}^f,
\end{aligned} \tag{1.59}$$

where

$$\mathbf{r}^s = -\mathbf{r}^f = \left[\frac{1}{\kappa}\overset{1}{\pi} - \bar{\rho}^f\left(1 - \frac{1}{\kappa}\right)\frac{\partial}{\partial t}\left(\mathbf{v}^f - \mathbf{v}^s\right)\right]. \tag{1.60}$$

Denoting

$$\bar{\rho}^f\left(1 - \frac{1}{\kappa}\right) = \rho_{12}, \quad \frac{1}{\kappa}\overset{1}{\pi} = \mathbf{R}^s, \quad \frac{1}{\kappa}\overset{1}{\pi} = \mathbf{R}^f,$$

Equations (1.59) can be written in the form

$$\begin{aligned}
div\,\mathbf{T}^s + \bar{\rho}^s\mathbf{b} + \mathbf{R}^s &= \bar{\rho}^s\frac{\partial}{\partial t}\mathbf{v}^s + \rho_{12}\frac{\partial}{\partial t}\left(\mathbf{v}^f - \mathbf{v}^s\right), \\
div\,\mathbf{T}^f + \bar{\rho}^f\mathbf{b} + \mathbf{R}^f &= \bar{\rho}^f\frac{\partial}{\partial t}\mathbf{v}^f - \rho_{12}\frac{\partial}{\partial t}\left(\mathbf{v}^f - \mathbf{v}^s\right),
\end{aligned} \tag{1.61}$$

which are identical to those derived by Biot (see Equation (1.56)).

All that shows that the results of the proposed theory for immiscible solid
- fluid composition ensure a proper transition to the well-known linear Biot
theory. They also allow one to express Biot's coupling parameter as

$$\rho_{12} = \bar{\rho}^f \left(1 - \frac{1}{\kappa}\right) = \rho^f f_v \left(1 - f_v/\lambda\right), \qquad (1.62)$$

and thus, to interpret its structural sense by means of the structural
permeability parameter λ (or κ) and the volume porosity f_v.

2. Constitutive description of fluid-saturated porous media with immiscible elastic constituents

Introduction

Constitutive modelling of fluid-saturated porous solids is a subject of great importance in many theoretical and practical engineering applications, particularly in such areas as geophysics, biomechanics, soil physics or petroleum engineering. Nonlinear description of such materials is based mostly upon the fundamental notions of the Classical Mixture Theory, [12, 11], and its reformulated form - the Theory of Interacting Continua, [21, 22]. In that theory, a solid-fluid composition is treated as superposition of two continua and the constituents are assumed to be completely miscible. Consequently, the microstructure of porous skeleton is not taken into account in formulation of the balance equations and constitutive relations.

There are works developing macro-continuum constitutive modelling of solid-fluid mixture that regard the immiscibility effect by incorporating in the description the volume porosity parameter characterizing the volume fractions of the constituents (see for example [9, 13, 20, 31, 41, 42, 43, 49]). Most of these papers have in common the fact they apply the principle of equipressence in formulation of constitutive relations which assumes that each constitutive quantity of particular component depends on a set of independent variables of the whole solid-fluid composition, and such description quickly become complex and unwieldy (see [1, 3, 19, 23, 30])

Here we develop, within the macro-continuum description, the nonlinear constitutive relations for fluid-saturated porous solid undergoing pure mechanical deformations, where the main consequences of the immiscibility, i.e. the skeleton pore structure characteristics and mutual independence of mechanical properties of individual constituents are taken into account. The components are assumed to be elastic and the skeleton pore structure has isotropic and homogeneous properties characterized by the volume porosity.

Considerations are based on the balance equation for the internal energy of the whole composition, which is required to be satisfied identically by the internal energy constitutive functions postulated for particular components, the functional forms of which reflect their individual features. In the analysis of the porous skeleton deformation process the notions of *the external (bulk) deformation* defined by the right Cauchy-Green deformation tensor and *the internal deformation* measured by the change of the effective skeleton mass density (or equivalently, volume porosity) are used, [17].

This enables one to obtain two nonlinear constitutive relations for stresses; one for porous skeleton and the other for pore fluid, and the relation for interface interaction force. Moreover, the additional relation is derived which is the condition of internal equilibrium for the solid-fluid composition. It relates the pore fluid pressure with independent variables describing the deformation state of porous skeleton.

The linear constitutive relations for saturated materials are established using as a starting point the derived equations of nonlinear theory. Such approach enables one to obtain the linear constitutive description of elastic solid-fluid composition in which the material constants are well defined and have clear physical meaning. It allows the reinterpretation of the Biot constitutive relations, [18].

2.1. Elastic porous solid filled with barotropic fluid

In this section we derive the nonlinear constitutive relation for the immiscible mixture of fluid-saturated porous solid. It is assumed that both constituents; the porous skeleton and the pore fluid have elastic properties and that mutual solid-fluid interactions on the interface is of mechanical type only. We confine our considerations to the pure elastic interactions, disregarding thermal and viscous effects in the composition.

We derive the general constitutive equations for the elastic porous solid filled with the barotropic fluid from the internal energy balance equation requiring it to be identically satisfied by the postulated functions for internal energies of individual constituents of the solid-fluid system.
Such method of derivation of constitutive relations is analogous to the classical approach used for the hyperelastic medium.

• *Constitutive postulates for fluid and porous skeleton internal energies*

The essence of immiscibility reflects the fact that during a deformation process the physical constituents of porous solid-fluid mixture retain their material integrity on the microscopic level and preserve their own mechanical properties during a deformation process. Therefore each constituent of such a medium in the local sense shall obey the constitutive relations for that constituent alone. It is also reasonable to define independently the internal energy for each constituent by means of the field quantities describing its own state of deformation.

In the case of the elastic (barotropic) inviscid pore fluid, its local state is defined by the effective fluid mass density p^f. Thus, the constitutive function for the fluid internal energy can be written in the form:

$$e^f = \hat{e}^f(\rho^f). \tag{2.1}$$

The local state of deformation of elastic porous skeleton filled with fluid, contrary to a non-porous material, have to be characterized by two kinds of independent variables describing, say, the internal and external deformations of the skeleton. The external deformation (bulk deformation of porous sample), is defined by the deformation gradient $\mathbf{F}_s = \mathbf{F}$, (similarly as in non-porous material). The internal deformation is connected with a change of the volume porosity f_v which is the quantity independent of the deformation gradient $\mathbf{F}$. In the description, this quantity plays a role of internal state variable and can be replaced equivalently by the skeleton effective mass density ρ^s. The density ρ^s is related to f_v and $\mathbf{F}$ by the skeleton mass continuity equation

$$\rho^s \left(1 - f_v\right) J = \rho_o^s \left(1 - f_v^o\right), \tag{2.2}$$

where J=det($\mathbf{F}$) and ρ_o^s, f_v^o stand for the values of ρ^s, f_v, respectively, in the reference configuration.

Taking into account the above discussion and satisfying the principle of objectivity, the constitutive relation for the internal energy of porous elastic skeleton can be proposed in the form

$$e^s = \hat{e}^s(\mathbf{C}, \rho^s), \tag{2.3}$$

where $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ is the right Cauchy-Green deformation tensor.

The constitutive functions for the fluid (2.1) and porous skeleton (2.3) describe entirely the energetic state of elastic porous solid filled with fluid undergoing finite deformations.

- *Constitutive relations for stresses*

To establish constitutive stress-strain relations for each constituent of the fluid-porous solid immiscible mixture, and relations describing mutual solid-fluid interactions, we apply the approach characteristic for the hyperelastic material. We introduce relations (2.1) and (2.3) to the internal energy balance equation for the whole porous solid-fluid mixture which must be identically satisfied for an arbitrary mechanical process. In such a case the energy balance equation and the continuity equations for particular components play the role of constraints that restrict changes of quantities describing mechanical processes in the considered medium. Thus, the energy balance equation can be written in the following form

$$
tr\left[\left(2\bar{\rho}^s \mathbf{F}\frac{\partial e^s}{\partial \mathbf{C}}\mathbf{F}^T - (1-f_v)\,p^f\mathbf{I} - \mathbf{T}^s\right)\mathbf{L}^s\right] - tr\left[\left(p^f f_v\mathbf{I} + \mathbf{T}^f\right)\mathbf{L}^f\right] +
$$
$$
\left(-\frac{p^f}{(\rho^s)^2} + \frac{\partial e^s}{\partial \rho^s}\right)\bar{\rho}^s\frac{D^s\rho^s}{Dt} + \left(p^f\,grad\,(f_v) + \boldsymbol{\pi}\right)\cdot\left(\mathbf{v}^s - \mathbf{v}^f\right) = 0,
$$

$$\tag{2.4}$$

where p^f is defined by

$$
p^f = \left(\rho^f\right)^2 \frac{de^f}{d\rho^f} \tag{2.5}
$$

and is the effective pore pressure. Quantity I stands for the identity second order tensor. Since equation (2.4) is the linear function of independent quantities:

$$
\mathbf{L}^s, \mathbf{L}^f, \frac{D^s\rho^s}{Dt}, \mathbf{v}^s - \mathbf{v}^f,
$$

it will be identically satisfied if the corresponding coefficients are equal to zero. Therefore we obtain

$$
\mathbf{T}^s = -(1-f_v)\,p^f\mathbf{I} + 2\bar{\rho}^s\mathbf{F}\frac{\partial e^s}{\partial \mathbf{C}}\mathbf{F}^T, \tag{2.6}
$$

$$
\mathbf{T}^f = -f_v p^f\mathbf{I}, \tag{2.7}
$$

$$
p^f = (\rho^s)^2\,\frac{\partial e^s}{\partial \rho^s}, \tag{2.8}
$$

$$\pi = -p^f grad(f_v). \tag{2.9}$$

Equations (2.6) and (2.7) (together with (2.5)) are the constitutive relations for partial stresses of the elastic porous skeleton and the barotropic pore fluid, respectively. From (2.6), it is seen that the stress in the skeleton is composed of two parts; the first part is due to the presence of the pore fluid in the skeleton pores and the second part is due to the (bulk, external) deformation of porous solid.

Introducing the effective stress tensor in the skeleton by the definition,

$$\mathbf{T}^{*s} = \mathbf{T}^s / (1 - f_v) \ , \tag{2.10}$$

from (2.3) we obtain the constitutive equation for the effective skeleton stresses

$$\mathbf{T}^{*s} = -p^f \mathbf{I} + 2\rho^s \mathbf{F} \frac{\partial e^s}{\partial \mathbf{C}} \mathbf{F}^T, \tag{2.11}$$

equivalent to (2.6).

Relation (2.9) defines the solid-fluid interaction force. It is seen that, despite of the fluid inviscosity, such force exists and is connected with the non-homogeneity of the skeleton pore structure. Equation (2.8) relates the fluid pore pressure p^f with the deformation tensor $\mathbf{C}$ and the skeleton mass density ρ^s, the quantities which define the deformation state of porous skeleton. It is the condition for *internal mechanical equilibrium* between porous skeleton and fluid filling pores. This condition can be considered as the equation describing variations of the skeleton mass density ρ^s .

The stress-strain relations (2.5), (2.11), the internal equilibrium condition (2.8) and the interaction force (2.9) form a complete set of constitutive equations for fluid-saturated porous solid undergoing finite elastic deformations. They describe, in general, anisotropic mechanical properties of the skeleton the pore structure of which is characterized by the volume porosity parameter.

2.2. Linear constitutive relations

The obtained set of nonlinear constitutive equations of fluid-filled porous solid is an appropriate basis for derivation of the linear constitutive relations describing small deformations of a porous medium around its equilibrium state.

The linear constitutive relations regarding both anisotropic and isotropic elastic properties of porous skeleton can be obtained by the linearization of equations (2.5), (2.8) and (2.11). However, it is more convenient to derive first the constitutive relations for the skeleton of anisotropic, mechanical properties and then, in the next step, to reduce the obtained relations to the isotropic case. Such approach provides the consistent linear constitutive description of mechanical behaviour of porous solid-fluid composition with the simple physically motivated interpretation of interactions between both constituents and with the precisely defined material constants.

• *Porous medium with anisotropic skeleton*

Since fluid does not have the natural stress-free states, both physical constituents of fluid-filled porous medium possess some initial stress state in the reference configuration that we identify with the porous medium equilibrium state. Assuming that the medium in the reference configuration is homogeneous, its initial state will be characterized by the following set of quantities:

$$\mathbf{T}_o^{*s}, \ \rho_o^s, \ f_v^o, \ p_o^f, \rho_o^f \ ,$$

that are related to each other by equations (2.5), (2.8) and (2.11), written for the reference configuration.

