



**Poromechanical coupling
in saturated geomaterials.
A homogenization based study**

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The note presents a few developments in the field of the saturated geomaterials behavior obtained using the micro-macro approach. The main ideas of the micro-macro approach, i.e. an averaging method and homogenization technique, are exposed. Special attention is paid to the homogenization technique for periodic structures. Different processes taking place in saturated porous media as: filtration, consolidation and sorption are then studied using the homogenization method. The effect of porous material microstructure on the values of poroelasticity material constants is clearly pointed out. The effective stress concept is validated in the inelastic behaviour range for the case of micro-homogeneous skeleton material.

Key words: *porous medium, effective stress, poroelasticity, sorption, homogenization*

1. Introduction

It is obvious that the mathematical description of physical processes occurring in a porous material depends on the observation scale. For instance, while a sample of a given material in a natural observation scale might be treated as a homogeneous one then microscopically, due to a presence of pores, it is clearly heterogeneous. Thus, a description of such material in the natural observation scale is, within the framework of continuous mechanics, a certain approximation as well as any such-scale laboratory investigation is in fact a relationship between averaged physical fields observed during the experiment.

Generally, when one consider porous material saturated or not with fluid, two different modeling approaches are possible. The first, the so-called phenomenological one, ignores the distances of the order of pore diameters (microscopic distances) and takes into account only the macroscopic distances. The porous medium is treated as the continuous one. When the porous medium is filled with a fluid then, within this approach, each point of the space is occupied by the skeleton and the fluid simultaneously and both the phases are treated as continuous. Hence, the multiphase medium is modeled as a system of overlapping continua. The conservation laws are introduced directly at the macroscopic level analogous to the laws for a single-phase continuum medium, supplemented however with additional terms expressing interactions between the phases.

The other kind of modeling is based on the so-called micro-macro approach, i.e. a passage with a mathematical description from micro- to a macro-scale. The starting point of this approach is a micro-scale description, i.e. a mathematical description of the process investigated formulated at the scale of heterogeneities. At this scale each distinct space point is occupied by only one distinct component which is considered as a continuous medium with its own strain-stress relation, balance equation and boundary conditions on the phase separation surface. Finally, the macroscopic description is formulated by using an appropriate averaging operator or as a limit obtained by letting

the microscale tend to zero.

These notes intend to give a brief summary of a few developments in the field of the saturated geomaterials behavior obtained using the micro-macro approach. The chapter is written in the following sequence. In the subsequent section different formulations of the micro-macro approach are discussed first and then the so-called asymptotic homogenization technique is presented in details. The main ideas of the asymptotic homogenization method are exposed and explained based on a solution of one-dimensional diffusion problem. Necessity of normalization of the micro-scale description is pointed out by investigation some features of homothetic transformation. The next section is concerned with the poroelasticity theory founded by Biot. Using the asymptotic homogenization technique the poroelasticity equations are recovered and the effect of porous material microstructure on the values of poroelasticity material constants is carefully studied. The mathematical modeling of sorption and sorption swelling phenomena in saturated with gas geomaterials is a subject of the next section. Once more the macroscopic description is formulated based on the asymptotic homogenization technique. This section presents also the material constants identification. Recent results concerning the so-called stress- and strain equivalence principles for saturated porous material end the paper.

Throughout the paper the index notation with Einstein's convention of summation over repeated indices is used, as well as the classical sign convention of the continuum mechanics (tension as positive).

2. Homogenization method

The aim of the micro-macro approach is to formulate a macroscopic description of a process considered based on knowledge of a mathematical description of the process at the heterogeneities scale (so-called microscopic description). The notions of the micro- and macroscopic description are understood as the description that takes into account the heterogeneous structure of the medium and that for an equivalent homogenous medium, respectively. Roughly speaking, the microscopic description can be treated, in some sense, as the exact one whereas the macroscopic - as some approximation, useful for engineering calculations. Furthermore, using the micro-macro approach one assumes at the same time a possibility of defining, for a given heterogeneous material, an equivalent homogeneous medium. Therefore, such heterogeneous medium is very often called as the micro-heterogeneous one in order to emphasize its heterogeneous structure and at the same time a possibility of defining for it the equivalent homogeneous medium. In practice, this is known as the macro-homogeneity condition, which is classically assumed in geomechanics, for instance.

Methodologically, one can distinguish at least two different formulation of the micro-macro approach. The first one is based on a notion of the so-called representative volume element (REV) and consists in an averaging process of the physical field of the microscopic description, i.e.:

$$\langle u \rangle (x) = \int_{V_{REV}} u(y)m(x-y)dy \quad (2.1)$$

where: $u(y)$ is a certain physical field of the microscopic description whereas $\langle u \rangle (x)$ is assumed to be the corresponding physical field of the macroscopic description, $m(x-y)$ is the so-called weight function. Concluding, this micro-macro passage consists in smoothing of the rapidly oscillating physical field of the microscopic description by using the averaging operator.

The second formulation is known as the homogenization. The micro-macro passage is realized, within this formulation, by considering a family functions u^ε parameterized by a certain scale parameter $\varepsilon > 0$ representing, for instance, the typical size of a pore (Fig. 1). The macroscopic description is obtained by determining the limit at $\varepsilon \rightarrow 0$, i.e.:

$$u(x) = \lim_{\varepsilon \rightarrow 0} u^\varepsilon(x) \quad (2.2)$$

and finding differential equations that the limit $u(x)$ satisfies. Concluding, the macroscopic description is obtained, within the homogenization approach, by letting the scale parameter tend to zero.

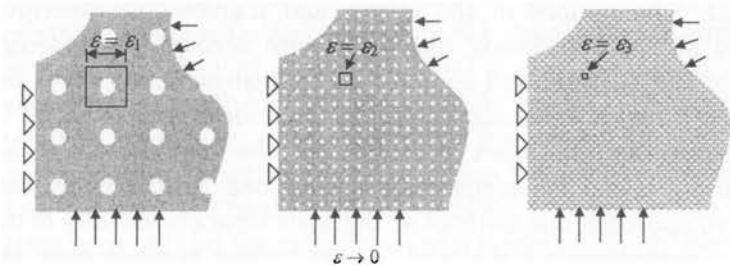


Figure 1: Schematic view at the homogenization concept

2.1. One-dimensional diffusion problem

In order to clarify the homogenization idea let us consider a simple one-dimensional diffusion problem (Fig. 2). It is assumed that a sample of length L is composed of N the same base cells, each of length $l=L/N$. Furthermore, the value of a diffusion coefficient within the base cell is supposed to be described by a following distribution function:

$$D(x) = \frac{D_o}{1 + \cos^2\left(\frac{\pi x}{l}\right)}, \quad (2.3)$$

where: D_o - a distribution parameter.

Hence, for the sample composed of N base cells, the distribution of diffusion coefficient within the sample is described by:

$$D_N(x) = \frac{D_o}{1 + \cos^2\left(\frac{N \pi x}{L}\right)}. \quad (2.4)$$

The above function depends on the number N of base cells of which the sample is composed. The homogenization method uses, however, the microscopic description parameterized by a scale parameter ε representing, for instance, a ratio between a characteristic length of heterogeneity l (size of the base cell) and a characteristic length of the sample L , so $\varepsilon=l/L$ ($= 1/N$). Using this scale parameter the relation (2.4) can be expressed as:

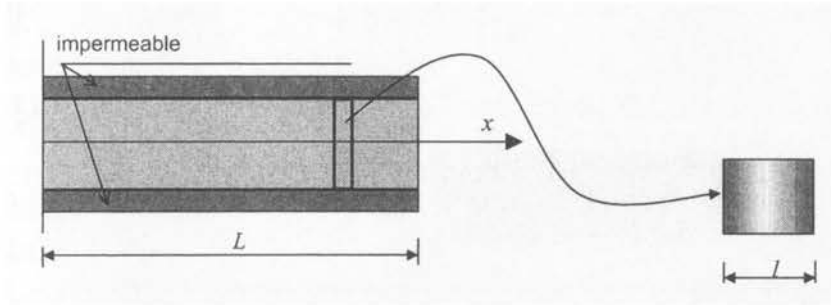


Figure 2: One-dimensional diffusion problem

$$D^\varepsilon(x) = \frac{D_o}{1 + \cos^2\left(\frac{\pi x}{L\varepsilon}\right)}. \tag{2.5}$$

and finally, the parameterized description of the stationary diffusion process, within the sample considered, becomes as:

$$\frac{d}{dx} \left(D^\varepsilon(x) \frac{dC^\varepsilon(x)}{dx} \right) = 0, \tag{2.6}$$

where: $C^\varepsilon(x)$ represents a mass concentration within the sample at the point x . The superscript ε denotes that this concentration corresponds to an actual assumed ε -value or equivalently to an actual number of the base cells of which the sample is composed.

Assuming the boundary conditions as: $C^\varepsilon(0) = C_o$ and $C^\varepsilon(L) = 0$ one gets:

$$D^\varepsilon(x) \frac{dC^\varepsilon(x)}{dx} = A, \text{ and } C^\varepsilon(x) = A \int_0^x \frac{dx}{D^\varepsilon(x)} + C_o \tag{2.7}$$

where the value of A is uniquely determined by the boundary conditions.

Using the relation (2.5) for $D^\epsilon(x)$, the solution can be finally written as:

$$C^\epsilon(x) = C_o \left(1 - \frac{x}{L}\right) - \epsilon \frac{2C_o}{34\pi} \sin\left(\frac{2\pi}{L} \left(\frac{x}{\epsilon}\right)\right). \tag{2.8}$$

The above equation implies also the following relations, i.e.:

$$\frac{dC^\epsilon(x)}{dx} = -\frac{C_o}{L} \left\{1 + \frac{1}{3} \cos\left(\frac{2\pi}{L} \left(\frac{x}{\epsilon}\right)\right)\right\}, \tag{2.9}$$

$$Q^\epsilon = -D^\epsilon(x) \frac{dC^\epsilon(x)}{dx} = \frac{2}{3} D_o \frac{C_o}{L}, \tag{2.10}$$

where Q^ϵ denotes a flux of diffusing mass.

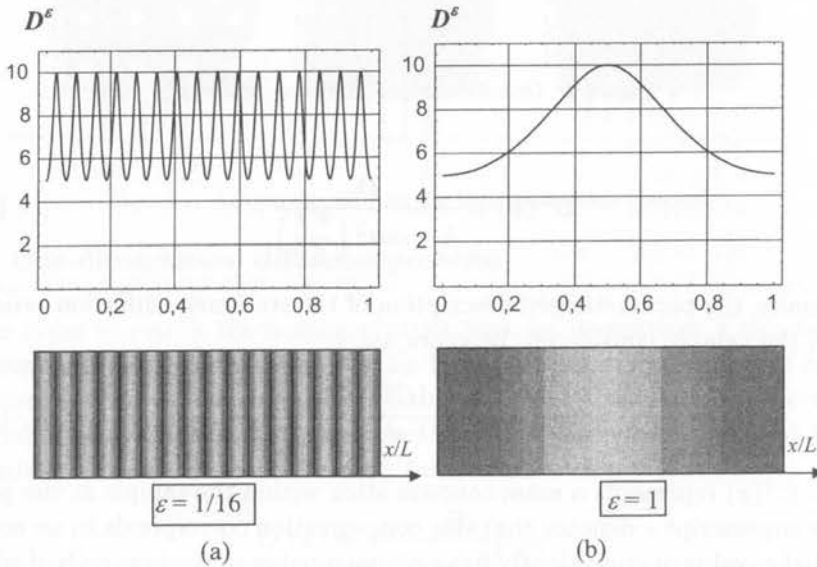


Figure 3: Fluctuations of diffusion coefficient along the sample ($D_o = 10$ is assumed): (a) sample consists of 16 periodic base cells, (b) sample consists of 1 periodic base cell

The relation (2.8) may be rewritten in the form:

$$C^\epsilon(x) = C(x) + \epsilon C_1 \left(\frac{x}{\epsilon}\right). \tag{2.11}$$

where

$$C(x) = \lim_{\epsilon \rightarrow 0} C^\epsilon(x) = C_o \left(1 - \frac{x}{L}\right), \tag{2.12}$$

and

$$C_1(y) = -\frac{2 C_o}{3 4\pi} \sin\left(\frac{2\pi}{L}y\right), \tag{2.13}$$

The function $C(x)$ describes the macroscopic behavior of the function $C^\epsilon(x)$, whereas the function $C_1(y)$, called as the corrector, describes the local fluctuations of the function $C^\epsilon(x)$. By virtue of the equations (2.10) and

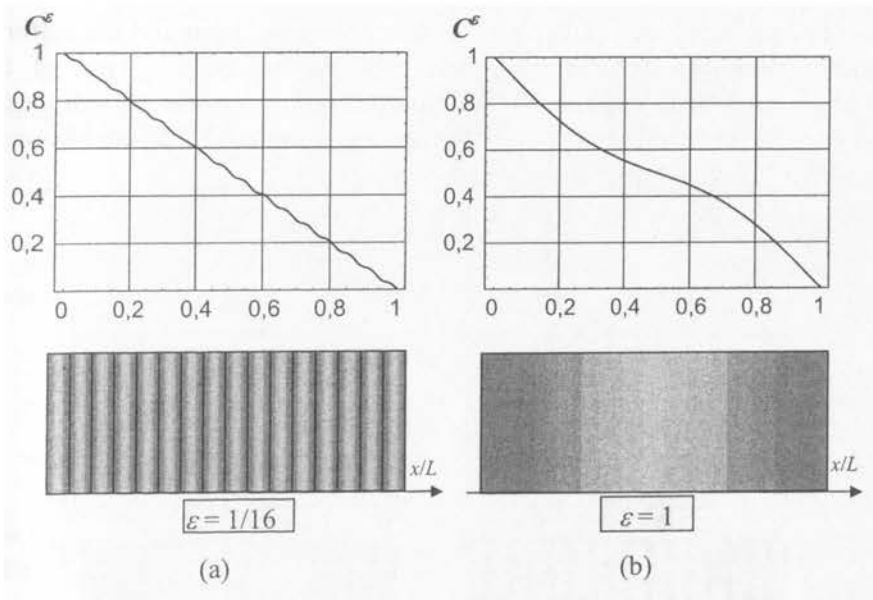


Figure 4: Mass concentration within the sample ($C_o = 1$ is assumed): (a) sample consists of 16 periodic base cells, (b) sample consists of 1 periodic base cell

(2.12), one gets also:

$$Q = \lim_{\epsilon \rightarrow 0} Q^\epsilon = -\left(\frac{2}{3}D_o\right) \frac{dC(x)}{dx}. \tag{2.14}$$

The above expression can be interpreted as the macroscopic constitutive equation since it defines the relation between the macroscopic flux and the macroscopic concentration gradient. The coefficient $\frac{2}{3}D_o$ represents the so-called effective diffusion coefficient and it is the material property of the

equivalent homogeneous medium. It can be shown that:

$$D^{\text{eff}} = \frac{2}{3}D_o = \frac{1}{\frac{1}{\varepsilon L} \int_0^{\varepsilon L} \frac{dx}{D^\varepsilon(x)}} = \frac{1}{\frac{1}{L} \int_0^L \frac{1 + \cos^2(\frac{\pi}{L}y)}{D_o} dy}. \quad (2.15)$$

Finally, the macroscopic description of one-dimensional diffusion process has the form:

$$\frac{d}{dx} \left(D^{\text{eff}} \frac{dC(x)}{dx} \right) = 0. \quad (2.16)$$

The relations (2.14) and (2.15), as the constitutive equation and the material property definition respectively, could not depend on the type of the boundary conditions applied in the homogenization process, as well as the other possible external excitation of the process considered. The next example considered below assesses the above statements.

