AMAS Course on Reliability-Based Design and Optimisation
RBO'03 – (pp.345–405) – Warsaw, September 15-18, 2003

Objective functions for reliability-oriented structural optimization

H. STREICHER and R. RACKWITZ

Technische Universität München
Munich, Germany

Objective functions for discounted cost optimization based on a continuous renewal model for a series of cases are presented. They include failures and subsequent renewals by crossings of loading processes or random disturbances out of safe states of structural components, failures due to aging, non-constant benefit and damage functions, finite renewal times, repeated reconstructions at renewals and inspection and repair. A method for reliability-oriented time-variant structural optimization of separable (independent) series systems using first order reliability methods (FORM) in standard space is developed generalizing theories proposed earlier for component problems and time-invariant series system problems in a special one-level approach. Certain improvements by taking account of dependencies among failure modes are also proposed. Numerical Laplace transforms are proposed for the treatment of aging components. The optimization problem is solved by a newly developed gradient-based algorithm. A one-level optimization is proposed by adding the first-order Kuhn–Tucker optimality conditions for the design points of the series system reliability problem as constraints to the cost optimization problem using first order reliability methods (FORM) in standard space. Some algorithmic details are discussed. The approach is demonstrated at examples.

Key words: *structural reliability, one-level optimization, time-variant problems, outcrossing approach, series systems.*

1. Introduction

The calculation of failure probabilities or reliability indices for given sets of basic variables or random processes, limit state functions and deterministic parameters is well known. It requires solution of an optimization task if modern FORM/SORM is used. The determination of a certain design parameter set, e.g., initial cost or weight of a structure, in order to maximize benefits or to make efficient use of resources is much more difficult and involves another optimization task. Reliability-oriented optimization of design parameters is more expensive than simple reliability analysis. Both tasks can, however, be combined in the inverse problem of finding optimal designs with or without reliability restrictions. In this one-level approach the first-order Kuhn–Tucker optimality conditions of the reliability problem(s) are added as constraints to the overall cost optimization problem. Techniques have been developed so far for time-invariant and time-variant component problems [27, 28, 30] and for time-invariant series system problems [29] based on this concept. Time-variant series systems are first dealt with in [54]. Although conceptionally similar, each of the cases requires special handling of the details.

Time-variant optimization concepts making use of a simple renewal model have been proposed as early as 1971 by Rosenblueth/Mendoza [45] with special reference to earthquake resistant design. More generality has been added by Hasofer [20] and Rosenblueth [46] and lately in [21] and [40]. In this paper the classical renewal model is briefly reviewed. The tools of Laplace transforms are found to be extremely useful. An attempt is then made to further generalize the model to cover new fields of application, for example, finite renewal times, repeated reconstruction at renewal, series systems and inspection and repair. This requires new computational methods which are developed to a certain extent.

The paper is organized as follows. A review of the readily available reliability models is given first. Then, the basics of renewal theory as needed for the further developments are presented. Section 4 is devoted to the detailed discussion of the renewal model in the context of cost-benefit optimization of technical systems including some new results. Section 5 summarizes the different types of possible constraints and gives a brief overview on some recent results on suitable public risk acceptance criteria. Section 6 then focuses on the details of the techniques of cost-benefit optimization in a one-level approach. Several illustrative examples conclude the paper.

2. Review of Reliability models

It is assumed that classical FORM/SORM is used. Safe and failure domains are separated by differentiable limit state surfaces $h(\mathbf{x}, \mathbf{p}, t) = 0$ where $\mathbf{x}$ is a n-dimensional vector of uncertain (process) variables with continuous distribution function $F_X(\mathbf{x}, \mathbf{p}, t)$, $\mathbf{p}$ = a parameter vector and t = time. Also, it is assumed that a unique probability distribution transformation $\mathbf{x} = \mathbf{T}(\mathbf{u})$ exists where $\mathbf{u}$ is an independent standard normal vector so that $g(\mathbf{u}, \mathbf{p}, t) = 0$ (see [22] or approximate transformations in [10, 60]). Finally, it is assumed that a unique β–point for each failure mode exists, i.e. for which $\beta_k = \|\mathbf{u}^*\| = \max\{\|\mathbf{u}\|\}$ for $\{\mathbf{u} : g_k(\mathbf{u}, \mathbf{p}, t) \leq 0\}$. Note that $\beta_k > 0$ for $g_k(\mathbf{0}, \mathbf{p}, t) > 0$ and $\beta_k \leq 0$ for $g_k(\mathbf{0}, \mathbf{p}, t) \leq 0$. It follows that the instantaneous componental failure probability is [36]

$$P_{f,k}(t) = \int\limits_{g_k(\mathbf{u},\mathbf{p},t)\leq 0} \varphi(\mathbf{u})\, d\mathbf{u} \approx \Phi(-\beta_k(t))C, \tag{2.1}$$

where C is a correction either determined by SORM [3] or any other suitable method for an improvement of FORM results such as importance sampling [24].

Time-variant problems are substantially more complicated and computationally also more difficult. Failure time distributions are required and the failure probability is defined as:

$$P_f(0,t) = P(T \leq t) = F_T(t).$$

For strictly monotonic cumulative damage phenomena $F_T(t)$ can be computed from

$$F_T(t) = P_f(t) = P\left(g(\mathbf{X},t) \leq 0\right)) \approx \Phi(-\beta(t))C, \tag{2.2}$$

where $\beta(t)$ is the usual geometric reliability index and C is a correction factor. Quite generally, one ignores such corrections. The failure density is $f(t) = -\varphi(\beta(t))\frac{d\beta(t)}{dt}$ and $r(t) = -\frac{\varphi(\beta(t))}{\Phi(\beta(t))}\frac{d\beta(t)}{dt}$ the risk function which according to our assumption is increasing. Mean and variance of the failure times can be computed from

$$E[T^k] = \int\limits_0^\infty kt^{k-1}(1 - F_T(t))dt \tag{2.3}$$

for $T \geq 0$. For $k = 1$ we obtain the mean $E[T]$ and for $k = 2$ the second moment $E[T^2]$, respectively, and therefore $Var\,[T] = E[T^2] - E[T]^2$.

Because straightforward analytical results for failure time (first passage) distributions under random process loading are almost entirely missing one uses the so-called outcrossing approach for time-variant problems instead. One can derive an upper bound

$$P_f(0,t) = P(T \leq t) \leq P_f(0) + E\left[N^+(0,t)\right] \tag{2.4}$$

with the mean number of outcrossings $E\left[N^+(0,t)\right]$ given as

$$E\left[N^+(0,t)\right] = \int\limits_0^t \nu^+(\tau)d\tau, \tag{2.5}$$

and where $\nu^+(\tau)$ is the outcrossing rate defined as

$$\nu^+(\tau) = \lim_{\vartheta \to 0} \frac{P(\{\mathbf{X}(\tau) \in \bar{V}(\tau)\} \cap \{\mathbf{X}(\tau+\vartheta) \in V(\tau+\vartheta)\}}{\vartheta}. \tag{2.6}$$

$V(.)$ and $\bar{V}(.)$ denote failure and safe domain, respectively. Clearly, for stationary processes we have $E\,[N^+(0,t)] = \nu^+ t$. The vector $\mathbf{X}(\tau)$ collects all simple random variables and random processes. Under certain conditions (strong mixing of the outcrossing process) an important asymptotic result has been proven [9]

$$P_f(0,t) = P(T \le t) \sim 1 - \exp\left[-E\left[N^+(0,t)\right]\right], \tag{2.7}$$

i.e., the exponential distribution for failure times which will turn out to be of utmost importance in the following. In this model outcrossings form a Poissonian point process.

In the stationary case, arbitrary limit state function and loading by a combination of a vectorial rectangular wave renewal Gaussian process with jump rates λ_j and a vectorial differentiable Gaussian process with covariance function matrix $\mathbf{R}(\tau)$ the outcrossing rate according to FORM is [58, 5]:

$$\nu^+(\mathbf{p}) = \left(\sum_{i=1}^{n_J} \lambda_i \Phi_2(\beta(\mathbf{p}), -\beta(\mathbf{p}); \rho_i(\mathbf{p})) + \omega_0 \frac{\varphi(\beta(\mathbf{p}))}{\sqrt{2\pi}}\right). \tag{2.8}$$

$\Phi_2(.,.;.)$ is the bivariate standard normal integral with correlation coefficient $\rho_i = 1 - \alpha_i^2$, $\beta(\mathbf{p}) = \|\mathbf{u}^*\|$ and ω_0 is the central frequency with which the process outcrosses the limit state function (i.e.: $\omega_0^2 \approx \mathbf{n}(\mathbf{u}^*,\mathbf{p})^T \ddot{\mathbf{R}}(0)\mathbf{n}(\mathbf{u}^*,\mathbf{p})$, $\mathbf{n}(\mathbf{u}^*,\mathbf{p}) = -\alpha^T(\mathbf{u}^*,\mathbf{p}) = -\frac{\mathbf{u}^*}{\beta(\mathbf{p})}$, $\ddot{\mathbf{R}}(0) = \mathbf{E}\left[\dot{\mathbf{U}}\dot{\mathbf{U}}^T\right]$). A combination of a rectangular wave renewal process and a differentiable process is possible because crossings are assumed to be regular processes, i.e. processes for which more than one crossings in a short time interval have probability→ 0. Eq. (2.8) may be multiplied by a SORM-correction factor $C_{SORM}(\mathbf{p})$ involving curvature information of $g(\mathbf{u},\mathbf{p},t) = 0$ in $\mathbf{u}^*$ [4, 38, 39, 40]. For brevity of notation this is not done herein. Rectangular wave renewal processes must have independent components but can have arbitrary distribution functions. Certain non-normal differentiable processes can also be handled after a suitable probability distribution transformation [10, 60]. Outcrossing rates have also been established for non-stationary problems which are not given here.

If in some application one is forced to use a time-variant reliability method for non-stationary problems the asymptotic life time distribution is

$$F(t) = 1 - \exp\left[-\int_0^t \nu^+(\tau) d\tau\right] \tag{2.9}$$

with density

$$f(t) = \nu^+(t) \exp\left[-\int_0^t \nu^+(\tau)d\tau\right]. \tag{2.10}$$

Instead of Eq. (2.9) it is frequently better to use the well-known upper bound, at least for aging problems

$$F_T(t) = P_f(t) \le P_f(0) + \int_0^t \nu^+(\tau)d\tau \le 1, \tag{2.11}$$

where $P_f(0) = 0$ in many cases. The corresponding density is

$$f_T(\tau) \approx P_f(0)\delta(0) + \nu^+(\tau). \tag{2.12}$$

This density should be close to the exact one for aging problems but is less suitable for the stationary case. The advantage of these formulations is that well-known FORM/SORM-methodology is applicable, at least for more complicated problems and for outcrossing rates $\nu^+(\tau)$ depending on a random vector $\mathbf{R}$ [49].

3. Elements of Renewal Theory

Renewal processes generate a sequence of points whose interarrival times are independent. They have proven to be a very powerful tool in reliability theory. Assume that a component is replaced after failure by a new, stochastically identical component.