In order to obtain the linear constitutive relations we introduce the incremental form of $\mathbf{T}^{*s}$, p^f, $\mathbf{C}$, ρ^s, ρ^f, as follows

$$\mathbf{T}^{*s} = \mathbf{T}_o^{*s} + \Delta \mathbf{T}^{*s}, \quad p^f = p_o^f + \Delta p^f,$$
$$\rho^s = \rho_o^s + \Delta \rho^s, \quad \rho^f = \rho_o^f + \Delta \rho^f, \tag{2.12}$$
$$\mathbf{C} = \mathbf{I} + 2\mathbf{E},$$

where

$$\mathbf{E} = (\mathbf{H} + \mathbf{H}^T)/2 \ ,$$

is the infinitesimal strain tensor of the skeleton and $\mathbf{H} = \mathbf{F} - \mathbf{I}$ is the displacement gradient. Then, the relations (2.11), (2.8) and (2.5) after expansion of the internal energy functions with respect to increments and after omission of all nonlinear terms, take forms

$$\Delta \mathbf{T}^{*s} + \Delta p^f \mathbf{I} = \mathbf{S}^* : \mathbf{E} + \mathbf{K}^* \frac{\Delta \rho^s}{\rho_o^s} + \left(\mathbf{T}_o^{*s} + p_o^f \mathbf{I} \right) \frac{\Delta \rho^s}{\rho_o^s} + 2 p_o^f \mathbf{E} + \mathbf{H} \mathbf{T}_o^{*s} + \mathbf{T}_o^{*s} \mathbf{H}^T,$$
$$\tag{2.13}$$

$$\Delta p^f = \mathbf{K}^* : \mathbf{E} + \left(K_c^* + 2p_o^f \right) \frac{\Delta \rho^s}{\rho_o^s}, \tag{2.14}$$

$$\Delta p^f = K^f \frac{\Delta \rho^f}{\rho_o^f}, \tag{2.15}$$

where quantities

$$\mathbf{S}^* = 4\rho_o^s \frac{\partial^2 e^s}{\partial \mathbf{C}^2} \Big|^o, \quad \mathbf{K}^* = 2 \left(\rho_o^s \right)^2 \frac{\partial^2 e^s}{\partial \rho^s \partial \mathbf{C}} \Big|^o \tag{2.16}$$

$$K_c^* = \left(\rho_o^s \right)^3 \frac{\partial^2 e^s}{\partial \left(\rho^s \right)^2} \Big|^o,$$

are the effective material constants of the porous skeleton. The parameter

$$K^f = \rho_o^f a_o^2, \tag{2.17}$$

is the fluid volumetric modulus of elasticity, and

$$a_0 = \left(2\rho_o^f \frac{de^s}{d\rho^f} \Big|^o + \left(\rho_o^f \right)^2 \frac{d^2 e^s}{d \left(\rho^f \right)^2} \right)^{1/2},$$

stands for the velocity of wave propagation in a bulk fluid.

From definitions (2.16) and (2.17) it is seen that anisotropic elastic properties of porous skeleton filled with fluid are characterized by two tensors $\mathbf{S}^*$, $\mathbf{K}^*$ and one scalar material constant K_c^* whereas the properties of pore fluid are characterized by only one scalar constant K^f. The fourth order tensor $\mathbf{S}^*$ appearing in the skeleton stress relation (2.13) represents the tensor of elastic constants of the porous skeleton undergoing small external deformations at constant effective skeleton mass density ρ^s (pure external deformations). The material constant K_c^* in (2.11) is the volumetric modulus of elasticity of the skeleton material corresponding to the internal deformation caused by changes of the pore pressure p^f at constant deformation tensor $\mathbf{C}$ ($\mathbf{E} = \mathbf{0}$, pure internal deformation). The second order tensor $\mathbf{K}^*$, as it is seen from the definition $(2.16)_2$ and the relations (2.13) and (2.14) as well, is the tensor characterizing the coupling between two independent kinds of deformations measured by the tensor $\mathbf{C}$ and the increment of ρ^s. Due to the symmetry of tensor $\mathbf{C}$, tensor $\mathbf{K}^*$ is also symmetric and tensor $\mathbf{S}^*$ has the same symmetry properties as that one in the classical elasticity.

• *Porous medium with isotropic skeleton*

The constitutive relations (2.13) - (2.15) will describe an elastic behaviour of the isotropic porous body if their form is invariant under any orthogonal transformation of the dependent and independent variables

$$\Delta p^f, \Delta \rho^f, \Delta \rho^s, \Delta \mathbf{T}^{*s}, \mathbf{E}.$$

Therefore considering arbitrary orthogonal transformations of these variables the following condition should be satisfied

$$\mathbf{Q} * \mathbf{S}^* = \mathbf{S}^*, \quad \mathbf{Q}\mathbf{K}^*\mathbf{Q}^T = \mathbf{K}^*, \quad \mathbf{Q}\mathbf{T}_o^{*s}, \mathbf{Q}^T = \mathbf{T}_o^{*s}, \qquad (2.18)$$

where $\mathbf{Q}$ ($\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$) is the orthogonal tensor and $\mathbf{Q}*$ is a linear operator defined by identity

$$\mathbf{Q} * (\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \otimes \mathbf{v}_4) = \mathbf{Q}\mathbf{v}_1 \otimes \mathbf{Q}\mathbf{v}_2 \otimes \mathbf{Q}\mathbf{v}_3 \otimes \mathbf{Q}\mathbf{v}_4,$$

in which $\otimes$ denotes the tensorial product of vectors.

The isotropy condition (2.18) reduces the quantities $\mathbf{S}^*$, $\mathbf{K}^*$ and $\mathbf{T}_o^{*s}$ to the following form

$$\mathbf{S}^* = \lambda^* \mathbf{I} \otimes \mathbf{I} + 2\mu^* \mathbf{J}, \quad \mathbf{K}^* = K^* \mathbf{I}, \quad \mathbf{T}_o^{*s} = -p_o^s \mathbf{I} \qquad (2.19)$$

where quantities λ^* and μ^* are the effective Lame constants of porous skeleton measured at constant effective mass density of skeleton material, and p_o^s is the initial stress in the skeleton. The fourth order unit tensor $\mathbf{J}$ is defined as the identity operator for the second order tensor $\mathbf{A}$ ($\mathbf{J}:\mathbf{A}=\mathbf{A}$).

Assuming that the initial stress in porous skeleton is equal to the initial pore fluid pressure $\left(p_o^s = p_o^f\right)$ and using representations (2.19), we obtain

$$\Delta \mathbf{T}^{*s} + \Delta p^f \mathbf{I} = 2\mu^* \mathbf{E} + \left(\lambda^* tr\,(\mathbf{E}) + K^* \frac{\Delta \rho^s}{\rho_o^s}\right) \mathbf{I} \qquad (2.20)$$

$$\Delta p^f = K^* tr\,(\mathbf{E}) + (K_c^* + 2p_o^s) \frac{\Delta \rho^s}{\rho_o^s} \qquad (2.21)$$

The relation (2.15) remains unchanged.

Equations (2.20), (2.21) and (2.15) form the complete set of the linear constitutive relations for the isotropic, elastic porous solid filled with fluid.

They contain five independent material constants

$$\mu^*, \lambda^*, K^*, K_c^*, K^f,$$

the first four of which characterize elastic properties of porous skeleton and only the last one describes the mechanical properties of the pore fluid.

It is worth to underline that the above description reduces to the much more simple three parameter (μ^*, λ^*, K^f) constitutive theory when the incompressibility of the skeleton material $(\rho^s = \rho_o^s)$ is assumed. In that case relation (2.10) does not appear in the description.

2.3. Comparison with the Biot relations

The obtained, in the former section, complete set of constitutive relations for the isotropic, elastic porous solid filled with fluid enables one to compare this description with the constitutive relations for saturated porous media proposed by Biot.

To derive linear stress-strain relations for elastic deformations of porous solid filled with fluid Biot considered small departures of the medium state from its equilibrium state at rest. Assuming that the potential energy of such elastic deformations for the whole porous solid-fluid aggregate is given by the quadratic forms of the skeleton and the fluid strain components he obtained relations for stress tensors $(\boldsymbol{\sigma}^s, \boldsymbol{\sigma}^f)$ of individual constituents for the isotropic case in the following form

$$\boldsymbol{\sigma}^s = 2N\mathbf{E} + \left(Atr(\mathbf{E}) + Q\varepsilon^f\right)\mathbf{I}, \tag{2.22}$$

$$\boldsymbol{\sigma}^f = \left(Qtr(\mathbf{E}) + R\varepsilon^f\right)\mathbf{I}, \tag{2.23}$$

where ε^f is the dilatation of the pore fluid and N, A, Q, R are the material coefficients of the aggregate.

Taking into account the fact that equations (2.22) and (2.23) contain four material coefficients while in the description presented above appear five material constants, it can be stated that both descriptions are not equivalent. To compare these two descriptions we consider in detail the stress tensors $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^f$. Their derivation, suggests that they should be interpreted as increments of the partial stress tensors with respect to their values in the equilibrium state.

However, for the increments of the partial stresses, we have

$$\Delta \mathbf{T}^s = \Delta \left((1 - f_v) \mathbf{T}^{*s} \right) \approx (1 - f_v^o) \Delta \mathbf{T}^{*s} + p_o^f \Delta f_v \mathbf{I} \quad ,$$

$$\Delta \mathbf{T}^f = -\Delta \left(f_v p_f \right) \mathbf{I} \approx - \left(f_v^o \Delta p^f + p_o^f \Delta f_v \right) \mathbf{I}$$

and constitutive relations of the form (2.22) and (2.23) can be derived using equations (2.15), (2.20) and (2.21). Applying the incremental linear form of the components continuity equations, we obtain

$$\Delta \mathbf{T}^s = 2 \bar{\mu} \mathbf{E} + \left(\beta_{11} tr \left(\mathbf{E} \right) + \beta_{12} \varepsilon^f \right) \mathbf{I}, \tag{2.24}$$

$$\Delta \mathbf{T}^f = - \left(\beta_{21} tr \left(\mathbf{E} \right) + \beta_{22} \varepsilon^f \right) \mathbf{I}, \tag{2.25}$$

where

$$\beta_{11} = \bar{\lambda} - K + \left[(1 - f_v^o)^2 K^f + f_v^o \left(K + (1 - f_v^o) p_o^f \right) \right] (K_z - K) / M$$

$$\beta_{12} = f_v^o (1 - f_v^o) K^f \left(K_z - K - (1 - f_v^o) p_o^f \right) / M,$$

$$\beta_{21} = f_v^o (1 - f_v^o) \left(K^f - p_o^f \right) (K_z - K) / M, \tag{2.26}$$

$$\beta_{22} = f_v^o K^f \left(f_v^o K_z + p_o^f (1 - f_v^o)^2 \right) / M,$$

$$M = (1 - f_v^o)^2 K^f + f_v^o K_z, \quad K_z = K_c + 2 p_0^f \left(1 - f_v^0 \right), \tag{2.27}$$

and where

$$\left\{ \bar{\mu}, \ \bar{\lambda}, \ K, \ K_c \right\} = \left\{ \mu^*, \ \lambda^*, \ K^*, \ K_c^* \right\} (1 - f_v^o),$$

$\bar{\mu}, \bar{\lambda}, K, K_c$, are the partial elastic constants of porous skeleton.

Comparing the relations (2.24) and (2.25) with (2.22) and (2.23) we find that

$$\beta_{12} \neq \beta_{21}$$

and both pairs of equations are not equivalent. It turns out that the symmetry of coefficients appear in the constitutive relations for quantities

$$\Delta \mathbf{T}^s - p_o^f \Delta f_v \mathbf{I}, \quad \Delta \mathbf{T}^f + p_o^f \Delta f_v \mathbf{I},$$

or in particular case when $p_o^f = 0$.
If we identify these quantities with the Biot stress tensors taking

$$\sigma^s = \Delta \mathbf{T}^s - p_o^f \Delta f_v \mathbf{I}, \quad \sigma^f = \Delta \mathbf{T}^f + p_o^f \Delta f_v \mathbf{I}, \tag{2.28}$$

we obtain the following expressions for the Biot material coefficients

$$
\begin{aligned}
A &= \bar{\lambda} - K + \left((1 - f_v^o)^2 K^f + f_v^o K\right)(K_z - K)/M, \\
Q &= f_v^o (1 - f_v^o) K^f (K_z - K)/M, \\
R &= (f_v^o)^2 K^f K_z/M, \quad N = \bar{\mu},
\end{aligned}
\tag{2.29}
$$

where M and K_z are given by (2.27). Equalities (2.28) are additionally justified by relations

$$
div\left(\boldsymbol{\sigma}^s\right) = div\left(\Delta \mathbf{T}^s\right) + \boldsymbol{\pi}, \quad div\left(\boldsymbol{\sigma}^f\right) = div\left(\Delta \mathbf{T}^f\right) - \boldsymbol{\pi}
\tag{2.30}
$$

where

$$
\boldsymbol{\pi} = -p_o^f grad\left(f_v\right)
$$

is the linear form of the interaction force (2.9).

Equalities (2.28) and their divergent form (2.30) prove that Biot's stress tensors $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^f$ do not represent the surface force only. Their divergence is equal to the sum of divergence of partial stress tensor and interaction force between fluid and skeleton resulting from the inhomogeneity of pore structure during a deformation process.

3. Effective stress - strain ralations for fluid - porous solid immiscible mixture

The principle of effective stresses for porous media filled with fluid plays an important role especially in soil mechanics, rock and concrete mechanics as well as in mechanics of materials such as ceramics or powder metals. The effective stress is the stress that controls the strain, volume change, and strength behaviour of a given porous medium, independent of the magnitude of the pore pressure.