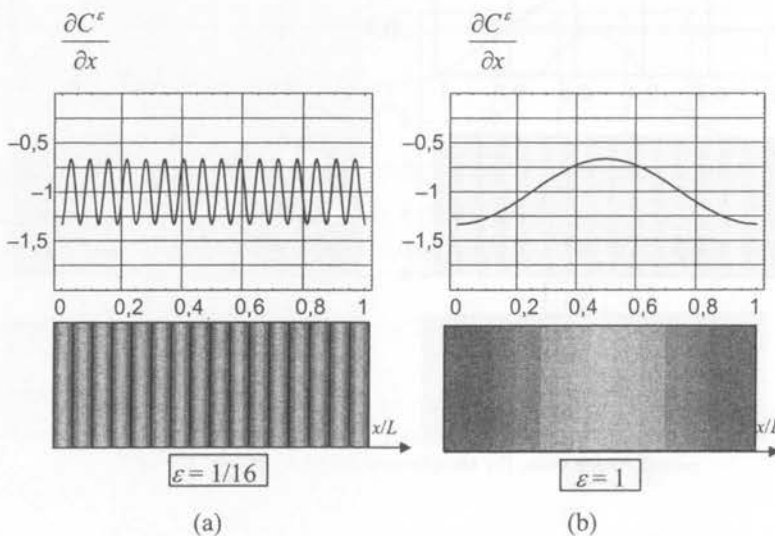


Figure 5: Fluctuations of concentration gradient within the sample ($C_o = 1$ is assumed): (a) sample consists of 16 periodic unit cells, (b) sample consists of 1 periodic unit cell

The microscopic description (2.6) is modified by adding an additional source term, i.e.:

$$\frac{d}{dx} \left(D^\varepsilon(x) \frac{dC^\varepsilon(x)}{dx} \right) + f_o = 0. \quad (2.17)$$

The distribution function of the diffusion coefficient $D^\epsilon(x)$ is, of course, assumed to be the same as before, as well as the boundary conditions. Finally, one gets:

$$\begin{aligned}
 C^\epsilon(x) = & \frac{3}{4} \frac{f_o}{D_o} x (L - x) + C_o \left(1 - \frac{x}{L}\right) + \\
 & + \epsilon \left\{ -\frac{f_o}{D_o} \frac{L}{4\pi} x \sin \left(\frac{2\pi x}{L \epsilon}\right) - \frac{1}{6\pi} \left(C_o - \frac{3}{4} \frac{f_o}{D_o} L^2\right) \sin \left(\frac{2\pi x}{L \epsilon}\right) \right\} + \\
 & - \epsilon^2 \left\{ \frac{1}{8} \frac{f_o}{D_o} \frac{L^2}{\pi^2} \left(\cos \left(\frac{2\pi x}{L \epsilon}\right) - 1\right) \right\}.
 \end{aligned}
 \tag{2.18}$$

The above relation leads to:

$$Q^\epsilon(x) = -D^\epsilon(x) \frac{dC^\epsilon(x)}{dx} = - \left(\frac{2}{3} D_o\right) \left(-\frac{3}{2} \frac{f_o x}{D_o} + \frac{3}{4} \frac{f_o L}{D_o} - \frac{C_o}{L}\right). \tag{2.19}$$

By letting the scale parameter tend to zero, the appropriate macroscopic fields are obtained, i.e.:

$$C(x) = \lim_{\epsilon \rightarrow 0} C^\epsilon(x) = \frac{3}{4} \frac{f_o}{D_o} x (L - x) + C_o \left(1 - \frac{x}{L}\right), \tag{2.20}$$

$$Q(x) = \lim_{\epsilon \rightarrow 0} Q^\epsilon(x) = - \left(\frac{2}{3} D_o\right) \frac{dC(x)}{dx}. \tag{2.21}$$

It is clearly seen that the constitutive equation (2.21) is of the same form as before, as well as the value of the effective diffusion coefficient. The additional excitation does not affect the homogenization result.

The one-dimensional diffusion problem considered enables to formulate the following statements, i.e.:

Remark 1

Parameterized function $C^\epsilon(x)$, being the solution of the microscopic description, exhibits the asymptotic character with respect to the parameter ϵ , (see the equations. (2.11) and (2.18)), i.e.

$$C^\epsilon(x) = C^{(a)}(x) + \epsilon C^{(1)}\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 C^{(2)}\left(\frac{x}{\epsilon}\right), \tag{2.22}$$

Remark 2

All the terms of the above asymptotic development are periodic with respect to the variable $y = x/\varepsilon$ with a period $Y=l/\varepsilon=L$. This means that the assumed periodic structure of the sample induces periodic fluctuations of the parameterized field $C^\varepsilon(x)$.

Remark 3

The arguments x and $y = x/\varepsilon$ of the function $C^\varepsilon(x)$ can be treated as the independent space variables and then the function $C^\varepsilon(x)$ can be interpreted as the locally periodic one, i.e. periodic with respect to the y - variable. Calculating a space derivative one should, however, take into account that $\frac{dy}{dx} = \varepsilon^{-1}$ and hence:

$$\frac{dC^\varepsilon(x)}{dx} = \frac{\partial C^\varepsilon(x, y)}{\partial x} + \varepsilon^{-1} \frac{\partial C^\varepsilon(x, y)}{\partial y} \quad (2.23)$$

Remark 4

The macroscopic description (so called the homogenized limit) is obtained by letting the scale parameter ε tend to zero or equivalently by letting the number of base cells of which the sample is composed tend to infinity. The increase of the base cells number in the sample with fixed length is realized, within homogenization, by homothetic transformation of the base cell.

The above statements are the basis of the general methodology known as the two-scale asymptotic expansion technique.

2.2. Asymptotic homogenization method

The central assumption of this homogenization technique is the condition that the medium exhibits a periodic structure, i.e. it is a periodicity - generated from a single element, the so-called base cell or unit cell (Fig. 6). The necessary condition for homogenization to be possible is the so-called separation of the scales, i.e. if l denotes a characteristic length of the unit cell and L one of volumetric dimension of the medium being considered, the following must be satisfied: $\varepsilon=l/L \ll 1$.

The asymptotic homogenization method seeks for the solution of the microscopic description assuming that the parameterized field being considered exhibits the asymptotic character with respect to the scale

parameter ε , i.e.:

$$u^\varepsilon(x) = u^{(0)}(x, y) + \varepsilon u^{(1)}(x, y) + \varepsilon^2 u^{(2)}(x, y) + \dots + \text{ where } y = \frac{x}{\varepsilon}. \quad (2.24)$$

Furthermore, it is postulated that each term $u^{(i)}(x, y)$ of the asymptotic development is Y -periodic in y which means that they take equal values on the opposite site of the unit cell, i.e.

$$u^{(i)}(x, y + Y) = u^{(i)}(x, y). \quad (2.25)$$

The arguments x and y are treated as the independent space variables, so the spatial derivative operator is modified as:

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \varepsilon^{-1} \frac{\partial}{\partial y_i}. \quad (2.26)$$

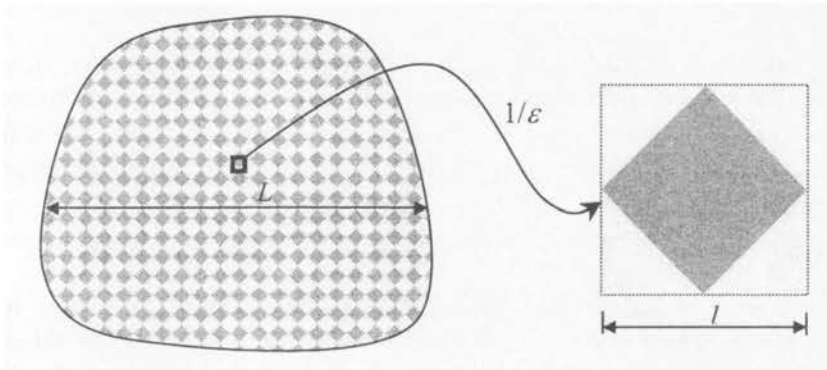


Figure 6: The periodic structure and the base cell of a medium.

The method consists in incorporating the expansion (2.24) into the microscopic description and in identifying terms with the same powers of ε . As the result, the sequence of equations for each term of the expansion is obtained. When imposed on the particular terms of $u^{(i)}(x, y)$, the periodicity condition make us look for the solutions of thus-obtained the so-called cell-boundary value problem. In consequence, this process results in the sought macroscopic description of the issue being considered.

Noticeably, the process we described is relatively simple, being at the same time very effective, which is proved by many important results obtained

in this way. It has to be marked, however, that the result obtained using the asymptotic homogenization method is, from the mathematics point of view, only formal. It is obtained by postulating the asymptotic character of the parameterized solution of the microscopic description which is not a priori guaranteed. So the result, to be mathematically rigorous, has to be supplemented by a proof that:

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon(x) = u^{(0)}(x) \quad (2.27)$$

In most problems, a mathematical proof of the above convergence is available using, for instance, the so-called *energy method* of Tartar (see for more details the appendix in [11]).

A certain variant of the two-scale asymptotic expansion method is the so-called two-scale convergence method. This method gives a rigorous deductive procedure for obtaining the macroscopic equations along with the convergent theorem.

An objection which rises as to the applicability of the asymptotic homogenization method to any porous media is the periodicity assumption of the structure. It appears, however, that independently of whether the medium is random or periodic, the form of the macroscopic equivalent description remains unchanged ([18] and [19]).

Remark

The base cell is subjected, during the homogenization process, to a homothetic transformation. This transformation does not affect the values of dimensionless microstructure parameters, as for instance: a volume fraction, a relative distances between the components, shape of the grains (Fig. 7).

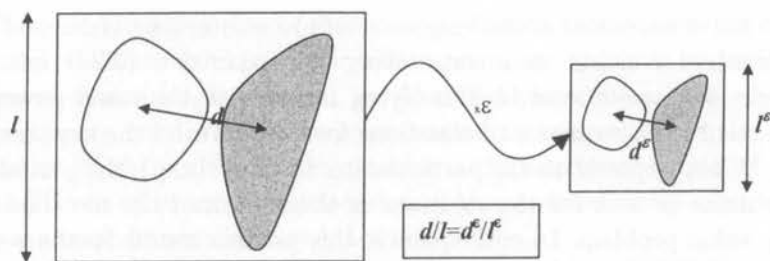


Figure 7: Homothetic transformation of the base cell.

The homothetic transformation affects however the values of microstructure parameters with metric units, as for instance: a specific surface of porous medium (Fig. 8) or diameters of pores. So, if a physical process depends on the microstructure parameters sensitive for the homothetic transformation then its effective parameters depend also on the absolute size of the base cell. It is also necessary to normalize all the equations of the microscopic description, before applying the homogenization procedure described above ([15]).

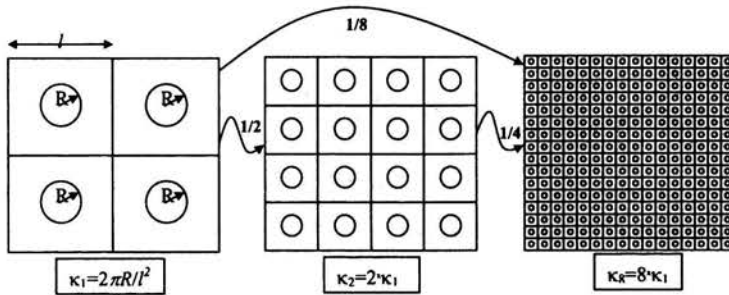


Figure 8: Variation of a porous medium specific surface due to homothetic transformation.

3. Poroelasticity and homogenization

The poroelasticity theory founded by Biot has largely been used in modeling of porous media behavior. Many papers deal with the physical interpretation of the material coefficients with respect to laboratory measurements and their relations with the elastic constants of the bulk frame and skeleton material ([1, 3, 23]). In many cases, however, discussions are limited to isotropic saturated porous media. Only a few of investigations (for instance [23] and [6]) have been devoted to anisotropic cases but generally the link between microstructural geometry and macroscopic poroelastic parameters has not been established in a systematic way. On the other hand, laboratory tests performed on saturated rocks have clearly shown strong connections between the microstructure of porous material and the overall poroelastic responses ([20]). For instance, a strong correlation has been found between the Biot's effective stress coefficient and the growth of microcracks.

The purpose of this chapter is to investigate, using the asymptotic homogenization method, the influence of microstructure on overall poroelastic behavior of saturated porous media. As the Biot's poroelasticity theory gives the fundamentals for modeling of poromechanical coupling, it appears essential to study at first correlation between microstructural parameters and material coefficients involved in Biot's theory.

Firstly, the Biot's poroelasticity theory is recovered using the asymptotic homogenization method. Basic relations about the overall poroelastic parameters obtained from the homogenization method are then used to study the influence of the microstructure on the values of poroelasticity material coefficients.

3.1. Homogenization process

We consider a consolidation process of two-phase medium composed of a porous skeleton V_s and an incompressible Newtonian liquid - V_l (Fig. 9). It

is assumed that a deformation process of the skeleton is quasi-static and a liquid flow in pores is slow and quasi-permanent. In addition, the skeleton is postulated to be periodic and composed of a linearly elastic material.

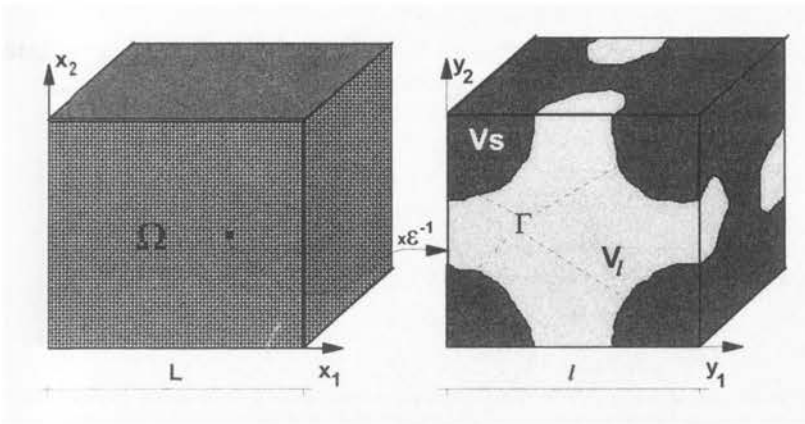


Figure 9: Two-phase medium considered: Ω - a bulk volume of the medium, $V = V_s + V_l$ - a unit cell volume.

3.1.1. Local description

With the above assumptions, the microscopic description is given by:

1. for liquid in the pores:

- balance equations (the small Reynolds' number is assumed)

$$\sigma_{ij,i}^l = 0, \text{ in } V_l, \tag{3.1}$$

- mass conservation law

$$v_{i,i} = 0, \text{ in } V_l, \tag{3.2}$$

- constitutive equations for an incompressible Newtonian liquid

$$\sigma_{ij}^l = -p\delta_{ij} + \mu^l (v_{i,j} + v_{j,i}), \text{ in } V_l. \tag{3.3}$$

2. for skeleton material:

- balance equations

$$\sigma_{ij,i}^s = 0, \text{ in } V_s, \tag{3.4}$$

- constitutive equations of the linear elasticity

$$\sigma_{ij}^s = C_{ijkh} e_{kh}(u), \text{ where } e_{ij}(u) = \frac{1}{2} (u_{i,j} + u_{j,i}), \text{ in } V_s. \quad (3.5)$$

The set (3.1)-(3.5) is completed with the boundary conditions on the interface Γ between the solid and fluid, i.e. (see Fig. 9):

- continuity of the stress vector

$$\sigma_{ij}^s N_i - \sigma_{ij}^l N_i = 0, \text{ on } \Gamma, \quad (3.6)$$

- continuity of the velocity vector

$$\dot{u}_i - v_i = 0 \text{ on } \Gamma, \quad (3.7)$$

where:

- σ_{ij}^s - stress tensor component in the skeleton material,
- C_{ijkh} - stiffness tensor component of the skeleton material,
- u_i - displacement vector component in the skeleton material,
- σ_{ij}^l - stress tensor component in the liquid,
- p - value of a liquid pressure,
- v_i - velocity vector component in the liquid,
- μ^l - a liquid viscosity,
- N_i - component of a unit normal N to the common surface Γ ,
- δ_{ij} - Kronecker's symbol.

The flow of the Newtonian liquid through a network of pores strongly depends on a absolute value of a pore diameter. Therefore, the filtration process is sensitive for a homothetic transformation of the base cell. This implies a necessity of the microscopic description normalization, before performing the homogenization process.

