AMAS COURSE ON RELIABILITY-BASED DESIGN AND OPTIMISATION
RBO'03 – (PP.211–240) – WARSAW, SEPTEMBER 15-18, 2003

Algorithms for reliability-based optimal design

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Algorithms for solving three classes of reliability-based optimal design problems are presented. The algorithms address design problems for structural components, series systems, and a portfolio of series systems, where the objective and/or constraint functions involve probability terms. Due to lack of continuous differentiability of the probability terms or their approximations, the posed problems cannot be solved by standard optimization algorithms. The proposed approach employs reformulations of the problems, in which probability terms are replaced by better-behaving functions. The reformulated problems can be solved by existing semi-infinite optimization algorithms. It is shown that the reformulated problems produce solutions that are identical to those of the original problems when the limit-state functions are affine, or when first-order reliability approximations are used. Improved solutions for higher-order reliability approximations are obtained by adjusting a set of parameters in the algorithm. An important advantage of the approach is that the required reliability and optimization calculations are completely decoupled, thus allowing flexibility in the choice of the optimization algorithm and the reliability method. Numerical examples demonstrate applications of the proposed algorithms.

1. Introduction

Uncertainties and optimization are two major considerations in structural design. Uncertainties, arising from randomness in structural materials and applied loads as well as from errors in behavioral models, are inevitable and must be properly accounted for in the design of structures to assure safety and reliability. Optimization in the design of structures is desirable in order to maximize benefits and to make effective use of resources. Thus, optimal design under uncertainty is a topic of significant practical interest in structural engineering. Due to the challenges present in both probabilistic analysis and optimal design of structures, the combined problem poses significant difficulties as well as opportunities for research and innovation.

The typical single-objective optimal design problem involves an objective function that is to be minimized (or maximized), and one or more equality or inequality constraints, which define the feasible domain of the design variables. Under conditions of uncertainty, probabilistic terms may enter the objective, the constraints or both. Furthermore, the probabilistic terms may involve various measures, such as statistical moments of structural response or probabilities associated with various structural performance events.

In this paper, we present a summary of algorithms developed by the authors for solving single-objective design optimization problems involving failure probabilities (complements of reliability) as constraints, in the objective function, or both as constraints and in the objective function. Both structural component and series system problems are considered. For more detailed background on the development of these algorithms, including proofs of the various statements, the reader should consult Royset *et al.* (2001, 2002, 2003). A comprehensive review of other works in reliability-based optimal design is presented in Royset *et al.* (2002) and will not be repeated here. However, two distinguishing characteristics of the approach presented here relative to the reliability-based optimal design algorithms developed or used by other researchers and practitioners are that:

(a) in the proposed approach the computations for reliability and optimization are decoupled, thus allowing maximum latitude in the choice of algorithms for solving these sub-problems,

(b) the developed algorithms have proven convergence properties under certain conditions.

The paper begins with a brief review of the relevant reliability methods followed by the definition of three classes of optimal design problems. Each problem is then considered and applicable algorithms are developed and applied to example problems.

2. Structural reliability

Let $\mathbf{x}$ be an n-dimensional vector of deterministic, real-valued design variables, e.g., member sizes, maintenance times, extent of future repair. Following the well-established theory of structural reliability (Ditlevsen and Madsen 1996), we express the system reliability of a structural design by means of a set of continuously differentiable limit-state functions $G_k(\mathbf{x}, \mathbf{v})$, $k \in \mathbf{K} = \{1, 2, \ldots, K\}$, involving $\mathbf{x}$ and an m-dimensional vector $\mathbf{v}$ of realizations of random variables $\mathbf{V}$. The event $\{G_k(\mathbf{x}, \mathbf{V}) \leqslant 0\}$ defines the failure of the structure in its k-th mode, a "component" event. The failure of the structure as a "series system" occurs if any of the component events $\{G_k(\mathbf{x}, \mathbf{V}) \leqslant 0\}$, $k \in \mathbf{K}$, occurs.

Several computational reliability methods require a bijective transformation of realizations $\mathbf{v}$ of the random vector $\mathbf{V}$ into realizations $\mathbf{u}$ of a standard normal random vector $\mathbf{U}$. Such transformations can be defined under weak assumptions. For a given design vector $\mathbf{x}$, let $T_{\mathbf{x}}(\mathbf{v})$ be this transformation. Replacing $\mathbf{v}$ by $T_{\mathbf{x}}^{-1}(\mathbf{u})$ gives the equivalent limit-state functions $g_k(\mathbf{x}, \mathbf{u})$, $k \in \mathbf{K}$, defined by $g_k(\mathbf{x}, \mathbf{u}) = G_k(\mathbf{x}, T_{\mathbf{x}}^{-1}(\mathbf{u}))$. Since structures usually posses high reliability, any realistic design should be safe at the mean point and hence $g_k(\mathbf{x}, \mathbf{0}) > 0$ for all problems of interest here.

The failure probability of the structural system is defined by

$$p(\mathbf{x}) = \int_{\Omega(\mathbf{x})} \varphi(\mathbf{u})\, d\mathbf{u}, \tag{2.1}$$

where $\varphi(\mathbf{u})$ is the m-dimensional standard normal probability density function and

$$\Omega(\mathbf{x}) = \bigcup_{k \in \mathbf{K}} \{\, \mathbf{u} \in \mathrm{R}^m \mid g_k(\mathbf{x}, \mathbf{u}) \leqslant 0\}, \tag{2.2}$$

is the failure domain. The failure probability for the k-th component, $p_k(\mathbf{x})$, is defined as in (2.1) with the integration domain replaced by $\Omega_k(\mathbf{x}) = \{\mathbf{u} \in \mathrm{R}^m \mid g_k(\mathbf{x}, \mathbf{u}) \leqslant 0\}$. We define the *critical component* to be the component with the largest failure probability.

Various reliability methods are available for estimating the probabilities $p_k(\mathbf{x})$ and $p(\mathbf{x})$. In the first-order reliability method (FORM), an approximation to $p_k(\mathbf{x})$ is obtained by linearizing the limit-state function $g_k(\mathbf{x}, \mathbf{u})$ at the point in the set $\{\mathbf{u} \in \mathrm{R}^m \mid g_k(\mathbf{x}, \mathbf{u}) = 0\}$ closest to the origin, i.e., at

$$\mathbf{u}_k^*(\mathbf{x}) \in \arg\min_{\mathbf{u} \in \mathrm{R}^m} \{\, \|\mathbf{u}\| \mid g_k(\mathbf{x}, \mathbf{u}) = 0\}. \tag{2.3}$$

Such closest points are referred to as *design points*. The corresponding approximation of the component failure probability takes the form

$$p_k(\mathbf{x}) \approx \Phi(-\beta_{1,k}(\mathbf{x})), \tag{2.4}$$

where $\beta_{1,k}(\mathbf{x}) = \|\mathbf{u}_k^*(\mathbf{x})\|$ is the first-order reliability index and $\Phi(\cdot)$ is the standard normal cumulative distribution function. Equality holds in (2.4) when $g_k(\mathbf{x}, \mathbf{u})$ is affine in $\mathbf{u}$, i.e., when $g_k(\mathbf{x}, \mathbf{u}) = b_{0,k}(\mathbf{x}) + \mathbf{b}_k(\mathbf{x})^{\mathrm{T}}\mathbf{u}$ for some positive valued function $b_{0,k}(\mathbf{x})$ and vector-valued function $\mathbf{b}_k(\mathbf{x})$. For a series system with affine component limit-state functions, the failure probability is obtained as the probability content in a polyhedral domain in the standard normal space. For non-affine component limit-state functions, the polyhedral domain defined by linearization of the individual component limit states provides a first-order approximation to the series system probability.

In the second-order reliability method (SORM), an approximation to $p_k(\mathbf{x})$ is obtained by replacing the limit-state function $g_k(\mathbf{x}, \mathbf{u})$ by a quadratic approximation in $\mathbf{u}$ at the design point $\mathbf{u}_k^*(\mathbf{x})$. The expression for the second-order approximation of $p_k(\mathbf{x})$ involves $\beta_{1,k}(\mathbf{x})$ and the principal curvatures of the surface $\{\mathbf{u} \mid g_k(\mathbf{x}, \mathbf{u}) = 0\}$ at the design point. A second-order approximation to the series system probability may be obtained by adjusting the distances to the faces of the first-order polyhedral approximation from $\beta_{1,k}(\mathbf{x})$ to $\beta_{2,k}(\mathbf{x})$ such that $\Phi(-\beta_{2,k}(\mathbf{x}))$ equals the SORM approximation of $p_k(\mathbf{x})$.

Another approach for computing failure probabilities is by Monte Carlo simulation, including various forms of importance sampling (see Ditlevsen and Madsen 1996). The probability of failure in (2.1) is written in the form

$$p(\mathbf{x}) = \int_{\Omega(\mathbf{x})} I(\mathbf{x}, \mathbf{w}) \frac{\varphi(\mathbf{w})}{h(\mathbf{w})} h(\mathbf{w})\, d\mathbf{w}, \tag{2.5}$$

where $h(\mathbf{w})$ is a suitable sampling density and $I(\mathbf{x}, \mathbf{w}) = 1$ for $\mathbf{w} \in \Omega(\mathbf{x})$ and $I(\mathbf{x}, \mathbf{w}) = 0$ for $\mathbf{w} \notin \Omega(\mathbf{x})$. It follows that $p(\mathbf{x})$ is equal to the expectation of $I(\mathbf{x}, \mathbf{W})\varphi(\mathbf{W})/h(\mathbf{W})$, where $\mathbf{W}$ is a random vector with probability density function $h(\mathbf{w})$. An estimate of the failure probability is then obtained as the sample mean of $I(\mathbf{x}, \mathbf{W})\varphi(\mathbf{W})/h(\mathbf{W})$, given by $\sum_{i=1}^{N} I(\mathbf{x}, \mathbf{w}_i)\varphi(\mathbf{w}_i)/(h(\mathbf{w}_i)N)$, for a set of randomly generated realizations $\mathbf{w}_i$, $i = 1, \ldots, N$, of $\mathbf{W}$ in accordance with the density $h(\mathbf{w})$. The estimate is unbiased provided $h(\mathbf{w}) \neq 0$ wherever $I(\mathbf{x}, \mathbf{w})\varphi(\mathbf{w}) \neq 0$. Furthermore, the simulation is efficient (i.e., the estimate converges rapidly) if $h(\mathbf{w})$ is selected to be nearly proportional to $I(\mathbf{x}, \mathbf{w})\varphi(\mathbf{w})$. It is evident that to maintain efficiency, the sampling density $h(\mathbf{w})$ may have to be changed with $\mathbf{x}$. The conventional Monte Carlo approach corresponds to the selection $h(\mathbf{w}) = \varphi(\mathbf{w})$.