The commonly discussed expression for the effective stress σ^{eff} is proposed as the difference between the total stress σ and the fraction of the neutral stress (pore water pressure) p^f i.e.

$$\sigma^{eff} = \sigma - \eta p^f \mathbf{I} \tag{3.1}$$

where η expresses the fraction of the pressure that should be employed to make the written equation to satisfy the effective stress definition cited above.

The formulation of the concept of effective stress is most often attributed to Terzaghi. Basing on the one-dimensional consolidation problem for saturated clay he realized that any change in linear and volumetric strains of porous media is controlled by the stress which is difference between total stress acting on the element of soil and the neutral stress p^f,

$$\sigma^{eff} = \sigma - p^f \tag{3.2}$$

and therefore in Terzaghi's original expression the value of $\eta = 1$.

Although Terzaghi's expression (3.2) works well for most geotechnical application, it is not adequate for some porous media such as concrete and rock and for porous materials under very high pressure conditions.

Here we consider within macro-continuum description of fluid saturated porous media a stress-strain relation satisfying the effective stress concept in

which the stress is determined by strain measures of porous solid only.

In the considerations the components are assumed to be elastic and in the reference state pore structure has isotropic and homogeneous properties in the macroscopic sense.

This enables one to define the internal energy for each component independently by the field quantities describing its own state of deformation. Then, the constitutive relations for each component derived from the balance equation for the internal energy of the whole composition are basic relations used to formulate the effective stresses.

In such a case considering the relation between incremental stresses and the porous solid strains in the vicinity of an initial stress state the effective stress can be derived and the expression for η parameter can be established.

3.1. Constitutive relations for the elastic fluid-porous solid immiscible mixture

The starting point for the considerations is the macroscopic non-linear constitutive description of an elastic porous skeleton filled with barotropic fluid.

Applying the method of derivation of the constitutive relations analogous to the approach used for the hyperelastic medium we propose the fluid internal energy e^f and the internal energy of porous elastic solid e^s in the form, [17],

$$e^f = \hat{e}^f(\rho^f); \quad e^s = e^s(\mathbf{C}, \rho^s),$$

that should satisfy the balance equation for the internal energy of the whole system. We obtain the constitutive relations for partial stresses of the elastic porous skeleton and the barotropic pore fluid respectively,

$$\mathbf{T}^s = -(1 - f_v)\, p^f \mathbf{I} + 2\overline{p}^s \mathbf{F} \frac{\partial \hat{e}^s}{\partial \mathbf{C}} \mathbf{F}^T, \tag{3.3}$$

$$\mathbf{T}^f = -f_v p^f \mathbf{I} \tag{3.4}$$

and the condition for internal mechanical equilibrium between porous solid and fluid

$$p^f = (\rho^s)^2 \frac{\partial \hat{e}^s}{\partial \rho^s} \tag{3.5}$$

relating the pore pressure p^f with the deformation state of porous skeleton.

3.2. The effective stress when material of porous skeleton is incompressible

The stress analysis of saturated porous media when the solid material of skeleton is incompressible, can be applicable in many practical problems. The incompressibility condition has following form

$$\rho^s = \rho^s_o$$

and thus the internal energy of porous solid depends only on tensor $\mathbf{C}$, i.e.

$$e^s = \hat{e}^s\left(\mathbf{C}\right).$$

In such a case the condition (3.5) does not appear in the set of constitutive relations, and we obtain the non-linear stress-strain relation

$$\mathbf{T}^s + \mathbf{T}^f + p^f \mathbf{I} = 2\overline{\rho}^s \mathbf{F} \frac{\partial \hat{e}^s}{\partial \mathbf{C}} \mathbf{F}^T \tag{3.6}$$

that corresponds to the effective stress principle. The LHS of (3.6) is the effective stress

$$\mathbf{T}^s + \mathbf{T}^f + p^f \mathbf{I} = \mathbf{T}^c + p^f \mathbf{I} \tag{3.7}$$

which is identical to that proposed by Terzaghi.

3.3. Linear stress - strain relation for the effective stress expression

To find the expression for the effective stress when both constituents of fluid - porous solid mixture are compressible we consider small incremental stresses and strains in the vicinity of a given initial stress state.
Assuming that the porous medium in the reference configuration is homogeneous and isotropic, its initial state will be characterized by the following set of quantities:

$$\mathbf{T}^s_o = -\left(1 - f^o_v\right) p^s_o \mathbf{I}, \quad \rho^s_o, \; f^o_v, \; p^f_o, \; \rho^f_o.$$

By the linearization procedure applied to eqs. (2.5), (2.6) and (2.8) we obtain the corresponding set of the linear constitutive relations in the following incremental form

$$\Delta\mathbf{T}^s + \left(1 - f^o_v\right) p^f \mathbf{I} = 2\left[\bar{\mu} - \left(1 - f^o_v\right)\left(p^s_o - p^f_o\right)\right]\mathbf{E} + $$
$$+ \left\{\left[\bar{\lambda} + \left(1 - f^o_v\right) p^s_o\right] tr\mathbf{E} + \left[K + \left(1 - f^o_v\right) p^f_o\right]\frac{\Delta\rho^s}{\rho^s_o}\right\}\mathbf{I} \tag{3.8}$$

$$\left(1 - f_v^o\right) \Delta p^f = K tr \mathbf{E} + \left[K_c + 2\left(1 - f_v^o\right) p_o^f\right] \frac{\Delta \rho^s}{\rho_o^s}, \tag{3.9}$$

$$\Delta p^f = K^f \frac{\Delta \rho^f}{\rho_o^f}, \tag{3.10}$$

where $\mathbf{E}$ is the infinitesimal strain tensor of the skeleton and $\bar{\mu}$, $\bar{\lambda}$, K, K_c and K^f are independent material constants defined in section 2.3. Combining eqs. (3.8) and (3.9) and making use of the linear form of the skeleton continuity equation, we obtain the expression relating the increments of total stresses and pore pressure to small deformations of porous skeleton

$$\begin{aligned}
\Delta \mathbf{T}^c &+ \left[1 - \left(1 - f_v^o\right) \alpha^I\right] \Delta p^f \mathbf{I} = \\
&= 2\left[\bar{\mu} + \left(1 - f_v^o\right)\left(p_o^f - p_o^s\right)\right] \mathbf{E} + \\
&+ \left[\bar{\lambda} - \left(1 - f_v^o\right)\left(p_o^f - p_o^s\right) - K\alpha^I\right] tr\left(\mathbf{E}\right) \mathbf{I},
\end{aligned} \tag{3.11}$$

where

$$\alpha^I = \frac{K}{K_c + 2\left(1 - f_v^o\right) p_o^f}.$$

For the hydrostatic case equation (3.11) reduces to

$$\begin{aligned}
\Delta \sigma^c &+ \left[1 - \left(1 - f_v^o\right) \alpha^I\right] \Delta p^f = \\
&\left[\frac{2\bar{\mu} + 3\bar{\lambda}}{3} - \frac{1}{3}\left(1 - f_v^o\right)\left(p_o^f - p_o^s\right) - K\alpha^I\right] tr\left(\mathbf{E}\right)
\end{aligned} \tag{3.12}$$

where $\Delta \sigma^c$ is the increment of the spherical part of the total stress, i.e.

$$\Delta \sigma^c = \frac{1}{3} tr\left(\Delta \mathbf{T}^c\right).$$

From equation (3.12) (as well as from (3.11)) one can find that for the isotropic, elastic fluid-porous solid mixture

$$\Delta \sigma^{eff} = \Delta \sigma^c + \left[1 - \left(1 - f_v^o\right) \alpha^I\right] \Delta p^f \tag{3.13}$$

and the η - parameter gets the form

$$\eta = 1 - \left(1 - f_v^o\right) \alpha^I. \tag{3.14}$$

It is worth to note that, in general, stress-strain relation (3.11) or (3.12) is influenced by the initial pressure conditions and is characterized by

four material constants. It can be also seen that in the case of very low compressibility of skeleton material ($K_c \to \infty$), $\eta = 1$, and the effective stress (3.13) takes form that of Terzaghi. The same effect is observed for the very high value of initial pore pressure.

To provide the determination of the effective stress and an measurement of the coefficient α^I one can consider jacketed and unjacketed compressibility test (at simplified initial condition: $p_o^s = p_o^f$).

• *Jacketed compressibility test*

The stress conditions are: $\Delta\sigma = \Delta p_c^f$, $\Delta p^f = 0$, where Δp_c^f is the increment of confining preassure. In such a case from (3.9) and (3.12) one can find

$$\alpha^I = \frac{K}{K_c + 2(1 - f_v^o)p_o^f} = \left.\frac{\varepsilon_{ms}}{\varepsilon}\right|_J \tag{3.15}$$

$$\frac{\Delta p_c^f}{\varepsilon} = \frac{2\mu + 3\lambda}{3} - \mathbf{K}\alpha^I \tag{3.16}$$

where ε_{ms} and ε stand for dilatations of matrix material and porous skeleton, respectively.

The expression (3.15) provides an interpretation of the α^I quantity.

Another form of (3.15) can be written in terms of compressibilities of matrix material and porous sample. We have

$$\alpha^I = \left.\frac{\varepsilon_{ms}/\Delta p_c^f}{\varepsilon/\Delta p_o^f}\right|_J = \left.\frac{C_{ms}}{C}\right|_J \leq 1. \tag{3.17}$$

The relation (3.16) determines the elastic coefficient of porous sample in the jacketed test that can be also represented by

$$K_s = \left.\frac{1}{C}\right|_J = 1/\kappa \tag{3.18}$$

where κ is the coefficient of jacketed compressibility introduced by Biot and Willis, [3].

• *Unjacketed compressibility test*

The stress conditions are: $\Delta\sigma = -\Delta p_c^f$, $\Delta p^f = 0$. With such conditions from (3.12) we can determine unjacketed compressibility coefficient δ

analogous to that defined by Biot and Willis

$$\delta = -\frac{\varepsilon_{ms}}{\Delta p_c^f}\bigg|_{UN} = C\bigg|_J (1 - f_v^o)\alpha^I = \kappa(1 - f_v^o)\alpha^I. \tag{3.19}$$

Combining (3.14) and (3.19) we find the expression

$$\eta = 1 - (1 - f_v^o)\alpha^I = 1 - \frac{\delta}{\kappa}, \tag{3.20}$$

that proves the derived here effective stress coefficient is equivalent to that proposed by Biot and Willis, [3].

3.4. Conclusions

From the foregoing discussion it is seen that the derived results for the effective stresses allows to obtain the expressions corresponding to some limiting cases:

- material of porous matrix is incompressible

$$\alpha^I = 0 \quad \Longrightarrow \quad \eta = 1,$$

  which is the result analogous to that of Terzaghi;
- deformation of porous sample at the constant porosity,

$$f_v = \text{const.} \quad \Longrightarrow \quad \alpha^I = 1 \quad \Longrightarrow \quad \eta = f_v;$$

- deformation at very high pore pressure

$$p_o^f \uparrow \quad \Longrightarrow \quad \alpha^I \downarrow \quad \Longrightarrow \quad \eta = 1,$$

  (compressibility of matrix material becomes exhausted);
- very low porosity

$$f_v \downarrow \quad \Longrightarrow \quad \alpha^I \to 1 \quad \Longrightarrow \quad \eta = 0.$$

4. Anisotropic Consolidation

In a process of consolidation of fluid - saturated porous solid there appear stresses in the solid skeleton, stresses in the fluid and diffusive forces resulting from the solid-fluid couplings. The permeability of a porous body has its origin in an interpenetrated microchannels system. It has an oriented character and can not be specified solely by a scalar quantity, therefore, to characterize this system a second order, symmetric tensor of structural permeability $\mathbf{P}$, introduced in section 1.1, should be used. The structural permeability tensor is responsible not only for the geometrical structure of pores but determines as well the overall anisotropy of a porous material.

To determine response of a porous, consolidating material it is necessary to specify the constitutive relations concerning the following quantities related to the current configuration:

$\mathbf{T}^s$: stresses in the solid skeleton,

$\mathbf{T}^f$: stresses in the fluid filling the pores,

$\mathbf{P}$: structural permeability tensor, thus law for the permeability variation due to the deformation induced,

τ : diffusive force due to fluid flow with respect to the skeleton. The force acts in addition to the static stress of mutual interaction of the constituents.

The dependent variables specified earlier are vector or tensor valued tensor functions of independent variables. In general, the independent variables are:

$\bar{\rho}^f = \rho^f f_v$: partial density of the fluid, where ρ^f denotes effective density of the fluid constituent, f_v stands for volume porosity,

$\mathbf{E}$: strain tensor of the porous skeleton,

$\mathbf{P}_o$: initial value of the structural permeability tensor,

$\mathbf{D}^f$: rate of deformation of the fluid constituent,

$\mathbf{u} = \mathbf{v}^f - \mathbf{v}^s$: velocity vector of the fluid with respect to the solid, where on the right hand side are the particle velocities of the fluid and solid, respectively.