The normalization affects only the constitutive equations of the liquid and finally leads to (see for more details: [2] or [15]):

$$\sigma_{ij}^l = -p\delta_{ij} + \varepsilon^2 \mu^l (v_{i,j} + v_{j,i}) \quad (3.8)$$

Using the above equation and making some manipulations on the set (3.1)- (3.7), the following normalized form of the microscopic description is obtained, i.e.:

$$\varepsilon^2 \mu v_{i,jj} - p_{,i} = 0, \text{ on } V_l, \tag{3.9}$$

$$v_{i,i} = 0, \text{ on } V_l, \tag{3.10}$$

$$\left[C_{ijkh} e_{kh}(u) \right]_{,i} = 0, \text{ on } V_s, \tag{3.11}$$

$$\left[C_{ijkh} e_{kh}(u) + p \delta_{ij} - \varepsilon^2 \mu^l (v_{i,j} + v_{j,i}) \right] N_i = 0, \text{ on } \Gamma, \tag{3.12}$$

$$\dot{u}_i - v_i = 0, \text{ on } \Gamma. \tag{3.13}$$

3.1.2. Asymptotic developments and averaging process

Introducing into the equations (3.9) - (3.13) the asymptotic developments for v , p and u , i.e.:

$$v_i(x, y, t) = v_i^{(0)}(x, y, t) + \varepsilon v_i^{(1)}(x, y, t) + \varepsilon^2 v_i^{(2)}(x, y, t) + \dots \quad y = \frac{x}{\varepsilon}, \tag{3.14}$$

$$p(x, y, t) = p^{(0)}(x, y, t) + \varepsilon p^{(1)}(x, y, t) + \varepsilon^2 p^{(2)}(x, y, t) + \dots \quad y = \frac{x}{\varepsilon}, \tag{3.15}$$

$$u_i(x, y, t) = u_i^{(0)}(x, y, t) + \varepsilon u_i^{(1)}(x, y, t) + \varepsilon^2 u_i^{(2)}(x, y, t) + \dots \quad y = \frac{x}{\varepsilon} \tag{3.16}$$

and taking into account the transformation rule (2.26) for the spatial derivative operator, one gets:

$$-\varepsilon^{-1} \frac{\partial p^{(0)}}{\partial y_i} + \varepsilon^0 \left[\mu \frac{\partial}{\partial y_k} \left(\frac{\partial}{\partial y_k} v_i^{(0)} \right) - \frac{\partial p^{(0)}}{\partial x_i} - \frac{\partial p^{(1)}}{\partial y_i} \right] + \varepsilon^1 [\dots] + \dots = 0, \text{ in } V_l, \tag{3.17}$$

$$\varepsilon^{-1} \frac{\partial v_i^{(0)}}{\partial y_i} + \varepsilon^0 \left[\frac{\partial v_i^{(0)}}{\partial x_i} + \frac{\partial v_i^{(1)}}{\partial y_i} \right] + \varepsilon^1 [\dots] + \dots = 0, \text{ in } V_l, \quad (3.18)$$

$$\begin{aligned} & \varepsilon^{-2} \frac{\partial}{\partial y_i} \left\{ C_{ijkh} e_{kh}^y(u^{(0)}) \right\} \\ & + \varepsilon^{-1} \left\{ \frac{\partial}{\partial y_i} \left[C_{ijkh} e_{kh}^x(u^{(0)}) + C_{ijkh} e_{kh}^y(u^{(1)}) \right] + \frac{\partial}{\partial x_i} \left[C_{ijkh} e_{kh}^y(u^{(0)}) \right] \right\} + \\ & + \varepsilon^0 \left\{ \frac{\partial}{\partial x_i} \left[C_{ijkh} e_{kh}^x(u^{(0)}) + C_{ijkh} e_{kh}^y(u^{(1)}) \right] + \right. \\ & \quad \left. + \frac{\partial}{\partial y_i} \left[C_{ijkh} e_{kh}^x(u^{(1)}) + C_{ijkh} e_{kh}^y(u^{(2)}) \right] \right\} + \\ & + \varepsilon^1 [\dots] + \dots = 0, \quad \text{in } V_s, \end{aligned} \quad (3.19)$$

$$\begin{aligned} & \varepsilon^{-1} \left[C_{ijkh} e_{kh}^y(u^{(0)}) \right] N_i + \varepsilon^0 \left[C_{ijkh} e_{kh}^x(u^{(0)}) + C_{ijkh} e_{kh}^y(u^{(1)}) + p^{(0)} \delta_{ij} \right] N_i + \\ & + \varepsilon^1 \left[C_{ijkh} e_{kh}^x(u^{(1)}) + C_{ijkh} e_{kh}^y(u^{(2)}) + p^{(1)} \delta_{ij} - \mu^l \left(\frac{\partial v_i^{(0)}}{\partial y_j} + \frac{\partial v_j^{(0)}}{\partial y_i} \right) \right] N_i \\ & + \varepsilon^2 [\dots] + \dots = 0, \quad \text{on } \Gamma \end{aligned} \quad (3.20)$$

$$\varepsilon^0 \left[\dot{u}_i^{(0)} - v_i^{(0)} \right] + \varepsilon^1 \left[\dot{u}_i^{(1)} - v_i^{(1)} \right] + \varepsilon^2 [\dots] + \dots = 0, \text{ on } \Gamma. \quad (3.21)$$

The following notations has been above applied, i.e.:

$$e_{ij}^x(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad e_{ij}^y(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} \right). \quad (3.22)$$

Grouping the terms with the same powers of ε , we get sets of equations to be satisfied by the consecutive terms of the asymptotic expansions.

I local problem: Eq. (3.17) at ε^{-1}

$$\frac{\partial p^{(0)}}{\partial y_i} = 0, \quad \text{in } V_l. \quad (3.23)$$

The above equation together with the periodicity condition imply:

$$p^{(0)}(x, y, t) = p^{(0)}(x, t), \quad \text{in } V_l. \tag{3.24}$$

II local problem: Eq. (3.19) at ε^{-2} and Eq. (3.20) at ε^{-1} :

$$\frac{\partial}{\partial y_i} \left[C_{ijkh} e_{kh}^y \left(u^{(0)} \right) \right] = 0, \quad \text{in } V_s, \tag{3.25}$$

$$\left[C_{ijkh} e_{kh}^y \left(u^{(0)} \right) \right] N_i = 0, \quad \text{on } \Gamma. \tag{3.26}$$

The solution can be written as:

$$u^{(0)}(x, y, t) = u^{(0)}(x, t), \quad \text{in } V_s. \tag{3.27}$$

III local problem: Eq. (3.17) at ε^0 , Eq. (3.18) at ε^{-1} and Eq. (3.21) at ε^0 :

$$\mu \frac{\partial}{\partial y_j} \left(\frac{\partial}{\partial y_j} v_i^{(0)} \right) - \frac{\partial p^{(0)}}{\partial x_i} - \frac{\partial p^{(1)}}{\partial y_i} = 0, \quad \text{in } V_l, \tag{3.28}$$

$$\frac{\partial v_i^{(0)}}{\partial y_i} = 0, \quad \text{in } V_l, \tag{3.29}$$

$$\dot{u}_i^{(0)} - v_i^{(0)} = 0, \quad \text{on } \Gamma. \tag{3.30}$$

Existence and uniqueness of the above local problem is preserved by Lax-Milgram lemma. Since $p^{(0)}(x, t)$ and $u^{(0)}(x, t)$ do not depend on the y -variable, therefore:

$$v_i^{(0)}(x, y, t) - \dot{u}_i^{(0)} = -\frac{k_{ij}(y)}{\mu^l} \frac{\partial p^{(0)}}{\partial x_j}, \quad \text{in } V_l \tag{3.31}$$

$$p^{(1)}(x, y, t) = \chi_i(y) \frac{\partial p^{(0)}}{\partial x_i} + \overline{p^{(1)}(x)}, \quad \text{in } V_l, \tag{3.32}$$

where: $\overline{p^{(1)}(x)}$ is an arbitrary function independent on y -variable, $k_{ij}(y)$ and $\chi_i(y)$ are the tensorial and vectorial fields, being the solution of the following local boundary-value problem, i.e.:

$$\frac{\partial}{\partial y_l} \left(\frac{\partial k_{ij}(y)}{\partial y_l} \right) + \delta_{ij} - \frac{\partial \chi_j}{\partial y_i} = 0, \quad \text{in } V_l, \tag{3.33}$$

$$\frac{\partial k_{ij}(y)}{\partial y_i} = 0, \quad \text{in } V_l, \quad (3.34)$$

$$k_{ij}(y) = 0, \quad \text{on } \Gamma. \quad (3.35)$$

Volume averaging of the equation (3.31) results in the well-known Darcy law, i.e.:

$$V_i - n\dot{u}_i^{(0)}(x, t) = -K_{ij} \frac{\partial p^{(0)}(x, t)}{\partial x_j}, \quad \text{in } \Omega, \quad (3.36)$$

where:

n - porosity of the medium considered,

K_{ij} - the filtration tensor,

V_i - component of a filtration velocity defined as:

$$V_i = \frac{1}{\|V\|} \int_{V_i} v_i^{(0)} dV, \quad (3.37)$$

$$K_{ij} = \frac{1}{\|V\|} \int_{V_i} k_{ij}(y) dV. \quad (3.38)$$

IV local problem: Eq. (3.19) at ε^{-1} , Eq. (3.20) at ε^0 (the result (3.27) has been also taken into account):

$$\frac{\partial}{\partial y_i} \left[C_{ijkh} e_{kh}^x(u^{(0)}) + C_{ijkh} e_{kh}^y(u^{(1)}) \right] = 0, \quad \text{in } V_s, \quad (3.39)$$

$$\left[C_{ijkh} e_{kh}^x(u^{(0)}) + C_{ijkh} e_{kh}^y(u^{(1)}) + p^{(0)} \delta_{ij} \right] N_i = 0, \quad \text{on } \Gamma. \quad (3.40)$$

Similarly as in the *II local problem*, the above boundary-value problem is a classical problem of the elasticity theory. The fields $u^{(0)}(x, t)$ and $p^{(0)}(x, t)$ as y -independent are treated as known, whereas the field $u^{(1)}(x, y, t)$ is a solution we are looking for. The above set is linear, hence:

$$u_i^{(1)} = \xi_i^{jk}(y) e_{jk}^x(u^{(0)}) + \eta_i(y) p^{(0)} + \overline{u^{(1)}}(x), \quad \text{in } V_s, \quad (3.41)$$

where:

$\xi_i^{jk}(y)$ -component of a displacement vector corresponding to $e_{jk}^x(u^{(0)}) = 1$ and $p^{(0)} = 0$,

$\eta_i(y)$ -component of a displacement vector corresponding to $p^{(0)} = 1$ and $e_{jk}^x(u^{(0)}) = 0$,

$\overline{u^{(1)}}(x)$ - an arbitrary function of x -argument.

V local problem: Eq. (3.19) at ε^0 , Eq. (3.20) at ε^1 :

$$\frac{\partial}{\partial x_i} \left[C_{ijkh} e_{kh}^x(u^{(0)}) + C_{ijkh} e_{kh}^y(u^{(1)}) \right] + \frac{\partial}{\partial y_i} \left[C_{ijkh} e_{kh}^x(u^{(1)}) + C_{ijkh} e_{kh}^y(u^{(2)}) \right] = 0, \quad \text{in } V_s \tag{3.42}$$

$$\left[C_{ijkh} e_{kh}^x(u^{(1)}) + C_{ijkh} e_{kh}^y(u^{(2)}) + p^{(1)} \delta_{ij} - \mu^l \left(\frac{\partial v_i^{(0)}}{\partial y_j} + \frac{\partial v_j^{(0)}}{\partial y_i} \right) \right] N_i = 0, \quad \text{on } \Gamma. \tag{3.43}$$

Volume average of the equation (3.42), after some transformations, results in:

$$\frac{\partial}{\partial x_i} \left[\left\langle h(y) \left(C_{ijkh} e_{kh}^x(u^{(0)}) + C_{ijkh} e_{kh}^y(u^{(1)}) \right) - (1 - h(y)) p \delta_{ij} \right\rangle \right] = 0, \quad \text{in } \Omega. \tag{3.44}$$

Defining a total stress tensor for the two-phase medium considered as:

$$\sigma_{ij}^T = h(y) \sigma_{ij}^s - (1 - h(y)) p^{(0)} \delta_{ij}, \quad \text{where } h(y) = \begin{cases} 1 & \text{if } y \in V_s \\ 0 & \text{if } y \in V_l \end{cases}, \quad \text{in } V, \tag{3.45}$$

one gets a macroscopic balance equation:

$$\frac{\partial \langle \sigma_{ij}^T \rangle}{\partial x_i} = 0, \quad \text{in } \Omega. \tag{3.46}$$

as well as the macroscopic constitutive equations for the two-phase medium considered, i.e.:

$$\langle \sigma_{ij}^T \rangle = C_{ijkh}^{\text{eff}} e_{kh}^x (u^{(0)}) - \alpha_{ij} p^{(0)}, \quad \text{in } \Omega, \quad (3.47)$$

where the effective parameters are defined by the following relations:

$$C_{ijkh}^{\text{eff}} = \langle h(y) (C_{ijkh} + C_{ijlm} e_{lm}^y (\xi^{kh})) \rangle, \quad \text{in } \Omega, \quad (3.48)$$

$$\alpha_{ij} = n\delta_{ij} - \langle h(y) C_{ijkh} e_{kh}^y (\eta) \rangle, \quad \text{in } \Omega. \quad (3.49)$$

The symbol $\langle \rangle$ denotes the volume average, i.e.:

$$\langle \varphi \rangle = \frac{1}{\|V\|} \int_V \varphi dV. \quad (3.50)$$

VI local problem: Eq. (3.18) at ε^0 and Eq.(3.21) at ε^1 :

$$\frac{\partial v_i^{(0)}}{\partial x_i} + \frac{\partial v_i^{(1)}}{\partial y_i} = 0, \quad \text{in } V_i, \quad (3.51)$$

$$\dot{u}_i^{(1)} - v_i^{(1)} = 0, \quad \text{on } \Gamma. \quad (3.52)$$

Finally, after the averaging process, the above system results in the macroscopic mass conservation law for an incompressible liquid, i.e.:

$$\frac{\partial (V_i - n\dot{u}_i^{(0)})}{\partial x_i} = -\gamma_{ij} e_{ij}^x (u^{(0)}) + \beta \dot{p}^{(0)}, \quad \text{in } \Omega \quad (3.53)$$

where the effective properties are given by the following relations:

$$\gamma_{ij} = n\delta_{ij} - \langle h(y)e_{kh}^y (\xi^{ij}) \rangle, \quad \beta = \langle h(y)e_{ii}^y (\eta) \rangle, \quad \text{in } \Omega. \quad (3.54)$$

3.1.3. The macroscopic description

The governing set of the macroscopic description consists of:

- **filtration law**

$$V_i - n\dot{u}_i^{(0)}(x, t) = -K_{ij} \frac{\partial p^{(0)}(x, t)}{\partial x_j}, \quad \text{in } \Omega, \quad (3.55)$$

- **mass conservation law for a filtrating liquid:**

$$\frac{\partial (V_i - n\dot{u}_i^{(0)})}{\partial x_i} = -\alpha_{ij} \dot{e}_{ij}^x(u^{(0)}) + \beta \dot{p}^{(0)}, \quad \text{in } \Omega, \quad (3.56)$$

- **balance equations for a two-phase medium:**

$$\frac{\partial \langle \sigma_{ij}^T \rangle}{\partial x_i} = 0, \quad \text{in } \Omega, \quad (3.57)$$

- **constitutive equations:**

$$\langle \sigma_{ij}^T \rangle = C_{ijkh}^{\text{eff}} e_{kh}^x(u^{(0)}) - \alpha_{ij} p^{(0)}, \quad \text{in } \Omega. \quad (3.58)$$

The governing set of equations obtained above is the same as that of the Biot's poroelasticity theory. It has to be marked however that the Darcy law was not assumed but obtained from the equations describing the flow of Newtonian liquid through a network of pores.

3.2. Study of poroelasticity material coefficients as response of microstructure

In order to study the relations between microstructure and the values of macroscopic poroelastic coefficients, the properties of the solutions $\xi^{kh}(y)$ and $\eta(y)$ have to be examined.

The fields $\xi^{kh}(y)$ and $\eta(y)$ are the solutions of the following boundary value problems stated for the base cell, respectively, i.e.:

$$\begin{aligned} \frac{\partial}{\partial y_i} \left[C_{ijkh} + C_{ijlm} e_{lm}^y(\xi^{kh}) \right] &= 0, \quad \text{in } V_s, \\ C_{ijlm} e_{lm}^y(\xi^{kh}) N_i &= -C_{ijkh} N_i, \quad \text{on } \Gamma \end{aligned} \quad (3.59)$$

$$\begin{aligned} \frac{\partial}{\partial y_i} \left[C_{ijlm} e_{lm}^y(\eta) \right] &= 0, \quad \text{in } V_s, \\ C_{ijlm} e_{lm}^y(\eta) N_i &= -\delta_{ij} N_i, \quad \text{on } \Gamma. \end{aligned} \quad (3.60)$$

In a following, a variational formulation of the above boundary-value problems will be used in order to examine the solutions considered.

Let $(H_{\#}^1(V)/\mathbb{R})^3$ to be a Hilbert space of vector-valued functions having square-integrable derivatives, defined by

$$(H_{\#}^1(V)/\mathbb{R})^3 = \left\{ v, v \in [H^1(V)]^3; v_i - Y\text{periodic} \right\}$$

Then, the corresponding variational formulations of the systems (3.59) and (3.60) read:

• **boundary-value problem (3.59)**

find $\xi^{kh}(y) \in (H_{\#}^1(V)/\mathbb{R})^3$ so that:

$$\forall w \in (H_{\#}^1(V)/\mathbb{R})^3, \quad B(\xi^{kh}, w) = - \int_{V_s} C_{ijkh} e_{ij}^y(w) dV, \quad (3.61)$$

• **boundary-value problem (3.60)**

find $\eta(y) \in (H_{\#}^1(V)/\mathbb{R})^3$ so that:

$$\forall w \in (H_{\#}^1(V)/\mathbb{R})^3, \quad B(\eta, w) = - \int_{V_s} e_{ii}^y(w) dV. \quad (3.62)$$

The bilinear form is defined by:

$$B(v, w) = \int_{V_s} C_{ijkh} e_{ij}^y(v) e_{kh}^y(w) dV. \quad (3.63)$$

It is worth noting that due to the ellipticity and the symmetry of the elastic stiffness tensor of the skeleton material, the existence and uniqueness of the solutions of the variational formulations (3.61) and (3.62) are proved, as well as the symmetry of the bilinear form (3.63), i.e. $B(v, w) = B(w, v)$.

Before proceeding with the further analysis and in order to make it tractable, some simplifying assumptions regarding the microstructural properties of porous medium are now introduced. In fact, the main interest of this work is to verify the role of pore geometry, i.e. the shape, the size

and the arrangement. So, it is assumed that the skeleton material is isotropic and homogeneous at the microscopic (grain) level, in the following. This means that we are focusing on the influences of the pore geometry and the average values of the elastic constants of the skeleton material only. However, the assumption of the micro-isotropy does not imply a micro-isotropy of the porous material. On the contrary, the porous material will exhibit a structural anisotropy due to directional arrangement of pores and fissures.