At first we determine the number of renewals in a given time interval. The components have independent identically distributed life times $\tau_i, \tau_2, \ldots, \tau_n$ and the failure times are $T_1 = \tau_1$, $T_2 = T_1 + \tau_2, \ldots$, $T_n = T_{n-1} + \tau_n$ (see Fig. 1). Obviously,

$$P(N(t) > n) = P(T_n \le t) = P\Big(\sum_{i=1}^{n} \tau_i \le t\Big) = F_n(t) \tag{3.1}$$

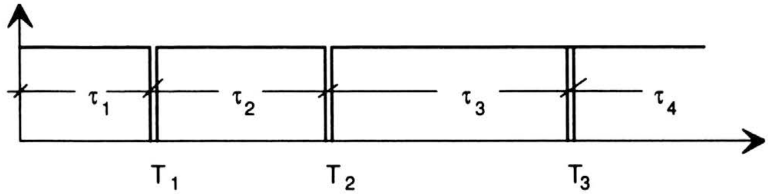

FIGURE 1. Trajectory of renewal process.

and

$$P(N(t) = n) = F_n(t) - F_{n+1}(t), \tag{3.2}$$

where $F_i(t) = P(\sum_{j=1}^{i} \tau_j \leq t)$. Assume that the random life times have finite moments up to arbitrary order. The mean number of failures in $[0, t]$ then is

$$\begin{aligned} H(t) = E[N(t)] &= \sum_{n=1}^{\infty} n\ P(N(t) = n) = \sum_{n=1}^{\infty} n(F_n(t) - F_{n+1}(t)) \\ &= \sum_{n=1}^{\infty} F_n(t) = \sum_{n=1}^{\infty} \int_0^t f_n(u)du = \int_0^t h(u)du \end{aligned} \tag{3.3}$$

$F_{n+1}(t)$ can be replaced by $\int F_n(t-u)dF(u)$. Therefore, in terms of an integral equation it is equivalently

$$H(t) = F(t) + \sum_{n=1}^{\infty} \int_0^t F_n(t-u)dF(u) = F(t) + \int_0^t H(t-u)dF(u) \tag{3.4}$$

This equation is called *renewal equation*. It is mainly of theoratical interest. The function $H(t) = E[N(t)]$ is denoted by renewal function. Its derivative

$$h(t) = \lim_{t \to 0^+} \frac{P(\text{one or more renewals in } [t, t+\Delta t])}{\Delta t} = \frac{\partial H(t)}{\partial t} = \sum_{n=1}^{\infty} f_n(t) \tag{3.5}$$

is denoted by renewal density or renewal intensity but in the context of reliability problems also by failure rate. The quantity $h(t)$ can be determined easily only in some special cases. For a exponential reliability function $R(t) = \exp[-\lambda t]$ one determines $H(t) = \lambda t$ and $h(t) = \lambda$ as one can easily verify from $H(t) = \sum_{n=1}^{\infty} n\ P(N(t) = n) = \sum_{n=1}^{\infty} n \frac{(\lambda t)^n}{n!} \exp[-\lambda t] = \lambda t$. For normally distributed renewal times we have

$$h(t) = \sum_{n=1}^{\infty} \frac{1}{\sigma\sqrt{n}} \varphi\Big(\frac{t - nm}{\sigma\sqrt{n}}\Big), \tag{3.6}$$

where m is the mean failure time and σ its standard deviation making use of the fact that a normal distribution is maintained under convolution. Since failure time distributions are valid only for $t \geq 0$ a more appropriate assumption is the Γ–distribution for integer k, also stable under convolution, and $m = k/\lambda$ and $\sigma = \sqrt{k}/\lambda$

$$h(t) = \sum_{n=1}^{\infty} \frac{\lambda^{nk} t^{nk-1}}{\Gamma(nk)} \exp\left[-\lambda t\right] = \frac{\lambda}{k} \sum_{j=1}^{k-1} \epsilon(k)^j \exp\left[\lambda t(\epsilon(k)^j - 1)\right], \tag{3.7}$$

where $\epsilon(k) = \cos(2\pi/k) + i\sin(2\pi/k)$ for $k > 1$. The renewal density has a characteristic damped oscillation type curve (see Fig. 2). The oscillations are larger for larger coefficients of variation of the individual failure times but, as illustrated, damp out rather soon.

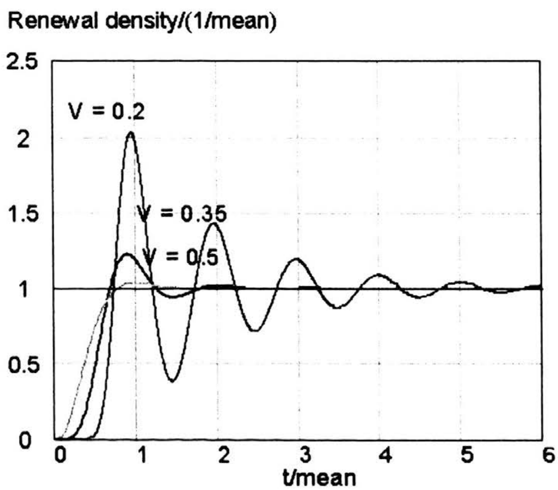

FIGURE 2. Renewal intensity divided by asymptotic value versus time divided by mean failure time.

For $H(t)$ there are some simple bounds which can easily be verified:

$$F(t) \leq H(t) \leq \frac{F(t)}{1 - F(t)}. \tag{3.8}$$

Further, several important asymptotic results have been proven [2]:

$$\lim_{t\to\infty} \frac{H(t)}{t} = \frac{1}{m}, \tag{3.9}$$

$$\lim_{t\to\infty} [H(t+\tau) - H(t)] = \frac{\tau}{m}, \tag{3.10}$$

$$\lim_{t\to\infty} h(t) = \frac{1}{m} \quad \text{if} \quad tf(t) \to 0 \text{ for } t \to \infty, \tag{3.11}$$

$$\lim_{t\to\infty} [H(t) - \frac{t}{m}] = \frac{\sigma^2}{2m^2} - \frac{1}{2}. \tag{3.12}$$

Equation (3.9) coincides with intuition. The mean number of failures is inversely proportional to the mean value of life times m. The second statement (3.10) has a similar interpretation and is stronger than (3.9). The third and forth statements concern stationarity. Equation (3.12) with σ the standard deviation of the failure times provides a better estimate for the renewal function. From Fig. 2 one concludes that the asymptotic result Eq. (3.11) holds in good approximation for all renewal processes unless the coefficient of variation of the interarrival times is very small.

The foregoing describes a so-called *ordinary renewal process*. For the *modified renewal process* it is assumed that the first time to failure has another distribution than all the other failure times. This generalization is useful for aging components and whose age is known. For the *equilibrium renewal process* it is assumed that renewals have occurred already for infinitely long time. The zero of the time axis, therefore, falls randomly in between two consecutive renewals.

In the following we will extensively work with Laplace transforms. Laplace transforms are defined by $f^*(\gamma) = \int_0^\infty e^{-\gamma t} f(t)dt$. If $f(t), t \geq 0$, is a probability density, it is $f^*(0) = 1$, $f^*(\infty) = 0$ and $0 < f^*(\gamma) \leq 1$ for all $\gamma \geq 0$. The Laplace transform can also be written as $f^*(\gamma) = E_T\left[e^{-\gamma T}\right]$. In the transformed space one can easily show that there is $h^*(\gamma) = f^*(\gamma)g^*(\gamma)$ for $h(t) = \int_0^\infty f(t-\tau)g(\tau)d\tau$. For independent, identically distributed interarrival times of failures the density of the time to the n-th event $f_n(t)$ then is $f_n(t) = \int_0^\infty f_{n-1}(t-\tau)f(\tau)d\tau$ and, therefore $f_n^*(\gamma) = f_{n-1}^*(\gamma)f^*(\gamma) = f^*(\gamma)^{n-1}f^*(\gamma)$. It is seen that the convolution operations necessary, for example, in Eq. (3.5) can be performed very easily. In particular, the Laplace transform of the renewal intensity in Eq. (3.5) is

$$h^*(\gamma) = \int_0^\infty \sum_{n=1}^\infty f_n(t) \exp\left[-\gamma t\right] dt = \sum_{n=1}^\infty f^*(\gamma)f^*(\gamma)^{n-1} = \frac{f^*(\gamma)}{1-f^*(\gamma)}. \tag{3.13}$$

Finally, the asymptotic result for the renewal density in Eq. (3.11) is restated in terms of Laplace transform (see [8], p. 55):

$$\lim_{t\to\infty} h(t) = \lim_{\gamma\to 0} \gamma h^*(\gamma) = \frac{1}{m} = \lambda \text{ for } f(t) \to_{t\to\infty} 0, \tag{3.14}$$

where m is the mean time between renewals (or failures) and λ is the failure rate.

4. Objective functions

4.1. General

A structural or any other technical facility is optimal if the following objective function is maximized:

$$Z(\mathbf{p}) = B(\mathbf{p}) - C(\mathbf{p}) - D(\mathbf{p}). \tag{4.1}$$

Without loss of generality it is assumed that all quantities in Eq. (4.1) can be measured in monetary units. $\mathbf{p}$ is the vector of all safety relevant parameters. $B(\mathbf{p})$ is the benefit derived from the existence of the facility, $C(\mathbf{p})$ is the cost of design and construction, usually decomposed into a cost C_0 independent of $\mathbf{p}$ and cost dependent on $\mathbf{p}$, and $D(\mathbf{p})$ is the cost in case of failure. Statistical decision theory dictates that expected values are to be taken [59]. In the following it is assumed that $B(\mathbf{p}), C(\mathbf{p})$ and $D(\mathbf{p})$ are differentiable in each component of $\mathbf{p}$. And it is reasonably assumed that $C(\mathbf{p})$ increases whereas $D(\mathbf{p})$ decreases in each component of $\mathbf{p}$.

The structure which eventually will fail or replaced after some time will have to be optimized at the decision point, i.e. at time $t = 0$. Therefore, all cost need to be discounted. A continuous discounting function is assumed which is accurate enough for all practical purposes

$$\delta(t) = \exp\left[-\gamma t\right], \tag{4.2}$$

where γ is the (tax-free) interest rate. For example, if failure with consequences D_0 occurs at time t (in years) the discounted damage is $D(t) = D_0 \exp\left[-\gamma t\right]$. If a yearly discount rate γ' is defined we have $\gamma = \ln(1 + \gamma')$. Also, it is assumed that construction cost $C(\mathbf{p})$ are without cost of financing. They can, however, be included easily.

In general, one has to distinguish between at least two replacement strategies; one where the facility is given up after service or failure and one where the facility is systematically replaced after failure or obsolescence. One third possibility is replacement after inspection and repair. Further we distinguish between structures which fail upon completion or never and structures which fail at a random point in time much later due to service loads, extreme external disturbances or deterioration. The first option implies that loads on the structure are time-invariant. At first sight there is no particular preference for either of the replacement strategies. For infrastructure facilities the second category is a natural strategy. Structures used only once, e.g., special auxiliary construction structures, boosters for space transport vehicles or devices exploiting limited deposits, might fall into the first category. In this paper focus is on time-variant problems and systematic reconstruction. For one mission structures the reader is referred to [21] and [40].