The constitutive relations will be looked for employing the following functional relations between the quantities involved into descriptions of a consolidating composite

$$\mathbf{T}^s = \mathbf{T}^s \left(\bar{\rho}^f, \mathbf{E}, \mathbf{P}_o \right) \tag{4.1}$$

$$\mathbf{T}^f = \mathbf{T}^f \left(\bar{\rho}^f, \mathbf{E}, \mathbf{P}_o, \mathbf{D}^f \right) \tag{4.2}$$

$$\mathbf{P} = \mathbf{P}(\bar{\rho}^f, \mathbf{E}, \mathbf{P}_o) \tag{4.3}$$

$$\tau = \tau(\bar{\rho}^f, \mathbf{P}, \mathbf{u}). \tag{4.4}$$

Purposely, we restrict the analysis to a reasonable situation when the fluid strain rate $\mathbf{D}^f$ does not influence stresses in the skeleton.

To specify effective constitutive relations the principles governing the tensor function representation will be employed, [52].

• *Tensor Function Representations*

Constitutive relations (4.1 - 4.4) for a consolidating solid with an *anisotropic structure* of the skeleton will be specified using the representation theorems for non-linear tensor functions. They have to satisfy the symmetries imposed by the arrangement of pores and by the symmetries of the solid constituent. We consider situations when the fluid and material of the skeleton are isotropic.

Let a dependent tensor $\mathbf{A}$, say, (in our case second order tensor or vector) be a function of r independent tensor variables $\mathbf{B}_r$, (scalars, vectors, second order tensors) i.e. $\mathbf{A} = \mathrm{f}(\mathbf{B}_r)$. This functional relation can be represented as a sum of a specified number of products of tensor generators $\mathbf{G}_i$ second order tensors or vectors, and scalar functions α_i,

$$\mathbf{A} = f(\mathbf{B}_r) = \alpha_i \mathbf{G}_i, \quad i = 1, \ldots, n \tag{4.5}$$

The number of generators is limited to a set of independent ones whereas α_i are functions of an irreducible set of invariants of the argument tensors. These invariants form the integrity basis. When specifying (4.1 - 4.4) we limit our considerations to polynomial representations. The tensor valued tensor functions (4.1 - 4.3) involve either two or three symmetric second order tensor arguments and scalars. In the case of three independent second order tensor

variables, the generators are:

$$
\begin{aligned}
\mathbf{G}_i : \quad & \mathbf{I}, \ \mathbf{B}_1, \ \mathbf{B}_2, \ \mathbf{B}_3, \ \mathbf{B}_1^2, \ \mathbf{B}_2^2, \ \mathbf{B}_3^2, \\
& (\mathbf{B}_1\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_1), \ (\mathbf{B}_1\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_1), \ (\mathbf{B}_2\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_2), \\
& (\mathbf{B}_1^2\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_1^2), \ (\mathbf{B}_1^2\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_1^2), \ (\mathbf{B}_2^2\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_2^2), \\
& (\mathbf{B}_1^2\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_1^2), \ (\mathbf{B}_1^2\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_1^2), \ (\mathbf{B}_2^2\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_2^2),
\end{aligned} \tag{4.6}
$$

and the set of independent invariants can be selected as follows

$$
\begin{aligned}
& tr\mathbf{B}_1, \ tr\mathbf{B}_2, \ tr\mathbf{B}_3, \ tr\mathbf{B}_1^2, \ tr\mathbf{B}_2^2, \ tr\mathbf{B}_3^2, \ tr\mathbf{B}_1^3, \\
& tr\mathbf{B}_2^3, \ tr\mathbf{B}_3^3, \ tr\mathbf{B}_1\mathbf{B}_2, \ tr\mathbf{B}_1\mathbf{B}_3, \ tr\mathbf{B}_2\mathbf{B}_3, \ tr\mathbf{B}_1^2\mathbf{B}_2, \\
& tr\mathbf{B}_1^2\mathbf{B}_3, \ tr\mathbf{B}_2^2\mathbf{B}_3, \ tr\mathbf{B}_1\mathbf{B}_2^2, \ tr\mathbf{B}_1\mathbf{B}_3^2, \ tr\mathbf{B}_2\mathbf{B}_3^2 \\
& tr\mathbf{B}_1^2\mathbf{B}_2^2, \ tr\mathbf{B}_1^2\mathbf{B}_3^2, \ tr\mathbf{B}_2^2\mathbf{B}_3^2, \ tr\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3.
\end{aligned} \tag{4.7}
$$

The vector valued function (4.4) involves one symmetric second order tensor argument and a vector. In such a case, the vector generators are:

$$
\mathbf{G}_i : \quad \mathbf{u}, \ \mathbf{Bu}, \ \mathbf{B}^2\mathbf{u}, \tag{4.8}
$$

and the following set of independent invariants can be taken

$$
tr\mathbf{B}, tr\mathbf{B}^2, tr\mathbf{B}^3, \mathbf{u} \cdot \mathbf{u}, \mathbf{u} \cdot \mathbf{Bu}, \mathbf{u} \cdot \mathbf{B}^2\mathbf{u}. \tag{4.9}
$$

These quantities will be used when devising constitutive relations for a porous, permeable solid. To the invariants forming functional bases enters, moreover, the scalar representing the density of fluid included into (4.1 - 4.4). The expressions concerning $\mathbf{T}^s$ and $\mathbf{P}$ involve two tensor variables. Therefore the set of generators and invariants can be obtained directly from (4.6) and (4.7), respectively, putting $\mathbf{B}_3 = 0$.

• *Stresses in the Skeleton*

Employing (4.5) and (4.6) in (4.1) the polynomial representation of the constitutive relation concerning the stress in the skeleton is

$$
\begin{aligned}
\mathbf{T}^s =& \varphi_0\mathbf{I} + \varphi_1\mathbf{E} + \varphi_2\mathbf{P}_0 + \varphi_3\mathbf{E}^2 + \varphi_4\mathbf{P}_0^2 \\
& + \varphi_5(\mathbf{EP}_0 + \mathbf{P}_0\mathbf{E}) + \varphi_6(\mathbf{E}^2\mathbf{P}_0 + \mathbf{P}_0\mathbf{E}^2) + \varphi_7(\mathbf{EP}_0^2 + \mathbf{P}_0^2\mathbf{E}) ,
\end{aligned} \tag{4.10}
$$

where $\varphi_0, ..., \varphi_7$ are scalar functions of the corresponding invariants and $\bar{\rho}^f$ which can be expressed in terms of ε^f representing the fluid dilatation. For the considered case the invariants (4.7) are

$$tr\mathbf{E}, \ tr\mathbf{E}^2, \ tr\mathbf{E}^3, \ tr\mathbf{P}_0, \ tr\mathbf{P}_0^2, \ tr\mathbf{P}_0^3,$$
$$tr\mathbf{EP}_0, \ tr\mathbf{EP}_0^2, \ tr\mathbf{E}^2\mathbf{P}_0, \ tr\mathbf{E}^2\mathbf{P}_0^2. \tag{4.11}$$

It should be pointed out that the structural permeability tensor $\mathbf{P}_0$ alone cannot result in stresses (4.11). Thus functions φ_2 and φ_4 must vanish for an undeformed state.

Let us consider the most general constitutive relation (4.10) restricted to the form linear in terms of all the tensor variables involved but including possible couplings. In the first place

$$\varphi_3 = \varphi_4 = \varphi_6 = \varphi_7 = 0. \tag{4.12}$$

Moreover, suppose that the non-zero valued scalar functions appearing in (4.10) are polynomial in terms of the scalar arguments listed in (4.11). Therefore the skeleton stress-strain relation (4.1) eventually becomes

$$\mathbf{T}^s = (\alpha_1 tr\mathbf{E} + \alpha_2\varepsilon^f + \alpha_3 tr\mathbf{EP}_0)\mathbf{I} + (\alpha_5 + \alpha_6 tr\mathbf{P}_0)\mathbf{E}$$
$$+ (\alpha_7\varepsilon^f + \alpha_8 tr\mathbf{E})\mathbf{P}_0 + \alpha_9(\mathbf{EP}_0 + \mathbf{P}_0\mathbf{E}). \tag{4.13}$$

where $\alpha_1, ... \alpha_9$ are material constans.

This is the expression governing the skeleton stress in the case of anisotropic porous material of permeability specified by the channel arrangements $\mathbf{P}_0$. The relation (4.13) indicates that the principal directions of stress and strain do not coincide due to the anisotropy introduced by the permeability tensor. It is also to be noticed that the last two terms in (4.13) indicate the appearance of shear stress due to volumetric changes of the skeleton. Similar phenomena were observed experimentally and the analysis employing the tensor functions representations gives such a result as a natural consequence of the oriented permeability.

Going back to the specification of (4.1) one can in advance take

$$\varphi_2 = \varphi_4 = 0.$$

In consequence in (4.13) $\alpha_7 = \alpha_8 = 0$ and the influence of hydrostatic fluid pressure on shear stresses in the skeleton is ignored.

If the structural permeability tensor is isotropic, thus

$$\mathbf{P}_0 = \lambda \mathbf{I} \tag{4.14}$$

then

$$\mathbf{T}^s = (\bar{\alpha}_1 tr\mathbf{E} + \bar{\alpha}_2 \varepsilon^f)\mathbf{I} + \bar{\alpha}_3 \mathbf{E}, \tag{4.15}$$

where $\bar{\alpha}_i$ are material constants and this is the form of equation governing the considered skeleton stresses in the infinitesimal elastic theory of isotropic consolidation.

• *Stresses in the fluid*

In a permeable anisotropic solid the general functional relation for stresses in a viscous fluid is given in (4.2). We restrict our analysis to the case linear in $\mathbf{E}$, $\mathbf{P}_0$ and $\mathbf{D}^f$. Taking into account (4.5) and (4.6) in (4.2) the constitutive relation takes the form

$$\mathbf{T}^f = \psi_0\mathbf{I} + \psi_1\mathbf{E} + \psi_2\mathbf{P}_0 + \psi_3\mathbf{D}^f + \psi_4(\mathbf{EP}_0 + \mathbf{P}_0\mathbf{E}) + \psi_5(\mathbf{P}_0\mathbf{D}^f + \mathbf{D}^f\mathbf{P}_0). \tag{4.16}$$

Assuming further that the scalar functions entering (4.16) are polynomial in terms of the invariants involved as given in (4.7) and recalling the fact that the tensor $\mathbf{P}_0$ alone does not provoke stresses we may write

$$
\begin{aligned}
\mathbf{T}^f =& (\beta_0 + \beta_1\varepsilon^f + \beta_2 tr\mathbf{E} + \beta_3 tr\mathbf{P}_0 + \beta_4 tr\mathbf{D}^f + \\
& + \beta_5 tr\mathbf{EP}_0 + \beta_6 tr\mathbf{P}_0\mathbf{D}^f)\mathbf{I} + (\beta_7 + \beta_8 tr\mathbf{P}_0)\mathbf{E} + \\
& + (\beta_9\varepsilon^f + \beta_{10} tr\mathbf{E} + \beta_{11} tr\mathbf{D}^f)\mathbf{P}_0 + (\beta_{12} + \beta_{13} tr\mathbf{P}_0)\mathbf{D}^f + \\
& + \beta_{14}(\mathbf{EP}_0 + \mathbf{P}_0\mathbf{E}) + \beta_{15}(\mathbf{P}_0\mathbf{D}^f + \mathbf{D}^f\mathbf{P}_0).
\end{aligned}
\tag{4.17}
$$

The condition that at $\mathbf{E} = \mathbf{0}$, $\mathbf{D}^f = \mathbf{0}$ and $\varepsilon^f = 0$ it is $\mathbf{T}^f = \mathbf{0}$ and taking into account the fact that if $\mathbf{D}^f = \mathbf{0}$ the fluid cannot carry shear stresses, notwithstanding $\mathbf{E}$ and $\mathbf{P}_0$, we have

$$
\begin{aligned}
&\beta_0 + \beta_3 tr\mathbf{P}_0 = 0 \\
&\beta_7 = \beta_8 = \beta_9 = \beta_{10} = \beta_{14} = 0.
\end{aligned}
\tag{4.18}
$$

Hence the partial stresses in a viscous, compressible fluid are

$$
\begin{aligned}
\mathbf{T}^f =& (\beta_1\varepsilon^f + \beta_2 tr\mathbf{E} + \beta_4 tr\mathbf{D}^f + \beta_5 tr\mathbf{EP}_0 + \beta_6 tr\mathbf{P}_0\mathbf{D}^f)\mathbf{I} \\
& + (\beta_{11} tr\mathbf{D}^f)\mathbf{P}_0 + (\beta_{12} + \beta_{13} tr\mathbf{P}_0)\mathbf{D}^f + \beta_{15}(\mathbf{P}_0\mathbf{D}^f + \mathbf{D}^f\mathbf{P}_0).
\end{aligned}
\tag{4.19}
$$

This is the most general linear expression for the fluid stress in terms of the fluid deformation rate tensor, strain in the porous skeleton and the anisotropic properties due to the tensor of structural permeability $\mathbf{P}_0$.