According to the above assumption one can write:

$$C_{ijkl} = \frac{E}{2 - \nu} \left(\frac{\nu}{1 - 2\nu} \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right), \quad (3.64)$$

where: E and ν are the Young's modulus and the Poisson's ratio, respectively. Substituting the equation (3.64) into the variational formulation (3.61) leads to the equivalent variational formulation for the solution $\xi_i^{kh}(y)$

$$\int_{V_s} a_{ijlm} e_{ij}^y(\xi^{kh}) e_{lm}^y(w) dV = - \int_{V_s} a_{ijkh} e_{ij}^y(w) dV, \quad (3.65)$$

where:

$$a_{ijkh} = \frac{\nu}{1 - 2\nu} \delta_{ij} \delta_{kh} + \delta_{ik} \delta_{jh} + \delta_{ih} \delta_{jk}. \quad (3.66)$$

It is obvious that (3.65) and (3.66) imply an independence of the solution $\xi^{kh}(y)$ on values of the Young's modulus. Furthermore, the following statements can be formulated, i.e.:

$$\xi_i^{kh}(y) = f_i^{kh}(y, \nu, \text{microstructure geometry}), \quad (3.67)$$

as well as ([14]):

$$\alpha_{ij} = n \delta_{ij} - g_{ij}(\nu, \text{microstructure geometry}). \quad (3.68)$$

The β - coefficient can be determined through the components of the α_{ij} - tensor and the values of E and ν . As a direct consequence of (3.61), the following relation can be written:

$$B(\xi^{kh} \delta_{kh}, \eta) = - \int_{V_s} C_{ijkl} \delta_{kl} e_{ij}^y(\eta) dV. \quad (3.69)$$

Since:

$$C_{ijkh} \delta_{kh} = \frac{E}{1-2\nu} \delta_{ij}, \quad (3.70)$$

therefore, after some manipulations one gets:

$$\beta = \frac{n - \frac{1}{3} \alpha_{ij} \delta_{ij}}{\frac{E}{3(1-2\nu)}}. \quad (3.71)$$

The denominator of the equation (3.71) represents the bulk modulus of the skeleton material and it is denoted as K_s .

The specific role of structural parameters will be detailed through numerical simulations performed for some typical pores geometries. Before that, however, bounds on values of macroscopic poroelastic constants will be presented, i.e. (details are in Lydzba&Shao, [14]):

$$n \leq \frac{1}{3} \alpha_{ij} \delta_{ij} \leq 1. \quad (3.72)$$

$$\frac{n-1}{K_s} \leq \beta \leq 0, \quad (3.73)$$

where:

$$K_s = \frac{E}{3(1-2\nu)}. \quad (3.74)$$

NUMERICAL EVALUATION OF POROELASTIC CONSTANTS

In order to precise influences of some relevant structural parameters on the values of macroscopic poroelastic coefficients, numerical calculations have been performed for some simplified pore geometries (Fig. 10).

The geometries I and II (Fig. 10) can be treated, in some sense, as an idealization of rock like porous materials whereas the geometry III - as some simplification of the granular medium microstructure.

Influence of Poisson's ratio

The first calculation concerns an estimation of the Poisson's ratio influence on the values of α_{ij} and β . The numerical simulation has been performed for the porosity value equated to $n = 0.102$ and $m = a_1/a_2 = 2$ (Fig. 10).

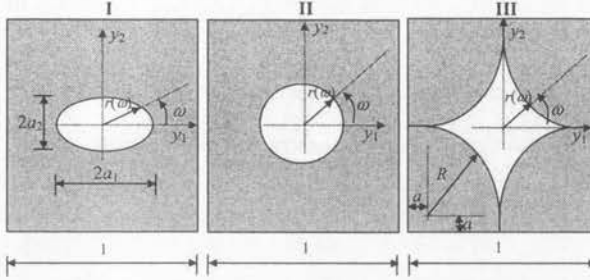


Figure 10: Simplified geometries of the base cell taken for the calculation $m = a_1/a_2$

The results are presented in Figures (11) -(14).

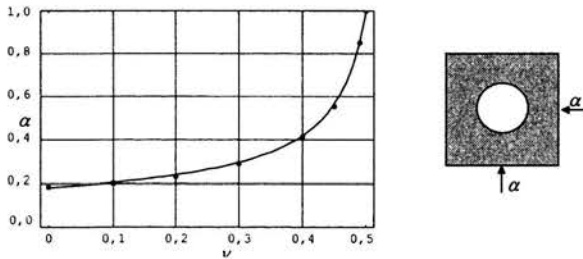


Figure 11: Values of the coefficient α versus Poisson's ratio

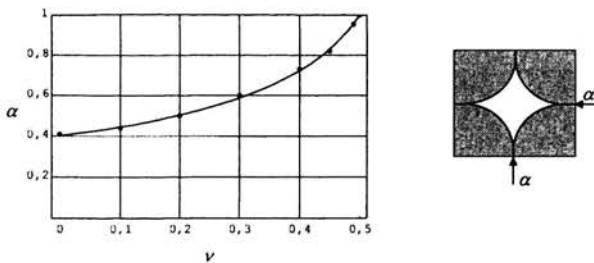


Figure 12: Values of the coefficient α versus Poisson's ratio

As expected, the geometry I exhibiting a structural anisotropy gives a clear difference between the two principal values of Biot's coefficient tensor

α_{11} and α_{22} . For ellipse-like pores the highest value of Biot's coefficients is obtained in the direction normal to the largest axis of the elliptic pore. With an increase of the Poisson's ratio this feature becomes less marked and disappears at $\nu = 0,5$. This is in a full agreement with the well-known property of the tensor α_{ij} that it becomes the unity tensor in the case of an incompressible skeleton material.

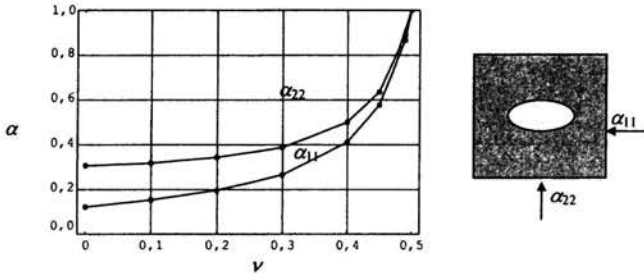


Figure 13: Values of the coefficient α versus Poisson's ratio

The influence of the microstructure geometry on values of the coefficient β is presented in the Figure 14.

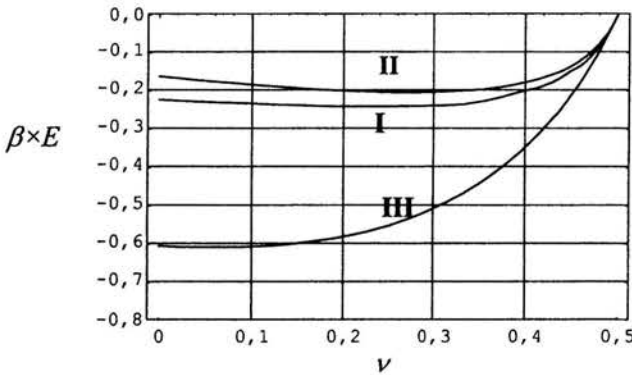


Figure 14: Dependence of the coefficient β on the value of Poisson's ratio of skeleton material for three microstructures considered; E - Young's modulus of the skeleton material

Influence of microstructure's geometry

The influence of the porosity variation is now analyzed. This time the value of the Poisson's ratio is kept constant and a moderate value $\nu = 0.2$ is used. At first, the geometries II and III have been considered. These

geometries imply the macro-isotropy. The results obtained for the periodic medium can be therefore compared with the corresponding bounds for the statistically homogeneous and isotropic stochastic medium. The well-known Hashin-Shtrikman bound for the two phase medium has been used, i.e.:

$$0 \leq K_o \leq K_s + \frac{n}{\frac{1}{K_s} + \frac{1-n}{K_s + G}} \tag{3.75}$$

For the plane problem, the value $\nu = 0,2$ implies $G = \frac{3}{4}K_s$ which results in:

$$\frac{K_o}{K_s} \leq \frac{3(1-n)}{3+4n} \tag{3.76}$$

The appropriate bound for the scalar Biot's coefficient α are determined from the well-known relation, i.e.:

$$\alpha = 1 - \frac{K_o}{K_s} \tag{3.77}$$

which together with (3.76) lead to:

$$1 \geq \alpha \geq 1 - \frac{3(1-n)}{3+4n} \tag{3.78}$$

The numerical results obtained for the geometries II and III are compared with the Hashin-Shtrikman bounds in the Fig. 15. Figure 15 shows that

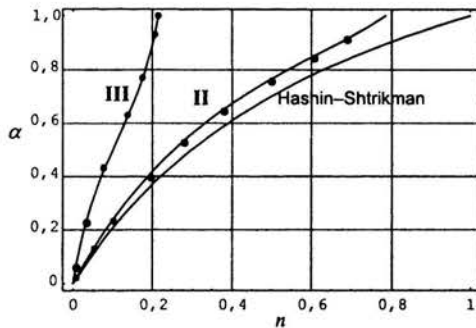


Figure 15: Dependence of the coefficient α on porosity ($\nu = 0.2$)

porosity increase leads to the increase of the Biot's effective stress coefficient. Furthermore, for an arbitrary shape of pores a critical porosity value exists

at which the value of the Biot's coefficient reaches the unity. For the geometry III, being an idealization of a granular medium, this critical value is a lit bit greater than 20% - the porosity value typical for real soils. It means that even for porous media involving a compressible skeleton material, the Terzaghi's effective stress concept can be recovered for a specific microstructural geometry. An influence of the porosity variation on the value of the effective bulk modulus is presented in the figure 16. The figure 17 presents the evolution of the coefficient β with the porosity variation.

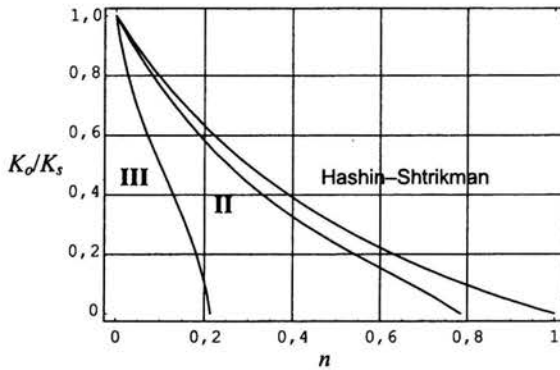


Figure 16: The bulk modulus value as a function of the porosity ($\nu = 0.2$)

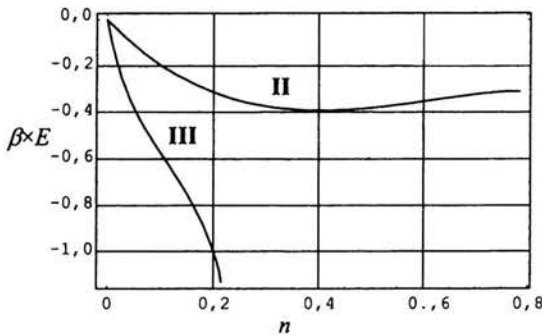


Figure 17: Value of the coefficient β versus porosity: E - Young's modulus of the skeleton material

The results of numerical simulation performed for the geometry I are

presented in the figures (18)-(23). The parametric analysis has been extended on the parameter m (Fig. 10) in order to verify the influence of the pore shape on the values of the parameters studied.

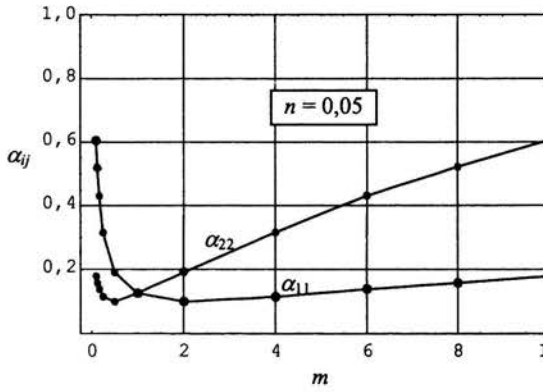


Figure 18: Values of the tensor α_{ij} versus parameter m ($n = 0.05$)

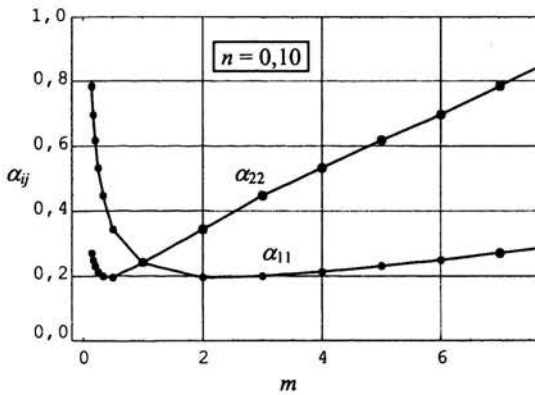


Figure 19: Values of the tensor α_{ij} versus parameter m ($n = 0.10$)

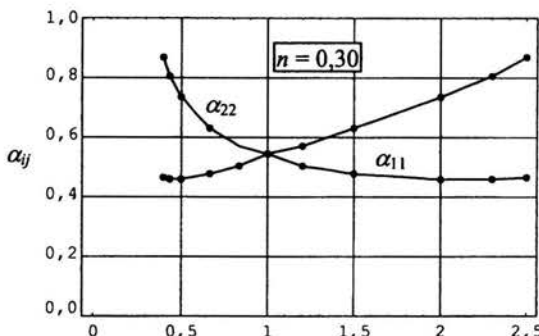


Figure 20: Values of the tensor α_{ij} versus parameter m ($n = 0.30$)

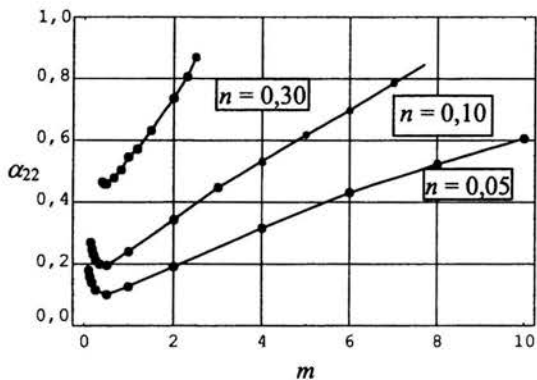


Figure 21: Values of component α_{22} versus parameter m for different assumed values of n

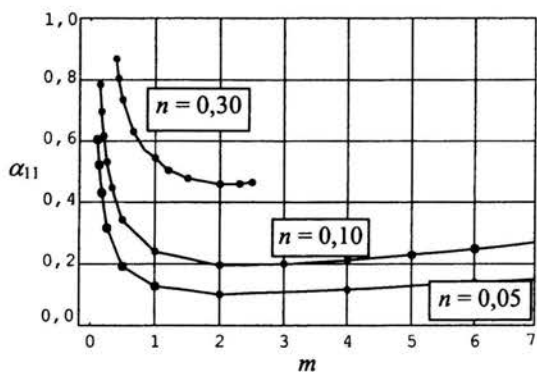


Figure 22: Values of component α_{11} versus parameter m for different assumed values of n

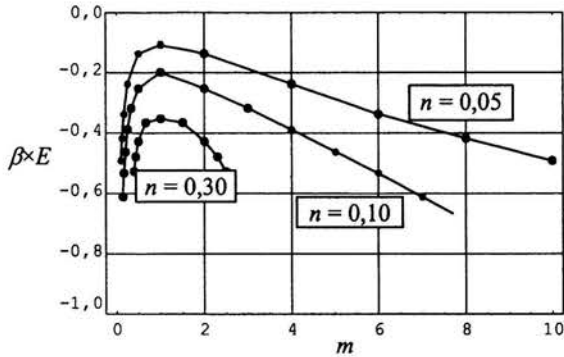


Figure 23: Values of the β - coefficient versus parameter m for different assumed values of n

3.3. Conclusions

The Biot's poroelasticity theory has been recovered from the description valid at the pore level. At this level it has been assumed that the liquid flow was governed by the Navier-Stokes equations for an incompressible Newtonian fluid and the skeleton was linearly elastic. The macroscopic description has been obtained through the passage from micro-to macro-scale.

The use of the asymptotic homogenization method allowed also for a more in-depth interpretation of the physical processes occurring in the saturated porous material. Particularly, the macroscopic filtration law, the so-called Darcy law, has been obtained and not assumed. The relations between the microstructure of saturated porous medium and the poroelastic parameters have been established. The numerical simulation performed for the simplified microstructures quantified the influence of the porous medium porosity on the values of the poroelastic material coefficients.