An important requirement in all gradient-based optimization algorithms is the existence of at least first-order derivatives of the objective and constraint functions with respect to the design variables $\mathbf{x}$. In a reliability-based optimal design problem, this translates into the requirement of differentiability of the failure probability, or the employed approximations thereof, with respect to the design variables. Unfortunately, none of the reliability approximations described above are guaranteed to be differentiable with respect to $\mathbf{x}$. For example, one can easily show that the FORM and SORM approximations of the failure probability for the limit-state function $g(x, \mathbf{u}) = 5 - 0.2(u_1 - x)^2 - u_2$ are not differentiable at $x = 0$. Similarly, the estimate of the failure probability obtained by a simulation method naturally fluctuates, with each sample leading to varying accuracy of finite difference estimates of the gradient of the sample mean of $I(\mathbf{x}, \mathbf{W})\varphi(\mathbf{W})/h(\mathbf{W})$. Even for a given sample, the gradient of the sample mean of $I(\mathbf{x}, \mathbf{W})\varphi(\mathbf{W})/h(\mathbf{W})$ is not differentiable everywhere with respect to $\mathbf{x}$ due to the non-smoothness of $I(\mathbf{x}, \mathbf{u})$ as a function of $\mathbf{x}$. In a recent paper, Royset and Polak (2004) have shown that the component failure probability computed by a new simulation method is differentiable under certain conditions. However, for the more general case of a series system, even the exact failure probability can be non-differentiable. One example is the series system with the component limit-state functions $g_1(x, u_1, u_2) = 3 - u_1$, $g_2(x, u_1, u_2) = 3 - u_2$ and $g_3(x, u_1, u_2) = 3 - u_2 - x$, whose exact failure probability is not differentiable at $x = 0$.

The difficulty with differentiability means that optimization problems involving the failure probability in the objective function or the constraints may not be solvable by standard nonlinear optimization algorithms (e.g., NLPQL by Schittkowski (1985), LANCELOT by Conn *et al.* (1992), and NPSOL by Gill *et al.* (1998)). Ironically, most existing literature on reliability-based optimal design employs FORM approximations or simulation techniques and standard nonlinear optimization algorithms. This does not mean that the solutions reported in the literature by use of these methods are necessarily wrong, but that the algorithms employed in these applications are not robust for the given problem and may fail to reach a solution for other similar problems. In short, standard nonlinear optimization algorithms do not appear to be suitable for the solution of reliability-based optimal design problems obtained by the use of the probability approximations described above. The algorithms presented in this paper circumvent this problem by a reformulation that replaces the probability terms with other better-behaving functions. The reformulation does not lead to optimization problems that can be solved by standard nonlinear optimization algorithms, but the prob-

lems can be solved by so-called semi-infinite optimization algorithms. These algorithms are well-known in the optimization literature.

3. Definition of optimization problems

This paper addresses three classes of optimization problems denoted as $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$. The series-system versions of these problems are denoted as $\mathbf{P}_{1,\text{sys}}$, $\mathbf{P}_{2,\text{sys}}$ and $\mathbf{P}_{3,\text{sys}}$. For $\mathbf{P}_3$, a version applicable to a "portfolio" of series systems, e.g., a group of bridges, is also formulated and denoted as $\mathbf{P}_{3,\text{por}}$. To define these problems, let $c_0(\mathbf{x})$ be the initial cost of the design, $c_k(\mathbf{x})$, $k \in \mathbf{K}$, be the cost associated with the failure of component k, and

$$\mathbf{X} = \{\mathbf{x} \in \mathrm{R}^n \mid f_j(\mathbf{x}) \leqslant 0,\ j = 1, \ldots, q\}, \tag{3.1}$$

with $f_j(\mathbf{x})$ being continuously differentiable functions describing deterministic constraints. Problems $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are defined as follows:

$$\mathbf{P}_1 = \min_{\mathbf{x}\in\mathrm{R}^n} \left\{ c_0(\mathbf{x}) \;\middle|\; p_k(\mathbf{x}) \leqslant \hat{p}_k,\ k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}, \tag{3.2}$$

$$\mathbf{P}_2 = \min_{\mathbf{x}\in\mathrm{R}^n} \left\{ \max_{k\in\mathbf{K}} p_k(\mathbf{x}) \middle| \mathbf{x} \in \mathbf{X} \right\}, \tag{3.3}$$

$$\mathbf{P}_3 = \min_{\mathbf{x}\in\mathrm{R}^n} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) p_k(\mathbf{x}) \;\middle|\; p_k(\mathbf{x}) \leqslant \hat{p}_k,\ k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}. \tag{3.4}$$

As can be seen, $\mathbf{P}_1$ minimizes the cost of the design subject to the constraints that individual component failure probabilities are less than $\hat{p}_k$, $\mathbf{P}_2$ minimizes the failure probability of the critical component, and $\mathbf{P}_3$ minimizes the sum of the initial cost and the expected cost of failure of the components, assuming the component failure costs are additive, subject to constraints on individual component failure probabilities. All three problems are also subject to the deterministic constraints $f_j(\mathbf{x}) \leqslant 0$, $j = 1, \ldots, q$. The series system versions of these problems are defined as

$$\mathbf{P}_{1,\text{sys}} = \min_{\mathbf{x}\in\mathrm{R}^n} \left\{ c_0(\mathbf{x}) \;\middle|\; p(\mathbf{x}) \leqslant \hat{p},\ p_k(\mathbf{x}) \leqslant \hat{p}_k,\ k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}, \tag{3.5}$$

$$\mathbf{P}_{2,\text{sys}} = \min_{\mathbf{x}\in\mathrm{R}^n} \left\{ p(\mathbf{x}) \;\middle|\; \mathbf{x} \in \mathbf{X} \right\}, \tag{3.6}$$

$$\mathbf{P}_{3,\text{sys}} = \min_{\mathbf{x}\in\mathrm{R}^n} \left\{ c_0(\mathbf{x}) + c(\mathbf{x}) p(\mathbf{x}) \;\middle|\; p(\mathbf{x}) \leqslant \hat{p},\ p_k(\mathbf{x}) \leqslant \hat{p}_k,\ k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}. \tag{3.7}$$

As can be seen, in $\mathbf{P}_{1,\text{sys}}$ a constraint on the system failure probability is added, whereas in $\mathbf{P}_{2,\text{sys}}$ it is the system probability that is minimized. In $\mathbf{P}_{3,\text{sys}}$, the expected failure cost is in terms of the system failure, and the system failure probability is included in the constraint set definition. To define the portfolio version of $\mathbf{P}_3$, let the superscript $^{(l)}$, $l \in \mathbf{L} = \{1, \ldots, L\}$, define the l-th series structural system among a portfolio of L such systems. $\mathbf{P}_{3,\text{por}}$ is then defined as

$$\mathbf{P}_{3,\text{por}} = \min_{\mathbf{x}\in R^n} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}) + \sum_{l=1}^{L} c^{(l)}(\mathbf{x}) p^{(l)}(\mathbf{x}) \;\middle|\; p^{(l)}(\mathbf{x}) \leqslant \hat{p}^{(l)},\ l \in \mathbf{L},\ \mathbf{x} \in \mathbf{X} \right\}. \tag{3.8}$$

Here, $c_0^{(l)}(\mathbf{x})$ is the initial cost of the l-th series structural system, and $c^{(l)}(\mathbf{x})$ is cost associated with failure of the l-th series structural system. For the sake of simplicity in the notation, in the above we have removed the constraints on the individual components of the series systems. This problem aims to minimize the portfolio cost of the design plus expected cost of system failures, subject to constraints on the individual system probabilities.

In this paper, all the cost, limit-state and constraint functions are assumed to be continuously differentiable. Additionally, we assume that the interval (for m =1), area (for m = 2), volume (for m = 3), etc., in which the limit-state function vanishes, have length, area, volume, etc., equal to zero, respectively. This is normally satisfied in realistic design problems. The precise mathematical statement of this assumption can be found as Assumption 1(iii) in Royset *et al.* (2003).

4. Probabilistic constraints: problems $\mathbf{P}_1$ and $\mathbf{P}_{1,\text{sys}}$

4.1. Approximating problems

Consider the probabilistic constraint $p_k(\mathbf{x}) \leqslant \hat{p}_k$ in (3.2). The failure probability $p_k(\mathbf{x})$ can rarely be computed exactly and some approximation must be employed. As mentioned above, the existing reliability methods for estimating the failure probability are not suitable for use with standard nonlinear optimization algorithms due to the lack of differentiability. However, the FORM is a motivation for the following approximation originally found in Kirjner-Neto *et al.* (1998) and Der Kiureghian and Polak (1998).

In view of (2.4), the classic FORM approximation to the constraint $p_k(\mathbf{x}) \leqslant \hat{p}_k$ is

$$\beta_{1,k}(\mathbf{x}) \geqslant \beta_{0,k}, \tag{4.1}$$

where

$$\beta_{0,k} = -\Phi^{-1}(\hat{p}_k). \tag{4.2}$$

By definition, (4.1) implies

$$\min_{\mathbf{u}\in\mathrm{R}^m} \{\|\mathbf{u}\| \mid g_k(\mathbf{x},\mathbf{u}) = 0\} \geqslant \beta_{0,k}. \tag{4.3}$$

Structures tend to be highly reliable, and hence we can assume $g_k(\mathbf{x},\mathbf{0}) > 0$ for all relevant designs $\mathbf{x}$. Consequently, the equality in (4.3) can be replaced by an inequality. Under this assumption, an equivalent but mathematically more convenient expression for the constraint (4.3) is

$$\min_{\mathbf{u}\in\mathrm{R}^m} \{g_k(\mathbf{x},\mathbf{u}) \mid \|\mathbf{u}\| \leqslant \beta_{0,k}\} \geqslant 0. \tag{4.4}$$

At first glance, the expression in (4.4) does not appear more advantageous than the one in (4.3). However, the left hand-side of (4.4) can be interpreted as a so-called standard min-function, while $\beta_{1,k}(\mathbf{x})$ is a generalized min-function. Standard min-functions have been studied extensively in the literature, and there is a variety of efficient and robust algorithms available for solving optimization problems involving such functions. On the other hand, generalized min-functions are significantly more difficult to deal with. Note that standard min-functions are not differentiable everywhere even if $g_k(\mathbf{x},\mathbf{u})$ is differentiable. This fact is incorporated into the algorithms in the literature for solving optimization problems with min-functions.