Taking into account both the material properties and the couplings disclosed, Eq. (4.19) in the final form involves 9 material constants.

When the fluid is inviscid necessarily

$$\mathbf{T}^f = \sigma\mathbf{I} \tag{4.20}$$

independently of the structural permeability and the deformation of the solid. Therefore

$$\beta_{11} = \beta_{12} = \beta_{13} = \beta_{15} = 0 \,,$$

and eventually

$$\mathbf{T}^f = (\beta_1\varepsilon^f + \beta_2 tr\mathbf{E} + \beta_5 tr\mathbf{E}\mathbf{P}_0)\mathbf{I}. \tag{4.21}$$

It gives the fluid pressure in an anisotropic porous solid, which reduces further if $\mathbf{P}_0$ is of the form (4.14). With reference to (4.21) we can remark that the skeleton deformation influences the stress in fluid and thus volumetric coupling is seen.

Equation (4.19), being the general one, exhibits the effect of viscous coupling influencing stresses in the fluid. Specific assumptions regarding the couplings will result in particular theories.

• *The diffusive force*

The diffusive force (4.4) generated by the fluid flow is associated with the relative velocity $\mathbf{u}$ as follows from (4.8), namely

$$\tau = (\gamma_0 + \gamma_1\mathbf{P} + \gamma_2\mathbf{P}^2)\mathbf{u} \,. \tag{4.22}$$

It is therefore not coaxial to the vector of fluid flow whereas the scalar functions γ_i depend on the invariants given in (4.9) and on $\bar{\rho}^f$,

$$\gamma_i = \gamma_i(\bar{\rho}^f, \, tr\mathbf{P}, \, tr\mathbf{P}^2, \, tr\mathbf{P}^3, \, \mathbf{u}\cdot\mathbf{u}).$$

For small relative velocity of the fluid it is reasonable to suppose that

$$\varepsilon^f \approx tr\mathbf{E} = \mathrm{O}(\mathbf{E}),$$

and

$$trP = trP_0 + O(E), \quad trP^2 = trP_0^2 + O(E), \quad trP^3 = trP_0^3 + O(E).$$

Hence the diffusive force (4.22) becomes

$$\boldsymbol{\tau} = (\gamma_0 I + \gamma P_0) u \tag{4.23}$$

whereas

$$\gamma_0 = \eta_1 trP_0 + \eta_2 trP_0^2 + \eta_3 trP_0^3, \quad \gamma_1 = \gamma = const.$$

Purposely the higher invariants of P_0 are not neglected as no restriction regarding smallness of the structural permeability was made. In (4.23) the higher order terms have been neglected as well as the condition that $\boldsymbol{\tau} = 0$ if $P_0 = 0$ has been used.

It is interesting to use in the equation of motion the obtained expression (4.23). For clarity we neglect the inertia forces and the equation for the fluid motion is

$$div T^f + \bar{\rho}^f b + \pi + \tau = 0, \tag{4.24}$$

where b stands for the unit body force and π is the fluid force exerted at rest. Definition of this equilibrium contact force is the following

$$\boldsymbol{\pi} = p^f \; grad(f_v)$$

and its justification can be found in the section 2.1.

Introducing the diffusive force (4.23) into (4.24) one eventually obtains the generalized Darcy law

$$p_{,i} + \rho^f b_i = k_{ij} u_j,$$

where the permeability tensor is

$$k_{ij} = - \left[\frac{\gamma_0}{f_v} \delta_{ij} + \frac{\gamma}{f_v} (P_0)_{ij} \right].$$

It is seen that the structural permeability tensor influences the flow through an anisotropic porous medium. The anisotropic filtration receives thus a structural justification.

• *The permeability variation*

To specify changes of the structural permeability tensor we employ the same technique of the tensor function representations. It will appear that such a technique is applicable not only to the constitutive equations involving kinematical and dynamical quantities but also to provide a law of variation of the structural permeability in the process of deformation. This procedure will as well disclose the couplings occuring in porous solids during deformation. The representation concerning variation of permeability has the form analogous to (4.1) and using (4.5) and (4.6) can be written as

$$
\begin{aligned}
\mathbf{P} = \ & (\delta_0 + \delta_1\varepsilon^f + \delta_2 tr\mathbf{E} + \delta_3 tr\mathbf{P}_0 + \delta_4 tr\mathbf{E}\mathbf{P}_0)\mathbf{I} + (\delta_5 + \delta_6 tr\mathbf{P}_0)\mathbf{E} \\
& + (\delta_7 + \delta_8\varepsilon^f + \delta_9 tr\mathbf{E})\mathbf{P}_0 + \delta_{10}(\mathbf{E}\mathbf{P}_0 + \mathbf{P}_0\mathbf{E})
\end{aligned}
$$

This relation is subjected to initial conditions. When $\varepsilon^f = 0$ and $\mathbf{E} = \mathbf{0}$ it is $\mathbf{P} = \mathbf{P}_0$. It thus follows that

$$
\delta_0 + \delta_3 tr\mathbf{P}_0 = 0, \quad \delta_7 = 1.
$$

Eventually variation of the structural permeability with deformation is governed by the relation

$$
\begin{aligned}
\mathbf{P} = \ & (\delta_1\varepsilon^f + \delta_2 tr\mathbf{E} + \delta_4 tr\mathbf{E}\mathbf{P}_0)\mathbf{I} + (\delta_5 + \delta_6 tr\mathbf{P}_0)\mathbf{E} \\
& + (1 + \delta_8\varepsilon^f + \delta_9 tr\mathbf{E})\mathbf{P}_0 + \delta_{10}(\mathbf{E}\mathbf{P}_0 + \mathbf{P}_0\mathbf{E})
\end{aligned}
\tag{4.25}
$$

where δ_i are constants. It is to be noticed that the terms $\delta_1\varepsilon^f\mathbf{I}$ and $\delta_8\varepsilon^f\mathbf{P}_0$ are responsible for the permeability change due solely to the fluid pressure in the pores.

By a similar procedure it is possible to consider the case when the fluid viscosity is accounted for and to disclose the appearing couplings.

• *Transverse isotropy*

We specialize the obtained results to the case of transverse isotropy which often occurs in stratified or sedimentary mineral deposits or rocks. In this case the material properties do not vary under any orthogonal transformation around an axis perpendicular to the strata.

Choosing the coordinate system so that the structural permeability tensor is

$$
\mathbf{P}_0 = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
\tag{4.26}
$$

the invariants take the form

$$tr\mathbf{P}_0 = 2\lambda_0 + \lambda_3, \quad tr\mathbf{EP}_0 = \lambda_0(E_{11} + E_{22}) + \lambda_3 E_{33}.$$

The mixed tensors appearing in (4.13) have the components

$$\mathbf{EP}_0 = \begin{bmatrix} \lambda_0 E_{11} & \lambda_0 E_{12} & \lambda_3 E_{13} \\ \lambda_0 E_{21} & \lambda_0 E_{22} & \lambda_3 E_{23} \\ \lambda_0 E_{31} & \lambda_0 E_{32} & \lambda_3 E_{33} \end{bmatrix}, \quad \mathbf{P}_0\mathbf{E} = \begin{bmatrix} \lambda_0 E_{11} & \lambda_0 E_{12} & \lambda_0 E_{13} \\ \lambda_0 E_{21} & \lambda_0 E_{22} & \lambda_0 E_{23} \\ \lambda_3 E_{31} & \lambda_3 E_{32} & \lambda_3 E_{33} \end{bmatrix}.$$

Eventually the relations (4.13), if explicited, are

$$
\begin{aligned}
T_{11}^s &= [\alpha_1 + \alpha_3\lambda_0 + \alpha_8\lambda_0 + \alpha_5 + \alpha_6(2\lambda_0 + \lambda_3) + 2\alpha_9\lambda_0]\,E_{11} + \\
&+ (\alpha_1 + \alpha_3\lambda_0 + \alpha_8\lambda_0)E_{22} + (\alpha_1 + \alpha_3\lambda_0 + \alpha_8\lambda_0)E_{33} + \\
&+ (\alpha_2 + \alpha_7\lambda_0)\varepsilon^f,
\end{aligned}
$$

$$
\begin{aligned}
T_{22}^s &= (\alpha_1 + \alpha_3\lambda_0 + \alpha_8\lambda_0)E_{11} + [\alpha_1 + \alpha_3\lambda_0 + \alpha_8\lambda_0 + \\
&+ \alpha_5 + \alpha_6(2\lambda_0 + \lambda_3) + 2\alpha_9\lambda_0]\,E_{22} + \\
&+ (\alpha_1 + \alpha_3\lambda_3 + \alpha_8\lambda_0)E_{33} + (\alpha_2 + \alpha_7\lambda_0)\varepsilon^f,
\end{aligned}
$$

$$
\begin{aligned}
T_{33}^s &= (\alpha_1 + \alpha_3\lambda_0 + \alpha_8\lambda_3)E_{11} + (\alpha_1 + \alpha_3\lambda_0 + \alpha_8\lambda_3)E_{22} + \\
&+ [\alpha_1 + \alpha_3\lambda_3 + \alpha_8\lambda_3 + \alpha_5 + \alpha_6(2\lambda_0 + \lambda_3) + 2\alpha_9\lambda_3]\,E_{33} + \\
&+ (\alpha_2 + \alpha_7\lambda_3)\varepsilon^f,
\end{aligned}
$$

$$
\begin{aligned}
T_{12}^s &= [\alpha_5 + \alpha_6(2\lambda_0 + \lambda_3) + 2\alpha_9\lambda_0]\,E_{12}, \\
T_{13}^s &= [\alpha_5 + \alpha_6(2\lambda_0 + \lambda_3) + \alpha_9(\lambda_0 + \lambda_3)]\,E_{13}, \\
T_{23}^s &= [\alpha_5 + \alpha_6(2\lambda_0 + \lambda_3) + \alpha_9(\lambda_0 + \lambda_3)]\,E_{23}.
\end{aligned}
$$

It is seen that the coefficients of transverse isotropy are influenced by the structural permeability.

● *The Biot description*

It is worthwhile to relate the obtained results with the existing description of a linearly elastic anisotropic porous solid as developed by Biot, [8]. It was then assumed that the material is hyperelastic and that the stresses both

in the solid and in the fluid depend linearly on the associated strains. The energy

$$dV = \frac{1}{2}\sigma_{ij}E_{ij} + \sigma\varepsilon^f \tag{4.27}$$

was considered to be positive definite. In total such a relation involves 7 generalized stresses including $\boldsymbol{\sigma}$ and 7 generalized strains, including ε^f. The constitutive relation has the form

$$[\boldsymbol{\sigma}] = [\mathbf{C}] \cdot [\mathbf{E}]^T, \tag{4.28}$$

where $\mathbf{C}$ has 28 independent constants.

The equation of fluid motion used in this theory is

$$grad(\sigma) + \bar{\rho}^f \mathbf{b} = \mathbf{k}\mathbf{u}$$

where $\mathbf{k}$ is a symmetric second order tensor of the permeability and has no connection with the $[\mathbf{C}]$ specifying the solid material directional characteristics. To make the discussion specific we consider a transversely isotropic solid. We may write

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{12} \\ \sigma \end{pmatrix} \begin{pmatrix} 2N+A & A & F & 0 & 0 & 0 & M \\ & 2N+A & F & 0 & 0 & 0 & M \\ & & C & 0 & 0 & 0 & Q \\ & & & L & 0 & 0 & 0 \\ & & & & L & 0 & 0 \\ & & & & & c_{55} & 0 \\ & & & & & & R \end{pmatrix} \begin{pmatrix} E_{11} \\ E_{22} \\ E_{33} \\ E_{13} \\ E_{23} \\ E_{12} \\ \varepsilon^f \end{pmatrix} \tag{4.29}$$

The constant c_{55} is not independent but is expressible in terms of N and A.

The material constants appearing in (4.29) can be related to the results presented earlier. Eventually one obtains the following relations

$$\begin{aligned} 2N &= \alpha_5 + \alpha_6(2\lambda_0 + \lambda_3) + 2\alpha_9\lambda_0 \quad A = \alpha_1 + \alpha_3\lambda_0 + \alpha_8\lambda_0 \\ F &= \alpha_1 + \alpha_3\lambda_3 + \alpha_8\lambda_0 \\ C &= \alpha_1 + (\alpha_3 + \alpha_8)\lambda_3 + \alpha_5 + \alpha_6(2\lambda_0 + \lambda_3) + 2\alpha_9\lambda_3 \\ L &= \alpha_5 + \alpha_6(2\lambda_0 + \lambda_3) + \alpha_9(\lambda_0 + \lambda_3) \\ M &= \alpha_2 + \alpha_7\lambda_0, \quad Q = \alpha_2 + \alpha_7\lambda_3 \\ R &= \beta_1 \text{ (entering the stresses in fluid)} \\ c_{55} &= \alpha_5 + \alpha_6(2\lambda_0 + \lambda_3) + 2\alpha_9\lambda_0. \end{aligned}$$

It is seen from obtained results that the structural permeability influences the material response. The arrangement of channel system makes that the overall response is anisotropic.