4. Phenomena of sorption and sorption swelling

Sorption is often the consequence of the contact of two volumetric phases. It consists in sorption of one phase by the other. In the case of the contact between a solid body and gas, the character of this phenomenon depends on the structure and surface type of the solid body. Two extreme cases can be distinguished (Zarzycki & Chacuk [24]):

- A solid body is characterized by a flat, nonporous surface. The process takes place on the surface of the solid phase and it consists in the formation of autonomous sorption phase. This phenomenon is called as the adsorption;
- Micro-porous structure with the pore sizes comparable to the gas molecule diameters is characteristic of a solid body. Gas sorption occurs in the whole volume of the solid body and the process is called as the absorption.

Desorption is a reverse process of sorption. It consists in the release of the gas sorbed by the solid body.

The our study is limited to a system of bituminous coal and carbon dioxide. Heterogeneous and complex porous structure of bituminous coal brings about that the sorption process taking place in the porous skeleton-gas system is of a very complex character. It is possible to distinguish three ranges of pores in the fabric of bituminous coal, which determine different mechanisms of gas storing (Lason, [12]):

- the area of network sorption: the radii of capillaries range from 0.3 to 0.5 nm and the absorption which occurs inside them resembles a phenomenon of dissolution;
- the area of micro-pores: the capillaries with radii up to 1.5 nm, in which the phenomenon of filling up to volume is observed;
- the area of meso- and macro-pores which comprise capillaries of greater radii.

Such complicated porous structure of bituminous coal results in various mass transport processes which take place simultaneously in the medium. Therefore, the internal geometry of porous medium considered is simplified in the following. It is assumed that the medium is composed of a solid part and the network of macro-pores filled by the gas in a free state. The solid part comprises, however, the porous matrix with the capillaries of three smaller dimensions where the gas occurs as the constrained phase. Furthermore, at the pore level, it is assumed that:

- flow of the gas in the macropores is described by the Navier-Stokes equations of a barotropic liquid;
- transport process of the constrained gas is governed by the Fick molecular diffusion law;
- mechanical behavior of the microporous skeleton with the constrained gas can be described by the equations of the uncoupled diffuso-elasticity theory (Nowacki ,[17]).

4.1. Homogenization process

4.1.1. Local description

With the above assumptions, the micro-scale description is given by:

1. Microporous skeleton with constrained gas:

- balance equations:

$$\sigma_{ij,i}^s = 0, \quad \text{in } V_s, \tag{4.1}$$

- constitutive equations of uncoupled diffuso-elasticity theory

$$\sigma_{ij}^s = a_{ijkl} e_{kl}(u) - \gamma_c C \delta_{ij}, \quad \text{in } V_s, \tag{4.2}$$

- mass conservation equation for the constrained gas:

$$\frac{\partial C}{\partial t} - DC_{,ii} = 0, \quad \text{in } V_s. \tag{4.3}$$

2. free gas in macropores:

- balance equations:

$$\sigma_{ij,i}^l = 0, \quad \text{in } V_l, \tag{4.4}$$

- constitutive equations of barotropic Newtonian liquid:

$$\sigma_{ij}^l = -p\delta_{ij} + \lambda v_{k,k} \delta_{ij} + \mu (v_{i,j} + v_{j,i}), \quad \text{in } V_l, \tag{4.5}$$

- mass conservation equation:

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0, \quad \text{in } V_l, \quad (4.6)$$

- the ideal gas law for isothermic processes:

$$\rho = \frac{\rho_a}{p_a} p, \quad \text{in } V_l, \quad (4.7)$$

where:

- v_i - component of free gas velocity vector,
- p - free gas pressure,
- ρ - free gas density,
- C - concentration of constrained gas,
- u_i - component of skeleton displacement vector,
- $\sigma_{ij}^s, \sigma_{ij}^l$ - stress tensor components for skeleton and free gas, respectively,
- a_{ijkh} - elasticity tensor component of solid skeleton,
- D - diffusion coefficient,
- γ_c - coefficient of sorption expansion,
- λ and μ - viscosity's coefficients,
- ρ_a - gas density at the atmospheric pressure,
- p_a - value of the atmospheric pressure,
- V_s - volume occupied by the microporous skeleton,
- V_l - volume occupied by the macropores.

The above system of equations is completed by the boundary conditions on the phase separation surface Γ , i.e.:

- continuity of the mass flux:

$$\left[\rho(v_i - \dot{u}_i) + DC_{,i} \right] N_i = 0, \quad \text{on } \Gamma, \quad (4.8)$$

- sorption isotherm equation:

$$C = F(p), \quad \text{on } \Gamma, \quad (4.9)$$

- an adhesion condition:

$$(v_i - \dot{u}_i) t_i = 0, \quad \text{on } \Gamma, \quad (4.10)$$

- continuity of the stress vector:

$$\left[\sigma_{ij}^s - \sigma_{ij}^l \right] N_i = 0, \quad \text{on } \Gamma. \tag{4.11}$$

Here N_i and t_i are the components of normal and tangent vector to the common surface Γ , respectively. In addition, the thermodynamic equilibrium between the phases is assumed at the initial instant.

4.1.2. Normalization

According to the homogenization principle, the macroscopic description is obtained by letting the scale parameter tend to zero. This implies, in the case one considers the flow process, necessity of normalization of the microscopic description. For the Newtonian liquid, as has been pointed out in the previous chapter, the normalization is equivalent to the following substitution: $\mu \rightarrow \varepsilon^2 \mu$ and $\lambda \rightarrow \varepsilon^2 \lambda$ (see Eq. (3.8)).

In order to normalize the diffusion process one has to distinguish two different cases, i.e.:

- case 1^o: microstructure of the porous medium considered and its diffusion property induce the same variation rate of sorbed gas concentration as the variation rate of free gas pressure;
- case 2^o: microstructure of the porous medium considered and its diffusion property bring about that the diffusion process is much slower than the filtration one.

The two cases distinguished above clearly show that the system: bituminous coal-carbon dioxide is characterized by two characteristic times, T_f and T_D , of the fluid filtration and fluid diffusion, respectively. The first case distinguished above implies $T_D/T_f = O(1)$ whereas the second one - $T_D/T_f = O(\varepsilon^{-1})$. During the homogenization process, the base cell of a medium considered is subjected to a homothetic transformation. This operation can not, of course, affect the proportion between the characteristic times distinguished.

Using the Fick law, the characteristic diffusion time can be estimated as:

$$V_D^\varepsilon C = O\left(D \frac{C}{\varepsilon l}\right) \Rightarrow T_D^\varepsilon = O\left(\frac{\varepsilon l}{V_D^\varepsilon}\right) = O\left(\frac{\varepsilon^2 l^2}{D}\right) \tag{4.12}$$

The characteristic filtration time is given by:

$$T_f^\varepsilon = O\left(\frac{\varepsilon l}{V_f^\varepsilon}\right), \quad (4.13)$$

where: l is a characteristic length of the base cell before its homothetic transformation, V_D^ε and V_f^ε denote mean values of diffusion and filtration velocities corresponding to the actual length of the base cell equated to εl , respectively.

Using Eq. (4.12) and Eq. (4.13), one gets:

$$\frac{T_D^\varepsilon}{T_f^\varepsilon} = O\left(V_f^\varepsilon l \frac{\varepsilon}{D}\right). \quad (4.14)$$

The consequence of the viscosities' scaling ($\mu \rightarrow \varepsilon^2 \mu$ and $\lambda \rightarrow \varepsilon^2 \lambda$) is an ε - independence of the filtration velocity V_f^ε . The constant characteristic times' proportion is preserved by scaling, similarly as viscosities, the diffusion coefficient. For the two cases distinguished above, one gets:

case 1. $\frac{T_D^\varepsilon}{T_f^\varepsilon} = O(1)$

$$\frac{T_D^\varepsilon}{T_f^\varepsilon} = O\left(V_f^\varepsilon l \frac{\varepsilon}{D}\right) = O(1) \Rightarrow D \rightarrow \varepsilon D, \quad (4.15)$$

case 2. $\frac{T_D^\varepsilon}{T_f^\varepsilon} = O(\varepsilon^{-1})$

$$\frac{T_D^\varepsilon}{T_f^\varepsilon} = O\left(V_f^\varepsilon l \frac{\varepsilon}{D}\right) = O(\varepsilon^{-1}) \Rightarrow D \rightarrow \varepsilon^2 D. \quad (4.16)$$

The above scaling, together with the viscosities' scaling, affect only the equations (4.3), (4.5) and (4.8) of the microscopic description. For the two cases distinguished they become:

case 1. $\frac{T_D^\varepsilon}{T_f^\varepsilon} = O(1)$:

$$\frac{\partial C}{\partial t} - \varepsilon D C_{,ii} = 0, \quad (4.17)$$

$$\sigma_{ij}^l = -p \delta_{ij} + \varepsilon^2 \lambda v_{k,k} \delta_{ij} + \varepsilon^2 \mu (v_{i,j} + v_{j,i}), \quad (4.18)$$

$$\left[\rho (v_i - \dot{u}_i) + \varepsilon D C_{,i} \right] N_i = 0; \quad (4.19)$$

case 2. $\frac{T_D^\varepsilon}{T_f^\varepsilon} = O(\varepsilon^{-1})$:

$$\frac{\partial C}{\partial t} - \varepsilon^2 DC_{,ii} = 0, \tag{4.20}$$

$$\sigma_{ij}^l = -p\delta_{ij} + \varepsilon^2 \lambda v_{k,k} \delta_{ij} + \varepsilon^2 \mu (v_{i,j} + v_{j,i}), \tag{4.21}$$

$$[\rho (v_i - \dot{u}_i) + \varepsilon^2 DC_{,i}] N_i = 0. \tag{4.22}$$

Finally, the microscopic description can be presented as:

- **Microporous skeleton with constrained gas:**

balance equations:

$$(a_{ijkh} e_{kh}(u) - \gamma_c C \delta_{ij})_{,i} = 0, \quad \text{in } V_s, \tag{4.23}$$

mass conservation equation for the constrained gas:

$$\frac{\partial C}{\partial t} - \varepsilon^m DC_{,ii} = 0, \quad \text{in } V_s, \tag{4.24}$$

- **free gas in macropores:**

balance equations:

$$\varepsilon^2 \mu v_{i,jj} + \varepsilon^2 (\lambda + \mu) v_{j,ji} - p_{,i} = 0, \quad \text{in } V_l, \tag{4.25}$$

mass conservation equation:

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0, \quad \text{in } V_l, \tag{4.26}$$

the ideal gas law for isothermic processes:

$$\rho = \frac{\rho_a}{p_a} p, \quad \text{in } V_l, \tag{4.27}$$

- **boundary condition on common surface Γ :**

continuity of mass flux:

$$[\rho (v_i - \dot{u}_i) + \varepsilon^m DC_{,i}] N_i = 0, \quad \text{on } \Gamma, \tag{4.28}$$

sorption isotherm equation:

$$C = F(p), \quad \text{on } \Gamma, \tag{4.29}$$

adhesion condition:

$$(v_i - \dot{u}_i) t_i = 0, \quad \text{on } \Gamma, \tag{4.30}$$

continuity of stress vector:

$$\left[a_{ijkh} e_{kh}(u) - \gamma_c C \delta_{ij} + p \delta_{ij} + \varepsilon^2 \lambda v_{k,k} \delta_{ij} + \varepsilon^2 \mu (v_{i,j} + v_{j,i}) \right] N_i = 0, \quad \text{on } \Gamma. \quad (4.31)$$

The substitution $m=1$ corresponds to the case 1., whereas $m=2$ -case 2.

4.1.3. Asymptotic developments and averaging process

Introducing the asymptotic developments for: v, p, ρ, u i C and modifying the spatial derivative operator according to (2.26) one gets:

$$\begin{aligned} & \varepsilon^{-2} \frac{\partial}{\partial y_i} \left[a_{ijkh} e_{kh}^y(u^{(0)}) \right] + \\ & + \varepsilon^{-1} \left\{ \frac{\partial}{\partial y_i} \left[a_{ijkh} e_{kh}^x(u^{(0)}) + a_{ijkh} e_{kh}^y(u^{(1)}) - \gamma_c C^{(0)} \delta_{ij} \right] + \right. \\ & \quad \left. + \frac{\partial}{\partial x_i} \left[a_{ijkh} e_{kh}^y(u^{(0)}) \right] \right\} + \\ & + \varepsilon^0 \left\{ \frac{\partial}{\partial x_i} \left[a_{ijkh} e_{kh}^x(u^{(0)}) + a_{ijkh} e_{kh}^y(u^{(1)}) - \gamma_c C^{(0)} \delta_{ij} \right] \right\} + \\ & + \varepsilon^0 \left\{ \frac{\partial}{\partial y_i} \left[a_{ijkh} e_{kh}^x(u^{(1)}) + a_{ijkh} e_{kh}^y(u^{(2)}) - \gamma_c C^{(1)} \delta_{ij} \right] \right\} + \dots = 0, \text{ in } V_s, \end{aligned} \quad (4.32)$$

$$\begin{aligned} & \varepsilon^0 \frac{\partial C^{(0)}}{\partial t} - \varepsilon^{m-2} D \frac{\partial}{\partial y_i} \left(\frac{\partial C^{(0)}}{\partial y_i} \right) + \\ & - \varepsilon^{m-1} D \left\{ \frac{\partial}{\partial y_i} \left(\frac{\partial C^{(0)}}{\partial x_i} + \frac{\partial C^{(1)}}{\partial y_i} \right) + \frac{\partial}{\partial x_i} \left(\frac{\partial C^{(0)}}{\partial y_i} \right) \right\} + \dots = 0, \quad \text{in } V_s, \end{aligned} \quad (4.33)$$

$$\begin{aligned} & -\varepsilon^{-1} \frac{\partial p^{(0)}}{\partial y_i} + \varepsilon^0 \left[-\frac{\partial p^{(0)}}{\partial x_i} - \frac{\partial p^{(1)}}{\partial y_i} + \right. \\ & \quad \left. + \mu \frac{\partial}{\partial y_j} \left(\frac{\partial v_i^{(0)}}{\partial y_j} \right) + (\lambda + \mu) \frac{\partial}{\partial y_i} \left(\frac{\partial v_j^{(0)}}{\partial y_j} \right) \right] + \dots = 0, \text{ in } V_l, \end{aligned} \quad (4.34)$$

$$\begin{aligned} & \varepsilon^{-1} \frac{\partial (\rho^{(0)} v_i^{(0)})}{\partial y_i} + \\ & + \varepsilon^0 \left[\frac{\partial \rho^{(0)}}{\partial t} + \frac{\partial (\rho^{(0)} v_i^{(0)})}{\partial x_i} + \frac{\partial (\rho^{(0)} v_i^{(1)})}{\partial y_i} + \frac{\partial (\rho^{(1)} v_i^{(0)})}{\partial y_i} \right] + \dots = 0, \quad \text{in } V_I, \end{aligned} \tag{4.35}$$

$$\varepsilon^0 \left(\rho^{(0)} - \frac{\rho_a}{p_a} p^{(0)} \right) + \varepsilon^1 \left(\rho^{(1)} - \frac{\rho_a}{p_a} p^{(1)} \right) + \dots = 0, \quad \text{in } V_I, \tag{4.36}$$

$$\begin{aligned} & \left[\varepsilon^0 \rho^{(0)} (v_i^{(0)} - \dot{u}_i^{(0)}) + \varepsilon^1 \left\{ \rho^{(0)} (v_i^{(1)} - \dot{u}_i^{(1)}) + \rho^{(1)} (v_i^{(0)} - \dot{u}_i^{(0)}) \right\} \right] N_i + \\ & + \left[\varepsilon^{m-1} D \frac{\partial C^{(0)}}{\partial y_i} + \varepsilon^m \left(D \frac{\partial C^{(0)}}{\partial x_i} + D \frac{\partial C^{(1)}}{\partial y_i} \right) \right] N_i = 0, \quad \text{on } \Gamma \end{aligned} \tag{4.37}$$

$$C^{(0)} = F(p^{(0)}), \quad \text{on } \Gamma, \tag{4.38}$$

$$\varepsilon^0 (v_i^{(0)} - \dot{u}_i^{(0)}) t_i + \varepsilon^1 (v_i^{(1)} - \dot{u}_i^{(1)}) t_i + \dots = 0, \quad \text{on } \Gamma, \tag{4.39}$$

$$\begin{aligned} & \varepsilon^{-1} \left[a_{ijkh} e_{kh}^y (u^{(0)}) \right] N_i \\ & + \varepsilon^0 \left[a_{ijkh} e_{kh}^x (u^{(0)}) + a_{ijkh} e_{kh}^y (u^{(1)}) - \gamma_c C^{(0)} \delta_{ij} + p^{(0)} \delta_{ij} \right] N_i \\ & + \varepsilon^1 \left[a_{ijkh} e_{kh}^x (u^{(1)}) + a_{ijkh} e_{kh}^y (u^{(2)}) - \gamma_c C^{(1)} \delta_{ij} + p^{(1)} \delta_{ij} \right. \\ & \quad \left. - \lambda \frac{\partial v_k^{(0)}}{\partial y_k} \delta_{ij} - \mu \left(\frac{\partial v_i^{(0)}}{\partial y_j} + \frac{\partial v_j^{(0)}}{\partial y_i} \right) \right] N_i = 0, \quad \text{on } \Gamma. \end{aligned} \tag{4.40}$$

Most of the local boundary-value problems, for the succeeding powers of ε , are similar to those considered in the previous chapter. In order to not repeat the steps presented before, the final solutions are therefore only stated.