4.2. Standard time-invariant case [45]

The objective function for failure during or immediately at the start of operation and abandonment after failure is:

$$Z(\mathbf{p}) = B^* R_f(\mathbf{p}) - C(\mathbf{p}) - HP_f(\mathbf{p}) = B^* - C(\mathbf{p}) - (B^* + H)P_f(\mathbf{p}), \quad (4.3)$$

where $R_f(\mathbf{p}) = 1 - P_f(\mathbf{p})$ is reliability and $P_f(\mathbf{p})$ failure probability. For failure at the start of operation and systematic reconstruction (until a realization survives) one has

$$\begin{aligned} Z(\mathbf{p}) &= B^* - C(\mathbf{p}) - (C(\mathbf{p}) + H) \sum_{i=1}^{\infty} i P_f(\mathbf{p})^i R_f(\mathbf{p}) \\ &= B^* - C(\mathbf{p}) - (C(\mathbf{p}) + H) \frac{P_f(\mathbf{p})}{1 - P_f(\mathbf{p}))}, \end{aligned} \quad (4.4)$$

because for independent failure events

$$\begin{aligned} \sum_{i=1}^{\infty} i P_f(\mathbf{p})^i R_f(\mathbf{p}) &= (1 - P_f(\mathbf{p})) \sum_{i=1}^{\infty} i P_f(\mathbf{p})^i \\ &= (1 - P_f(\mathbf{p})) \frac{P_f(\mathbf{p})}{(1 - P_f(\mathbf{p}))^2} = \frac{P_f(\mathbf{p})}{1 - P_f(\mathbf{p})}. \end{aligned}$$

This result for infinite sums will turn out to be very important in the following.

After failure one usually investigates the causes of failure and updates the design. Here, we assume that the design is already optimal so that there is no reason to change the design rules and new realizations are stochastically independent.

For a intended service time t_s the benefit term becomes

$$B(t_s) = \int_0^{t_s} b(t)\delta(t)dt. \quad (4.5)$$

For constant benefit rate $b(t) = b$ it is

$$B(t_s) = \frac{b}{\gamma}\left[1 - \exp\left[-\gamma t_s\right]\right], \tag{4.6}$$

and therefore for $t_s \to \infty$

$$B^* = \frac{b}{\gamma}. \tag{4.7}$$

4.3. Standard time-variant case [40]

For easy reference the standard case is first rederived for systematic reconstruction. For the moment, assume reconstruction times to be negligibly short. The times between failure (renewal) events have identical distribution function $F(t, \mathbf{p})$ with probability density $f(t, \mathbf{p})$ and are independent. The independence assumption needs to be verified carefully. Here again, one has to assume that loads and resistances in the system are independent for consecutive renewal periods and there is no change in the design rules after the first and all consecutive failures. Even if designs change failure time distributions must remain the same. For constant benefit per time unit $b(t) = b$ and $f_n(t, \mathbf{p})$ the density of the time to the n-th renewal an objective function can be derived by making use of the convolution theorem for Laplace transforms

$$\begin{aligned} Z(\mathbf{p}) &= \int_0^\infty b e^{-\gamma t} dt - C(\mathbf{p}) - (C(\mathbf{p}) + H) \sum_{n=1}^{\infty} \int_0^\infty e^{-\gamma t} f_n(t, \mathbf{p}) dt \\ &= \int_0^\infty b e^{-\gamma t} dt - C(\mathbf{p}) - (C(\mathbf{p}) + H) \sum_{n=1}^{\infty} f^*(\gamma, \mathbf{p})^{n-1} f^*(\gamma, \mathbf{p}) \\ &= \frac{b}{\gamma} - C(\mathbf{p}) - (C(\mathbf{p}) + H) \frac{f^*(\gamma, \mathbf{p})}{1 - f^*(\gamma, \mathbf{p})} \\ &= \frac{b}{\gamma} - C(\mathbf{p}) - (C(\mathbf{p}) + H) h^*(\gamma, \mathbf{p}), \end{aligned} \tag{4.8}$$

where $h^*(\gamma, \mathbf{p})$ is the Laplace transform of the renewal density (renewal intensity) $h(t, \mathbf{p}) = \sum_{k=1}^{\infty} f_k(t, \mathbf{p})$ (see Eq. (3.13)). H is the expected monetary loss in case of failure including direct failure cost, loss of business and, of course, the cost to reduce the risk to human life and limb. We may also include the cost of demolition in H.

In principle, renewal theory also allows for the case that the time to the first renewal is different from all others. This refinement by the modified renewal process is done here only for the case just considered, for the sake of easy notation in the following. Let $f_1(t)$ be the density of the time to first failure and $f(t)$ the density of all other failure times. It is, in fact, easy to show that

$$\begin{aligned} Z(\mathbf{p}) &= \int_0^\infty be^{-\gamma t}dt - C(\mathbf{p}) - (C(\mathbf{p}) + H)\sum_{n=1}^{\infty}\int_0^\infty e^{-\gamma t} f_n(t,\mathbf{p})dt \\ &= \int_0^\infty be^{-\gamma t}dt - C(\mathbf{p}) - (C(\mathbf{p}) + H)\sum_{n=1}^{\infty} f^*(\gamma,\mathbf{p})^{n-1} f_1^*(\gamma,\mathbf{p}) \\ &= \frac{b}{\gamma} - C(\mathbf{p}) - (C(\mathbf{p}) + H)\frac{f_1^*(\gamma,\mathbf{p})}{1 - f^*(\gamma,\mathbf{p})} \\ &= \frac{b}{\gamma} - C(\mathbf{p}) - (C(\mathbf{p}) + H)h_1^*(\gamma,\mathbf{p}). \end{aligned} \tag{4.9}$$

Laplace transforms are analytic only for a few failure models. For easy reference some important models are collected in Table 1 [40].

The one for a normal failure time distribution is especially important because it is also approximately the (two-sided) Laplace transform for an arbitrary failure time distribution with known mean m and standard deviation σ provided that $V = \frac{\sigma}{m}$ is small and $\gamma \leq 2m/\sigma^2$ [20]. For this we use the "cumulant-generating function". For the case of a Laplace transform we have

$$\ln f^*(\gamma) = K(-\gamma) = -m\gamma + \frac{1}{2}\sigma^2\gamma^2 + \sum_{n=3}^{\infty}\frac{K_n}{n!}(-\gamma)^n,$$

where $K(\theta)$ is the usual cumulant-generating function of $f(t)$. If we assume that γ is low enough to neglect powers of γ above 2, we have

$$f^*(\gamma) \approx \exp(-m\gamma + \frac{1}{2}\sigma^2\gamma^2). \tag{4.10}$$

which coincides with the two-sided Laplace transform for the untruncated normal distribution. If the failure times have an exponential distribution one obtains

$$h^*(\gamma,\mathbf{p}) = \frac{\lambda(\mathbf{p})}{\gamma}, \tag{4.11}$$

since $f^*(\gamma,\mathbf{p}) = \frac{\lambda(\mathbf{p})}{\gamma+\lambda(\mathbf{p})}$. This result is especially relevant because the parameter $\lambda(\mathbf{p})$ may be replaced asymptotically by the stationary outcrossing rate $\nu^+(\mathbf{p})$ frequently used in time-variant structural reliability analysis

TABLE 1. Laplace transform for some failure models.

Name	*Density function* $f(t)$	*Laplace transform* $f^*(\gamma)$
Deterministic	$\delta(a)$	$\exp\left[-a\gamma\right]$
Uniform	$\frac{1}{b-a}$	$\frac{\exp[-a\gamma]-\exp[-b\gamma]}{\gamma(b-a)}$
Exponential	$\lambda \exp\left[-\lambda t\right]$	$\frac{\lambda}{\gamma+\lambda}$
Gamma	$\frac{\lambda^k}{\Gamma(k)} t^{k-1} \exp\left[-\lambda t\right]$	$\left(\frac{\lambda}{\gamma+\lambda}\right)^k$
Rayleigh	$\frac{2t}{w^2} \exp\left[-\left(\frac{t}{w}\right)^2\right]$	$\frac{\gamma w^2 \sqrt{\pi}}{2} \operatorname{erf} c \left(\frac{1}{2}\gamma w\right) \exp\left[\frac{1}{4}\gamma^2 w^2\right] + w$
Normal $(t \geq 0)$	$\frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left[-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2\right] \frac{1}{\Phi(\frac{m}{\sigma})}$	$\exp\left[\frac{1}{2}\gamma\left(\gamma\sigma^2 - 2m\right)\right] \frac{1+\operatorname{erf}\left[\frac{1}{\sigma\sqrt{2}}(m-\gamma\sigma^2\right]}{2\Phi(\frac{m}{\sigma})}$
Inverse Normal	$\frac{t_0}{\sigma}(2\pi t^3)^{-1/2} \exp\left[-\frac{a^2 t}{2\sigma^2} + \frac{a t_0}{\sigma^2} - \frac{t_0^2}{2\sigma^2 t}\right]$	$\exp\left[-t_0\sqrt{\frac{a^2}{\sigma^4} + \frac{2\gamma}{\sigma^2}} - \frac{a}{\sigma^2}\right]$

(see Eq. (2.7)). If $\nu^+(\mathbf{p})$ depends on an uncertain parameter vector $\mathbf{R}$ and/or a random sequence $\mathbf{Q}$ one should use $E_{\mathbf{R},\mathbf{Q}}[\nu^+(\mathbf{p},\mathbf{R},\mathbf{Q})]$ instead:

$$Z(\mathbf{p}) = \frac{b}{\gamma} - C(\mathbf{p}) - (C(\mathbf{p}) + H)\frac{E_{\mathbf{R},\mathbf{Q}}\left[\nu^+(\mathbf{p},\mathbf{R},\mathbf{Q})\right]}{\gamma}. \tag{4.12}$$

It is seen that continuous discounting and continuous failure models lead to relatively simple, analytical results. Completely parallel results, however, can be obtained for discrete failure models and discrete discounting [57].

4.4. Random disturbances [20, 46]

If, at extreme loading events (e.g., flood, wind storm, earthquake, explosion) having a density $f(t)$ of interarrival times, (independent) failure occurs with probability $P_f(\mathbf{p})$,the density of times between failures is

$$g_n(t,\mathbf{p}) = \sum_{k=1}^{n} f_k(t)P_f(\mathbf{p})R_f(\mathbf{p})^{k-1}, \tag{4.13}$$

and after taking Laplace transforms [20, 46]:

$$g^*(\gamma,\mathbf{p}) = \sum_{n=1}^{\infty} f^*(\gamma)P_f(\mathbf{p})[f^*(\gamma)R_f(\mathbf{p})]^{n-1} = \frac{P_f(\mathbf{p})f^*(\gamma)}{1 - R_f(\mathbf{p})f^*(\gamma)} \tag{4.14}$$

with $R_f(\mathbf{p}) = 1 - P_f(\mathbf{p})$. The damage term becomes:

$$D(p) = (C(\mathbf{p}) + H)\frac{g^*(\gamma,\mathbf{p})}{1 - g^*(\gamma,\mathbf{p})} = (C(\mathbf{p}) + H)\frac{P_f(\mathbf{p})f^*(\gamma)}{1 - f^*(\gamma)}. \tag{4.15}$$

It may sometimes be realistic to change to a modified renewal process in which case $f^*(\gamma)$ in the numerator of Eq. (4.15) has to be replaced by $f_1^*(\gamma)$.