5. Matching conditions at the contact surface between fluid-saturated porous solid and bulk fluid

Introduction

Continuum description of mechanical behaviour of fluid-saturated porous solids being in contact with a fluid flowing through the porous solid boundary is of great importance in many applications, e.g. in acoustics (Gogete and Munjal, [24], Bourbie et al. [11], Kubik et al. [37]), in the porous bearing design (Joseph and Tao, [28], Rhodes and Roulcau, [46], Shir and Joseph, [48]), in glaciology (Hutter et al., [27], Wu et al., [56]), in modelling of lubrication of diarthrodial joints (Ateshian, [2], Hou et al., [26]) and others. The main open problem in this domain is the formulation of the appropriate boundary conditions to be imposed at the permeable boundary surface between a fluid-filled porous solid and the bulk fluid. A number of papers were devoted to the problem of finding such macroscopic conditions making use of discontinuity analysis (Bear and Bahmat, [4], Cieszko and Kubik, [16], Hou et al., [26], Hutter et al., [27], Raats, [44], Wilmanski, [55]) and micro-macro or semi-empirical analysis (Beavers and Joseph, [5], Levy and Sanchez-Palencia, [40], Taylor, [50]). In spite of great effort the problem of matching conditions for the tangential components of relative fluid velocities has not found yet satisfactory solution.

The classical conditions at the permeable boundary of porous material filled with viscous fluid surrounded by the bulk fluid are usually assumed to be the continuity of both, the fluid effective pressure and the fluid mass flux across the surface. Moreover, the tangential component of the bulk fluid velocity is assumed to be zero (Levy and Sanchez-Palencia, [40]). The first two conditions are commonly accepted, at least for the linear problems, while the adherence condition is rather restricted to very special cases of porous materials for which the migration of fluid particles across the boundary may be disregarded.

Experiments done by Beavers and Joseph, [5] have shown that the fluid
mass flow rate through the channel with a permeable lower wall is much more
greater than that predicted for channel with nonpermeable walls at which
the adherence flow condition is valid. Therefore instead of the fluid adherence
condition, these authors introduced the fluid slip velocity condition in the
following form

$$\frac{du}{dz}\Big|_{z=0} = \frac{\alpha}{\sqrt{k}}(u_s - q) \tag{5.1}$$

where u_s is the slip velocity of the bulk fluid at the permeable boundary, q is
the filter velocity of a fluid flowing through the porous skeleton and k is the
skeleton permeability. Parameter α is the dimensionless material constant
characterizing the skeleton pore structure within the boundary region and
being independent of the fluid viscosity.

Although the condition (5.1) gives a qualitative agreement between the
theoretical predictions and experimental measurements for the fluid flow
along permeable surface this condition is not sufficiently general. It does not
allow to determine the interaction forces exerted by the flowing bulk fluid
on the pore fluid and the porous skeleton. The only one condition is also
insufficient to solve boundary problems where the fluid flow within porous
medium is described by e.g. Brinkman, [14] type of equation.

It should be pointed out, that both experimental and theoretical
works were done (see Koplik et al., [32], Levy and Sanchez-Palencia, [40],
Richardson, [45], Saffman, [47], Taylor, [50]) to establish the validity of the
condition (5.1) without obtaining a satisfactory answer.

The main purpose of this part is to establish and to discuss the general
form of compatibility conditions matching macroscopic mechanical fields
at the contact surface between fluid-saturated porous solid and adjacent
bulk fluid. Special attention is paid to the derivation of conditions for the
tangential components of the fluid flow velocities at the contact surface and
to the verification of validity of the condition postulated by Beavers and
Joseph, [5].

Our approach to the problem of compatibility conditions is based on the
assumption that the only justified conditions for mechanical fields we can
formulate at the contact surface of both media are the balance equations
for the mass of fluid, the linear momentum and the mechanical energy of all
three components of the system. Therefore we identify, the general nonlinear
compatibility conditions with the balance equations at the contact surface.

Then, the linearization provides the right form of the linear compatibility conditions.

Here we use the general compatibility conditions formulated by authors in the paper by Cieszko and Kubik, [16] in order to prove that the macroscopic velocities of all three constituents of the system are of different values at the contact surface. This justify the existence of dissipation of mechanical energy at such surface due to the fluid viscosity. The general form of dissipation function is proposed allowing to derive two linear slip conditions for the tangential components of the relative fluid velocities. These conditions for the fluid flow within porous material described by Brinkman type equation take form similar to the condition (5.1) postulated by Bevers and Joseph, [5].

5.1. Nonlinear Compatibility Conditions for Mechanical Fields

We consider the system (see Fig. 4) composed of a fluid saturated deformable porous solid and a bulk fluid being in contact at the permeable boundary Γ. The material Γ-surface is formed by particles of the skeleton.
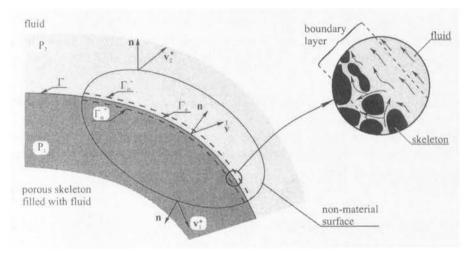

Figure 4: A system composed of a fluid-saturated deformable porous solid and a bulk fluid

The fluid filling pores and the bulk fluid are viscous and of the same physical properties. It is assumed that the macroscopic mechanical state of the bulk fluid is characterized by three field quantities: the velocity $\mathbf{v}$, mass density

ρ and the Cauchy stress tensor $\mathbf{T}$, that satisfy the dynamic equations for viscous fluid (see e.g. Landau and Liphshic, [38]).

The mechanical macroscopic behaviour of isotropic porous solid filled with fluid is described by the continuum immiscible mixture theory (Kubik, [36], Cieszko and Kubik, [17], [18]) in which the geometrical pore structure is characterized by two macroparameters, Kubik, [35]; the volume porosity f_v and the parameter κ. It is related to the Biot coupling parameter ρ_{12} and the tortuosity parameter α used by Johnson, [29], as follows

$$\frac{1}{\kappa} = 1 - \frac{\rho_{12}}{\bar{\rho}^f} = \alpha_T$$

where $\bar{\rho}^f$ is the fluid partial mass density.

The characteristic feature of this theory is that the description of kinematics and dynamics of the porous solid-fluid mixture is derived by applying the micro-macro transition assuring the full linear momentum and kinetic energy representation for both components at the macrolevel. As a result the fluid-porous solid mixture is considered as composed of two virtual components; the first one ($|^1$) being the skeleton and the fluid associated with it moving at the skeleton velocity $\overset{1}{\mathbf{v}}$, and the other being free fluid ($|^2$) moving at the velocity $\overset{2}{\mathbf{v}}$. Between these constituents the interchange of mass during a deformation process exists, due to change of the amount of the associated fluid.

The general compatibility conditions of mechanical fields for such system of two adjacent media are identified with the balance equations at the Γ-surface for mass of fluid, linear momentum and mechanical energy for all three components. Using the standard discontinuity analysis they take form, Cieszko and Kubik, [16],:

$$\rho \mathbf{u} \cdot \mathbf{n} = \overset{2}{\rho} \overset{2}{\mathbf{u}} \cdot \mathbf{n} , \qquad (5.2)$$

$$\rho \mathbf{u} \cdot \mathbf{n}(\mathbf{u} - \overset{2}{\mathbf{u}}) = \mathbf{t} - (\overset{1}{\mathbf{t}} + \overset{2}{\mathbf{t}}) , \qquad (5.3)$$

$$\rho \mathbf{u} \cdot \mathbf{n} \left[\frac{1}{2}(\mathbf{u}^2 - (\overset{2}{\mathbf{u}})^2) + e^{bf} - e^{pf} \right] = \mathbf{t} \cdot \mathbf{u} - \overset{2}{\mathbf{t}} \cdot \overset{2}{\mathbf{u}} - \vartheta \qquad (5.4)$$

where

$$\mathbf{u} = \mathbf{v} - \overset{1}{\mathbf{v}} \qquad \text{and} \qquad \overset{2}{\mathbf{u}} = \overset{2}{\mathbf{v}} - \overset{1}{\mathbf{v}}$$

are the velocities of the bulk and free fluid flow relative to the Γ-surface, respectively, and $\mathbf{t}, \overset{1}{\mathbf{t}}, \overset{2}{\mathbf{t}}$ are the partial stress vectors for the bulk fluid and

the virtual components. They satisfy Cauchy theorem and are related with the stress vectors $\mathbf{t}^s$, $\mathbf{t}^f$ for the physical components by

$$\overset{1}{\mathbf{t}} = \mathbf{t}^s + (1 - \kappa)\mathbf{t}^f , \qquad \overset{2}{\mathbf{t}} = \kappa \mathbf{t}^f .$$

Quantities e^{bf} and e^{pf} stand for the internal energies of the bulk fluid and pore fluid, respectively.

In relation (5.4) the function ϑ for the sink of mechanical energy at the Γ-surface is introduced. We will show that at this surface the dissipation of mechanical energy exists and is caused by the fluid viscosity.

5.2. Dissipation of the Mechanical Energy at the Γ-surface

It will be shown here that at the discontinuity surface all three macroscopic velocity fields of the constituents of the considered system have different values. These discontinuities, together with the assumption that the pore and bulk fluid are viscous, motivate the existence of a sink of mechanical energy on the Γ-surface.

• *Discontinuities of the velocity fields*

The normal components of the macroscopic velocities of particular constituents of the system under consideration are confined by the mass continuity equation for the fluid phase. Equation (5.2) due to the inequality

$$\rho_- = \rho \neq \overset{2}{\rho} = \kappa f_v \rho_+ \qquad (0 < \kappa f_v \leq f_v \leq 1) ,$$

shows that the normal component of the relative velocity of fluid flow through the surface of porous skeleton is discontinuous, i.e.

$$0 \neq \mathbf{u} \cdot \mathbf{n} \neq \overset{2}{\mathbf{u}} \cdot \mathbf{n} \neq 0 .$$

As a consequence we have

$$\mathbf{v} \cdot \mathbf{n} \neq \overset{1}{\mathbf{v}} \cdot \mathbf{n} \neq \overset{2}{\mathbf{v}} \cdot \mathbf{n} . \tag{5.5}$$

The inequality (5.5) proves that, in general case, the normal velocity components of particular constituents of the system differ from each other on the discontinuity surface Γ. Similar conclusion can be derived for tangential

components of these velocities considering the linear momentum balance (5.3), and the balance of total mechanical energy (5.4) when the experimental results of Beavers and Joseph [5] are taken into account.

In the case when the fluid flow along the Γ-surface takes place ($\mathbf{u} \cdot \mathbf{n} = \overset{2}{\mathbf{u}} \cdot \mathbf{n} = 0$) and disregarding the sink of mechanical energy ϑ, from Eqs (5.3) and (5.4) we obtain

$$\mathbf{t}_\tau = \overset{1}{\mathbf{t}}_\tau + \overset{2}{\mathbf{t}}_\tau \,, \tag{5.6}$$

$$\mathbf{t}_\tau \cdot \mathbf{u}_\tau = \overset{2}{\mathbf{t}}_\tau \cdot \overset{2}{\mathbf{u}}_\tau \tag{5.7}$$

where the subscript τ indicates vector components tangential to the Γ-surface.

When a fluid in the system is non-viscous, the conditions (5.6) and (5.7) do not impose any restriction on the tangential velocities $\mathbf{u}_\tau$ and $\overset{2}{\mathbf{u}}_\tau$. Since in such case there is no tangential interactions, i.e.

$$\mathbf{t}_\tau = \overset{2}{\mathbf{t}}_\tau = \overset{1}{\mathbf{t}}_\tau = \mathbf{0} \,,$$

the conditions (5.6) and (5.7) are satisfied identically, independently of the values of velocities $\mathbf{u}_\tau$ and $\overset{2}{\mathbf{u}}_\tau$.

For the viscous fluid, however, all the tangential stresses are not zero, and the conditions (5.6) and (5.7) confine both the values of tangential stresses as well as the tangential velocities $\mathbf{u}_\tau$ and $\overset{2}{\mathbf{u}}_\tau$.

Taking into account the experimental results of Beavers and Joseph [5] for viscous fluid flow along permeable boundary which prove that the tangential component of the bulk fluid velocity is nonzero, i.e.

$$\mathbf{u}_\tau \neq \mathbf{0} \,, \tag{5.8}$$

from Eqs. (5.6) and (5.7) we conclude that

$$\overset{2}{\mathbf{u}}_\tau \neq \mathbf{0} \,, \tag{5.9}$$

and also

$$\mathbf{u}_\tau \neq \overset{2}{\mathbf{u}}_\tau \,. \tag{5.10}$$

Otherwise the stress vectors $\mathbf{t}_\tau$ and $\overset{2}{\mathbf{t}}_\tau$ would have to be equal and in such a case the relation (5.6) yields $\overset{1}{\mathbf{t}} = \mathbf{0}$. It means that there is no interaction

between porous skeleton and the fluid flowing along the surface of the fluid-saturated porous medium what is physically not acceptable.