Case 1. $\frac{T_D^\varepsilon}{T_f^\varepsilon} = O(1)$, $m = 1$

The equations: (4.32) at ε^{-2} , (4.40) at ε^{-1} , (4.33) at ε^{-1} , (4.39) at ε^{-1} , (4.37) at ε^0 and (4.38) result in:

$$\begin{aligned} u^{(0)} &= u^{(0)}(x, t), \\ C^{(0)} &= C^{(0)}(x, t) = F(p^{(0)}), \\ p^{(0)} &= p^{(0)}(x, t), \\ \rho^{(0)} &= \rho^{(0)}(x, t). \end{aligned} \quad (4.41)$$

The next boundary-value problem is given by: (4.34) at ε^0 , (4.35) at ε^{-1} , (4.37) and (4.39) at ε^0 . Using (4.41), this problem can be transformed to the same form as **III local problem** analyzed in the chapter 3. Finally, one can write:

$$\langle v_i^{(0)} \rangle - n \dot{u}_i^{(0)} = -K_{ij} \frac{\partial p^{(0)}}{\partial x_j}, \quad (4.42)$$

As the consequence of the equations: (4.32) at ε^{-1} and (4.40) at ε^0 (the system (4.41) has been used also), the following system is obtained:

$$\begin{aligned} \frac{\partial}{\partial y_i} \left[a_{ijkh} e_{kh}^x(u^{(0)}) + a_{ijkh} e_{kh}^y(u^{(1)}) - \gamma_c F(p^{(0)}) \delta_{ij} \right] &= 0, \text{ in } V_s, \\ \left[a_{ijkh} e_{kh}^x(u^{(0)}) + a_{ijkh} e_{kh}^y(u^{(1)}) - \gamma_c F(p^{(0)}) \delta_{ij} + p^{(0)} \delta_{ij} \right] N_i &= 0, \text{ on } \Gamma. \end{aligned} \quad (4.43)$$

The above system is linear, therefore:

$$u_i^{(1)}(x, y) = \xi_i^{jk}(y) e_{jk}^x(u^{(0)}) - \eta_i(y) p^{(0)} + \eta_i \gamma_c F(p^{(0)}) + \bar{u}_i(x), \quad (4.44)$$

where: $\xi_i^{jk}(y)$ and $\eta_i(y)$ denote exactly the same solutions as formulated in the chapter before. The solution (4.44) has been written with the assumption that the parameter γ_c does not depend on the microscopic space variable y . Defining, similarly as in the chapter 3, a total stress tensor for the medium considered, i.e.:

$$\sigma_{ij}^T = h(y) \sigma_{ij}^{s(0)} + (1 - h(y)) p^{(0)} \delta_{ij}, \text{ where } h(y) = \begin{cases} 1 & \text{if } y \in V_s, \\ 0 & \text{if } y \notin V_s \end{cases}, \quad (4.45)$$

one gets:

$$\frac{\partial \langle \sigma_{ij}^T \rangle}{\partial x_i} = 0, \tag{4.46}$$

$$\langle \sigma_{ij}^T \rangle = A_{ijkh} e_{kh}^x (u^{(0)}) - \alpha_{ij} p^{(0)} - \gamma_{ij} F (p^{(0)}), \tag{4.47}$$

as the consequence of the equations: (4.32) at ε^0 , (4.40) at ε^1 , (4.44) and the averaging process used. The effective parameters are defined by the following relations, i.e.:

$$\begin{aligned} A_{ijkh} &= (1 - n) a_{ijkh} + a_{ijlm} \langle e_{lm}^y (\xi^{kh}) \rangle, \\ \alpha_{ij} &= n \delta_{ij} + a_{ijkh} \langle e_{kh}^y (\eta) \rangle, \\ \gamma_{ij} &= \gamma_c (1 - n) \delta_{ij} - \gamma_c a_{ijkh} \langle e_{kh}^y (\eta) \rangle. \end{aligned} \tag{4.48}$$

The macroscopic mass conservation equation for a free gas is obtained by a direct averaging of the equation (4.35) at ε^0 and using (4.37) at ε^1 . The final form can be presented as:

$$\begin{aligned} n \frac{\partial \rho^{(0)}}{\partial t} + \frac{\partial}{\partial x_i} \left[\rho^{(0)} \left(\langle v_i^{(0)} \rangle - n \dot{u}_i^{(0)} \right) \right] + (1 - n) \frac{\partial F (p^{(0)})}{\partial t} \\ + \rho^{(0)} \left[\alpha_{ij} \dot{e}_{ij}^x (\dot{u}^{(0)}) - \beta \dot{p}^{(0)} + \gamma_c \beta \dot{F} (p^{(0)}) \right] = 0. \end{aligned} \tag{4.49}$$

The coefficient β denotes exactly the same material constant as analyzed in the chapter 3.

Case 2.: $\frac{T_D^\varepsilon}{T_f^\varepsilon} = O(\varepsilon^{-1})$, $m = 2$.

Most of the boundary-value problems are exactly the same as those for the case 1. The first difference appears in the mass conservation equation, i.e. in the equation (4.33) at ε^0 , which together with the equation (4.38) lead to:

$$\frac{\partial C^{(0)}}{\partial t} - D \frac{\partial}{\partial y_i} \left(\frac{\partial C^{(0)}}{\partial y_i} \right) = 0, \quad \text{in } V_s, \tag{4.50}$$

$$C^{(0)} = F (p^{(0)}), \quad \text{on } \Gamma. \tag{4.51}$$

To solve it, the following substitution is applied:

$$U (x, y, t) \doteq C^{(0)} (x, y, t) - F (p^{(0)}), \tag{4.52}$$

By using the Laplace transform, we obtain:

$$s \mathcal{L}(U) - \frac{\partial}{\partial y_i} \left(D \frac{\partial \mathcal{L}(U)}{\partial y_i} \right) = -\mathcal{L} \left(\frac{\partial F(p^{(0)})}{\partial t} \right), \quad \text{in } V_s, \quad (4.53)$$

$$\mathcal{L}(U) = 0, \quad \text{on } \Gamma, \quad (4.54)$$

where: s is the complex Laplace variable and

$$\mathcal{L}(g(t)) = \int_0^{\infty} g(t) e^{-st} dt. \quad (4.55)$$

The equation (4.53) is linear, so the solution can be written as:

$$\mathcal{L}(U) = -\mathcal{L}(G(y, t)) \mathcal{L} \left(\frac{\partial F(p^{(0)})}{\partial t} \right), \quad (4.56)$$

where function $G(y, t)$ is a solution of the following local problem, i.e.:

$$\begin{aligned} s \mathcal{L}(G) - \frac{\partial}{\partial y_i} \left(D \frac{\partial \mathcal{L}(G)}{\partial y_i} \right) &= 1, \quad \text{in } V_s, \\ \mathcal{L}(G) &= 0, \quad \text{on } \Gamma. \end{aligned} \quad (4.57)$$

By virtue of the convolution theorem:

$$U(x, y, t) = - \int_0^t \frac{\partial F(p^{(0)})}{\partial \tau} G(y, t - \tau) d\tau, \quad (4.58)$$

and the substitution (4.52), one can write:

$$C^{(0)}(x, y, t) = F(p^{(0)}) - \int_0^t \frac{\partial F(p^{(0)})}{\partial t} G(y, t - \tau) d\tau. \quad (4.59)$$

Averaging the above equation, the final form of the solution is obtained, i.e.:

$$\langle C^{(0)}(x, y, t) \rangle = (1 - n) F(p^{(0)}) - \int_0^t \frac{\partial F(p^{(0)})}{\partial \tau} \langle G(y, t - \tau) \rangle d\tau. \quad (4.60)$$

The equation (4.60) shows that the macroscopic value of the sorbed gas concentration depends on a history of the first time-derivative of the sorption isotherm. Function $\langle G(y, t - \tau) \rangle$ represents a memory function.

The form of the relation (4.59) modifies, with respect to the case 1, the next local problem. This problem is given by the equations: (4.32) at ε^{-1} and (4.40) at ε^0 (the equation (4.59) has been used also)

$$\frac{\partial}{\partial y_i} \left[a_{ijkh} e_{kh}^x(u^{(0)}) + a_{ijkh} e_{kh}^y(u^{(1)}) - \gamma_c \left\{ F(p^{(0)}) - \int_0^t \frac{\partial F(p^{(0)})}{\partial \tau} G(y, t - \tau) d\tau \right\} \delta_{ij} \right] = 0, \quad \text{in } V_s, \tag{4.61}$$

$$\left[a_{ijkh} e_{kh}^x(u^{(0)}) + a_{ijkh} e_{kh}^y(u^{(1)}) - \gamma_c F(p^{(0)}) \delta_{ij} + p^{(0)} \delta_{ij} \right] N_i = 0, \quad \text{on } \Gamma. \tag{4.62}$$

The solution exhibits a quite similar form as the solution (4.44), i.e

$$u_i^{(1)}(x, y) = \xi_i^{jk}(y) e_{jk}^x(u^{(0)}) - \eta_i(y) p^{(0)} + \eta_i \gamma_c F(p^{(0)}) + \gamma_c \int_0^t \frac{\partial F(p^{(0)})}{\partial \tau} w_i(y, t - \tau) d\tau + \bar{u}_i(x), \quad \text{in } V_s, \tag{4.63}$$

The additional term is a consequence of the integral in the relation (4.59). The vectorial field is a solution, in the Laplace transform space, the following local problem:

$$\frac{\partial}{\partial y_i} [a_{ijlm} e_{lm}^y(\mathcal{L}[w(y, t)]) + \mathcal{L}[G(y, t)] \delta_{ij}] = 0, \quad \text{in } V_s, \tag{4.64}$$

$$a_{ijlm} e_{lm}^y(\mathcal{L}[w(y, t)]) N_i = 0, \quad \text{on } \Gamma. \tag{4.65}$$

The macroscopic balance equations for the medium considered is obtained by the averaging process of the equations: (4.32) at ε^0 , (4.40) at ε^1 and (4.63), i.e.:

$$\frac{\partial \langle \sigma_{ij}^T \rangle}{\partial x_i} = 0, \tag{4.66}$$

$$\langle \sigma_{ij}^T \rangle = A_{ijkh} e_{kh}^x(u) - \alpha_{ij} p^{(0)} - \gamma_{ij} F(p^{(0)}) + \int_0^t \frac{\partial F(p^{(0)})}{\partial \tau} m_{ij}(t - \tau) d\tau, \tag{4.67}$$

where the memory function $m_{ij}(t)$ is defined as:

$$m_{ij}(t) = \gamma_c \langle G(t) \rangle \delta_{ij} + \gamma_c a_{ijkh} \langle e_{kh}^y(w(t)) \rangle. \quad (4.68)$$

The last macroscopic equation is a mass conservation law for a free gas in macropores. It is a result of the averaging process of the equation (4.35) at ε^0 with using the equation (4.37) at ε^1 . The following form is obtained:

$$\begin{aligned} n \frac{\partial \rho^{(0)}}{\partial t} + \frac{\partial}{\partial x_i} \left[\rho^{(0)} \left(\langle v_i^{(0)} \rangle - n \dot{u}_i^{(0)} \right) \right] + \frac{\partial \langle C^{(0)} \rangle}{\partial t} + \\ + \rho^{(0)} \left(\alpha_{ij} \dot{e}_{ij}^x(u) - \beta \dot{p} + \gamma_c \beta \dot{F}(p) + \int_0^t \frac{\partial F(p(\tau))}{\partial \tau} \beta_m(t-\tau) d\tau \right) = 0, \end{aligned} \quad (4.69)$$

where $\beta_m(t)$ is an additional memory function defined as:

$$\beta_m(t) = -\gamma_c \langle e_{ii}^y(w) \rangle \quad (4.70)$$

Analyzing the case 2, the presentation has been limited to the relations which are different than corresponding ones obtained for the case 1. The Darcy's law is, of course, valid also for the case 2.

4.1.4. Governing set of the macroscopic equations

Case 2.

- constitutive equations

$$\langle \sigma_{ij}^T \rangle = A_{ijkh} e_{khx}(u) - \alpha_{ij} p - \gamma_{ij} F(p) + \int_0^t \frac{\partial F(p(\tau))}{\partial \tau} m_{ij}(t-\tau) d\tau, \quad (4.71)$$

- balance equations:

$$\frac{\partial \langle \sigma_{ij}^T \rangle}{\partial x_i} = 0, \quad (4.72)$$

- filtration law for a free gas:

$$\langle v_i \rangle - n \dot{u}_i = -K_{ij} \frac{\partial p}{\partial x_j}, \quad (4.73)$$

- mass conservation law for a free gas:

$$\begin{aligned}
 & n \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} \left[\rho (\langle v_i \rangle - n \dot{u}_i) \right] + \frac{\partial \langle C \rangle}{\partial t} + \\
 & + \rho \left(\alpha_{ij} \dot{e}_{ijx}(u) - \beta \dot{p} + \gamma_c \beta \dot{F}(p) + \int_0^t \frac{\partial F(p)}{\partial \tau} \beta_m(t - \tau) d\tau \right) = 0,
 \end{aligned}
 \tag{4.74}$$

where:

$$\langle C \rangle = (1 - n) F(p) - \int_0^t \frac{\partial F(p(\tau))}{\partial \tau} \langle G(t - \tau) \rangle d\tau. \tag{4.75}$$

Case 1. The set (4.71)-(4.75) with $m_{ij}(t)=0$, $\beta_m(t)=0$ and $\langle G(t) \rangle = 0$.

The macroscopic description presented above shows that the gas constrained in the microporous skeleton influences the filtration process and the mechanical behavior of the system considered. Constitutive equations obtained for the case 1 can be interpreted as mixed equations of the classical consolidation theory (Biot's model) and the theory of diffusio-elasticity. Depending on physicochemical properties of the multiphase medium considered and its internal geometry, it has been shown that the medium could exhibit a strong rheological character and then its strain-stress relation demonstrates a memory effect according to the well-known Boltzmann's principle of a heredity.

4.2. Parameters identification

The identification process is based on the relations obtained from the homogenization procedure, i.e. relations (4.48), (4.68) and (4.70), as well as the laboratory data reported in Czaplinski et.al., [8], Ceglarska-Stefańska & Jagiełło, [5] and Tarnowski, [22]. The determination process has been, however, limited only to these parameters which represent the effect of a sorption process, i.e. the memory functions: $m_{ij}(t)$, $\beta_m(t)$ and $\langle G(t) \rangle$ and the material constants: γ_{ij} and γ_c . The parameters: α_{ij} and β , the material coefficients of the Biot's theory, have been carefully examined in the previous chapter. The macroscopic isotropy of the medium has been postulated. Firstly, let us examine the possible relations between the "new" material

constants and the material constants of the Biot's theory. From the equations (4.48), it is immediately seen that the following relation is fulfilled, i.e.:

$$\frac{\gamma_{ij}}{\gamma_c} + \alpha_{ij} = \delta_{ij}. \quad (4.76)$$

The macroscopic isotropy of the medium implies:

$$\frac{\gamma}{\gamma_c} + \alpha = 1, \quad (4.77)$$

where: $\gamma_{ij} = \gamma \delta_{ij}$ and $\alpha_{ij} = \alpha \delta_{ij}$.

Using the identity $a_{ijkh} \delta_{ij} = 3K_s \delta_{kh}$, it can be proved that the definitions (4.68) and (4.70) imply:

$$\beta_m(t) = \frac{\gamma_c \langle G(t) \rangle - m(t)}{K_s}, \quad (4.78)$$

where: K_s denotes a bulk modulus of the skeleton material and $m_{ij}(t) = m(t) \delta_{ij}$ (macroscopic isotropy condition).

The relation (4.76) clearly shows that the coefficient γ_{ij} is correlated with the material constant of the Biot's theory. Furthermore, the equation (4.78) indicates that model considered is characterized by only two independent memory functions, i.e. $m(t)$ and $\langle G(t) \rangle$.

Memory function $\langle G(t) \rangle$

The identification is based on the theoretical consideration performed by Lydzba & Auriault, [13]. They assumed that the coal medium was composed of spherical grains made of homogeneous and microporous medium. It appeared that such microstructure is characterized by the following memory function, being the solution of the local problem (4.57), i.e.:

$$\langle G(t) \rangle = \sum_{i=1}^{\infty} \frac{1}{i^2 \pi} e^{-D(i\pi/R)^2 t}, \quad (4.79)$$

where: R denotes the assumed radii of a spherical grain, i - a natural number.