If, in particular, the loading events follow a stationary Poisson process with intensity λ we have

$$h^*(\gamma,\mathbf{p}) = \frac{\lambda P_f(\mathbf{p})}{\gamma}. \tag{4.16}$$

It is noted here that the memoryless nature of a Poisson process for the disturbances implies that $f^*(\gamma) = f_1^*(\gamma)$.

For uncertain H in the failure event (exceedance of different limit states) one replaces H by $E[H] = \sum p_i H_i$ (p_i = probability of exactly failure loss H_i, $\sum_i p_i = 1$). More generally, the damage term may be replaced by a so-called risk integral, i.e. by $\int (C(p) + H(x,p))f(x)h^*(x,\gamma,\mathbf{p})dx$ where x a damage parameter and $f(x)$ its probability density. If there are Poissonian disturbances of different kind whose failure leads to the same losses one replaces λ by $\sum \lambda_j$.

4.5. The case of a one-mission structure

The case of giving up the facility after failure or completion of the intended task is extensively treated in [40]. Here, we present only the main idea. Suppose that the structure is abandoned after the first failure and let $F_1(t)$ be the distribution function of the time T to first failure, with probability density function $f_1(t)$. Assume that the expected benefit per unit time, b, is constant during the life of the structure. Then the discounted expected benefit B for $t_s \to \infty$ is given by

$$B = \int_0^\infty \int_0^t b \exp(-\gamma\tau) d\tau f_1(t) dt = \frac{b}{\gamma} \int_0^\infty (1 - e^{-\gamma t}) f_1(t) dt = \frac{b}{\gamma} [1 - f_1^*(\gamma)], \tag{4.17}$$

where

$$f_1^*(\gamma) = \int_0^\infty e^{-\gamma t} f_1(t) dt \tag{4.18}$$

is the Laplace transform of the density function of the time to failure, with parameter γ, the interest rate. Similarly, the discounted expected cost D is given by

$$D = H \int_0^\infty e^{-\gamma t} f_1(t) dt = H f_1^*(\gamma). \tag{4.19}$$

The case of random disturbances is also interesting. Under the same assumptions as before one arrives at essentially the same result for $t_s \to \infty$

$$\begin{aligned} h_1^*(\gamma, \mathbf{p}) &= \sum_{n=1}^\infty f_1^*(\gamma) f_{n-1}^*(\gamma) P_f(\mathbf{p}) R_f(\mathbf{p}))^{n-1} \\ &= \sum_{n=1}^\infty f_1^*(\gamma) [f^*(\gamma)]^{n-1} P_f(\mathbf{p}) R_f(\mathbf{p})^{n-1} = \frac{P_f(\mathbf{p}) f_1^*(\gamma)}{1 - R_f(\mathbf{p}) f^*(\gamma)}. \end{aligned} \tag{4.20}$$

The present value of the damage is

$$D(\mathbf{p}) = H h_1^*(\gamma, \mathbf{p}), \tag{4.21}$$

and the benefit:

$$B = \frac{b}{\gamma}(1 - h_1^*(\gamma, \mathbf{p})). \tag{4.22}$$

For a Poissonian disturbance process one determines:

$$h^*(\gamma, \mathbf{p}) = \frac{P_f(\mathbf{p})\frac{\lambda}{\gamma+\lambda}}{1-(1-P_f(\mathbf{p}))\frac{\lambda}{\gamma+\lambda}} = \frac{P_f(\mathbf{p})\lambda}{\gamma + P_f(\mathbf{p})\lambda}. \tag{4.23}$$

However, in this case and $b = b(\tau)$ one can integrate directly:

$$\begin{aligned} B(t_s, \mathbf{p}) &= \int_0^{t_s}\int_0^{t} b(\tau)e^{-\gamma\tau}d\tau\lambda P_f(\mathbf{p})\exp\left[-\lambda P_f(\mathbf{p})t\right]dt \\ &= \frac{b}{\gamma+\lambda P_f(\mathbf{p})}\Bigg(1-\exp\left[-(\gamma+\lambda P_f(\mathbf{p}))t_s\right]\cdot \\ &\qquad \cdot\left((1+\frac{\lambda P_f(\mathbf{p})}{\gamma})\exp\left[\gamma t_s\right]-\frac{\lambda P_f(\mathbf{p})}{\gamma}\right)\Bigg), \end{aligned} \tag{4.24}$$

$$\begin{aligned} D(t_s, \mathbf{p}) &= H\int_0^{t_s} e^{-\gamma t}\lambda P_f(\mathbf{p})\exp\left[-\lambda P_f(\mathbf{p})t\right]dt \\ &= H\frac{\lambda P_f(\mathbf{p})}{\gamma+\lambda P_f(\mathbf{p})}\left(1-\exp\left[-(\gamma+\lambda P_f(\mathbf{p}))t_s\right]\right). \end{aligned} \tag{4.25}$$

This gives information how fast asymptotic conditions will be reached. For $t_s \to \infty$ we have

$$B^*(\mathbf{p}) = B(\infty, \mathbf{p}) = \frac{b}{\gamma+\lambda P_f(\mathbf{p})}, \tag{4.26}$$

$$D(\mathbf{p}) = H\frac{\lambda P_f(\mathbf{p})}{\gamma+\lambda P_f(\mathbf{p})}, \tag{4.27}$$

and therefore:

$$Z(\mathbf{p}) = \frac{b-\lambda P_f(\mathbf{p})H}{\gamma+\lambda P_f(\mathbf{p})} - C(\mathbf{p}). \tag{4.28}$$

4.6. Non-constant benefit function [21]

Assume that the benefit rate is not constant but an arbitrary function of time. At each failure (and renewal) it starts at $b(0)$. Following [21], let U_n be the time between the $n-1$-th and the n-th arrival and let

$$T_n = \sum_{r=1}^{n} U_r \tag{4.29}$$

be the time to the n-th arrival. We recollect that U_n is independent of T_{n-1} for $n = 2, 3, \ldots$. Given the U_n, the total discounted benefit B_T is given by

$$\begin{aligned} B_T &= \int_0^{U_1} e^{-\gamma t} b(t)dt + \sum_{n=2}^{\infty} \int_0^{U_n} b(t) e^{-\gamma(T_{n-1}+t)} dt \\ &= \int_0^{U_1} e^{-\gamma t} b(t)dt + \sum_{n=2}^{\infty} e^{-\gamma T_{n-1}} \int_0^{U_n} b(t) e^{-\gamma t} dt, \end{aligned} \tag{4.30}$$

and with

$$B_D(t) = \int_0^t e^{-\gamma u} b(u) du, \tag{4.31}$$

$$B_T = B_D(U_1) + \sum_{n=2}^{\infty} e^{-\gamma T_{n-1}} B_D(U_n). \tag{4.32}$$

Taking expectations it follows that

$$\begin{aligned} B = E(B_T) &= E[B_D(U_1)] + \sum_{n=2}^{\infty} E\left(e^{-\gamma T_{n-1}}\right) E[B_D(U_n)] \\ &= \int_0^{\infty} B_D(t) f(t) dt + \left(\sum_{n=2}^{\infty} \int_0^{\infty} e^{-\gamma t} f_{n-1}(t) dt \right) \int_0^{\infty} B_D(t) f(t) dt \end{aligned}$$

Using the results on Laplace transforms, we obtain

$$\begin{aligned} B &= \int_0^{\infty} B_D(t) f(t) dt + \left(\sum_{n=2}^{\infty} f^*(\gamma) [f^*(\gamma)]^{n-2} \right) \int_0^{\infty} B_D(t) f(t) dt \\ &= \int_0^{\infty} B_D(t) f(t) dt + \left[\frac{f^*(\gamma)}{1 - f^*(\gamma)} \right] \int_0^{\infty} B_D(t) f(t) dt \\ &= \frac{1}{1 - f^*(\gamma)} \int_0^{\infty} B_D(t) f(t) dt, \end{aligned} \tag{4.33}$$

and for a homogeneous Poissonian failure process with rate $\lambda(\mathbf{p})$:

$$B = \left(1 + \frac{\lambda(\mathbf{p})}{\gamma}\right) \int_0^{\infty} B_D(t) \lambda(\mathbf{p}) \exp\left[-\lambda(\mathbf{p}) t\right] dt. \tag{4.34}$$

4.7. Non-constant damage

Also, the damage term may depend on t. For example, the damage cost $H(t)$ can accumulate over time due to gradual storage of valuable goods or the reconstruction cost net of inflation can increase over time due to increasingly scarce resources. By generalizing a result in [16] it is possible to consider time-dependent damage cost $K(t,\mathbf{p}) = C(t,\mathbf{p}) + H(t)$ (see also [57])

$$D(\mathbf{p}) = \frac{\int\limits_0^\infty \exp\left[-\gamma t\right] K(t,\mathbf{p}) f(t,\mathbf{p}) dt}{1 - f^*(\gamma,\mathbf{p})}. \tag{4.35}$$

Clearly, the numerator is no more the classical Laplace transform of a failure density and the integral must remain finite.

4.8. Finite renewal times

Next we consider finite renewal times, i.e., finite reconstruction times, ignoring the rare case of failure under an external extreme loading event. During these times the facility cannot be used and it cannot fail. Let T_W be the (random) renewal times with density $f_W(t)$ and T_N be the (random) times of use with density $f_N(t)$. Therefore, $T = T_W + T_N$ is the time between failures (or renewals). An exact consideration is complicated. However, renewal theory shows that the availability of a system asymptotically equals:

$$A_W(\infty) = \lim_{t\to\infty} \frac{1}{t} \int\limits_0^t A_{inst}(x) dx = \frac{E\left[T_N\right]}{E\left[T_W\right] + E\left[T_N\right]}, \tag{4.36}$$

where $A_{inst}(x)$ is instantaneous availability. It follows that the benefit has to be multiplied by $A_W(\infty)$ so that:

$$Z(\mathbf{p}) \approx B\ A_W(\infty) - C(\mathbf{p}) - (C(\mathbf{p}) + H) A_W(\infty) h_A^*(\gamma,\mathbf{p}). \tag{4.37}$$

The renewal intensity $h_A^*(\gamma,\mathbf{p})$ is most easily be determined from the density of times between renewals $f_A(t)$ as a convolution of $f_W(t)$ and $f_N(t)$ whose Laplace transform simply is $f_A^*(t) = f_W^*(t)\ f_N^*(t)$. During a finite renewal the structure is supposed not to fail. Therefore, in first approximation the damage term is also multiplied by $A_W(\infty)$.