In view of the above discussion, we define the standard min-function

$$\psi_{k,s_k}(\mathbf{x}) = \min_{\mathbf{u}\in\mathrm{R}^m} \left\{ g_k(\mathbf{x},\mathbf{u}) \;\middle|\; \|\mathbf{u}\| \leqslant s_k \right\}, \tag{4.5}$$

where $s_k > 0$ is a parameter. Hence, we obtain the following approximation to $\mathbf{P}_1$:

$$\mathbf{P}_{1,\mathbf{s}} = \min_{\mathbf{x}\in\mathrm{R}^n} \left\{ c_0(\mathbf{x}) \;\middle|\; \psi_{k,s_k}(\mathbf{x}) \geqslant 0,\ k \in \mathbf{K}, \mathbf{x} \in \mathbf{X} \right\}. \tag{4.6}$$

The relation between $\mathbf{P}_1$ and $\mathbf{P}_{1,\mathbf{s}}$ is given in the following statement. Various versions of this statement are found in precise mathematical language with proofs in Kirjner-Neto *et al.* (1998), Der Kiureghian and Polak (1998), Polak *et al.* (2000), and Royset *et al.*(2002).

Statement about $\mathbf{P}_1$ *and* $\mathbf{P}_{1,\mathbf{s}}$: If the limit-state functions $g_k(\mathbf{x},\mathbf{u})$, $k \in \mathbf{K}$, are affine in their second argument, i.e., $g_k(\mathbf{x},\mathbf{u}) = b_{0,k}(\mathbf{x}) + \mathbf{b}_k(\mathbf{x})^{\mathrm{T}}\mathbf{u}$, *and* $\mathbf{s} = (s_1, s_2, \ldots s_K)$, with $s_k = -\Phi^{-1}(\hat{p}_k)$, $k \in \mathbf{K}$, then $\hat{\mathbf{x}}$ solves $\mathbf{P}_1$ if and only if it solves $\mathbf{P}_{1,\mathbf{s}}$.

An important consequence of the above finding is that, for non-affine limit-state functions, $\mathbf{P}_{1,\mathbf{s}}$ has solutions identical to those of $\mathbf{P}_1$, if the failure probability terms in the latter are expressed in terms of the FORM approximation. However, if higher order probability approximations are to be used, adjustments in the parameters s_k, $k \in \mathbf{K}$, must be made. Specifically, if at the solution $\hat{\mathbf{x}}$ of $\mathbf{P}_{1,\mathbf{s}}$ the FORM approximation for a component k is smaller than the corresponding higher-order probability approximation, such that the latter violates the probability constraint in (3.2), then problem $\mathbf{P}_{1,\mathbf{s}}$ must be re-solved using a larger value of s_k. Conversely, if the FORM approximation is larger than the higher-order probability approximation, then a smaller value of s_k may be used. This process is repeated until all the component probability constraints in (3.2) are satisfied for the desired probability approximation level. A recursive formula for these updates of parameters s_k is given below.

The above parameter-adjustment procedure can also be employed to solve the series system problem $\mathbf{P}_{1,\mathrm{sys}}$. It is well known (Ditlevsen and Madsen, 1996) that

$$\max_{k\in\mathbf{K}} p_k(\mathbf{x}) \leqslant p(\mathbf{x}) \leqslant \sum_{k\in\mathbf{K}} p_k(\mathbf{x}). \tag{4.7}$$

Hence, the failure probability of the series system is closely related to the failure probabilities of the components, particularly that of the critical component. In view of the right-hand side of (4.7), the constraint $p(\mathbf{x}) \leqslant \hat{p}$ can be satisfied by selecting a sufficiently large value of s_k for each component.

4.2. Algorithms

It is clear that when the limit-state functions $g_k(\mathbf{x}, \mathbf{u})$, $k \in \mathbf{K}$, are affine, $\mathbf{P}_1$ can be solved by applying an algorithm to $\mathbf{P}_{1,\mathbf{s}}$. The latter belongs to a well known class of optimization problems called semi-infinite (see e.g., Polak, 1997, or Royset *et al.*, 2002). Thus, it can be solved by any of a series of well-honed algorithms with guaranteed convergence properties. To obtain approximate solutions in the case of non-affine limit-state functions and/or problems involving series systems, we repeatedly solve the approximating problem $\mathbf{P}_{1,\mathbf{s}}$ as described below. This approach was originally proposed by Der Kiureghian and Polak (1998) and Polak *et al.* (2000) for problems with component failure probabilities, i.e., $\mathbf{P}_1$. In Royset *et al.* (2001) and (2002), this approach was extended to also address $\mathbf{P}_{1,\mathrm{sys}}$.

Algorithm 1 for solving $\mathbf{P}_1$

Data. Provide an initial design $\mathbf{x}_0$ and a sequence of strictly increasing integers $N_0, N_1, N_2, \ldots$.

Step 0. Set $i = 0$ and the parameters $(s_k)_0 = -\Phi^{-1}(\hat{p}_k)$, $k \in \mathbf{K}$.

Step 1. Set $\mathbf{x}_{i+1}$ to be the last iterate after N_i iterations of a semi-infinite optimization algorithm on the problem $\mathbf{P}_{1,\mathbf{s}_i}$, with $\mathbf{s}_i = ((s_1)_i, (s_2)_i, (s_K)_i)$, and initialization $\mathbf{x}_i$.

Step 2. Compute appropriate estimates $\tilde{p}_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$, of $p_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$.

Step 3. Update the components of $\mathbf{s}_{i+1}$ by setting

$$(s_k)_{i+1} = (s_k)_i \frac{\Phi^{-1}(\hat{p}_k)}{\Phi^{-1}(\tilde{p}_k(\mathbf{x}_{i+1}))}, \quad k \in \mathbf{K}. \tag{4.8}$$

Step 4. Replace i by $i + 1$ and go to Step 1.

With the phrase "appropriate estimate" of a failure probability in Step 2 of Algorithm 1, we mean that the failure probability estimate should be computed using the same reliability method (e.g., FORM, SORM, or Monte Carlo Simulation) and with the same level of accuracy as the one used to verify the final design obtained by Algorithm 1.

In the case of $\mathbf{P}_{1,\text{sys}}$, the following algorithm can be used:

Algorithm 2 for solving $\mathbf{P}_{1,\text{sys}}$

Data. Provide an initial design $\mathbf{x}_0$ and a sequence of strictly increasing integers $N_0, N_1, N_2, \ldots$.

Step 0. Set $i = 0$, and the parameters $(s_k)_0 = -\Phi^{-1}(\hat{p}_k)$, $k \in \mathbf{K}$.

Step 1. Set $\mathbf{x}_{i+1}$ to be the last iterate after N_i iterations of a semi-infinite optimization algorithm on the problem $\mathbf{P}_{1,\mathbf{s}_i}$, with $\mathbf{s}_i = ((s_1)_i, (s_2)_i, \ldots, (s_K)_i)$, and initialization $\mathbf{x}_i$.

Step 2. Compute appropriate estimates $\tilde{p}_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$, and $\tilde{p}(\mathbf{x}_{i+1})$ of $p_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$, and $p(\mathbf{x}_{i+1})$, respectively.

Step 3. Update the components of $\mathbf{s}_{i+1}$ by setting

$$(s_k)_{i+1} = (s_k)_i \max\left(\frac{\Phi^{-1}(\hat{p}_k)}{\Phi^{-1}(\tilde{p}_k(\mathbf{x}_{i+1}))}, \frac{\Phi^{-1}(\hat{p})}{\Phi^{-1}(\tilde{p}(\mathbf{x}_{i+1}))}\right), \tag{4.9}$$

for all $k = \hat{k}$ such that $\tilde{p}_{\hat{k}}(\mathbf{x}_{i+1}) = \max_{k \in \mathbf{K}} \tilde{p}_k(\mathbf{x}_{i+1})$. Otherwise, set

$$(s_k)_{i+1} = (s_k)_i \frac{\Phi^{-1}(\hat{p}_k)}{\Phi^{-1}(\tilde{p}_k(\mathbf{x}_{i+1}))}. \tag{4.10}$$

Step 4. Replace i by $i + 1$ and go to Step 1.

For the special case of $\mathbf{P}_{1,\text{sys}}$ with no component failure probability constraints, a slightly simplified algorithm can be found in Royset *et al.* (2002).

4.3. Example: reinforced concrete girder

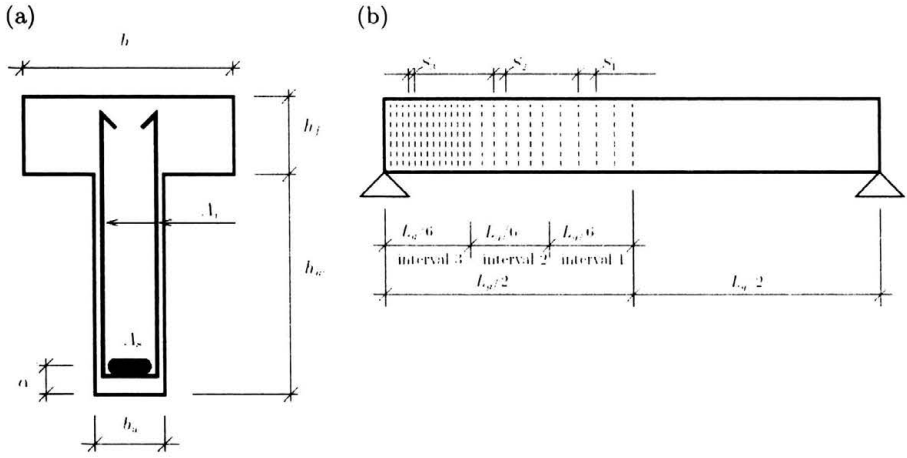

FIGURE 1. Example reinforced concrete girder: (a) cross section, (b) side view with shear reinforcement.

In this subsection, we present a summary of the design example found in Royset *et al.* (2002). Consider a highway bridge with reinforced concrete girders of the type shown in Fig. 1. The objective is to find the optimal design for one such girder using the material and load data from Lin and Frangopol (1996) and Frangopol *et al.* (1997). The nine design variables are collected in the vector

$$\mathbf{x} = (A_s, b, h_f, b_w, h_w, A_v, S_1, S_2, S_3)\,, \tag{4.11}$$

where A_s is the area of the tension steel reinforcement, b is the width of the flange, h_f is the thickness of the flange, b_w is the width of the web, h_w is the height of the web, A_v is the area of the shear reinforcement (twice the cross-section area of a stirrup), and S_1, S_2 and S_3 are the spacings of shear reinforcements in intervals 1, 2 and 3, respectively, see Fig. 1(b). The random variables describing the loading and material properties are collected in the 8-dimensional vector

$$\mathbf{V} = \left(f_y, f_c', P_D, M_L, P_{S1}, P_{S2}, P_{S3}, W\right), \tag{4.12}$$

where f_y is the yield strength of the reinforcement, f_c' is the compressive strength of concrete, P_D is the dead load excluding the weight of the girder, M_L is the live load moment, P_{S1}, P_{S2} and P_{S3} are the live load shear forces in intervals 1, 2 and 3, respectively, see Fig. 1(b), and W is the unit weight of concrete. Following Lin and Frangopol (1996), all the random variables are considered to be independent and normally distributed with the means

TABLE 1. Statistics of normal random variables in girder example.