During the laboratory tests performed by Czaplinski et.al. [8], the magnitude of carbon dioxide sorbed by the coal sample and coal sample swelling have been simultaneously measured. During this experiment the coal sample has been saturated with gas at constant pressure equated to 2 MPa. According to

the equations (4.75), the magnitude of the sorbed gas concentration is given by:

$$\langle C \rangle (t) = (1 - n) F(p) - \int_0^t \frac{\partial F(p(\tau))}{\partial \tau} \langle G(t - \tau) \rangle d\tau. \quad (4.80)$$

Supposing, that the gas penetrated the macropores immediately during this experiment, the above relation implies:

$$\frac{M_z(t)}{M_z(t \rightarrow \infty)} = 1 - \frac{\langle G(t) \rangle}{1 - n}, \quad (4.81)$$

where: $M_z(t)$ denotes a mass of sorbed gas in the sample and $M_z(t \rightarrow \infty)$ represents the so-called equilibrium value of sorbed gas.

The memory function considered has been assumed, according to the equation (4.79), as:

$$\frac{\langle G(t) \rangle}{1 - n} = g_1 e^{-\frac{t}{T_1}} + g_2 e^{-\frac{t}{T_2}} + \dots \quad (4.82)$$

where: T_1, T_2, \dots and g_1, g_2, \dots are the parameters to be determined. Verification of the model proposed against the laboratory results is presented in the Figure 24. The really satisfactory result is obtained. The following values of the constants have been determined: $g_1 = 0,3086, g_2 = 0,4735, T_1 = 2246 \text{ s}$ and $T_2 = 278 \text{ s}$.

Memory function $m(t)$

According to conditions of the experiment discussed above, the constitutive equations (4.71) imply:

$$-p = K_o \varepsilon_v(t) - \alpha p - \gamma F(p) + \int_0^t \frac{\partial F(p(\tau))}{\partial \tau} m(t - \tau) d\tau, \quad (4.83)$$

where: ε_v denotes a volumetric strain of the sample and K_o represents the effective bulk modulus of the sample.

Using the well-known relation:

$$\alpha = 1 - \frac{K_o}{K_s} \quad (4.84)$$

as well as (4.77), the equation (4.83) can be rewritten as:

$$\varepsilon_v(t) = -\frac{p}{K_s} + \frac{\gamma_c}{K_s} F(p) - \frac{1}{K_o} \int_0^t \frac{\partial F(p(\tau))}{\partial \tau} m(t-\tau) d\tau. \quad (4.85)$$

Once more supposing, that the gas penetrated the macropores immediately,

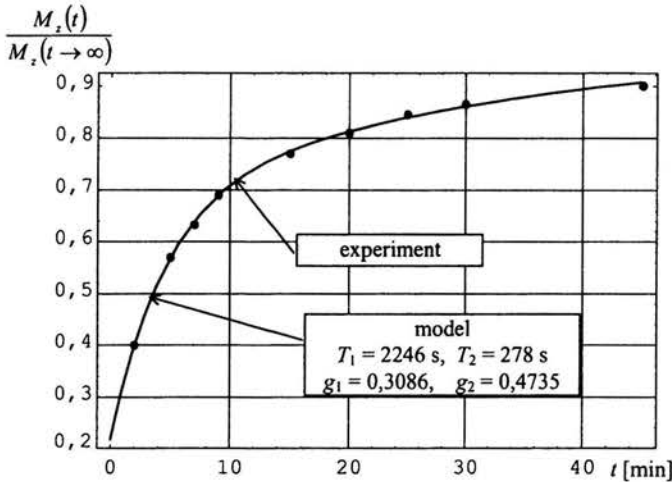


Figure 24: Relative value of a gas being sorbed versus time. Free gas pressure $p = 2$ MPa.

the above equation implies:

$$\varepsilon_v(t) = -\frac{p}{K_s} + \frac{\gamma_c}{K_s} F(p) \left[1 - \frac{K_s}{\gamma_c K_o} m(t) \right]. \quad (4.86)$$

Denoting by ρ_z - specific density, without any sorbed gas, of microporous skeleton and by M_s - mass of coal sample also without any sorbed gas, the equation (4.86) can be expressed as:

$$\varepsilon_v(t) = -\frac{p}{K_s} + \lambda \frac{M_z(t \rightarrow \infty)}{M_s} \left[1 - \frac{K_s}{\gamma_c K_o} m(t) \right], \quad (4.87)$$

where:

$$\lambda = \frac{\gamma_c \rho_z}{K_s}, \quad (4.88)$$

can be interpreted as an effective extension coefficient due to sorption process, whereas $M_z(t \rightarrow \infty)/M_s$ as an equilibrium value of sorbed gas in unit mass

of the skeleton. From the experimental data it has been determined that $M_z(t \rightarrow \infty)/M_s = 0,0589$.

Assuming, similarly as for the memory function $\langle G(t) \rangle$, the approximation to be valid:

$$\frac{K_s}{\gamma_c K_o} m(t) = m_1 e^{-\frac{t}{T_1}} + m_2 e^{-\frac{t}{T_2}}, \tag{4.89}$$

the following values have been estimated, i.e.: $\lambda = 0,2136$, $m_1 = 0,0803$ and $m_2 = 0,7978$. The values of T_1 and T_2 have been kept as constants. Furthermore, a typical value of the skeleton bulk modulus equated to $K_s = 8000$ MPa has been assumed.

Figure 25 presents verification of the function proposed against the laboratory data. Once more the result is really satisfactory.

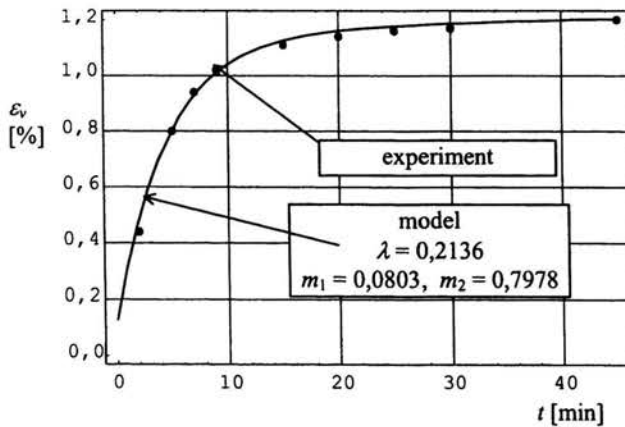


Figure 25: Sorption swelling versus time. Free gas pressure $p = 2$ MPa

Sorption isotherm $F(p)$ The most used sorption isotherm equations are:

- Langmuir's equation:

$$\frac{M_z}{M_s} = \frac{B_1 p / p_a}{B_2 + p / p_a}, \tag{4.90}$$

- Freundlich's equation

$$\frac{M_z}{M_s} = k \left(\frac{p}{p_a} \right)^{1/n_1}, \tag{4.91}$$

where $F(p) = \rho_z M_z / M_s$.

Verification of the sorption isotherms described by (4.90) and (4.91) against the laboratory data is plotted in the Figure 26. It is obvious that the Langmuir's equation is a proper isotherm for the system: bituminous coal-carbon dioxide. Determination of the sorption isotherm equation ends the identification process. Concluding, it has to be marked that the model proposed very well verifies the experimental data.

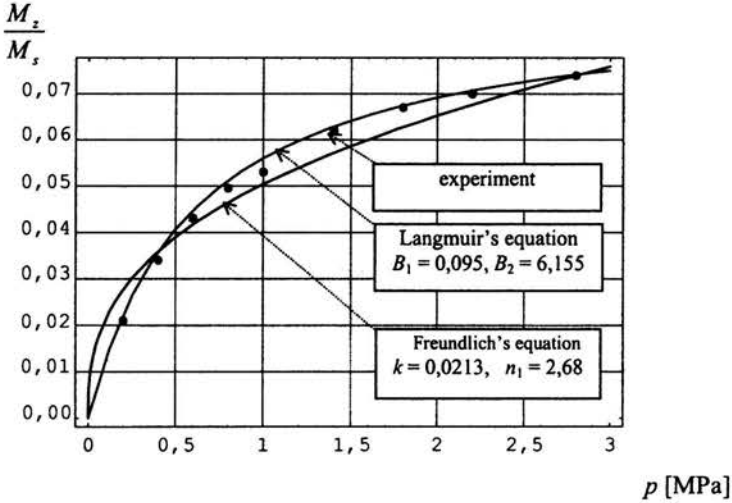


Figure 26: Sorption isotherm

5. Stress and strain equivalence for saturated porous media

Mathematical modeling of saturated porous media is usually simplified using the effective stress concept. This concept provides a possibility to extend the constitutive equations and complementary plastic laws of dry material to saturated media by using the strain and stress equivalence principles (R. de Boer & P.V. Lade, [7]). These principles can be formulated as, [17]:

Strain Equivalence: the overall stress-strain relation for the saturated porous medium is obtained from the corresponding relation of the dry porous medium by substituting the effective stress tensor for the applied nominal stress; **Stress Equivalence:** the overall yield function for the saturated porous medium is obtained from the overall yield function of the dry porous medium by substituting the effective stress tensor for the applied nominal stress.

The validity of the strain equivalence principle in the elastic domain has been confirmed from theoretical point of view in the chapter 3. The validity of the effective stress concept in the inelastic behavior range is still an open problem, particularly for cohesive materials like rocks and concrete. Recently Coussy has proposed a so-called plastic effective stress tensor for the modeling of porous media inelastic behavior, [4]. The assumption used by Coussy in [4] needs, however, to be checked. More strict results, based on a homogenization approach have been presented in [9, 10, 14]. It has been shown, that the failure criterion of a saturated porous medium can be expressed by that for dry material replacing the usual stress by the effective one. The appropriate form of the effective stress depends on the kind of failure criterion used at the micro-level.

The chapter presents a generalization of these works to the full plastic-hardening domain. The results have been obtained by the author jointly with prof. J.F. Shao and published in [16]. During the analysis, the porous medium is assumed to be composed of a micro-homogeneous and micro-isotropic

skeleton material. It is also assumed that the macroscopic behavior of the dry material is entirely determined. This lets us to proceed with the so-called comparison analysis which consists in a comparison of appropriately stated local problems for dry and saturated material, respectively. Any prerequisite assumptions concerning pores geometry and arrangement are not needed.

5.1. Stress localization law

For a clearness of description, the simplified notations are adopted in the following, i.e.: the macro-strain tensor, previously represented by $e_{ij}^x(u^{(0)})$, is now denoted by E_{ij} whereas the micro-strain tensor, formerly $e_{ij}^y(u)$, as $e_{ij}(u)$. In addition, the term $p^{(0)}$ is simply denoted by p whereas the second term of the asymptotic development of the displacement field u , i.e. $u^{(1)}$, is now distinguished by a symbol $u_i^*(y)$. The local elasto-plastic problem, stated for a unit cell, is concluded from the IV local problem presented in the chapter 3 (Eqs. (3.39)-(3.40)), for saturated as well as dry state. The dry state is obtained by letting $p = 0$. It has to marked, however, that the elasto-plastic problem involves, in addition, the plastic strain and yield function.

Dry material: For the porous medium without any liquid in the pores, a local boundary value problem with a prescribed distribution of plastic strain inside the unit cell can be presented as:

$$\left\{ \begin{array}{ll} \sigma_{ij,i}^s = 0 & \text{in } V_s \\ \sigma_{ij}^s n_i = 0 & \text{on } \Gamma \\ \sigma_{ij}^s(y) = C_{ijkl} [e_{kh}(u) - \varepsilon_{kh}^p] & \text{in } V_s \\ u_i(y) = E_{ij} y_j + u_i^*(y) & \text{in } V_s \\ f(\sigma_{ij}^s) \leq 0, u_i^*(y) - Y \text{ periodic, } \sigma_{ij}^s n_i - \text{antiperiodic} \end{array} \right. \quad (5.1)$$

where $\sigma_{ij}^s(y)$ are components of the solid micro-stress tensor, $u_i(y)$ - the displacement field, $u_i^*(y)$ - the fluctuating part of the displacement field, E_{ij} - the solid macro-strain tensor (treated as given), $e_{ij}(u)$ - the solid micro-strain tensor and $\varepsilon_{ij}^p(y)$ - the plastic micro-strain tensor (considered as given), C_{ijkl} - the elastic stiffness tensor of the skeleton material. The function $f(\sigma_{ij}^s) \leq 0$ represents the plastic yield criterion (at microscopic level). V_s is the volume of the solid material within the unit cell, Γ - the phase separation surface.

Using notions of a plastic macro-strain tensor E_{ij}^p and a self-equilibrated

residual stress field $\sigma_{ij}^{res}(y)$, the above local problem can be transformed to:

$$\sigma_{ij}^s(y) - \sigma_{ij}^{res}(y) = \left[C_{ijkh} + C_{ijlm}e_{lm}(\xi^{kh}) \right] (E_{kh} - E_{kh}^p), \tag{5.2}$$

and, finally, to the stress localization law (Suquet, [21]):

$$\sigma_{ij}^s(y) - \sigma_{ij}^{res}(y) = L_{ijkh}(y) \Sigma_{kh}^s; \quad L_{ijkh}(y) = \left[C_{ijpq} + C_{ijlm}e_{lm}(\xi^{pq}) \right] S_{pqkh}^{hom} \tag{5.3}$$

The tensor $\Sigma_{ij}^s \left(= \frac{1}{\|V\|} \int_{V_s} \sigma_{ij}^s(y) dy \right)$ is the solid macro-stress tensor, S_{pqkh}^{hom} is the overall elastic compliance tensor, $L_{ijkh}(y)$ are components of the stress localization operator and $\xi_i^{kh}(y)$ are components of the fluctuating part of the displacement field at $\left\{ \varepsilon_{ij}^p(y) = 0, E_{ij} = \delta_{ik}\delta_{jh} \right\}$, δ_{ij} is the Kronecker symbol and $\|V\|$ denotes a measure of the unit cell volume.

The plastic macro-strain tensor E_{ij}^p and residual stress field $\sigma_{ij}^{res}(y)$ are determined by the following formulae (Suquet, [21]):

$$E_{ij}^p = \frac{1}{\|V\|} \int_{V_s} L_{khi j}(y) \varepsilon_{kh}^p(y) dy \tag{5.4}$$

$$\sigma_{ij}^{res}(y) = - \int_{V_s} R_{ijkh}(y, y') \varepsilon_{kh}^p(y') dy' \tag{5.5}$$

where $R_{ijkh}(y, y')$ depends only on the geometry of microstructure within the unit cell.

Fully saturated material: For the porous material saturated with a liquid pressure p and with a prescribed distribution of plastic strain $\bar{\varepsilon}_{ij}^p(y)$, a local boundary value problem reads:

$$\left\{ \begin{array}{ll} \sigma_{ij,i}^s = 0 & \text{in } V_s \\ \sigma_{ij}^s n_i = -p \delta_{ij} n_i & \text{on } \Gamma \\ \sigma_{ij}^s(y) = C_{ijkh} \left[e_{kh}(\bar{u}) - \bar{\varepsilon}_{kh}^p \right] & \text{in } V_s \\ \bar{u}_i(y) = \bar{E}_{ij} y_j + \bar{u}_i^*(y) & \text{in } V_s \\ f(\sigma_{ij}^s) \leq 0, \bar{u}_i^*(y) - Y \text{ periodic, } \sigma_{ij}^s n_i - \text{antiperiodic} \end{array} \right. \tag{5.6}$$

In the same manner as for the dry material, the above local problem can be transformed to:

$$\sigma_{ij}^s(y) - \bar{\sigma}_{ij}^{res}(y) = \left[C_{ijkh} + C_{ijlm}e_{lm}(\xi^{kh}) \right] \left[\bar{E}_{kh} - \bar{E}_{kh}^p \right] + C_{ijkh} e_{kh}(\eta) p \tag{5.7}$$

where: \bar{E}_{ij}^p and $\bar{\sigma}_{ij}^{res}(y)$ are again the linear functionals (5.4) and (5.5) of the plastic micro-strain field $\bar{\varepsilon}_{ij}^p(y)$, $\eta_i(y)$ are components of the fluctuating part of the displacement field at $\{\bar{\varepsilon}_{ij}^p(y) = 0, \bar{E}_{ij} = 0, p = 1\}$ (Auriault&Sanchez-Palencia, [1]).

Using the assumption of the micro-homogeneity and micro-isotropy of the skeleton material it can be shown that (Lydzba&Shao, [14]):

$$\eta_i(y) = \frac{\xi_i^{kh}(y) \delta_{kh}}{3K_s}. \quad (5.8)$$

The above relation enable to rewrite the equation (5.7) as:

$$\sigma_{ij}^s(y) - \bar{\sigma}_{ij}^{res}(y) + p\delta_{ij} = \left[C_{ijkh} + C_{ijlm}e_{lm}(\xi^{kh}) \right] \left[\bar{E}_{kh} - \bar{E}_{kh}^p + \frac{p\delta_{kh}}{3K_s} \right] \quad (5.9)$$

which after some manipulation results in the stress localization law for the saturated material:

$$\sigma_{ij}^s(y) - \bar{\sigma}_{ij}^{res}(y) + p\delta_{ij} = L_{ijkh}(y) \Sigma_{kh}^{eT} \quad (5.10)$$

where $\Sigma_{ij}^{eT} (= \frac{1}{\|V\|} \int_{V_s} (\sigma_{ij}^s(y) + p\delta_{ij}) dy)$ is the so-called Terzaghi's effective stress tensor, K_s is a bulk modulus of the skeleton material.