4.9. Repeated reconstruction at renewal

One also can consider possibly repeated failure at construction (or reconstruction), i.e., the facility is reconstructed after failure in the construction phase according to the same rules until it can be put into service. It is sufficient to consider the additional cost of reconstruction at a time which are (see also Eq. (4.4))

$$D_W(\mathbf{p}) = (C_W(\mathbf{p}) + H_W) \sum_{i=1}^{\infty} i P_{f,W}(\mathbf{p})^i R_{f,W}(\mathbf{p}) = (C_W(\mathbf{p}) + H_W) \frac{P_{f,W}(\mathbf{p})}{R_{f,W}(\mathbf{p})}.$$

The result is:

$$Z(\mathbf{p}) = B - \left(C(\mathbf{p}) + (C_W(p) + H_W) \frac{P_{f,W}(\mathbf{p})}{R_{f,W}(\mathbf{p})} \right) - (C(\mathbf{p}) + H) \left(1 + \frac{P_{f,W}(\mathbf{p})}{R_{f,W}(\mathbf{p})} \frac{(C_W(\mathbf{p}) + H_W)}{(C(\mathbf{p}) + H)} \right) h^*(\gamma, \mathbf{p}). \tag{4.38}$$

The additional factor reflects the fact that the reconstruction and damage cost $(C_W(\mathbf{p}) + H_W)$ can happen multiply with probability $P_{f,W}(\mathbf{p})$. Successful construction happens with probability $R_{f,W}(\mathbf{p}) = 1 - P_{f,W}(\mathbf{p})$. Note that the first construction also needs an additional term. It is useful to distinguish between reconstruction cost $C_W(\mathbf{p})$ and damage cost H_W in the erection phase and the reconstruction and damage cost $C(\mathbf{p}) + H$ during use of the facility.

This model also enables to estimate the length of the finite renewal time. If $E[T_{W,1}]$ is the expectation of independent, identically distributed reconstruction times, then, since $E[Y] = E\left[\sum_{i=1}^{N} X_i\right] = E[X]\, E[N]$ (N random and geometrically distributed according to $p(n) = P_{f,W}(\mathbf{p})^{n-1} R_{f,W}(\mathbf{p})$) we have $E[T_W] = E[T_{W,1}] / (1 - P_{f,W}(\mathbf{p}))$. In general, this time is only insignificantly larger than $E[T_{W,1}]$ for $(1 - P_{f,W}(\mathbf{p})) \lesssim 1$.

4.10. Independent failure modes and different failure causes

Assume for the moment two independent failure modes, denoted by "V_1" and "V_2", respectively, each requiring renewal after failure. The times between renewals then are distributed as $F(t) = 1 - (1 - F_{V_1}(t))(1 - F_{V_2}(t)) = 1 - \bar{F}_{V_1}(t)\bar{F}_{V_2}(t)$. The corresponding density is $f(t) = f_{V_1}(t)\bar{F}_{V_2}(t) + f_{V_2}(t)\bar{F}_{V_1}(t)$ and its Laplace transform is $f^{**}(\gamma, \mathbf{p}) = f^{**}_{V_1|\bar{V}_2}(\gamma) + f^{**}_{V_2|\bar{V}_1}(\gamma)$. It follows that

$$D(\mathbf{p}) = \frac{(C_1(\mathbf{p}) + H_1) f^{**}_{V_1|\bar{V}_2}(\gamma) + (C_2(\mathbf{p}) + H_2) f^{**}_{V_2|\bar{V}_1}(\gamma)}{1 - (f^{**}_{V_1|\bar{V}_2}(\gamma) + f^{**}_{V_2|\bar{V}_1}(\gamma))}. \tag{4.39}$$

This equation is derived as follows: Let $\theta_i = t_i - t_{i-1}$ be the times between renewals with density $f_{V_1,V_2}(t)$ and, for example, C_{V_1} and C_{V_2} the cost associated with the two types of renewals. Then, the expected cost is

$$D = E\left[\sum_{n=1}^{\infty}(C_{V_1} + C_{V_2})\exp\left[-\gamma\sum_{k=1}^{n}\theta_k\right]\right]$$
$$= \sum_{n=1}^{\infty} E\left[\exp\left(-\gamma\theta\right)\right]^{n-1} E\left[(C_{V_1} + C_{V_2})\exp\left[-\gamma\theta\right]\right]$$
$$= \frac{E\left[(C_{V_1} + C_{V_2})\exp\left[-\gamma\theta\right]\right]}{1 - E\left[\exp\left(-\gamma\theta\right)\right]} = \frac{C_{V_1} f^{**}_{V_1|\bar{V}_2} + C_{V_2} f^{**}_{V_2|\bar{V}_1}(\gamma)}{1 - (f^{**}_{V_1|\bar{V}_2}(\gamma) + f^{**}_{V_2|\bar{V}_1}(\gamma))}.$$

Here, we distinguish between ordinary Laplace transforms $f^*(\gamma)$ for densities and modified Laplace transforms $f^{**}(\gamma)$ for which $f^{**}(\gamma) \leq f^*(\gamma)$. One can generalize to more (independently) caused renewals:

$$D(\mathbf{p}) = \frac{\sum_{i=1}^{s} C_i(\mathbf{p}) f^{**}_{V_i|\cap_{j\neq i}\bar{V}_j}(\gamma)}{1 - \sum_{i=1}^{s} f^{**}_{V_i|\cap_{j\neq i}\bar{V}_j}(\gamma)} \leq \frac{\sum_{i=1}^{s} C_i(\mathbf{p}) f^{*}_{V_i}(\gamma)}{1 - \sum_{i=1}^{s} f^{*}_{V_i}(\gamma)} \tag{4.40}$$

with

$$f(t) = \sum_{i=1}^{s} f_i(t) \prod\nolimits_{j\neq i} \bar{F}_j(t),$$

and therefore

$$f^{**}_{V_i|\cap_{j\neq i}\bar{V}_j}(\gamma) = \int_0^{\infty} \exp\left[-\gamma t\right] f_i(t) \prod\nolimits_{j\neq i} \bar{F}_j(t) dt.$$

Frequently, the upper bound can be used and this has also been proposed in [29] and [55]. For independent Poissonian failure modes one can show easily that the upper bound is the exact result.

4.11. Obsolescence

At this point it is useful to introduce obsolescence as an important cause for renewal. Obsolescence occurs if the technical facility no more fulfills its function. For example, a bridge may become too narrow for the increasing traffic, a fabrication hall is replaced because the machinery inside this hall has to be modernized and restructured, certain vehicles are put out of service because they become too uncomfortable, too uneconomical or unserviceable because of outdated equipment. Usually, this happens in spite of full structural integrity. In fact, most structures will be replaced not because they fail or deteriorate but because they become obsolete. Unfortunately, very few data are available about this well-known fact [25]. Obsolescence is almost always completely independent of the structural state. But this is just the case dealt with in the foregoing section where one of the failure modes, i.e. cause for renewal, is treated as obsolescence. With A denoting all cost for demolishment and removal of debris it is:

$$D(\mathbf{p}) = \frac{(C(\mathbf{p}) + H) f^{**}_{V_1|\bar{A}_2}(\gamma) + (C(\mathbf{p}) + A) f^{**}_{A_2|\bar{V}_1}(\gamma)}{1 - (f^{**}_{V_1|\bar{A}_2}(\gamma) + f^{**}_{A_2|\bar{V}_1}(\gamma))}. \tag{4.41}$$

Table 2 collects the modified Laplace transforms for a few obsolescence models given exponential times between failures.

TABLE 2. Modified Laplace transforms for some obsolescence models given that failure is caused by a Poisson process with parameter λ.

$f_A(t)$	$f^{**}_{A\|\bar{V}}(\gamma) = \int\limits_0^\infty \exp[-\gamma t] f_A(t) \exp[-\lambda t]\, dt$
$f_V(t)$	$f^{**}_{V\|\bar{A}}(\gamma, \mathbf{p}) = \int\limits_0^\infty \exp[-\gamma t] \lambda \exp[-\lambda t]\, \bar{F}_A(t) dt$
$\delta(a)$	$\exp[-(\gamma+\lambda)a]$
$\lambda \exp[-\lambda t]$	$\frac{\lambda}{\gamma+\lambda}(1 - \exp[-(\gamma+\lambda)a])$
$\beta \exp[-\beta t]$	$\frac{\beta}{\gamma+\lambda+\beta}$
$\lambda \exp[-\lambda t]$	$\frac{\lambda}{\gamma+\lambda+\beta}$
$\frac{2t}{w^2} \exp\left[-\frac{t^2}{w^2}\right]$	$\frac{w(\gamma+\lambda)\sqrt{\pi}}{2}\left(\mathrm{erf}\left(\frac{w}{2}(\gamma+\lambda)\right) - 1\right) \exp\left[\frac{w^2}{4}(\gamma+\lambda)^2\right] + 1$
$\lambda \exp[-\lambda t]$	$\frac{w\lambda\sqrt{\pi}}{2} \mathrm{erf}\left[\frac{w}{2}(\gamma+\lambda)\right] \exp\left[\frac{w^2}{4}(\gamma+\lambda)^2\right]$
$\frac{1}{\sqrt{2\pi\sigma}}\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right]$	$\exp\left[\frac{1}{2}(\gamma+\lambda)((\gamma+\lambda)\sigma^2 - 2\mu)\right] \Phi\left(\frac{(\gamma+\lambda)\sigma-\mu}{\sigma}\right)$
$\lambda \exp[-\lambda t]$	$\frac{\lambda \exp[-(\gamma+\lambda)\mu]}{(\gamma+\lambda)\sigma}\left(\begin{array}{c}\exp\left[\frac{(\gamma+\lambda)^2\sigma^2}{2}\right]\Phi\left(\frac{\mu}{\sigma} - \sigma^2(\gamma+\lambda)\right) \\ - \exp[-\mu(\gamma+\lambda)]\,\Phi\left(-\frac{\mu}{\sigma}\right)\end{array}\right)$

4.12. Dependent failure modes

An improvement for dependent failure modes is easiest for the stationary case in Eq. (4.16). Here, one replaces the upper bound solution $\lambda \sum_{k=1}^{s} P_{f,k}$ by the discount factor involving either one minus the probability of survival in all modes or the upper and/or lower bound for a union of failure events [12], i.e. by

$$P(\bigcup_{k=1}^{s} V_k) \approx \lambda(1 - \Phi_s(\beta; \mathbf{R})) \begin{cases} \le \lambda \sum_{k=1}^{s} \left(P_{f,k} - \max_{j<k}\{P_{f,k\cap j}\}_{k>1}\right), \\ \ge \lambda \sum_{k=1}^{s} \max\left(0, P_{f,k} - \sum_{j=1}^{k}\{P_{f,k\cap j}\}\right), \end{cases} \tag{4.42}$$

with $P_{f,k} = P(\{\mathbf{U} \in V_k\} = P(g_k(\mathbf{U}) \leq 0) \approx P(\alpha_k^T \mathbf{U} + \beta_k \leq 0)$ and $P_{f,k\cap j} \approx P(\{\alpha_k^T \mathbf{U} + \beta_k \leq 0\} \cap \left\{\alpha_j^T \mathbf{U} + \beta_j \leq 0\right\})$ requiring either the computation of the s-dimensional multinormal integral or the computation of $(s-1)^2/2$ bivariate normal integrals.