Variable	Mean		c.o.v.
f_y	413.4	MPa	0.150
f'_c	27.56	MPa	0.150
P_D	13.57	kN/m	0.200
M_L	929	kNm	0.243
P_{S1}	138.31	kN	0.243
P_{S2}	183.39	kN	0.243
P_{S3}	228.51	kN	0.243
W	22.74	kN/m^3	0.100

and coefficients of variation as listed in Table 1. In the remainder of this section, the random variables and their realizations are denoted with the same symbol. Let the girder length be $L_g = 18.30\,\text{m}$, and the distance from the bottom fiber to the centroid of the tension reinforcement be $\alpha = 0.1\,\text{m}$, see Fig. 1.

The objective is to design the girder according to the specifications in AASHTO (1992). However, these specifications do not lead to well-defined optimization problems for two reasons. First, some of the constraints are not continuous functions, but of the form $f(\mathbf{x}) \leqslant 1$ whenever $h(\mathbf{x}) \leqslant 0$ and otherwise $f(\mathbf{x}) \leqslant 2$, where $f(\mathbf{x})$ and $h(\mathbf{x})$ are some continuous functions. Second, $h(\mathbf{x})$ may also depend on the random variables of the problem. In the following, the first difficulty is overcome by considering different cases. For example, Case 1 has the constraints $f(\mathbf{x}) \leqslant 1$ and $h(\mathbf{x}) \leqslant 0$, while Case 2 has the constraints $f(\mathbf{x}) \leqslant 2$ and $h(\mathbf{x}) \geqslant 0$. The optimal design for each case is found independently, and the design with the smallest value of the objective function is our solution. The second difficulty is overcome by replacing any random variables in the definition of $h(\mathbf{x})$ by their mean values. In Royset *et al.* (2002), it is shown that the AASHTO specifications can be formulated as four cases, each with 28 constraints. The reader is referred to Royset *et al.* (2002) for the details.

As in Lin and Frangopol (1996), we assume that the reinforced concrete girder fails if it exceeds its flexure capacity or its shear capacity in one of three sections of the girder (see Fig. 1(b)). Hence, the reliability of the girder is defined by a series structural system with four components. The limit-state functions associated with the four failure modes are given in Royset *et al.* (2002).

Suppose that the objective is to minimize the material cost of the reinforced concrete girder subject to a constraint on the system failure probability, i.e., a design problem of the type $\mathbf{P}_{1,\text{sys}}$. Let $C_\text{s} = 50$ and $C_\text{c} = 1$ be the unit costs of steel reinforcement and concrete per cubic meter, respectively. As in Lin and Frangopol (1996), we define the objective function to be

$$\begin{aligned} c_0(\mathbf{x}) &= 0.75 C_\text{s} L_\text{g} A_\text{s} \\ &\quad + C_\text{s} n_\text{s} A_\text{v} (h_\text{f} + h_\text{w} - \alpha + 0.5 b_\text{w}) \\ &\quad + C_\text{c} L_\text{g} (b h_\text{f} + b_\text{w} h_\text{w}), \end{aligned} \tag{4.13}$$

where $n_\text{s} = L_\text{g}(1/S_1 + 1/S_2 + 1/S_3)/3$ is the total number of stirrups. In (4.13), the first term represents the cost of the bending reinforcement. The factor 0.75 appears due to the assumption that the total amount of bending reinforcement is placed only within a length $L_\text{g}/2$ centered at the middle point of the girder, and the remaining part is reinforced with $0.5A_\text{s}$. The second and third terms in (4.13) represent the costs of shear reinforcement and concrete, respectively. Let the constraint on the system failure probability be $p(\mathbf{x}) \leqslant 0.001350$.

This problem is solved by using Algorithm 2 for $\mathbf{P}_{1,\text{sys}}$. The results after 25 iterations (beyond which little change in the design is observed) are given in Table 2, where the design vector $\mathbf{x}_i$, the objective $c_0(\mathbf{x}_i)$, and the system failure probability $p(\mathbf{x}_i)$ are listed. The system failure probability is evaluated using Monte Carlo simulation with a c.o.v. of 0.01.

A direct comparison with Lin and Frangopol (1996) is not possible because of different assumptions regarding α (see Fig. 1(a)) and the fact that we have introduced additional constraints to eliminate the possibility of an unrealistic geometric shape of the girder. However, the design in Table 2 with a cost of 13.6 appears to be better than the one reported in Lin and Frangopol (1996), which has a cost of 16.7.

TABLE 2. Results for $\mathbf{P}_{1,\text{sys}}$ design of reinforced concrete girder.

Design variable	Value at iteration 25	Design variable	Value at iteration 25
A_s	0.009832 m^2	S_1	0.508 m
b	0.418 m	S_2	0.224 m
h_f	0.415 m	S_3	0.140 m
b_w	0.196 m	$p(\mathbf{x}_{25})$	0.001310
h_w	0.785 m	$c_0(\mathbf{x}_{25})$	13.664
A_v	0.0001859 m^2		

5. Probabilistic objective function: problems $\mathbf{P}_2$ and $\mathbf{P}_{2,\mathrm{sys}}$

5.1. Approximating problems

We now consider the problems $\mathbf{P}_2$ and $\mathbf{P}_{2,\mathrm{sys}}$, where the failure probability is not involved in the constraint set definition. In Der Kiureghian and Polak (1998), Polak *et al.* (2000), Royset *et al.* (2001), and Royset *et al.* (2002), we find that approximating problems for $\mathbf{P}_2$ and $\mathbf{P}_{2,\mathrm{sys}}$ can be constructed in a manner similar to those for $\mathbf{P}_1$ and $\mathbf{P}_{1,\mathrm{sys}}$, as described above.

In view of the equivalence between (4.3) and (4.4), it is clear that under fairly general conditions a larger value of

$$\beta_{1,k}(\mathbf{x}) = \min_{\mathbf{u}\in\mathrm{R}^m} \{ \|\mathbf{u}\| \mid g_k(\mathbf{x},\mathbf{u}) = 0 \},$$

corresponds to a larger value of $\psi_{k,s}(\mathbf{x})$ in (4.5) for a given s, and vice versa. Hence, instead of finding the design $\mathbf{x}$ with the largest $\beta_{1,k}(\mathbf{x})$, which would be $\mathbf{P}_2$ with a FORM approximation of the failure probability, we find the design $\mathbf{x}$ with the largest $\psi_{k,s}(\mathbf{x})$ for a given value of the parameter s. Thus, we avoid using the generalized min-function $\beta_{1,k}(\mathbf{x})$ and instead use an approximation in terms of the standard min-function $\psi_{k,s}(\mathbf{x})$. As mentioned above, this change results in the fact that we now have well-honed optimization algorithms available for solving an approximation to our optimal design problem. Additionally, as in Algorithms 1 and 2, adjustments in the parameter s may be used to improve the approximation.

In view of the above discussion, we define the following approximation to $\mathbf{P}_2$:

$$\mathbf{P}_{2,s} = \max_{\mathbf{x}\in\mathbf{X}} \left\{ \min_{k\in\mathbf{K}} \psi_{k,s}(\mathbf{x}) \right\}, \tag{5.1}$$

where $\psi_{k,s}(\mathbf{x})$ is given in (4.5). Note that in $\mathbf{P}_{2,s}$ the parameter s is a scalar, while in $\mathbf{P}_{1,\mathbf{s}}$ there are K parameters s_k. The relation between $\mathbf{P}_2$ and $\mathbf{P}_{2,s}$ is given in the following statement. Various versions of this statement are found in precise mathematical language with proofs in Der Kiureghian and Polak (1998), Polak *et al.* (2000), and Royset *et al.*(2002).

Statement about $\mathbf{P}_2$ *and* $\mathbf{P}_{2,s}$: If the limit-state functions $g_k(\mathbf{x},\mathbf{u})$, $k \in \mathbf{K}$, are affine in their second argument, i.e., $g_k(\mathbf{x},\mathbf{u}) = b_{0,k}(\mathbf{x}) + \mathbf{b}_k(\mathbf{x})^{\mathrm{T}}\mathbf{u}$, then $\hat{\mathbf{x}}$ solves $\mathbf{P}_2$ if and only if $\hat{\mathbf{x}}$ solves $\mathbf{P}_{2,s}$ for an arbitrary $s > 0$.

A geometric interpretation of problems $\mathbf{P}_2$ and $\mathbf{P}_{2,s}$ helps us understand the argument behind the above statement and the situation for non-affine limit-state functions. The distance from the origin in the $\mathbf{u}$-space to the

nearest point in $\Omega(\mathbf{x})$ is usually given by $\beta_{1,\hat{k}}(\mathbf{x})$, where $\hat{k}$ is the index of the critical component. With the FORM approximation, $\mathbf{P}_2$ finds the optimal design by maximizing this distance. In contrast, $\mathbf{P}_{2,s}$ finds the optimal design by maximizing $\min_{k\in\mathbf{K}} \psi_{k,s}(\mathbf{x})$, i.e., by maximizing the minimum value of the limit-state functions within the ball of radius s. This maximum may occur at a point $\hat{\mathbf{u}}_{\hat{k}}(\mathbf{x})$, which, in general, is different from $\mathbf{u}^*_{\hat{k}}(\mathbf{x})$ as defined in (2.3), see Fig. 2. For affine limit-state functions, it is clear from Fig. 2 that vectors $\hat{\mathbf{u}}_{\hat{k}}(\mathbf{x})$ and $\mathbf{u}^*_{\hat{k}}(\mathbf{x})$ are collinear and the two maximizations produce identical designs for an arbitrary $s > 0$, hence the above statement on the equivalence of $\mathbf{P}_2$ and $\mathbf{P}_{2,s}$ for affine limit-state functions. For non-affine limit-state functions, we see from Fig. 2 that the two approaches would produce identical designs if s is taken equal to $\beta_{1,\hat{k}}(\mathbf{x}^*)$, where $\mathbf{x}^*$ is a solution of $\mathbf{P}_2$. It is also clear from the geometry in Fig. 2 that the solution of $\mathbf{P}_{2,s}$ for a non-affine limit-state function would tend to be insensitive to the value of s in the neighborhood of $\beta_{1,\hat{k}}(\mathbf{x}^*)$. Hence, a rough estimate of s is usually sufficient. Furthermore, owing to the close relation between the FORM approximation and the exact failure probability, minimizing the FORM approximation produces designs, which tend to minimize the exact failure probability as well. Additionally, owing to the dominance of the critical failure mode in the series system failure probability (see (4.7)), we can conclude that a solution to $\mathbf{P}_{2,s}$, with s close to $\beta_{1,\hat{k}}(\mathbf{x}^*)$, is a good approximation to the solution of $\mathbf{P}_{2,\text{sys}}$ as well.