The inequalities (5.8)-(5.10) rewritten in the form

$$\mathbf{v}_\tau \neq \overset{1}{\mathbf{v}}_\tau \neq \overset{2}{\mathbf{v}}_\tau \tag{5.11}$$

show that the tangential components of velocities of particular constituents of the system differ each other on the Γ-discontinuity surface.

The above considerations allow us to state that at the Γ-surface the macroscopic velocity fields of all constituents of the system have different values. It concerns both the normal and tangential components of the velocity vectors. Accounting for the viscosity of the fluid one can state that the relative motion of the constituents on the Γ-surface causes the viscous friction between constituents resulting in dissipation of mechanical energy. It should be pointed out here that this dissipation process is different from the well known volume dissipation appearing in both saturated porous medium and in the viscous bulk fluid. Therefore, the function ϑ requires a separate constitutive postulate.

5.3. Specification of the dissipation function

Taking into account the fact that the dissipation of mechanical energy on the Γ-surface results from the discontinuity of velocity fields of particular constituents of the system and assuming isotropy of the skeleton pore structure we postulate the constitutive function to be dependent on the velocities $\mathbf{v}$, $\overset{1}{\mathbf{v}}$, $\overset{2}{\mathbf{v}}$ and additionally on the Γ-surface orientation described by the normal vector $\mathbf{n}$. We have

$$\vartheta = \bar{\vartheta}(\mathbf{v}, \overset{1}{\mathbf{v}}, \overset{2}{\mathbf{v}}, \mathbf{n}) . \tag{5.12}$$

The form of constitutive function $\bar{\vartheta}$ is restricted by the requirement that it should be objective and positively definite.

The objectivity condition, (Leigh [39]), when applied to Eq. (5.12) shows that function $\bar{\vartheta}$ depends only on the relative velocities $\mathbf{u}$, $\overset{2}{\mathbf{u}}$ and the vector $\mathbf{n}$, and has to be isotropic function. We obtain relation

$$\vartheta = \bar{\vartheta}'(\mathbf{u}, \overset{2}{\mathbf{u}}, \mathbf{n}) , \tag{5.13}$$

and the isotropy condition for the dissipation function takes the form of identity

$$\bar{\vartheta}'(\mathbf{Qu}, \mathbf{Q}\overset{2}{\mathbf{u}}, \mathbf{Qn}) = \bar{\vartheta}'(\mathbf{u}, \overset{2}{\mathbf{u}}, \mathbf{n}) \qquad (5.14)$$

that have to be satisfied for any orthogonal tensor $\mathbf{Q}$ ($\mathbf{Q}^T = \mathbf{Q}^{-1}$) and any vectors $\mathbf{u}$, $\overset{2}{\mathbf{u}}$ and $\mathbf{n}$. According to the Cauchy representation theorem for the isotropic scalar-valued functions, (Leigh [39]), we find that $\bar{\vartheta}'(\mathbf{u}, \overset{2}{\mathbf{u}}, \mathbf{n})$ can be regarded as a function of the following five scalar products

$$\mathbf{I}: \quad \mathbf{u}\cdot\mathbf{u}, \quad \mathbf{u}\cdot\overset{2}{\mathbf{u}}, \quad \overset{2}{\mathbf{u}}\cdot\overset{2}{\mathbf{u}}, \quad \mathbf{u}\cdot\mathbf{n}, \quad \overset{2}{\mathbf{u}}\cdot\mathbf{n}.$$

Then the relation (5.13) takes form

$$\bar{\vartheta}'(\mathbf{u}, \overset{2}{\mathbf{u}}, \mathbf{n}) \equiv \bar{\vartheta}''(\mathbf{u}\cdot\mathbf{u}, \ \mathbf{u}\cdot\overset{2}{\mathbf{u}}, \ \overset{2}{\mathbf{u}}\cdot\overset{2}{\mathbf{u}}, \ \mathbf{u}\cdot\mathbf{n}, \ \overset{2}{\mathbf{u}}\cdot\mathbf{n}) = \bar{\vartheta}''(\mathbf{I}), \qquad (5.15)$$

and for any $\bar{\vartheta}''$ the condition (5.14) is satisfied identically. Taking the compatibility condition (5.2) into account, the number of arguments in Eq. (5.15) can be reduced by one, and we obtain

$$\vartheta = \bar{\vartheta}^*(\mathbf{u}\cdot\mathbf{u}, \ \mathbf{u}\cdot\overset{2}{\mathbf{u}}, \ \overset{2}{\mathbf{u}}\cdot\overset{2}{\mathbf{u}}, \ \mathbf{u}\cdot\mathbf{n}). \qquad (5.16)$$

After decomposition of $\mathbf{u}$ and $\overset{2}{\mathbf{u}}$ into their normal and tangential components, that is

$$\mathbf{u}_n = (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \quad \mathbf{u}_\tau = \mathbf{u} - \mathbf{u}_n, \quad \overset{2}{\mathbf{u}}_\tau = \overset{2}{\mathbf{u}} - (\overset{2}{\mathbf{u}}\cdot\mathbf{n})\mathbf{n}, \qquad (5.17)$$

we get the following representation

$$\vartheta = \hat{\vartheta}(\mathbf{u}_\tau^2, \ \mathbf{u}_\tau\cdot\overset{2}{\mathbf{u}}_\tau, \ \overset{2}{\mathbf{u}}_\tau\cdot\overset{2}{\mathbf{u}}_\tau, \ \mathbf{u}\cdot\mathbf{n}). \qquad (5.18)$$

The relation (5.18) is a general form of the constitutive equation for the dissipation of the mechanical energy at the interface between regions of fluid flow in the porous skeleton and the adjacent bulk fluid flow.

The only constraint for the function $\hat{\vartheta}$ results from the last requirement of its positive definitness. This restriction takes form of the following inequality

$$\hat{\vartheta}(\mathbf{u}_\tau^2, \ \mathbf{u}_\tau\cdot\overset{2}{\mathbf{u}}_\tau, \ \overset{2}{\mathbf{u}}_\tau\cdot\overset{2}{\mathbf{u}}_\tau, \ \mathbf{u}\cdot\mathbf{n}) > 0, \qquad (5.19)$$

that should be satisfied for any nonzero values of $\mathbf{u}$, $\overset{2}{\mathbf{u}}$ and in the case of no fluid flow ($\mathbf{u} = \overset{2}{\mathbf{u}} = \mathbf{0}$) it must be equal to zero

$$\hat{\vartheta}(0, 0, 0, 0) = 0. \qquad (5.20)$$

The inequality (5.19) results from the entropy production principle, [54], due to the dissipation of mechanical energy on the Γ-surface.

Since, in general case, the function $\hat{\vartheta}$ may not be an even function of the argument $\mathbf{u} \cdot \mathbf{n}$, it follows from Eq. (5.18) that the dissipation of mechanical energy can reach different values during the inflow and outflow of fluid from the porous skeleton. However, for small values of the velocities $\mathbf{u}_\tau$, $\overset{2}{\mathbf{u}}_\tau$ and $\mathbf{u} \cdot \mathbf{n}$, conditions (5.19) and (5.20) ensure that the constitutive function $\hat{\vartheta}$ has a minimum at $\mathbf{u} = \overset{2}{\mathbf{u}} = \mathbf{0}$, and therefore in such a case the dissipation function may be proposed in the quadratic form

$$\vartheta = \frac{\mu}{\sqrt{k}}(\alpha_1 \mathbf{u}_\tau^2 - 2\alpha_2 \mathbf{u}_\tau \cdot \overset{2}{\mathbf{u}}_\tau + \alpha_3 \overset{2}{\mathbf{u}}_\tau^2 + \epsilon(\mathbf{u} \cdot \mathbf{n})^2) , \qquad (5.21)$$

where the fluid viscosity μ and the permeability k of porous skeleton have been introduced in order the constant coefficients α_1, α_2, α_3 and ϵ to be dimensionless quantities. The coefficients α_1, α_2, α_3 and ϵ in dissipation function are macroscopic parameters characterizing the pore structure of permeable material within the boundary layer.

The choice of parameters μ and k is motivated by the fact that fluid viscosity and permeability of porous material are the main properties upon which dissipation of mechanical energy appearing in the boundary region can depend. As it will be shown later the dissipation function of the form (5.21) leads to the Beavers-Joseph type boundary condition (see (5.36)) in which material parameters are independent of the fluid viscosity. Such result is strongly supported by experimental observations, [5].

Considering the fluid flow tangential and normal to the Γ-surface it can be shown that the constitutive relation (5.21) will be positively definite when constant coefficients satisfy the following inequalities:

$$\alpha_1 > 0, \quad \alpha_3 > 0, \quad \alpha_2^2 < \alpha_1 \alpha_3, \quad \epsilon > 0 . \qquad (5.22)$$

5.4. Linear compatibility conditions

The equation (5.2), (5.3) and (5.4) together with the dissipation function (5.18) form a set of the general compatibility conditions for mechanical fields on the discontinuity Γ-surface. These equations are to be used to establish

the linear form of compatibility conditions which play an important role in a large number of linear problems of porous media.

The linear compatibility condition for the stress vectors of particular constituents of the system can be derived directly from the balance equation of linear momentum (5.3) disregarding the nonlinear terms of its LHS. This condition written for the normal and tangential components yields

$$t_n = \overset{1}{t_n} + \overset{2}{t_n} \,, \tag{5.23}$$

$$\mathbf{t}_\tau = \overset{1}{\mathbf{t}_\tau} + \overset{2}{\mathbf{t}_\tau} \,, \tag{5.24}$$

From Eqs. (5.23) and (5.24) it follows that in the linear case the force exerted by the bulk fluid on the permeable boundary is equilibrated by forces of both constituents of saturated porous solid.

The linearization of the balance equation (5.4) for mechanical energy on the Γ-surface, in contradistinction to the balance of linear momentum (5.3), is not straightforward. We can arrive, however, at linear relations under some assumptions. After the use of Eqs. (5.2) and (5.21) and disregarding the LHS of Eq. (5.4), we obtain

$$\left[\rho \left(\frac{t_n}{\rho} - \frac{\overset{2}{t_n}}{\overset{2}{\rho}} \right) - \frac{\mu}{\sqrt{k}} \epsilon u_n \right] u_n = \overset{2}{\mathbf{t}_\tau} \cdot \overset{2}{\mathbf{u}_\tau} - \mathbf{t}_\tau \cdot \mathbf{u}_\tau - \frac{\mu}{\sqrt{k}} (\alpha_1 u_\tau^2 - 2\alpha_2 \mathbf{u}_\tau \cdot \overset{2}{\mathbf{u}_\tau} + \alpha_3 \overset{2}{\mathbf{u}_\tau^2}) \,. \tag{5.25}$$

In the above equation the components of stress vectors of bulk and free fluid are related with the corresponding relative velocities.

Since, in the linear case, the normal and tangential components of any vector field are mutually independent, Eq. (5.25) will be satisfied if its both sides are identically equal to zero. According to the relations

$$\rho = \rho^- \,, \quad \overset{2}{\rho} = \kappa f_v \rho^+ \,, \quad t_n = -p^- \,, \quad \overset{2}{t_n} = -\kappa f_v p^+ \,, \tag{5.26}$$

where p^-, p^+ are fluid effective pressures at both sides of Γ-surface, from Eq. (5.25) we get

$$\rho^- \left(\frac{p^+}{\rho^+} - \frac{p^-}{\rho^-} \right) = \frac{\mu}{\sqrt{k}} \epsilon u_n \,, \tag{5.27}$$

$$\overset{2}{\mathbf{t}_\tau} \cdot \overset{2}{\mathbf{u}_\tau} - \mathbf{t}_\tau \cdot \mathbf{u}_\tau - \frac{\mu}{\sqrt{k}} (\alpha_1 u_\tau^2 - 2\alpha_2 \mathbf{u}_\tau \cdot \overset{2}{\mathbf{u}_\tau} + \alpha_3 \overset{2}{\mathbf{u}_\tau^2}) = 0 \,, \tag{5.28}$$

where superscripts $+$ and $-$ denote values of quantities on the positive and negative side of the Γ-surface, respectively. In order to obtain the linear form of Eq. (5.27) we assume that the fluid is barotropic, i.e.

$$p = p(\rho) \tag{5.29}$$

then for the small values of $p^+ - p^-$ we can write

$$p^+ - p^- = a_o^2(\rho^+ - \rho^-) \tag{5.30}$$

where

$$a_o = \left(\frac{dp}{d\rho}\right)^{\frac{1}{2}} |_{\rho=\rho_0}$$

is the velocity of wave front propagation in an undisturbed region of the bulk fluid. Under the above assumption, from Eq. (5.27) we obtain the linear compatibility condition

$$p^+ - p^- = \frac{\mu}{\sqrt{k}} \frac{\epsilon}{(1 - p_o/(\rho_o a_o^2))} u_n . \tag{5.31}$$

It relates the fluid pressures on both sides of the Γ-surface and the normal component of the relative fluid velocity.