5.2. Form of the effective stress

Following (Suquet, [21]), the closure of the elastic domain in the macro-stress space, at prescribed distribution of residual stress field, can be expressed as:

- *dry material*

$$E_D(\{\sigma_{ij}^{res}\}) = \left\{ \Sigma_{ij}^s \left| f(L_{ijkh}(y) \Sigma_{kh}^s + \sigma_{ij}^{res}(y)) \leq 0 \quad \forall y \in V_s \right. \right\}, \quad (5.11)$$

- *saturated material*

$$E_S(p, \{\bar{\sigma}_{ij}^{res}\}) = \left\{ \Sigma_{ij}^{eT} \left| f(L_{ijkh}(y) \Sigma_{kh}^{eT} - p\delta_{ij} + \bar{\sigma}_{ij}^{res}(y)) \leq 0 \quad \forall y \in V_s \right. \right\}. \quad (5.12)$$

The set $\mathbf{E}_S(p; \{\bar{\sigma}_{ij}^{res}\})$ denotes, at prescribed residual stress field and value of p , the closure of the elastic domain in the Terzaghi's effective stress space.

The above definitions clearly show that determination of the elastic domain, for both cases, requires the knowledge of the whole field of the residual stresses: $\{\sigma_{ij}^{res}(y)\}$ and $\{\bar{\sigma}_{ij}^{res}(y)_s\}$ for the dry material and for the saturated material, respectively.

In practice, some approximate models are used. For instance, assuming the strain-hardening rule to be valid for the dry material, the closure of the elastic domain is then approximated as:

$$\mathbf{E}_D(E_{ij}^p) = \left\{ \Sigma_{ij}^s \left| F_D \left(\Sigma_{ij}^s; \chi(E_{ij}^p) \right) \leq 0 \right. \right\} \tag{5.13}$$

where $F_D(\Sigma_{ij}^s; \chi(E_{ij}^p))$ represents the macroscopic loading function for the dry material.

In the following, the concept of effective stress is examined by considering two qualitatively different plastic yield criteria for the skeleton material, namely: a pressure insensitive and a pressure sensitive one.

The Von-Mises criterion: This criterion is pressure insensitive. Defining for the saturated material an equivalent stress as:

$$\sigma_{ij}^{eq}(y) = \sigma_{ij}^s(y) + p\delta_{ij} \tag{5.14}$$

one gets the following identities:

$$\left\{ \begin{array}{l} \sigma_{ij}^{eq}(y) = C_{ijkl} (e_{kh}^{eq}(\tilde{u}) - \bar{\epsilon}_{kh}^p) \quad \text{with } e_{kh}^{eq}(\tilde{u}) = e_{kh}(\tilde{u}) + \frac{p\delta_{kh}}{3K_s} \\ (\forall p) \quad f(\sigma_{ij}^s) = f(\sigma_{ij}^{eq}), \quad \frac{\partial f(\sigma_{ij}^s)}{\partial \sigma_{kh}^s} = \frac{\partial f(\sigma_{ij}^{eq})}{\partial \sigma_{kh}^{eq}} \end{array} \right. \tag{5.15}$$

The property (5.15_b) together with the definition of the elastic domain (5.12) imply:

$$\mathbf{E}_S(p, \{\bar{\sigma}_{ij}^{res}\}) = \left\{ \Sigma_{ij}^{eT} \left| f(L_{ijkl}(y)\Sigma_{kh}^{eT} + \bar{\sigma}_{ij}^{res}(y)) \leq 0 \quad \forall y \in V_s \right. \right\} \tag{5.16}$$

The identities (5.15) enable also to transform the local problem (Eqs. (5.6)) to the equivalent one:

$$\left\{ \begin{array}{ll} \sigma_{ij,i}^{eq} = 0 & \text{in } V_s \\ \sigma_{ij}^{eq} n_i = 0 & \text{on } \Gamma \\ \sigma_{ij}^{eq} (y) = C_{ijkh} [e_{kh}^{eq}(\tilde{u}) - \bar{\varepsilon}_{kh}^p] & \text{in } V_s \\ \tilde{u}_i (y) = E_{ij}^{eq} y_j + u_i^* (y) \text{ with } E_{ij}^{eq} = \bar{E}_{ij} + \frac{p\delta_{ij}}{3K_s} & \text{in } V_s \\ f(\sigma_{ij}^{eq}) \leq 0, u_i^* (y) - Y \text{ periodic, } \sigma_{ij}^{eq} n_i - \text{antiperiodic} & \end{array} \right. \quad (5.17)$$

Assuming, for the plastic deformation of the skeleton material to be governed by an associative plastic flow rule, the properties (5.15_b) lead to:

$$\dot{\varepsilon}_{ij}^p (y) = \begin{cases} \frac{\frac{\partial f}{\partial \sigma_{kl}^{eq}} C_{klmn} \dot{e}_{mn}^{eq} (y)}{\frac{\partial f}{\partial \sigma_{kl}^{eq}} C_{klmn} \frac{\partial f}{\partial \sigma_{mn}^{eq}}} \frac{\partial f}{\partial \sigma_{ij}^{eq}} & \text{for } f(\sigma_{ij}^{eq} (y)) = 0 \wedge \dot{f}(\sigma_{ij}^{eq} (y)) = 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.18)$$

The system of equations (5.17)-(5.18) represents the local elasto-plastic problem for the saturated material expressed by the equivalent fields introduced. It is of the same form as the corresponding local problem for the dry material, except it contains the equivalent strain and stress fields instead of solid's strain and stress as for the dry state.

Consider now the saturated porous material, initially free of the plastic micro-strain field, subjected to a history $\{\bar{E}_{ij}(t), p(t)\}$. The pair:

$$\{\bar{E}_{ij}(t), p(t)\} \mapsto \{e_{ij}(\tilde{u}(y, t)), \bar{\varepsilon}_{ij}^p(y, t)\} \quad (5.19)$$

characterizes a solution of the local elasto-plastic problem for the saturated material, i.e. the loading history and corresponding micro-strain fields induced in the unit cell. As a direct consequence of the system (Eqs.(5.17)-(5.18)), the following equivalences can be written:

$$\{E_{ij}^{eq}(t)\} \mapsto \{e_{ij}^{eq}(\tilde{u}(y, t)), \bar{\varepsilon}_{ij}^p(y, t)\} \quad (5.20)$$

or:

$$\{E_{ij}(t) = E_{ij}^{eq}(t)\} \mapsto \{e_{ij}(u(y, t)) = e_{ij}^{eq}(\tilde{u}(y, t)), \varepsilon_{ij}^p(y, t) = \bar{\varepsilon}_{ij}^p(y, t)\} \quad (5.21)$$

The latter relation represents a solution of the local problem for the dry material. It indicates that the solution for the saturated material can be recovered from the solution of the local problem for the dry material by imposing the history $\{E_{ij}(t)\} = \left\{ \bar{E}_{ij}(t) + \frac{p(t)}{3K_s} \delta_{ij} \right\}$. Furthermore, as a consequence of the functional relations (5.4)-(5.5) as well as the constitutive equations, one gets also:

$$\begin{cases} \sigma_{ij}^s(y, t) = \sigma_{ij}^{eq}(y, t); \sigma_{ij}^{res}(y, t) = \bar{\sigma}_{ij}^{res}(y, t) \quad \forall y \in V_s \\ E_{ij}^p(t) = \bar{E}_{ij}^p(t); \Sigma_{ij}^s(t) = \Sigma_{ij}^{eT}(t) \end{cases} \text{ for } E_{ij}(t) = \bar{E}_{ij}(t) + \frac{p(t)}{3K_s} \delta_{ij} \quad (5.22)$$

where the variables on the left side of the equations (5.22) correspond to the dry material whereas on the right side to the saturated material.

The final step consists in comparison of the corresponding definitions of the elastic domains (5.11) and (5.16), for dry and saturated states, respectively. Taking into account the relation (5.22) one gets:

$$E_S(p, \{\bar{\sigma}_{ij}^{res}\}) = E_D(\{\bar{\sigma}_{ij}^{res}\}) \quad (5.23)$$

which clearly indicates that the Terzaghi's effective stress fulfills the stress equivalence principle for the material considered. It is obvious that this statement can be extended also for porous materials composed of a uniform material obeying any arbitrary pressure insensitive yield criterion with an associative or a non-associative plastic flow rule. For the case of non-associative plasticity, the plastic potential has to be, however, of the form:

$$\tilde{g}(\sigma_{ij}) = aI_1 + g(J_2, J_3) \quad (5.24)$$

where I_1 is the first invariant of stress tensor, J_3 is the third invariant of deviatoric stress tensor, $g(J_2, J_3)$ is an arbitrary function of J_2 and J_3 , 'a' is a parameter (could be $a=0$). Such the form of the plastic potential allows to express the rate of the plastic micro-strain using the equivalent fields introduced above and the formula is of the same form as for the dry material. Therefore, the relations (5.22) as well as the identity (5.23) are still hold true.

According to the approximate models, if a description for the dry material uses, in a plastic range, the loading function (5.13) and a plastic flow rule as:

$$\dot{E}_{ij}^p = \lambda \frac{\partial G(\Sigma_{kh}^s)}{\partial \Sigma_{ij}^s} \quad (5.25)$$

therefore, it follows from (5.22) that:

$$\mathbf{E}_S(p, \bar{E}_{ij}^p) = \left\{ \Sigma_{ij}^{eT} \mid F_D(\Sigma_{ij}^{eT}; \chi(\bar{E}_{ij}^p)) \leq 0 \right\} \quad (5.26)$$

$$\dot{\bar{E}}_{ij}^p = \lambda \frac{\partial G(\Sigma_{kh}^{eT})}{\partial \Sigma_{ij}^{eT}} \quad (5.27)$$

The equations (5.26) and (5.27), for the saturated state, are as good approximation of plastic behavior as the approximation (5.13) and (5.25) for the dry state. The results (5.26) and (5.27) prove also that the Terzaghi's effective stress fulfills, in the plastic range, the stress as well as the strain equivalence principles for saturated media. It has to be marked, however, that the result has been obtained with the assumption that the yield function was the pressure insensitive.

The Coulomb-Mohr criterion: The criterion is a pressure-sensitive. This time a particular loading history is only considered, namely the case corresponding to a so-called drained condition, i.e. $\{\bar{E}_{ij}(t), p(t) = \text{const.}\}$. The equivalent stress is now defined as:

$$\sigma_{ij}^{eq} = \frac{\sigma_{ij}^s + p\delta_{ij}}{1 + p \operatorname{tg}(\varphi)/c}. \quad (5.28)$$

The stress fulfills the identity:

$$f(\sigma_{ij}^s) = (1 + p \operatorname{tg}(\varphi)/c) f(\sigma_{ij}^{eq}) \quad (5.29)$$

where: φ represents a friction angle ($\varphi \geq 0$), c is the internal cohesion of the skeleton material ($c > 0$ is assumed).

Since $p \geq 0$, the equation (5.29) implies:

$$f(\sigma_{ij}^s) \leq 0 \text{ if } f(\sigma_{ij}^{eq}) \leq 0; \quad \frac{\partial f(\sigma_{ij}^s)}{\partial \sigma_{kh}} = \frac{\partial f(\sigma_{ij}^{eq})}{\partial \sigma_{kh}^{eq}} \quad (5.30)$$

The relations (5.30) enable to transform the local problem (Eqs.(5.6)) to the equivalent one described by the system (5.17). The equations (5.17_c) and

(5.17_d) are now, however, expressed by the new equivalent variables, i.e.:

$$\left\{ \begin{array}{l} \sigma_{ij}^{eq}(y) = C_{ijkh} \left(e_{kh}^{eq}(\bar{u}) - \frac{\varepsilon_{kh}^p}{1 + ptg(\varphi)/c} \right) \quad \text{with } e_{kh}^{eq}(\bar{u}) = \frac{e_{kh}(\bar{u}) + \frac{p\delta_{kh}}{3K_s}}{1 + ptg\varphi/c} \\ \bar{u}_i(y) = E_{ij}^{eq} y_j + u_i^*(y) \quad \text{with } E_{ij}^{eq} = \frac{\bar{E}_{ij} + \frac{p\delta_{ij}}{3K_s}}{1 + ptg(\varphi)/c} \end{array} \right. \quad (5.31)$$

Furthermore, a rate of the plastic micro-strain can be written as (the associated plastic flow rule is assumed):

$$\frac{\dot{\varepsilon}_{ij}^p(y)}{1 + p \, tg(\varphi)/c} = \begin{cases} \frac{\frac{\partial f}{\partial \sigma_{kl}^{eq}} C_{klmnp} \dot{e}_{mn}^{eq}(y)}{\frac{\partial f}{\partial \sigma_{kl}^{eq}} C_{klmnp} \frac{\partial f}{\partial \sigma_{mn}^{eq}}} \frac{\partial f}{\partial \sigma_{ij}^{eq}} & \text{for } f(\sigma_{ij}^{eq}(y)) = 0 \wedge \dot{f}(\sigma_{ij}^{eq}(y)) = 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.32)$$

Once more, the local elasto-plastic problem for the saturated material, expressed by the equivalent fields, is of the same form as for the dry material. Therefore, considering the pair:

$$\{ \bar{E}_{ij}(t), p(t) = const. \} \mapsto \{ e_{ij}(\bar{u}(y, t)), \bar{\varepsilon}_{ij}^p(y, t) \} \quad (5.33)$$

one gets also:

$$\{ E_{ij}(t) = E_{ij}^{eq}(t) \} \mapsto \left\{ e_{ij}(u(y, t)) = e_{ij}^{eq}(\bar{u}(y, t)), \varepsilon_{ij}^p(y, t) = \frac{\bar{\varepsilon}_{ij}^p(y, t)}{1 + p \, tg(\varphi)/c} \right\}. \quad (5.34)$$

The relation (5.34) implies:

$$\left\{ \begin{array}{l} \sigma_{ij}^s(y, t) = \sigma_{ij}^{eq}(y, t); \quad \sigma_{ij}^{res}(y, t) = \frac{\bar{\sigma}_{ij}^{res}(y, t)}{1 + ptg\varphi/c} \quad \forall y \in V_s \\ E_{ij}^p(t) = \frac{\bar{E}_{ij}^p(t)}{1 + p \, tg(\varphi)/c}; \quad \Sigma_{ij}^s(t) = \Sigma_{ij}^{eq}(t) \end{array} \right. , \quad (5.35)$$

where the variables on the left side correspond to the dry material whereas on the right side to the saturated material. The macroscopic equivalent stress tensor is the volume averaged equivalent micro-stress tensor (5.28):

$$\Sigma_{ij}^{eq} = \frac{\Sigma_{ij}^{eT}}{1 + p \, tg(\varphi)/c} = \frac{1}{\|V\|} \int_{V_s} \sigma_{ij}^{eq}(y) \, dy. \quad (5.36)$$

Using above tensor, the elastic domain for the saturated material can be defined as:

$$\mathbf{E}^{eq}(p, \{\bar{\sigma}_{ij}^{res}\}) \left\{ \Sigma_{ij}^{eq} \left| f \left(L_{ijkl}(y) \Sigma_{kh}^{eq} + \frac{\bar{\sigma}_{ij}^{res}(y)}{1 + ptg(\varphi)/c} \right) \leq 0 \forall y \in V_s \right. \right\}, \quad (5.37)$$

which together with the definition of the elastic domain (5.11) and the relations (5.38) result in:

$$\mathbf{E}^{eq}(p, \{\bar{\sigma}_{ij}^{res}\}) = \mathbf{E}_D \left(\left\{ \frac{\bar{\sigma}_{ij}^{res}}{1 + ptg(\varphi)/c} \right\} \right) \quad (5.38)$$

The equivalent macro-stress defined by (5.36) fulfills the stress equivalence principle, at the drained condition. It is clear that this statement is also valid for a non-associative plastic flow rule described by the potential (5.24) with a function $g(J_2, J_3)$ being a homogeneous function of deviatoric stress of the degree one.

According to the approximate model, it follows from (5.35) that:

$$\mathbf{E}^{eq}(p, \bar{E}_{ij}^p) = \left\{ \Sigma_{ij}^{eq} \left| F_D \left(\Sigma_{ij}^{eq}; \chi \left(\frac{\bar{E}_{ij}^p}{1 + p tg(\varphi)/c} \right) \right) \leq 0 \right. \right\}, \quad (5.39)$$

$$\frac{\dot{\bar{E}}_{ij}^p}{1 + p tg(\varphi)/c} = \lambda \frac{\partial G(\Sigma_{kh}^{eq})}{\partial \Sigma_{ij}^{eq}}. \quad (5.40)$$

5.3. Conclusions

The effective stress concept has been validated for saturated porous media composed of micro-homogeneous and micro-isotropic skeleton material. Particularly, it has been shown that the appropriate form of the effective stress tensor depends on the form of yield function used at the micro-level. Two different form of the effective stress have been determined. It has to be marked, however, that the assumption of the skeleton micro-homogeneity is the crucial one. Without this assumption one can prove that the Terzaghi's effective stress gives only a potentially safe approximation of failure criterion for saturated porous medium. This case is carefully studied in Łydzba (2002).

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