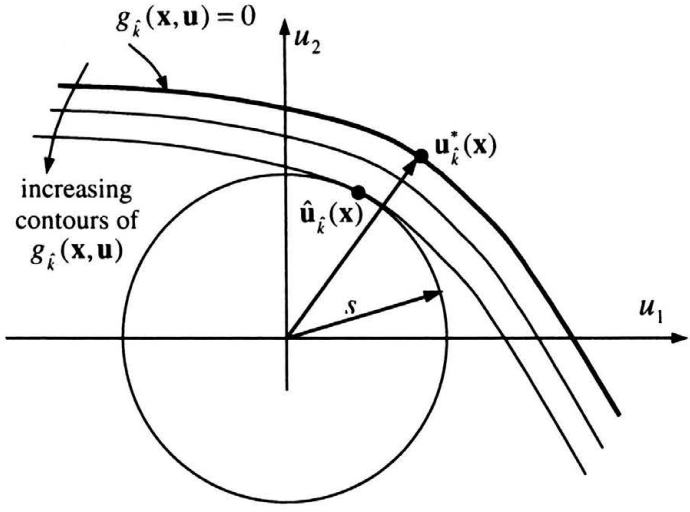

FIGURE 2. Geometric interpretation of problems P_2 and $\mathrm{P}_{2,s}$.

5.2. Algorithm

When the limit-state functions $g_k(\mathbf{x}, \mathbf{u})$, $k \in \mathbf{K}$, are affine, $\mathbf{P}_2$ can be solved by applying an algorithm to $\mathbf{P}_{2,s}$. Since $\mathbf{P}_{2,s}$ is a semi-infinite optimization problem (see e.g., Polak 1997), it can be solved by any of a series of well-honed algorithms with guaranteed convergence properties. To obtain approximate solutions in the case of non-affine limit-state functions and/or problems involving series systems, we repeatedly solve the approximating problem $\mathbf{P}_{2,s}$ as described below. This approach was originally proposed by Der Kiureghian and Polak (1998) and Polak *et al.* (2000) for problems with component failure probabilities, i.e., $\mathbf{P}_1$. In Royset *et al.* (2002), this approach was extended to also address $\mathbf{P}_{2,\text{sys}}$.

Algorithm 3 for solving $\mathbf{P}_2$ and $\mathbf{P}_{2,\text{sys}}$

Data. Provide an initial design $\mathbf{x}_0$, an integer N, and a parameter s_0, with value in the neighborhood of the first-order reliability index of the critical component for the anticipated optimal design.

Step 0. Set $i = 0$.

Step 1. Set $\mathbf{x}_{i+1}$ to be the last iterate after N iterations of a semi-infinite optimization algorithm on the problem $\mathbf{P}_{2,s_i}$, with initialization $\mathbf{x}_i$.

Step 2 . Compute appropriate estimates $\tilde{p}_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$, of $p_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$. If considering $\mathbf{P}_{2,\text{sys}}$, also compute the appropriate estimate $\tilde{p}(\mathbf{x}_{i+1})$ of $p(\mathbf{x}_{i+1})$.

Step 3 . Determine $\hat{k}_{i+1}$ (the index for the critical component) such that $\tilde{p}_{\hat{k}_{i+1}}(\mathbf{x}_{i+1}) = \max_{k \in \mathbf{K}} \tilde{p}_k(\mathbf{x}_{i+1})$ and compute the corresponding FORM reliability index $\beta_{1,\hat{k}_{i+1}}(\mathbf{x}_{i+1})$.

Step 4. Set $s_{i+1} = \beta_{1,\hat{k}_{i+1}}(\mathbf{x}_{i+1})$.

Step 5. The best estimate of the optimal design after $i+1$ iterations is $\hat{\mathbf{x}}_{i+1} \in \arg\min_{j=1,\ldots,i+1} \tilde{p}_{\hat{k}_j}(\mathbf{x}_j)$ (in case of $\mathbf{P}_2$) and $\hat{\mathbf{x}}_{i+1} \in \arg\min_{j=1,\ldots,i+1} \tilde{p}(\mathbf{x}_j)$ (in case of $\mathbf{P}_{2,\text{sys}}$).

Step 6. Replace i by $i+1$ and go to Step 1.

5.3. Example: offshore jacket structure

In this subsection we present the summary of a design example originally found in Royset *et al.* (2001) and (2002). Consider the idealized offshore jacket structure shown in Fig. 3, which is modeled as a plane truss with linear elastic members and supported by linear elastic springs representing the foundation flexibility.

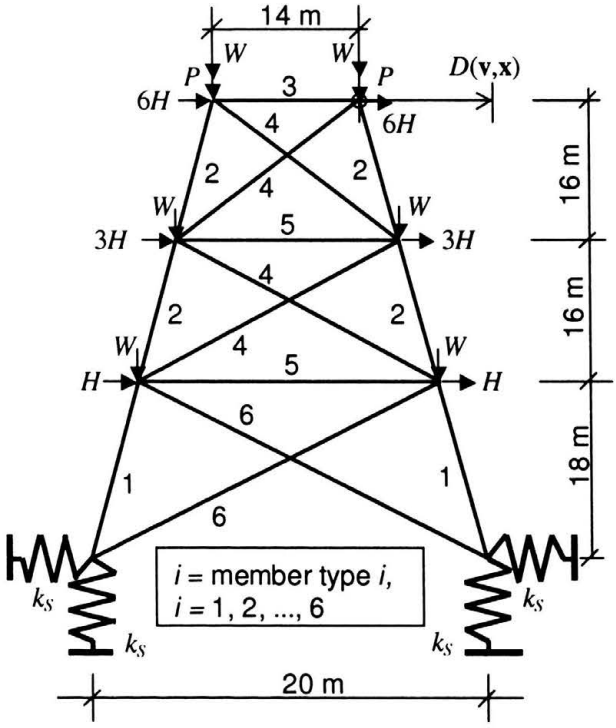

FIGURE 3. Offshore jacket structure.

The structure is subjected to combined wave and wind loads of magnitudes proportional to H, and gravity loads P and W, all applied at the nodes of the truss. The load magnitudes as well as the elastic modulus of the material, E, and the stiffnesses of the supporting springs, k_s, are considered to be statistically independent random variables with the distributions listed in Table 3. The structure has six different member types, each type having a

TABLE 3. Distributions of random variables for jacket structure.

Variable	Distribution	Mean		c.o.v.
H	Gumbel	70	kN	0.35
P	Gumbel	2,940	kN	0.10
W	Gumbel	20	kN	0.10
E	Lognormal	210	GPa	0.12
k_S	Lognormal	50,000	N/m	0.30

circular tubular cross section with an outside radius R_i and wall thickness t_i, $i = 1, \ldots, 6$. The ratio of the wall thickness to outside radius of each member is assumed to be a constant, $t_i/R_i = 0.05$. The task is to optimize the radii R_i to achieve maximum reliability with respect to a threshold of $D_0 = 0.20\,\mathrm{m}$ for the horizontal displacement at the top right node of the structure. The limit-state function is defined as

$$G(\mathbf{x}, \mathbf{v}) = D_0 - D(\mathbf{v}, \mathbf{x}), \tag{5.2}$$

where $\mathbf{v} = (H, P, W, E, k_s)$ is the vector of random variables, $\mathbf{x} = (R_1, R_2, R_3, R_4, R_5, R_6)$ is the vector of design parameters, and $D(\mathbf{v}, \mathbf{x})$ is the horizontal displacement of the structure at the top right node expressed as a function of $\mathbf{v}$ and $\mathbf{x}$. The function $D(\mathbf{v}, \mathbf{x})$ is not available in an explicit form and its evaluation requires structural analysis by means of, e.g., a finite element program. A Matlab (1999) structural analysis program was written for the present application to compute $D(\mathbf{v}, \mathbf{x})$ and its gradients with respect to $\mathbf{v}$ and $\mathbf{x}$.

The jacket structure is designed for maximum reliability, subject to the total volume of the material being limited to $6.25\,\mathrm{m}^3$. This implies the constraint function

$$f_1(\mathbf{x}) = \sum_{i=1}^{6} \sum_{j=1}^{N_i} L_{ij} \pi \left[R_i^2 - (R_i - t_i)^2 \right] - 6.25, \tag{5.3}$$

where L_{ij} is the length of the j-th member of type i and N_i is the number of members of type i. Each member of the truss is constrained to have a slenderness ratio (length divided by the radius of gyration of the cross section) not exceeding 130. For the circular tubular cross section, this implies $2L_{ij}/\sqrt{R_i^2 + (R_i - t_i)^2} \leqslant 130$. Let L_i be the maximum length of members of type i. Using $t_i = 0.05 R_i$, the above inequality leads to the constraint functions

$$f_{i+1}(\mathbf{x}) = \frac{2L_i}{130\sqrt{1 + 0.95^2}} - R_i, \quad i = 1, 2, \ldots, 6. \tag{5.4}$$

These are equivalent to the following constraints on the individual radii: $R_1 \geqslant 0.201\,\mathrm{m}$, $R_2 \geqslant 0.179\,\mathrm{m}$, $R_3 \geqslant 0.156\,\mathrm{m}$, $R_4 \geqslant 0.259\,\mathrm{m}$, $R_5 \geqslant 0.199\,\mathrm{m}$ and $R_6 \geqslant 0.291\,\mathrm{m}$.