Consider next a non-redundant series system under stationary renewal rectangular wave loading, i.e. a system where before the jump the process must be in the safe domain of all components and in the failure domain of at least one of the components after the jump and $V_S = \{\cup_{k=1}^s g_k(\mathbf{u}) \leq 0\}$. The outcrossing rate is [37]:

$$
\begin{aligned}
\nu_J^+(V_S) &= \sum_{i=1}^n \lambda_i \left[P\left(\bigcup_{k=1}^s \left\{ V_{ik}^+ \cap \bigcap_{j=1}^s \bar{V}_{ij} \right\} \right) \right] \\
&= \sum_{i=1}^n \lambda_i \left[P\left(\bigcup_{k=1}^s V_{ik}^+ \right) - P\left(\bigcup_{k=1}^s V_{ik}^+ \cap \bigcap_{j=1}^s V_{ij} \right) \right] \\
&\leq \sum_{k=1}^s \sum_{i=1}^n \lambda_i \left[\left(P\left(V_{ik}^+\right) - \max_{j<k} \left\{ P(\{V_{ik}^+\} \cap \left\{V_{ij}^+\right\}) \right\}_{k>1} \right) - A \right] \\
&\leq \sum_{k=1}^s \sum_{i=1}^n \lambda_i P\left(V_{ik}^+\right)
\end{aligned}
\tag{4.43}
$$

with

$$
A = \max \left\{ \begin{array}{l} P\left(\{V_{ik}^+\} \cap \left\{ \bigcap_{j=1}^s V_{ij} \right\} \right) \\ \quad - \sum_{\ell<k} P(\{V_{i\ell}^+\} \cap \{V_{ik}^+\} \cap \left\{ \bigcap_{j=1}^s V_{ij} \right\})_{k>1} \end{array}, 0 \right\}
\tag{4.44}
$$

and where $V_{ik}^+ = \left\{\mathbf{U}_i^+ \in V_k\right\}, V_{ik} = \{\mathbf{U}_i \in V_k\}, \bar{V}_{ij} = \left\{\mathbf{U}_i \in \bar{V}_j\right\}$ and where the subscript "$_{k>1}$" indicates that this term equals zero for $k = 1$. The computation of the correction term involves $s+1$- and $s+2$- dimensional normal integrals, respectively, if $V_k = \{g_k(\mathbf{u}) \approx \alpha_k^T \mathbf{u} + \beta_k \leq 0\}$. Dropping A leads to a less sharp bound and dropping the max-term produces the trivial upper bound. The events V_{rs} and the corresponding failure domains as well as the jump rates λ_i can be made dependent on t but there is no more any guarantee of sufficient accuracy.

Crossings by stationary Gaussian vector processes into time-invariant componential failure domains can also be considered, at least approximately.

For linearized limit state surfaces, i.e. for. $V_S = \{\cup_{k=1}^{s} g_k(\mathbf{u}) \leq 0\}$ with $\partial V_k = \{g_k(\mathbf{u}) \approx \alpha_k^T \mathbf{u} + \beta_k = 0\}$ and $g_k(\mathbf{0}) > 0$ for all $k = 1, \ldots, s$ one obtains after some computation [50]:

$$\begin{aligned} \nu_D^+(V_S) &= \sum_{k=1}^{s} \int_{\partial V_k} E\left[(-\alpha_k^T \dot{\mathbf{U}})^+ | \mathbf{U} = \mathbf{u}\right] \varphi_n(\mathbf{u}) ds(\mathbf{u}) \\ &= \sum_{k=1}^{s} \int_{\partial V_k} \Psi(m_k(\mathbf{u}), \sigma_k) \varphi_n(\mathbf{u}) ds(\mathbf{u}) \\ &\approx \sum_{k=1}^{s} \Psi(m_k(\mathbf{u}_k^*), \sigma_k) \varphi(\beta_k) \left[1 - \Phi_{s-1}(\mathbf{b}_k; \mathbf{B}_k)\right], \end{aligned} \tag{4.45}$$

with

$$\begin{aligned} \mathbf{b}_k &= \{\beta_r - \beta_k \alpha_r^T \alpha_k;\ 1 \leq r \leq s;\ r \neq k\}, \\ \mathbf{B}_k &= \{\alpha_r^T \alpha_t - (\alpha_r^T \alpha_k)(\alpha_t^T \alpha_k);\ 1 \leq r, t \leq s;\ r, t \neq k\}, \\ \Psi(m_k(\mathbf{u}_k^*), \sigma_k) &= E\left[(-\alpha_k^T \dot{\mathbf{U}})^+ | \mathbf{U} = \mathbf{u}\right] \\ &= \sigma_k \varphi\left(\frac{m_k(\mathbf{u}_k^*)}{\sigma_k}\right) + m_k(\mathbf{u}_k^*) \Phi\left(\frac{m_k(\mathbf{u}_k^*)}{\sigma_k}\right), \end{aligned}$$

where $(y)^+ = \max\{0, y\}$, $\Phi_0(.;.) = 0$, $\beta_r = \alpha_r^T \mathbf{u}_r^*$, $\beta_k = \alpha_k^T \mathbf{u}_k^*$, $\sigma_k^2 = \alpha_k^T (\ddot{\mathbf{R}} - \dot{\mathbf{R}}\dot{\mathbf{R}}^T)\alpha_k = \alpha_k^T \ddot{\mathbf{R}} \alpha_k$, $m_k(\mathbf{u}_k^*) = -\alpha_k^T \dot{\mathbf{R}} \mathbf{u}_k^* = 0$, and $2 \leq s \leq n$ as well as $\mathbf{u}_k^*$ the s different β–points. If $V_S(t) = \{\cup_{k=1}^{s} g_k(\mathbf{u}, t) \leq 0\}$ with $\partial V_k(t) = \{g_k(\mathbf{u}, t) \approx \alpha_k^T(t)\mathbf{u} + \beta_k(t) = 0\}$ one replaces $m_k(\mathbf{u}_k^*)$ by $m_k(\mathbf{u}_k^*) - \dot{a}_k$ where $\dot{a}_k = \dot{\beta}_k(t) + \sum_{j=1}^{n} \dot{\alpha}_{kj}$. Dropping the term $[1 - \Phi_{s-1}(\mathbf{b}_k; \mathbf{B}_k)]$ in Eq. (4.45) leads to the trivial upper bound.

For a combination of jump and differentiable processes we finally have:

$$\begin{aligned} \nu^+(V_S) \leq \sum_{k=1}^{s} \Bigg\{ \sum_{i=1}^{n_J} \lambda_i \left[\left(P\left(V_{ik}^+\right) - \max_{j<k} \left\{ P(V_{ik}^+ \cap V_{ij}^+) \right\}_{k>1} \right) - A \right] \\ + \Psi(m_k(\mathbf{u}_k^*), \sigma_k) \varphi(\beta_k) \left[1 - \Phi_{s-1}(\mathbf{b}_k; \mathbf{B}_k)\right] \Bigg\}. \end{aligned} \tag{4.46}$$

It is noted that the result is an upper bound to first order for both differentiable processes and for rectangular wave renewal processes. One can then use Eq. (2.12) with Eq. (6.4) in first approximation.

The case of monotonically decreasing state functions can be solved as follows. Assume that there are s time-dependent failure modes and whose state functions are given by $g_k(\mathbf{u},t) \approx \alpha_k^T(t)\mathbf{u}+\beta_k(t)$ so that $V_k(t) = P(T_k \leq t) = P(g_k(\mathbf{U},t) \leq 0) = P(Z_k \leq -\beta_k(t))$. The failure probability at time t_j then is $F(t) = P(\bigcup_{k=1}^{s}\{Z_k \leq -\beta_k(t)\}) = 1 - P(\bigcap_{k=1}^{s}\{Z_k \leq \beta_k(t)\}) \approx 1 - \Phi_s(\beta(t);\mathbf{R})$ where $\beta(t) = \{\alpha_k^T\mathbf{u}_k^*(t); k = 1,2,\ldots,s\}$, $\|\alpha_k\| = 1, k = 1,2,\ldots,s$; $\mathbf{u}_k^*(t) = \min\{\|\mathbf{u}\|\}$ for $\{\mathbf{u} : g_k(\mathbf{u},t) \leq 0\}$ and $\mathbf{R} = E[\mathbf{Z}\mathbf{Z}^T] = \{\rho_{ij}\} = \{\alpha_i^T\alpha_j; i,j = 1,2,\ldots,s\}$. In good approximation it is assumed that the matrix of correlation coefficients $\mathbf{R}$ varies little with time so that $\frac{\partial}{dt}\alpha_k(t) \approx 0$ and, hence, $\frac{\partial}{dt}\rho_{ij}(t) \approx 0$ and there is $g_k(\mathbf{0},t) > 0$ for all k. The general case of $\frac{\partial}{dt}\rho_{ij}(t) \neq 0$ is given in the Appendix. The failure density is

$$\begin{aligned}
f_s(t) &= \frac{d}{dt}(1 - \Phi_s(\beta(t);\mathbf{R})) = -\sum_{k=1}^{s}\frac{\partial}{\partial\beta_k(t)}\Phi_s(\beta(t);\mathbf{R})\frac{\partial\beta_k(t)}{\partial t} \\
&= -\sum_{k=1}^{s}\frac{\partial}{\partial\beta_k(t)}\int_{-\infty}^{\beta_k(t)}\Phi_{s-1}(\beta(t);\mathbf{R}\,|Z_k = \beta_k(t))\varphi_1(z_k)dz_k\frac{\partial\beta_k(t)}{\partial t} \\
&= -\sum_{k=1}^{s}\varphi_1(\beta_k(t))\Phi_{s-1}(\hat{\mathbf{c}}_k;\hat{\mathbf{R}}_k)\frac{\partial\beta_k(t)}{\partial t} \\
&= \sum_{k=1}^{s}\varphi_1(\beta_k(t))\Phi_{s-1}(\hat{\mathbf{c}}_k;\hat{\mathbf{R}}_k)\left(\frac{-\frac{\partial}{\partial t}g_k(\mathbf{u}^*,t)}{\|\nabla_{\mathbf{u}}g_k(\mathbf{u}^*,t)\|}\right) \\
&\leq \sum_{k=1}^{s}\varphi_1(\beta_k(t))\left(\frac{-\frac{\partial}{\partial t}g_k(\mathbf{u}^*,t)}{\|\nabla_{\mathbf{u}}g_k(\mathbf{u}^*,t)\|}\right),
\end{aligned}$$

with $\hat{\mathbf{c}}_k = \beta^k(t) - \beta_k(t)\rho_k^k$; and $\hat{\mathbf{R}}_k = \mathbf{R} - \rho_k^k(\rho_k^k)^T$ where ρ_k is the k-th column vector of $\mathbf{R}$ and the superscript means that the k-th row and column, respectively, are deleted from the original vector and matrix, respectively. This result is obtained from regression analysis. Note that $\hat{\mathbf{R}}_k$ needs to be re-normalised and therefore also $\hat{\mathbf{c}}_k$. The result $\frac{\partial}{\partial\beta_k(t)}\Phi_s(\beta(t);\mathbf{R}) = \varphi_1(\beta_k(t))\Phi_{s-1}(\hat{\mathbf{c}}_k;\hat{\mathbf{R}}_k)$ is due to [48]. Here, $s-1$-dimensional normal integrals have to be evaluated for each t. Suitable computation schemes for $\Phi_r(\mathbf{b};\mathbf{B})$ have been given in [19] and elsewhere. Due to the substantial numerical effort when computing multi-normal probabilities this scheme can only be applied to smaller systems. Dropping the terms $\Phi_{s-1}(\hat{\mathbf{c}}_k;\hat{\mathbf{R}}_k)$, i.e. the survival probabilities in the other failure modes, corresponds to the upper bound solution:

$$D = \sum_{i=1}^{s}(C_i + H_i)\frac{f_{1,i}^{**}(\gamma)}{1 - \sum_{i=1}^{s} f_{1,i}^{**}(\gamma)} \leq \sum_{i=1}^{s}(C_i + H_i)\frac{f_{1,i}^{*}(\gamma)}{1 - \sum_{i=1}^{s} f_{1,i}^{*}(\gamma)} \quad (4.47)$$

where

$$f_{1,i}^{**}(\gamma) = \int_0^\infty \exp[-\gamma t]\,\varphi_1(\beta_i(t))\Phi_{s-1}(\hat{\mathbf{c}}_i;\hat{\mathbf{R}}_i)\left(\frac{-\frac{\partial}{\partial t}g_i(\mathbf{u}^*,t)}{\|\nabla_{\mathbf{u}}g_i(\mathbf{u}^*,t)\|}\right)dt$$

$$\leq f_{1,i}^{*}(\gamma) = \int_0^\infty \exp[-\gamma t]\,\varphi_1(\beta_i(t))dt.$$

The trivial upper bound may be useful but its application is limited to smaller systems because Laplace transforms of densities must remain smaller than unity. Clearly, a trivial lower bound is formed by the largest member in the sum. A better lower bound is found by replacing $\Phi_{s-1}(\hat{\mathbf{c}}_k;\hat{\mathbf{R}}_k)$ by $\prod_{k=1}^{s}\Phi(\hat{c}_k)$ because $\Phi_m(\mathbf{x};\mathbf{R}) \leq \Phi_m(\mathbf{x};\mathbf{K})$ if for some ij there is $\{\rho_{ij}\} \leq \{\kappa_{ij}\}$ but $\mathbf{R} \geq \mathbf{0}$ and $\mathbf{K} \geq \mathbf{0}$ [51].