In the case when the RHS of Eq. (5.31) can be omitted, the condition (5.31) reduces to the classical one describing the fluid pressure continuity

$$p^+ = p^- = p . \tag{5.32}$$

Assumptions (5.29) and (5.30) and condition (5.31) allow the linearization of the fluid mass balance equation (5.2) at the Γ-surface. As a consequence, we obtain the compatibility condition for normal components of velocities of fluid flow across the Γ-surface

$$u_n = \kappa f_v \overset{2}{u}_n . \tag{5.33}$$

To obtain the compatibility conditions for tangential components of fluid velocities at the Γ-surface we consider condition (5.28). Assuming that tangential forces $\mathbf{t}_\tau$ and $\overset{2}{\mathbf{t}}_\tau$ depend linearly on the relative velocities $\mathbf{u}_\tau$ and $\overset{2}{\mathbf{u}}_\tau$, we can split up Eq. (5.28) into two independent equations

$$\mathbf{t}_\tau = \frac{\mu}{\sqrt{k}}(\alpha_1 \mathbf{u}_\tau - \alpha_2' \overset{2}{\mathbf{u}}_\tau) , \tag{5.34}$$

$$\overset{2}{\mathbf{t}_\tau} = \frac{\mu}{\sqrt{k}}(\alpha_2'' \mathbf{u}_\tau - \alpha_3 \overset{2}{\mathbf{u}}_\tau) \tag{5.35}$$

in which

$$\alpha_2' + \alpha_2'' = 2\alpha_2 \,.$$

Equations (5.34) and (5.35) are the two additional necessary linear compatibility conditions. Their functional form is determined by the constitutive function (5.21) of dissipation of mechanical energy of fluid in the boundary layer of porous solid.

Relations (5.23), (5.24), (5.31) (or (5.32)), (5.33)-(5.35) form the complete set of the linear compatibility conditions for mechanical fields at the contact surface between fluid-saturated porous solids and bulk fluid.

Taking into account the fact that tangential stresses in the bulk and free fluid are related with the corresponding velocity gradients via the constitutive equations, the compatibility conditions (5.34) and (5.35) can be transformed to the form of conditions for the velocities only. For the one-dimensional rectilinear fluid flow along a plane surface of porous solid, we have

$$\mathbf{t}_\tau = \mu \frac{\partial u_\tau}{\partial z}\tau \quad \text{and} \quad \overset{2}{\mathbf{t}_\tau} = \mu^* \frac{\partial \overset{2}{u}_\tau}{\partial z}\tau \,,$$

where τ is the unit vector tangential to the Γ-surface indicating the direction of fluid flow. The coordinate z is directed along the vector normal to the Γ-surface. The coefficient μ^* stands for the apparent viscosity of the fluid filling the porous skeleton corresponding to the coefficient introduced by Brinkman, [14]. In such a case, from Eqs. (5.34) and (5.35), we obtain

$$\frac{\partial u_\tau}{\partial z}\tau = \frac{1}{\sqrt{k}}(\alpha_1 \mathbf{u}_\tau - \alpha_2' \overset{2}{\mathbf{u}}_\tau) \,, \tag{5.36}$$

$$\frac{\mu^*}{\mu}\frac{\partial \overset{2}{u}_\tau}{\partial z}\tau = \frac{1}{\sqrt{k}}(\alpha_2'' \mathbf{u}_\tau - \alpha_3 \overset{2}{\mathbf{u}}_\tau) \tag{5.37}$$

Taking the above results into account, and the linear relationship between volume discharge and velocity of fluid flow through porous skeleton, we find that the Beavers-Joseph postulate (5.1) has the form that of (5.36). However, Beavers-Joseph postulate is not sufficiently general since the only one condition does not allow to solve boundary problems where the fluid flow within a porous medium is described by the second order differential equation e.g. the Brinkman type of equation, [14]. Moreover, the Beavers

and Joseph way of formulation of the slip-flow boundary condition does not give possibility to determine the interaction forces exerted by the flowing bulk fluid on the pore fluid and the porous skeleton at the permeable boundary of porous solid. It results from the fact that this condition is not explicitly related with the interaction forces. In our description the relative velocities $\mathbf{u}_\tau$ and $\overset{2}{\mathbf{u}}_\tau$ used in the compatibility conditions (5.36) and (5.37) are directly related with forces appearing at the boundary surface through the formulas(5.34) and (5.35). Consequently, the use of condition (5.24) allows the determination of interaction force between bulk fluid and saturated porous solid.

5.5. Solution of the Beavers-Joseph Flow Problem

In this section we use the compatibility conditions(5.36) and (5.37) to solve a problem of a viscous fluid flow along the surface of permeable porous material, so called the Beavers-Joseph problem. We compare this solution with that proposed by Beavers and Joseph and also with the selected results of experimental measurements performed by these authors, [5].

Let us consider laminar flow of viscous, incompressible fluid through a channel formed by an impermeable wall and a permeable, rigid, porous halfspace filled with fluid (Fig. 5).
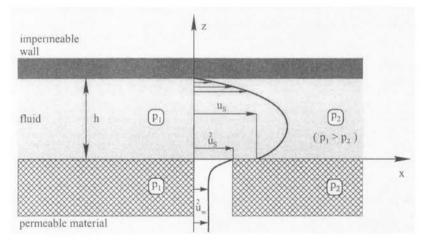

Figure 5: Laminar flow of viscous, incompressible fluid through a chanel

The flow induced by a constant pressure gradient dp/dx, is uni-directional and takes place in the channel as well as in the porous halfspace. We assume that the velocity field $u(z)$ of fluid in the channel is given by the reduced Navier-Stockes equation

$$\mu \frac{d^2 u}{dz^2} = \frac{dp}{dx} \qquad \text{for} \quad 0 < z < h \tag{5.38}$$

and the velocity field of fluid motion within the porous halfspace is described by equation of the Brinkman form, [14]

$$\mu^* \frac{d^2 \overset{2}{u}}{dz^2} - \frac{\mu}{k} \overset{2}{u} = \kappa f_v \frac{dp}{dx} \qquad \text{for} \quad z < 0 \,. \tag{5.39}$$

Although the fluid in the channel and pores of skeleton is the same, the viscosities μ^* and μ are of different values and the ratio μ^*/μ characterizes the structure of porous material and it is independent of the properties of fluid.

For the discussed problem the boundary condition at the surface of impermeable wall is the zero fluid velocity, i.e.

$$u|_{z=h} = 0 \,, \tag{5.40}$$

whereas at the surface of porous halfspace the compatibility conditions for velocities take form of relations (5.36) and (5.37). Solutions of equations (5.38) and (5.39), when conditions (5.40), (5.36) and (5.37) are taken into account give the following velocity fields

$$\frac{u}{\overset{2}{u}_\infty} = \left(1 - \frac{z}{h}\right)\left(\frac{z}{h}\frac{\hat\sigma^2}{2\kappa f_v} + \frac{u_s}{\overset{2}{u}_\infty}\right) \qquad \text{for} \quad 0 < z < h \,, \tag{5.41}$$

$$\frac{\overset{2}{u}}{\overset{2}{u}_\infty} = \left(\frac{\overset{2}{u}_s}{\overset{2}{u}_\infty} - 1\right)\exp\left(\frac{\hat\sigma}{\sqrt{\mu^*/\mu}}\frac{z}{h}\right) + 1 \qquad \text{for} \quad z < 0 \tag{5.42}$$

where

$$\hat\sigma = h/\sqrt{k}$$

is the dimensionless height of the channel and u_s, $\overset{2}{u}_s$ are the slip velocities of fluid at adequate sides of the surface of the porous halfspace. These velocities are linked to the parameters of the system by relations

$$\frac{u_s}{\overset{2}{u}_\infty} = \frac{\hat\sigma(\hat\sigma + 2\delta_1)}{2\kappa f_v(1 + \hat\sigma\delta_2)} \,, \tag{5.43}$$

$$\frac{\overset{2}{u}_s}{\overset{2}{u}_\infty} = \frac{1}{\alpha_3 + \sqrt{\mu^*/\mu}} \frac{\alpha_2''\hat{\sigma}^2 + 2\kappa f_v \sqrt{\mu^*/\mu}(1 + \alpha_1\hat{\sigma})}{2\kappa f_v(1 + \hat{\sigma}\delta_2)} \tag{5.44}$$

with

$$\delta_1 = \frac{\kappa f_v \alpha_2' \sqrt{\mu^*/\mu}}{\alpha_3 + \sqrt{\mu^*/\mu}}, \qquad \delta_2 = \alpha_1 - \frac{\alpha_2'\alpha_2''}{\alpha_3 + \sqrt{\mu^*/\mu}}. \tag{5.45}$$

In expressions (5.41)-(5.44) the reference velocity $\overset{2}{u}_\infty$ is introdused, i.e.

$$\overset{2}{u}_\infty = -\kappa f_v \frac{k}{\mu} \frac{dp}{dx},$$

that describes the fluid flow through the porous material in a large distance from the boundary of the considered halfspace.

Results from (5.43) and (5.44) show that the fluid at both sides of permeable surface has different velocities. It is worthfull to note that in the limit case when the impermeable wall approaches the surface of porous medium ($\hat{\sigma} = 0$) the velocity $\overset{2}{u}_s$ does not vanish and its finite value is

$$\frac{\overset{2}{u}_s^o}{\overset{2}{u}_\infty} = \frac{2\kappa f_v \sqrt{\mu^*/\mu}}{\alpha_3 + \sqrt{\mu^*/\mu}}.$$

Such effect is directly connected with the earlier introduced dissipation function for the boundary layer of porous material. The validity of this effect needs to be justified by experimental measurements.

Using the velocity field (5.41) and the velocity field for the fluid flow through the channel between two impermeable walls we can define the fractional increase

$$\Phi = \frac{M - M_o}{M_o}$$

in the mass flow rate through the channel with a permeable lower wall (M) over that it would be if the wall were impermeable (M_o). We obtain

$$\Phi = \frac{3(\hat{\sigma} + 2\delta_1)}{\hat{\sigma}(1 + \delta_2\hat{\sigma})}. \tag{5.46}$$

The above relation is more general than that derived by Beavers and Joseph, [5], i.e.

$$\Phi = \frac{3(\hat{\sigma} + 2\alpha)}{\hat{\sigma}(1 + \alpha\hat{\sigma})} \tag{5.47}$$

where the compatibility condition (5.1) was used. Expression (5.46) contains two parameters δ_1 and δ_2 that according to (5.45) are functions of the parameters characterizing pore structure of the interior and the surface layer of porous material while the expression (5.47) contains only one parameter α introduced by the compatibility condition (5.1).

The experimental measurements performed by Beavers and Joseph have shown the qualitative agreement of the experimental data with the theoretical predictions of relation (5.47). In some cases, however, the obtained quantitative agreement is not satisfactory. As an example, in Fig. 6 there are shown the results of measurements performed by Beavers and Joseph for the oil flow over specimens of granular material (aloxite) with two various permeabilities and the adequate theoretical curves obtained from (5.46) and (5.47).
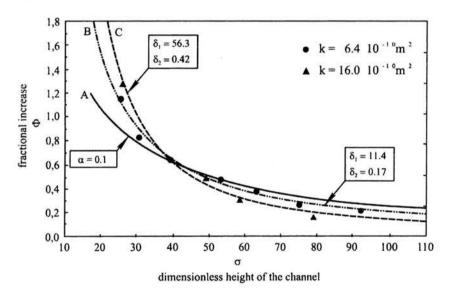

Figure 6: The results of measurements performed by Beavers and Joseph (1967)

The solid line A represents the theoretical predications obtained by Beavers and Joseph from relation (5.47) ($\alpha = 0.1$) for both kinds of porous specimens. It is seen that the considerable difference between theoretical and experimental results occurs. The dashed lines B and C in this figure are graphs obtained from the function (5.46) for parameters δ_1 and δ_2 determined for each porous specimens by the use of optimization procedure.

For specimens with permeability $k = 6.45 \times 10^{-10} m^2$ and $k = 16.0 \times 10^{-10} m^2$ these parameters take the following values; $\delta_1 = 11.4$, $\delta_2 = 0.17$ and $\delta_1 = 56.3$, $\delta_2 = 0.42$, respectively.

The above results have shown that the compatibility conditions (5.36) and (5.37) derived in this work assures the more precise description of fluid flow along a permeable wall of porous material in comparison to the condition (5.1) postulated by Beavers and Joseph.

5.6 Remarks

In the paper the problem of compatibility conditions for macroscopic mechanical fields at the contact surface between the fluid-saturated porous solid and the bulk fluid has been analyzed. Special attention was paid to the derivation of conditions for the tangential components of the fluid flow velocities at the contact surface. This allowed verification of accuracy of the condition postulated by Beavers and Joseph.

The analysis of general form of compatibility conditions, i.e. the balance equations for mass, linear momentum and mechanical energy at the discontinuity surface proved the existence of discontinuity of the macroscopic relative fluid velocities justifying the postulate of existence of the mechanical energy dissipation at this surface due to the fluid viscosity. It turned out that introduction of the term responsible for this effect into the balance equation of mechanical energy was the crucial point for the explanation of validity of the Beavers-Joseph type conditions.

The linearization of the balance equations at the contact surface provided the complete set of linear compatibility conditions and allowed to obtain the generalized solution of the Beavers-Joseph flow problem leading to the very well agreement with experimental results.

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