We design the jacket structure by solving a sequence of problems $\mathrm{P}_{2,s}$ with $K = 1$, the constraint functions $f_j(\mathbf{x})$, $j = 1, \ldots, 7$, and s values in the range 2 to 5, where we expect the first-order reliability index to lie. The results are summarized in Table 4. Included are the designs $\mathbf{x}_i$, the first-order reliability index $\beta_1(\mathbf{x}_i)$, and the failure probability estimates computed by Monte Carlo

simulation (with c.o.v. of 0.01). It is observed that the optimal solution as well as the failure probabilities are virtually invariant of the assumed value of s. It is noted that the first-order failure probability approximation $p(\mathbf{x}_i) \approx \Phi(-\beta_1(\mathbf{x}_i)) = 9.9611 \cdot 10^{-5}$ is quite close to the "exact" failure probability computed by Monte Carlo simulation. This indicates that the limit-state surface is nearly affine. It is, therefore, not surprising that the solution is invariant of s. Based on the results in Table 4, the solution for this design problem is $\hat{\mathbf{x}} = (0.354\,\text{m}, 0.240\,\text{m}, 0.156\,\text{m}, 0.259\,\text{m}, 0.199\,\text{m}, 0.291\,\text{m})$. It is noted that the constraint functions $f_j(\mathbf{x})$, $j = 1, 4, 5, 6$ and 7 are active at the solution point.

TABLE 4. Designs of the jacket structure for maximum reliability for different s values.

i	$R_1(m)$	$R_2(m)$	$R_3(m)$	$R_4(m)$	$R_5(m)$	$R_6(m)$	$\beta_1(\mathbf{x}_i)$	$p(\mathbf{x}_i)$	s_i
0	0.300	0.300	0.300	0.300	0.300	0.300	–	–	2.5
1	0.354	0.240	0.156	0.259	0.199	0.291	3.72	$9.574 \cdot 10^{-5}$	3.0
2	0.354	0.240	0.156	0.259	0.199	0.291	3.72	$9.574 \cdot 10^{-5}$	3.5
3	0.354	0.240	0.156	0.259	0.199	0.291	3.72	$9.574 \cdot 10^{-5}$	4.0
4	0.354	0.240	0.156	0.259	0.199	0.291	3.72	$9.574 \cdot 10^{-5}$	4.5
5	0.354	0.240	0.156	0.259	0.199	0.291	3.72	$9.574 \cdot 10^{-5}$	–

As noted, the limit-state surface for the above example is nearly affine in the standard normal space, in spite of the fact that $D(\mathbf{v}, \mathbf{x})$ is a non-affine function of $\mathbf{v}$ and that the random variables are non-normal. One way to impose greater deviation from affine limit-state functions is to use more strongly non-normal distributions and larger variances. To achieve this, we consider each random variable to be independently uniformly distributed with the bounds, means and c.o.v.'s listed in Table 5. We realize that the assumed distributions or the range of variations may not be realistic for an offshore jacket structure. Nevertheless, we use these values to check the

TABLE 5. Uniformly distributed random variables for modified jacket structure.

Variable	Range	Mean		c.o.v.
H	(-20 kN, 160 kN)	70	kN	0.74
P	(1 880 kN, 4 000 kN)	2 940	kN	0.21
W	(-10 kN, 50 kN)	20	kN	0.87
E	(130 GPa, 290 GPa)	210	GPa	0.22
k_S	(25 000 N/m, 75,000 N/m)	50 000	N/m	0.29

TABLE 6. Designs of the jacket structure for maximum reliability for different s values.

i	$R_1(m)$	$R_2(m)$	$R_3(m)$	$R_4(m)$	$R_5(m)$	$R_6(m)$	$\beta_1(\mathbf{x}_i)$	$p(\mathbf{x}_i)$	s_i
0	0.300	0.300	0.300	0.300	0.300	0.300	–	–	2.5
1	0.354	0.241	0.156	0.259	0.199	0.291	3.52	$7.532 \cdot 10^{-5}$	3.0
2	0.354	0.241	0.156	0.259	0.199	0.291	3.53	$7.235 \cdot 10^{-5}$	3.5
3	0.354	0.241	0.156	0.259	0.199	0.291	3.53	$7.235 \cdot 10^{-5}$	4.0
4	0.354	0.241	0.156	0.259	0.199	0.291	3.53	$7.235 \cdot 10^{-5}$	4.5
5	0.354	0.241	0.156	0.259	0.199	0.291	3.53	$7.235 \cdot 10^{-5}$	–

robustness of the proposed optimal design algorithm in terms of its sensitivity to the assumed value of s. Table 6 summarizes the results of the design of the jacket structure for the new set of random variables. We observe that now there is a significant difference between $p(\mathbf{x}_i)$ and $\Phi(-\beta_1(\mathbf{x}_i)) = 2.078 \cdot 10^{-4}$, indicating that the limit-state surface in the standard normal space is at least moderately non-affine. Nevertheless, the optimal solution and the reliability indices still remain practically invariant to the assumed value of s. This is a confirmation of our earlier conjecture that the solution of problem $\mathrm{P}_{2,s}$ for a non-affine limit-state surface is insensitive to the value of s in a broad neighborhood of the first-order reliability index. The design solution in this case is $\hat{\mathbf{x}} = (0.354\,\mathrm{m}, 0.241\,\mathrm{m}, 0.156\,\mathrm{m}, 0.299\,\mathrm{m}, 0.199\,\mathrm{m}, 0.291\,\mathrm{m})$, which is nearly the same as the solution for the previous case. Again, the constraint functions $f_j(\mathbf{x})$, $j = 1, 4, 5, 6$ and 7 are active at the solution point.

6. Probabilistic objective function and constraints: problems $\mathbf{P}_3$, $\mathbf{P}_{3,\mathrm{sys}}$ and $\mathbf{P}_{3,\mathrm{por}}$

6.1. Approximating problems

The problems $\mathbf{P}_3$, $\mathbf{P}_{3,\mathrm{sys}}$ and $\mathbf{P}_{3,\mathrm{por}}$, where the failure probabilities appear in both the constraint definition and the objective function, are more complicated than the problems discussed above. The approaches for solving $\mathbf{P}_1$ and $\mathbf{P}_2$ cannot simply be "combined" to create an approach for solving $\mathbf{P}_3$. In the approximation for $\mathbf{P}_2$, we replaced the failure probability by a function that had maxima approximately at the same designs as the minima for the failure probability on the given feasible set. The actual value of the failure probability was not involved in this approximation. In $\mathbf{P}_3$, $\mathbf{P}_{3,\mathrm{sys}}$ and $\mathbf{P}_{3,\mathrm{por}}$ we need an approximation of the failure probability to estimate the objective functions in these problems. In Royset *et al.* (2002), we find the following approach that builds on the approach for solving $\mathbf{P}_1$.

We first construct approximating problems for $\mathbf{P}_3$ by replacing the failure probabilities in the objective function of $\mathbf{P}_3$ with parameters. The parameters are included in an augmented design vector and, hence, their values are automatically determined by the optimization procedure.

Let $\bar{\mathbf{x}} = (\mathbf{x}, \mathbf{a})$ be an $(n+K)$-dimensional augmented design vector, where $\mathbf{x}$ is the original n-dimensional design vector and $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ is a K-dimensional vector of parameters. We define the problem

$$\hat{\mathbf{P}}_3 = \min_{(\mathbf{x},\mathbf{a})\in \mathrm{R}^{n+K}} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) a_k \;\middle|\; p_k(\mathbf{x}) = a_k, \right.$$
$$\left. 0 \leqslant a_k \leqslant \hat{p}_k,\ k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}. \quad (6.1)$$

Observe that the objective function in $\hat{\mathbf{P}}_3$ is equal to the one in $\mathbf{P}_3$ when $p_k(\mathbf{x}) = a_k$. Since in $\hat{\mathbf{P}}_3$ we only consider designs $\mathbf{x}$ such that $p_k(\mathbf{x}) = a_k$, $k \in \mathbf{K}$, and $0 \leqslant a_k \leqslant \hat{p}_k$, the following statement must hold:

Statement about $\mathbf{P}_3$ *and* $\hat{\mathbf{P}}_3$: The minimum value of problem $\mathbf{P}_3$ is equal to the minimum value of problem $\hat{\mathbf{P}}_3$.

The above statement is stated and proven formally in Royset *et al.* (2002). It is seen from (6.1) that $\hat{\mathbf{P}}_3$ is a minimization problem of a smooth objective function with failure probability equality constraints. This is similar to $\mathbf{P}_1$, but $\mathbf{P}_1$ contains inequality constraints. The above reformulation removes the failure probability in the objective function. However, the failure probability is still part of the constraint set definition. Hence, one more step is needed to reach an optimization problem that can be solved by semi-infinite optimization algorithms.

We proceed by constructing an approximating problem with min-function constraints. Let $\mathbf{t}$ be a K-dimensional vector of positive numbers. This parameter vector is similar in nature to $\mathbf{s}$ in $\mathbf{P}_{1,\mathbf{s}}$. However, as seen below, their numerical values tend to be different. We define the approximating problem

$$\hat{\mathbf{P}}_{3,\mathbf{t}} = \min_{\bar{\mathbf{x}}=(\mathbf{x},\mathbf{a})\in \mathrm{R}^{n+K}} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) a_k \;\middle|\; \hat{\psi}_{k,t_k}(\bar{\mathbf{x}}) \geqslant 0, \right.$$
$$\left. 0 \leqslant a_k \leqslant \hat{p}_k,\ k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}, \quad (6.2)$$

where

$$\hat{\psi}_{k,t_k}(\bar{\mathbf{x}}) = \min_{\mathbf{u}\in \mathrm{R}^m} \left\{ g_k\left(\mathbf{x}, -\Phi^{-1}(a_k) t_k \mathbf{u}\right) \mid \|\mathbf{u}\| \leqslant 1 \right\}. \quad (6.3)$$

Note that $\hat{\psi}_{k,t_k}(\bar{\mathbf{x}})$ is the minimum value of the limit-state function inside a ball of radius $-\Phi^{-1}(a_k)\,t_k$, while $\psi_{k,s_k}(\mathbf{x})$ is the minimum value of the limit-state function inside a ball of radius s_k. Hence, the radius of the ball associated with $\hat{\psi}_{k,t_k}(\bar{\mathbf{x}})$ varies with the argument $\bar{\mathbf{x}}$. The problem $\hat{\mathbf{P}}_{3,\mathbf{t}}$ is a semi-infinite optimization problem that can be solved by various algorithms (see, e.g., Polak, 1997, or Royset *et al.*, 2002).