For the general case, i.e. $\frac{\partial}{dt}\rho_{ij}(t) \neq 0$, we have:

$$f_s(t) = \frac{d}{dt}(1 - \Phi_s(\beta(t);\mathbf{R}(t))) = -\sum_{k=1}^{s}\left[\frac{\partial}{\partial\beta_k}\Phi_s(\beta(t);\mathbf{R}(t))\frac{\partial\beta_k(t)}{\partial t}\right.$$
$$\left.+\sum_{j=1}^{k-1}\frac{\partial}{\partial\rho_{kj}}\Phi_s(\beta(t);\mathbf{R}(t))\frac{\partial\rho_{kj}(t)}{\partial t}\right]. \quad (4.48)$$

The first term in the sum is given in Eq. (4.47), the second sum needs to be written out in more detail:

$$\frac{\partial}{\partial\rho_{kj}}\Phi_s\big(\beta(t);\mathbf{R}(t)\big)\frac{\partial\rho_{kj}(t)}{\partial t} = \frac{\partial^2}{\partial\beta_k\partial\beta_j}\Phi_s\big(\beta(t);\mathbf{R}(t)\big)\frac{\partial\rho_{kj}(t)}{\partial t}$$
$$= \frac{\partial^2}{\partial\beta_k\partial\beta_j}\int_{-\infty}^{\beta_j(t)}\int_{-\infty}^{\beta_k(t)}\Phi_{s-2}\big(\beta(t);\mathbf{R} \mid Z_k = \beta_k(t), Z_j = \beta_j(t)\big)$$
$$\cdot\varphi_2(z_k, z_j;\rho_{kj})\,dz_k\,dz_j\frac{\partial\rho_{kj}(t)}{\partial t}$$
$$= \varphi_2\big(\beta_k(t),\beta_j(t);\rho_{kj}(t)\big)\Phi_{s-2}\big(\check{\mathbf{c}}_k(t);\check{\mathbf{R}}_{kj}(t)\big)\frac{\partial\rho_{kj}(t)}{\partial t}$$

where

$$\check{\mathbf{c}}_k = \beta^{kj} - \left[\rho_k^{kj}\rho_j^{kj}\right]\begin{bmatrix}1 & \rho_{kj}\\ \rho_{kj} & 1\end{bmatrix}^{-1}\begin{bmatrix}\beta_k\\ \beta_j\end{bmatrix},$$

$$\check{\mathbf{R}}_{kj} = \mathbf{R}^{kj} - \left[\rho_k^{kj}\rho_j^{kj}\right]\begin{bmatrix}1 & \rho_{kj}\\ \rho_{kj} & 1\end{bmatrix}^{-1}\begin{bmatrix}(\rho_k^{kj})^T\\ (\rho_j^{kj})^T\end{bmatrix},$$

and $\frac{\partial \rho_{kj}(t)}{\partial t} = \frac{\partial}{\partial t}(\alpha_k(t)^T \alpha_j(t)) = \frac{\partial \alpha_k(t)^T}{\partial t}\alpha_j(t) + \alpha_k(t)^T \frac{\partial \alpha_j(t)}{\partial t}$. The notation follows the same rules as explained below Eq. (4.47). This result is due to Ditlevsen in this form [13]. The same reference also gives

$$\frac{\partial \alpha_k(t)}{\partial t} = -(\mathbf{I} + \beta_k(t)\mathbf{G}_k)\frac{1}{\|\nabla_u g_k(\mathbf{u},t)\|}\frac{\partial \nabla_u\, g_k(\mathbf{u},t)}{\partial t}$$

with

$$\frac{\partial \nabla_u\, g_k(\mathbf{u},t)}{\partial t} = \left\{\frac{\partial^2 g_k(\mathbf{u},t)}{\partial u_i \partial t}; i = 1, \ldots, n\right\}$$

and

$$\mathbf{G}_k = \frac{1}{\|\nabla_u g_k(\mathbf{u},t)\|}\left\{\frac{\partial^2 g_k(\mathbf{u},t)}{\partial u_i \partial u_j} - \frac{1}{\|\nabla_u g_k(\mathbf{u},t)\|}\right.$$
$$\left.\cdot\left(\frac{\partial g_k(\mathbf{u},t)}{\partial u_i}\frac{\partial \|\nabla_u g_k(\mathbf{u},t)\|}{\partial u_j} + \frac{\partial g_k(\mathbf{u},t)}{\partial u_j}\frac{\partial \|\nabla_u g_k(\mathbf{u},t)\|}{\partial u_i}\right);\ i,j \neq k\right\}_{\mathbf{u}=\mathbf{u}_k^*}.$$

For almost plane failure surfaces the second order derivatives occurring in the last equation are almost zero and we have approximately

$$\frac{\partial \alpha_k(t)}{\partial t} \approx -\frac{1}{\|\nabla_u g_k(\mathbf{u},t)\|}\frac{\partial \nabla_u\, g_k(\mathbf{u},t)}{\partial t}.$$

For a combination of cases treated in Eq. (4.47) with those in Eq. (4.46) one usually has to resort to the upper bound Eq. (4.47).

On similar lines one could attempt to consider the case when the failure density must be computed using Eq. (2.10). We refrain here from presenting results because the formulae become very complicated and may, nevertheless, have rather limited practical application.

4.13. Inspection and repair of aging components

In the literature maintenance cost frequently have been assumed to increase continuously with time. More realistic in the structures area is the case where maintenance cost are the sum of inspection and possible repair cost. Assume inspections at regular intervals $a, 2a, 3a, \ldots$. Repairs occur only at these points in time (or with some delay, say at $a+\Delta, 2a+\Delta, 3a+\Delta, \ldots$). Inspections and repairs occur only if renewals have not occurred before due to obsolescence or failure. Assume further that repairs, if undertaken, restore the properties of a component to its original (stochastic) state, i.e. repairs are equivalent to renewals. Inspection and repair times are assumed negligibly short. Of course, it makes only sense to consider aging components with increasing risk function $r(t)$.

Consider first the case with only one failure mode. A renewal occurs either after failure or at times $a, 2a, 3a, \ldots$ .and renewal (repair) times are negligibly short. In [1] this is denoted by age replacement. Then, we obviously have [18]:

$$Z(\mathbf{p},a) = B - C(\mathbf{p}) - \frac{(C(\mathbf{p}) + H) f_V^{***}(\gamma, \mathbf{p},a) + I_1(\mathbf{p}) \exp[-\gamma a] \bar{F}_V(\mathbf{p},a)}{1 - (f_V^{***}(\gamma, \mathbf{p},a) + \exp[-\gamma a] \bar{F}_V(\mathbf{p},a))} \tag{4.49}$$

with $I_1(\mathbf{p})$ the cost of repair and $I_1(\mathbf{p}) < (C(\mathbf{p}) + H)$.

If there are regular inspections there is not necessarily a repair because inspections are uncertain (or the signs of deterioration are vague). Denote the failure model for the aging component by "V" whereas "A" stands for any other (independent) failure mode (or obsolescence as another cause for renewal). Then, inspection and repair cost must also be included in the damage term:

$$Z(\mathbf{p},a) = B(\mathbf{p},a) - C(\mathbf{p}) - D(\mathbf{p},a). \tag{4.50}$$

Including now one failure mode "V" with subsequent renewal and obsolescence "A"

$$D(\mathbf{p},a) = \frac{ND}{D} \tag{4.51}$$

where

$$\begin{aligned} ND = (C(\mathbf{p}) + A)(f_{A|\bar{V}}^{***}(\gamma, a) + A11) + (C(\mathbf{p}) + H)(f_{V|\bar{A}}^{***}(\gamma, \mathbf{p},a) + A12) \\ + I_0((1 - P_R(a)) \exp[-\gamma a] \bar{F}_A(a) \bar{F}_V(\mathbf{p},a) + A21) \\ + (I_0 + I_1(\mathbf{p}))(P_R(a) \exp[-\gamma a] \bar{F}_A(a) \bar{F}_V(\mathbf{p},a) + A22), \end{aligned}$$

$$\begin{aligned} D = 1 - \Big(f_{A|\bar{V}}^{***}(\gamma, a) + A11 + f_{V|\bar{A}}^{***}(\gamma, \mathbf{p},a) + A12 \\ + P_R(a) \exp[-\gamma a] \bar{F}_A(a) \bar{F}_V(\mathbf{p},a) + A22 \Big), \end{aligned}$$

$$A11 = \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} (1 - P_R(ja)) f_{A|\bar{V}}^{****}(\gamma, \mathbf{p}, (n-1)a \le t \le na),$$

$$A12 = \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} (1 - P_R(ja)) f_{V|\bar{A}}^{****}(\gamma, \mathbf{p}, (n-1)a \le t \le na),$$

$$A21 = \sum_{n=2}^{\infty} (1 - P_R(na)) \prod_{j=1}^{n-1} (1 - P_R(ja)) \exp[-\gamma(na)] \bar{F}_A(na) \bar{F}_V(\mathbf{p},na),$$

$$A22 = \sum_{n=2}^{\infty} P_R(na) \prod_{j=1}^{n-1} (1 - P_R(ja)) \exp[-\gamma na] \bar{F}_A(na) \bar{F}_V(\mathbf{p},na).$$

Here the following notation is introduced:

$P_R(a)$ = probability of repair after inspection,

$\bar{P}_R(a) = 1 - P_R(a)$ = probability of no repair after inspection,

a = deterministic inspection interval,

I_0 = cost per inspection,

$I_1(\mathbf{p})$ = repair cost,

$f^{***}_{X|\bar{Y}}(\gamma, \mathbf{p}, a) = \int_0^a \exp[-\gamma t] f_X(t)\bar{F}_Y(t)dt \leq \int_0^a \exp[-\gamma t] f_X(t)dt$ = incomplete, modified Laplace transform of $f_X(t)$,

$f^{****}_{X|\bar{Y}}(\gamma, \mathbf{p}, (n-1)a \leq t \leq na) = \int_{(n-1)a}^{na} \exp[-\gamma t] f_X(t)\bar{F}_Y(t)dt.$

Moreover, one has to extend the renewal interval to $2a, 3a, \ldots$ if an inspection is not followed by repair. The terms $A11$, $A12$, $A21$ and $A22$ vanish for $P_R(a) \to 1$ and are significant only for relatively small a. Note that the renewal cost $C(\mathbf{p})$ can also be different in the two cases.