In the same way that $\mathbf{P}_1$and $\mathbf{P}_{1,\mathbf{s}}$ were related, we find that $\hat{\mathbf{P}}_3$ and $\hat{\mathbf{P}}_{3,\mathbf{t}}$ are related:

Statement about $\hat{\mathbf{P}}_3$ *and* $\hat{\mathbf{P}}_{3,\mathbf{t}}$: If the limit-state functions $g_k(\mathbf{x},\mathbf{u})$, $k \in \mathbf{K}$, are affine in their second argument, i.e., $g_k(\mathbf{x},\mathbf{u}) = b_{0,k}(\mathbf{x}) + \mathbf{b}_k(\mathbf{x})^{\mathrm{T}}\mathbf{u}$, and $\mathbf{t} = (1,1,\ldots,1)$, then $\hat{\mathbf{x}}$ solves $\hat{\mathbf{P}}_3$ if and only if $\hat{\mathbf{x}}$ solves $\hat{\mathbf{P}}_{3,\mathbf{t}}$.

The mathematically precise statement and its proof can be found in Royset *et al.* (2002). In view of the two statements above, the original problem $\mathbf{P}_3$ is equivalent to $\hat{\mathbf{P}}_{3,\mathbf{t}}$, when the limit-state functions are affine. For non-affine limit-state functions, $\hat{\mathbf{P}}_{3,\mathbf{t}}$ is a first-order approximation to $\mathbf{P}_3$ with parameters $\mathbf{t}$, which can be adjusted to improve the approximation.

The situation for $\mathbf{P}_{3,\mathrm{sys}}$ and $\mathbf{P}_{3,\mathrm{por}}$ is similar to the one for $\mathbf{P}_3$. We first define

$$\hat{\mathbf{P}}_{3,\mathrm{por}} = \min_{(\mathbf{x},\mathbf{a})\in\mathrm{R}^{n+L}} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}) + \sum_{l=1}^{L} c^{(l)}(\mathbf{x})a_l \,\middle|\, p^{(l)}(\mathbf{x}) = a_l, \right.$$
$$\left. 0 \leqslant a_l \leqslant \hat{p}^{(l)},\ l \in \mathbf{L},\ \ \mathbf{x} \in \mathbf{X} \right\}. \quad (6.4)$$

The equivalence between $\mathbf{P}_{3,\mathrm{por}}$ and $\hat{\mathbf{P}}_{3,\mathrm{por}}$ is clear from the next statement, the proof of which can be found in Royset *et al.* (2002).

Statement about $\mathbf{P}_{3,por}$ *and* $\hat{\mathbf{P}}_{3,por}$: The minimum value of the problem $\mathbf{P}_{3,\mathrm{por}}$ is equal to the minimum value of the problem $\hat{\mathbf{P}}_{3,\mathrm{por}}$.

Next, we define the approximating problem

$$\hat{\mathbf{P}}_{3,\mathrm{por},\mathbf{t}} = \min_{\bar{\mathbf{x}}=(\mathbf{x},\mathbf{a})\in\mathrm{R}^{n+L}} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}) + \sum_{l=1}^{L} c^{(l)}(\mathbf{x})a_l \,\middle|\, \hat{\psi}_{t_l}^{(l)}(\bar{\mathbf{x}}) \geqslant 0, \right.$$
$$\left. 0 \leqslant a_l \leqslant \hat{p}^{(l)},\ l \in \mathbf{L},\ \ \mathbf{x} \in \mathbf{X} \right\}, \quad (6.5)$$

where

$$\hat{\psi}_{t_l}^{(l)}(\bar{\mathbf{x}}) = \min_{k\in\mathbf{K}_l} \min_{\mathbf{u}\in\mathrm{R}^m} \left\{ g_k^{(l)}(\mathbf{x}, -\Phi^{-1}(a_l)t_l\mathbf{u}) \,\middle|\, \|\mathbf{u}\| \leqslant 1 \right\}. \quad (6.6)$$

We are not able to prove equivalence between $\hat{\mathbf{P}}_{3,\mathrm{por}}$ and $\hat{\mathbf{P}}_{3,\mathrm{por},\mathbf{t}}$ similar to that between $\hat{\mathbf{P}}_3$ and $\hat{\mathbf{P}}_{3,\mathbf{t}}$ for affine limit-state functions. However, if all the limit-state functions $g_k^l(\mathbf{x},\mathbf{u})$ are affine in their respective second arguments, then $\hat{\psi}_{t_l{}^l}(\bar{\mathbf{x}}) \geqslant 0$ implies that the critical failure component, say $\hat{k}_l$, of the l-th structure has failure probability $p_{\hat{k}_l}^{(l)}(\mathbf{x}) \leqslant \Phi\left(-\Phi^{-1}(a_l)t_l\right)$. Hence, when $t_l = 1$, $p_{\hat{k}_l}^{(l)}(\mathbf{x}) \leqslant a_l$. Due to the close relation between the failure probability of the critical component and the failure probability of the series system, see (4.7), we can adjust t_l such that $p^l(\mathbf{x}) \approx a_l$ whenever $\hat{\psi}_{t_l}^{(l)}(\bar{\mathbf{x}}) = 0$. Hence, $\hat{\mathbf{P}}_{3,\mathrm{por},\mathbf{t}}$ is a good approximation to $\hat{\mathbf{P}}_{3,\mathrm{por}}$ for a suitable selection of $\mathbf{t}$.

In view of the above discussion, we can approximately solve $\mathbf{P}_{3,\mathrm{por}}$ by solving the semi-infinite optimization problem $\hat{\mathbf{P}}_{3,\mathrm{por},\mathbf{t}}$. We present algorithms for $\mathbf{P}_3$ and $\mathbf{P}_{3,\mathrm{por}}$ in the next section. Since $\mathbf{P}_{3,\mathrm{sys}}$ is very similar to $\mathbf{P}_{3,\mathrm{por}}$ (set L = 1 in $\mathbf{P}_{3,\mathrm{por}}$ and add component failure probability constraints), we do not discuss $\mathbf{P}_{3,\mathrm{sys}}$ separately. It is straightforward to develop an algorithm for $\mathbf{P}_{3,\mathrm{sys}}$ based on the ones for $\mathbf{P}_{3,\mathrm{por}}$ and $\mathbf{P}_1$.

6.2. Algorithms

When the limit-state functions $g_k(\mathbf{x},\mathbf{u})$, $k \in \mathbf{K}$, are affine, $\mathbf{P}_3$ can be solved by applying an algorithm to $\hat{\mathbf{P}}_{3,\mathbf{t}}$. Since $\hat{\mathbf{P}}_{3,\mathbf{t}}$ is a semi-infinite optimization problem, it can be solved by any of a series of well-honed algorithms with guaranteed convergence properties. To obtain approximate solutions in the case of non-affine limit-state functions, we repeatedly solve the approximating problem $\hat{\mathbf{P}}_{3,\mathbf{t}}$ as described below.

Algorithm 4 for solving $\mathbf{P}_3$

Data. Provide an initial design $\mathbf{x}_0$ and a sequence of strictly increasing integers $N_0, N_1, N_2, \ldots$.

Step 0. Set $i = 0$, $\mathbf{a}_0 = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_K)$, $\mathbf{t}_0 = (1, 1, \ldots, 1)$ and $\bar{\mathbf{x}}_0 = (\mathbf{x}_0, \mathbf{a}_0)$.

Step 1. Set $\bar{\mathbf{x}}_{i+1}$ to be the last iterate after N_i iterations of a semi-infinite optimization algorithm on the problem $\hat{\mathbf{P}}_{3,\mathbf{t}_i}$, with initialization $\bar{\mathbf{x}}_i$.

Step 2. Compute appropriate estimates $\tilde{p}_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$, of $p_k(\mathbf{x}_{i+1})$, $k \in \mathbf{K}$.

Step 3. Update the components of $\mathbf{t}_{i+1}$ by setting

$$(t_k)_{i+1} = (t_k)_i \frac{\Phi^{-1}\left((a_k)_{i+1}\right)}{\Phi^{-1}\left(\tilde{p}_k(\mathbf{x}_{i+1})\right)}, \quad k \in \mathbf{K}. \tag{6.7}$$

Step 4. Replace i by $i+1$ and go to Step 1.

In a similar manner, we obtain the following algorithm for solving $\mathbf{P}_{3,\mathrm{por}}$.

Algorithm 5 for solving $\mathbf{P}_{3,\text{por}}$

Data. Provide an initial design $\mathbf{x}_0$ and a sequence of strictly increasing integers $N_0, N_1, N_2, \ldots$.

Step 0. Set $i = 0$, $\mathbf{a}_0 = \left(\hat{p}^{(1)}, \hat{p}^{(l)}, \ldots, \hat{p}^{(L)}\right)$, $\mathbf{t}_0 = (1, 1, \ldots, 1)$, and $\bar{\mathbf{x}}_0 = (\mathbf{x}_0, \mathbf{a}_0)$.

Step 1. Set $\bar{\mathbf{x}}_{i+1}$ to be the last iterate after N_i iterations of a semi-infinite optimization algorithm on the problem $\hat{\mathbf{P}}_{3,\mathbf{t}_i}$, with initialization $\bar{\mathbf{x}}_i$.

Step 2. Compute appropriate estimates $\tilde{p}^{(l)}(\mathbf{x}_{i+1})$, $l \in \mathbf{L}$, of $p^{(l)}(\mathbf{x}_{i+1}), l \in \mathbf{L}$.

Step 3. Update the components of $\mathbf{t}_{i+1}$ by setting

$$(t_l)_{i+1} = (t_l)_i \frac{\Phi^{-1}\left((a_l)_{i+1}\right)}{\Phi^{-1}\left(\tilde{p}^{(l)}(\mathbf{x}_{i+1})\right)}, \quad l \in \mathbf{L}. \tag{6.8}$$

Step 4. Replace i by $i + 1$ and go to Step 1.

Typically, structural systems are reliable, and hence the failure probabilities tend to be small. In implementation of Algorithms 4 and 5, numerical difficulties caused by the potential difference in the orders of magnitude of the components of $\mathbf{x}$ and $\mathbf{a}$ can be avoided by use of the transformation $b_k = -\Phi^{-1}(a_k)$, $k \in \mathbf{K}$. Then the optimization in Algorithm 4 is over the vector $(\mathbf{x}, \mathbf{b})$, where $\mathbf{b} = (b_1, b_2, \ldots, b_K)$. A similar transformation can be employed in Algorithm 5.

6.3. Example: reinforced concrete girder

Suppose that the objective is to minimize the initial cost plus the expected cost of failure of the reinforced concrete girder described above. Additionally, we assume a constraint on the system failure probability, i.e., a design problem of the type $\mathbf{P}_{3,\text{sys}}$, with no component failure probability constraints. We summarize this example, which was originally presented in Royset *et al.* (2002).

Let the initial cost of the design be as described in (4.13). The cost of failure is assumed to be $c(\mathbf{x}) = 500c_0(\mathbf{x})$. Let the constraint on the system failure probability be $p(\mathbf{x}) \leqslant 0.001350$, with no constraints on the component failure probabilities.

We solve this instance of $\mathbf{P}_{3,\text{sys}}$ by using Algorithm 5 and the results after 25 iterations (beyond which little change in the design is observed) are given in Table 7, where the design vector $\mathbf{x}_{25}$, the auxiliary design variable a_{25}, the objective $c_0(\mathbf{x}_{25}) + c(\mathbf{x}_{25})\, p(\mathbf{x}_{25})$, and the system failure probability $p(\mathbf{x}_{25})$ are listed. The system failure probability is evaluated using Monte Carlo simulation with a c.o.v. of 0.01.

Note that the system failure probability constraint is not active in this example. Comparing the results in Tables 2 and 7, we see that the initial cost of the design, $c_0(\mathbf{x}_{25})$, has increased significantly in account of the expected failure cost in the objective function.

TABLE 7. Results for $\mathbf{P}_{3,\text{sys}}$ design of reinforced concrete girder.

Design variable	Value at iteration 25	Design variable	Value at iteration 25
A_s	0.0116 m^2	S_2	0.226 m
b	0.492 m	S_3	0.142 m
h_f	0.415 m	$p(\mathbf{x}_{25})$	0.000188
b_w	0.196 m	$c_0(\mathbf{x}_{25})$	15.558
H_w	0.785 m	$c(\mathbf{x}_{25})p(\mathbf{x}_{25})$	1.459
A_v	0.000227 m^2	Objective	17.017
S_1	0.502 m		

Now suppose that the girder is subject to corrosion of its longitudinal reinforcement. We adopt a corrosion model similar to that used in Frangopol *et al.* (1997), where the diameter $D_b(T)$ of a longitudinal reinforcement bar at time T is given by

$$D_b(T) = \begin{cases} D_{b0} - 2\nu(T - T_I) & \text{for } T > T_I, \\ D_{b0} & \text{for } T \leqslant T_I, \end{cases} \tag{6.9}$$

with D_{b0} being the initial diameter, ν being the corrosion rate, and T_I being the corrosion initiation time. The factor 2 in (6.9) takes into account that the reinforcement bar is subject to corrosion from all sides. We assume $T_I = A + Bc_a$, where A is a lognormal random variable with mean 5 years and c.o.v. equal to 0.20, representing the time it takes to initiate corrosion with a 10 mm concrete cover, B is a lognormal random variable with mean 300 years/m and c.o.v. equal to 0.20, representing the additional time it takes to initiate corrosion per meter additional concrete cover, and c_a is the concrete cover in meters in addition to the 10 mm minimum cover. The additional concrete cover c_a is considered a design variable and is included in the design vector $\mathbf{x}$. We assume that the corrosion rate ν is lognormally distributed with mean 0.000040 m/years and c.o.v. 0.30. All the random variables are assumed to be statistically independent and lognormally distributed with the parameters as in Table 8.

As seen from Eq. (6.9), the area of bending reinforcement is reduced over time. Hence, the reinforced concrete girder is now a time-varying structure.

TABLE 8. Statistics of lognormal random variables describing corrosion.

Variable	Mean		c.o.v.
A	5	years	0.20
B	300	years/m	0.20
ν	0.000040	m/years	0.30

Since the area of the bending reinforcement is monotonically decreasing over time, it is reasonable to assume that the failure probability in a given time period is approximately equal to the failure probability at the end of the time period. Based on this assumption and a time period of 60 years, limit-state functions can be defined corresponding to the four failure modes of the girder. Details about this can be found in Royset *et al.* (2002). We obtain a design problem of the form, where the initial cost now is

$$c_0(\mathbf{x}) = 0.75C_s L_g A_s + C_s n_s A_v (h_f + h_w - \alpha + 0.5b_w) + C_c L_g (bh_f + b_w h_w) + C_c L_g b_w c_a, \quad (6.10)$$

and the cost of failure is $500c_0(\mathbf{x})$. Let the constraint on the system failure probability be, with no constraints on the component failure probabilities. The deterministic constraints defining $\mathbf{X}$ are as above except that we also include the two constraints $c_a \leqslant 0.05$ and $c_a \geqslant 0$.

We solve this instance of $\mathbf{P}_{3,\text{sys}}$ by means of Algorithm 5. Starting from the solution of the previous case, after 5 iterations (beyond which little change in the design is observed), the results in Table 9 are obtained. The system failure probability is evaluated using Monte Carlo simulation with c.o.v. 0.01.

We see in Table 9 that there is a discrepancy between a_5 and $p(\mathbf{x}_5)$. This is caused by the fact that a_5 is only approximately equal to $p(\mathbf{x}_5)$ whenever $\hat{\psi}_t^{(}\bar{\mathbf{x}}) = 0$. As the computations progress in Algorithm 5, the parameter t is automatically modified in a way such that the discrepancy between a_i and $p(\mathbf{x}_i)$ is reduced. Note that the constraint associated with maximum concrete cover is active, i.e., the use of maximum concrete cover is most cost efficient. Comparing the results in Table 7 with those in Table 9, we see that the objective function (total expected cost of the design) is higher in the latter case in account of the corrosion effect.

Now suppose it is decided to maintain the structure in intervals of 20 years, i.e., at 20 and 40 years after its construction. The time of maintenance can be incorporated as a design variable, but in this example we have fixed those times for simplicity. Let m_1 and m_2 be two design variables characterizing the maintenance effort at 20 years and 40 years, respectively.

TABLE 9. Results for $\mathbf{P}_{3,\mathrm{sys}}$ design of deteriorating girder.

Design variable	Value at iteration 5	Design variable	Value at iteration 5
A_s	0.0161 m^2	S_3	0.154 m
b	0.686 m	c_a	0.050 m
h_f	0.415 m	a_5	0.000246
b_w	0.197 m	$p(\mathbf{x}_5)$	0.000261
h_w	0.785 m	$c_0(\mathbf{x}_5)$	20.434
A_v	0.000255 m^2	$c(\mathbf{x}_5)p(\mathbf{x}_5)$	2.514
S_1	0.549 m	Objective	22.948
S_2	0.246 m		

Let $m_i = 0$ denote no maintenance, and $m_i = 1$ denote full maintenance, i.e., restoration to the initial state of the structure. Furthermore, we consider m_1 as the fraction of the aging of the structure from initial construction ($T = 0$) to the first maintenance action ($T = 20$ years), which is restored to its initial condition. Thus, $40 - 20m_1$ years is the effective age of the structure before the second maintenance action at $T = 40$ years. Similarly, m_2 is the fraction of the aging of the structure from initial construction ($T = 0$) to the second maintenance action ($T = 40$ years), which is mitigated by the second maintenance effort, i.e., $20 + (40 - 20m_1)(1 - m_2)$ years is the effective age of the structure at $T = 60$ years. We add the two variables m_1 and m_2 to the vector of design variables, i.e.,

$$\mathbf{x} = (A_s, b, h_f, b_w, h_w, A_v, S_1, S_2, S_3, c_a, m_1, m_2)\,. \tag{6.11}$$

We ensure the safety of the girder by imposing the constraint that the system failure probability over the 60 years lifetime be less than 0.00135. This probability is obtained as the probability of the union of the failure events during the intervals 0-20 years, 20-40 years and 40-60 years. For the reasons mentioned earlier, the event of failure within each interval is identical to the failure event at the end of the interval. The design is subject to the deterministic constraints as above with the additional constraints $m_i \leqslant 1$ and $0 \leqslant m_i$, $i = 1, 2$. Let the initial cost of the structure be as in (6.10), the cost of failure be $c(\mathbf{x}) = 500c_0(\mathbf{x})$, and the cost of maintenance be

$$c_m(\mathbf{x}) = c_y\,(20m_1 + (40 - 20m_1)\,m_2)\,, \tag{6.12}$$

where $c_y = 0.15$ represents the cost of complete restoration of the girder after a year's worth of corrosion. Note that the factor in front of m_2 represents the

effective age of the structure at 40 years. We solve this particular instance of $\mathbf{P}_{3,\text{sys}}$ by using the Algorithm 5. Starting from the solution point of the previous case, after 6 iterations the results in Table 10 are obtained. The system failure probability is evaluated using Monte Carlo simulation with c.o.v. 0.01.

TABLE 10. Results for $\mathbf{P}_{3,\text{por}}$ design of maintenance plan.

Design variable	Value at iteration 6	Design variable	Value at iteration 6
A_s	0.0144 m^2	c_a	0.050 m
b	0.612 m	m_1	0.4684
h_f	0.415 m	m_2	0.5316
b_w	0.196 m	a_6	0.000195
h_w	0.785 m	$p(\mathbf{x}_6)$	0.000198
A_v	0.000255 m^2	$c_0(\mathbf{x}_6)$	18.678
S_1	0.550 m	$c(\mathbf{x}_6)p(\mathbf{x}_6)$	1.824
S_2	0.247 m	$c_m(\mathbf{x}_6)$	1.699
S_3	0.155 m	Objective	22.201

Comparing the results in Tables 9 and 10, we see that the expected total cost of the design is smaller for the case with the option of maintenance (Table 10) than for the case without this option (Table 9). Also in the case with maintenance, there is a significant decrease in the initial cost, at the expense of a subsequent maintenance cost. The optimal solution suggests a larger maintenance effort at 40 years than at 20 years.

7. Conclusions

Five algorithms are developed for solving three classes of optimal structural design problems with functions representing the failure probability in the objective function and the constraint set definition. The failure probabilities can describe component or series structural system failures. Based on a first-order approximation to the failure probability, we have constructed approximating problems that can be solved repeatedly to obtain an approximation to a solution of the original design problem. By the use of higher-order reliability methods in the iterative scheme, e.g., second-order or Monte Carlo simulation, the approximating solution can be made to satisfy failure probability constraints for corresponding reliability measures.

The algorithms have stronger convergence properties than other algorithms found in the literature. Hence, the proposed algorithms are expected to be numerically more efficient and robust than algorithms based on heuristics. A significant advantage of the new algorithms is the flexibility in the selection of the reliability method. The approximating problems are semi-infinite optimization problems that can be solved using algorithms from the literature.

Numerical examples demonstrate that the new algorithms can be used in design and maintenance planning and with models involving both time-invariant and time-variant failure probabilities.

Acknowledgements

Financial support from the Taisei Chair in Civil Engineering at UC Berkeley, the National Science Foundation under grant ECS-9900985, and the UC Berkeley Space Sciences Laboratory and Lockheed Martin Advanced Technology Center Mini-grant Program is acknowledged.

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