If the benefit is constant in time we simply have $B(\mathbf{p},a) = \frac{b}{\gamma}$. For non-constant benefit $b(t)$ it is in analogy with Eq. (4.51):

$$B(\mathbf{p}, a) = \frac{NB}{D} \tag{4.52}$$

where

$$NB = \int_0^a B_D(t) f_A(t)\bar{F}_V(\mathbf{p}, t)dt + B11$$
$$+ \int_0^a B_D(t) f_V(t, \mathbf{p})\bar{F}_A(t)dt + B12 + B_D(a)\bar{F}_A(a)\bar{F}_V(\mathbf{p}, a) + B2,$$

$$B11 = \sum_{n=2}^{\infty} \int_{(n-1)a}^{na} B^*_D(t) \prod_{j=1}^{n-1}(1 - P_R(ja)) f_A(t)\bar{F}_V(\mathbf{p}, t)dt,$$

$$B12 = \sum_{n=2}^{\infty} \int_{(n-1)a}^{na} B^*_D(t) \prod_{j=1}^{n-1}(1 - P_R(ja)) f_V(t, \mathbf{p})\bar{F}_A(t)dt,$$

$$B2 = \sum_{n=2}^{\infty} B^*_D(na) \prod_{j=1}^{n-1}(1 - P_R(ja))\bar{F}_A(na)\bar{F}_V(\mathbf{p}, na),$$

with

$$B_D(t) = \int_0^t \exp[-\gamma t]\, b(t)dt, \qquad B_D^*(t) = \int_{(n-1)a}^t \exp[-\gamma t]\, b(t)dt.$$

The denominator in Eq. (4.52) is the same as in Eq. (4.51). Some extra consideration can also include delay times Δ of repairs.

The repair probability depends on the magnitude of a suitable damage indicator. For cumulative damage phenomena $P_R(a, \mathbf{p})$ increases with a. For example, $P_R(a, \mathbf{p}) = P(S(a, X, \mathbf{p}) > s_c)$ with $S(a, X, \mathbf{p})$ a monotonically increasing damage indicator and X a random variable taking into account of all uncertainties during inspection. Frequently, the length of inspection intervals is taken as an optimization parameter. The case without inspection and $P_R(a, \mathbf{p}) = 1$ is already dealt with in the literature [18, 57]. Repair after inspection is interpreted as preventive renewal (replacement of an aging component after a finite time of use a). Renewal after failure is called corrective renewal. It must be mentioned that optimal inspection/repair intervals do not always exist. Preventive renewals must, in fact, be substantially cheaper than corrective renewals. Also, the repair probability must be sufficiently high at a.

4.14. Block replacements

In some cases an aging component of a system will be replaced whenever it fails but at some time all components in a system will be replaced simultaneously independent of their history. This replacement strategy is denoted by block replacements. Block replacements can have organizational reasons. In [1] it is shown that although the number of block replacements will be stochastically larger than for age replacements the mean number of failures, i.e. renewals with large consequences, per unit time is smaller.

Assume that the block replacements occur at times $a, 2a, 3a$. Then, making use of the Laplace transform for deterministic renewals (see Table 1)

$$\begin{aligned} D(\mathbf{p}, a) &= \frac{C(\mathbf{p})e^{-\gamma a} + (C(\mathbf{p}) + H)\sum_{n=1}^{\infty} \int_0^a e^{-\gamma t} f_n(t)dt}{1 - e^{-\gamma a}} \\ &= \frac{C(\mathbf{p})e^{-\gamma a} + (C(\mathbf{p}) + H)\int_0^a e^{-\gamma t} h(t)dt}{1 - e^{-\gamma a}} \end{aligned} \tag{4.53}$$

where $h(t) = \sum_{k=1}^{\infty} f_k(t)$ is the renewal intensity after failures with $f_k(t)$ the density of the time until the k-th renewal. Obviously, the second term in the numerator takes account of the renewals after failure between the interval $[0, a]$. As mentioned only very few analytical results for $h(t)$ exist. But one can derive an upper bound in using

$$f_k(t) = \int_0^t f_{k-1}(t-\tau)f(\tau)d\tau,$$

interchanging the order of integration in line 2 and making the substitution $v = t - \tau$ in line 3:

$$\begin{aligned}
&\int_0^a e^{-\gamma t}\int_0^t f_{k-1}(t-\tau)f(\tau)d\tau dt \\
&= \int_0^a\int_0^t e^{-\gamma t} f_{k-1}(t-\tau)f(\tau)d\tau dt = \int_0^a\int_\tau^a e^{-\gamma t} f_{k-1}(t-\tau)f(\tau)dt d\tau \\
&= \int_0^a f(\tau)\int_0^{a-\tau} e^{-\gamma(\tau+v)} f_{k-1}(v)dv d\tau = \int_0^a e^{-\gamma\tau} f(\tau)\int_0^{a-\tau} e^{-\gamma v} f_{k-1}(v)dv d\tau \\
&= \int_0^a e^{-\gamma\tau} f(\tau)\left(\int_0^a e^{-\gamma v} f_{k-1}(v)dv - \int_{a-\tau}^a e^{-\gamma v} f_{k-1}(v)dv\right)d\tau \\
&\qquad\qquad \le \int_0^a e^{-\gamma\tau} f(\tau)d\tau \int_0^a e^{-\gamma v} f_{k-1}(v)dv.
\end{aligned}$$

Applying this scheme k-times yields

$$\int_0^a e^{-\gamma t} f_k(t)dt \le \left(\int_0^a e^{-\gamma t} f(t)dt\right)^k = (f^{**}(\gamma, a))^k$$

For $a \to \infty$ the exact Laplace transform of the k-fold convolution of $f(t)$ with itself is obtained. Therefore,

$$\begin{aligned}
D(\mathbf{p}, a) &= \frac{C(\mathbf{p})e^{-\gamma a} + (C(\mathbf{p}) + H)\sum_{k=1}^{\infty}\int_0^a e^{-\gamma t} f_k(t)dt}{1 - e^{-\gamma a}} \\
&\le \frac{C(\mathbf{p})e^{-\gamma a} + (C(\mathbf{p}) + H)\frac{f^{**}(\gamma,a)}{1-f^{**}(\gamma,a)}}{1 - e^{-\gamma a}}
\end{aligned}$$

where

$$f^{**}(\gamma, a) = \int_0^a e^{-\gamma t} f(t) dt$$

is the incomplete Laplace transform of $f(t)$.

For constant b we have $B = \frac{b}{\gamma}$, as before. On similar lines one can also derive the benefit term for non-constant benefit function $b(t)$. Unfortunately, it will in turn be only an upper bound so that it is not given herein.

4.15. Serviceability losses

For completeness a simple serviceability model is also derived. Loss of serviceability, in general, has two effects: (1) repair is necessary at usually much lower cost than a repair corresponding to a complete renewal, and (2) loss of benefit after the event and during repair. In some cases serviceability losses including cost of repair and loss of benefit form a large part of the total cost of a project. We make use of simple asymptotic arguments. If the occurrence process of serviceability losses also forms a renewal process with mean interarrival time m_S the additional discounted repair cost are $C_S = C_S(p_S)/(\gamma m_S)$ to be added to $C(\mathbf{p})$ where p_S is a parameter controlling the frequency and (mean) duration of downtimes. Also, for constant benefit outside the downtimes due to serviceability failure the benefit must be reduced by the (asymptotic) availability $A_S(\infty) = \frac{E[T_N]}{E[T_D]+E[T_N]}$ where T_D is the (random) downtime during repair and T_N the time of use of the facility (see also section 4.9). Therefore, in first approximation as a generalization of Eq. (4.8) for Poissonian failures and systematic reconstruction and where we have taken into account that during downtimes there cannot be failure (if this is appropriate):

$$Z(\mathbf{p}) \approx \frac{b}{\gamma} A_S(\infty) - C(\mathbf{p}) - C_S(p_S)\frac{1}{\gamma m_S} - (C(\mathbf{p}) + H) A_S(\infty)\frac{\lambda(\mathbf{p})}{\gamma}. \quad (4.54)$$

If failure can occur also during down times, e.g., by an external disturbance, the term $A_S(\infty)$ must be deleted from the damage term. However, classical reliability theory has developed more sophisticated models for this case.

4.16. A note on interest rates

In passing it is appropriate to make a few further comments on discounting. In consideration of the time horizon for structural and other technical facilities of 20 to more than 100 years the interest rate used should be a long term average net of in/deflation. In accordance with economic theory benefits and (expected) cost should be discounted by the same rate as done above. Different parties, e.g., the owner, operator or the public, may, however, use different rates. While the owner or operator may take interest rates from the financial market the assessment of the interest rate for an optimization in the name of the public is difficult. The benefit a society derives from the economic activities of its members is approximately the sum of economic growth rate per capita ζ and demographic growth rate n so that $\beta \approx \zeta + n$ (see [43] for further discussion). The requirement that the objective function must be non-negative leads immediately to the conclusion that the interest rate must have an upper bound $\gamma_{\max}$ depending on the benefit rate $b = \beta C(\mathbf{p})$ (see [21]). For the model in Eq. (4.8) with Eq. (4.16) we have

$$\frac{\beta C(\mathbf{p})}{\gamma} - C(\mathbf{p}) - (C(\mathbf{p})+H)\frac{\lambda P_f(\mathbf{p})}{\gamma} = 0, \tag{4.55}$$

and, therefore, by solving for γ and given (optimal) $\mathbf{p} = \mathbf{p}^*$

$$\gamma < \gamma_{\max} < \beta - \lambda P_f(\mathbf{p}) \left(1+\frac{H}{C(\mathbf{p})}\right) \tag{4.56}$$

implying $\gamma < \beta$ for $\lambda P_f(\mathbf{p}) \ll \beta$. It follows that the benefit rate β must be slightly larger than $\gamma_{\max}$. From Eq. (4.55) one also concludes that there must be $\gamma > 0$ because the limit $\gamma \to 0^+$ is $\pm\infty$ or at least undefined.

5. Constraint functions

5.1. Normal constraints

The set of constraints generally consists of three groups: reliability constraints, deterministic constraints and simple bounds on design parameters. Usually, reliability constraints have a form of inequalities enforcing optimal design to satisfy assumed minimal level of reliability. For so called elemental formulation of a problem reliability constraints have the form:

$$\beta_i(\mathbf{p}) \geq \beta_i^{\min}; i = 1, \ldots, s \tag{5.1}$$

where each constraint corresponds to the beta index of a single failure mode (failure element). The system formulation exists as well and has a very similar form:

$$\beta_S = -\Phi^{-1}(P(\bigcup_{i=1}^{s} g_i(\mathbf{u}, \mathbf{p}) \leq 0)) \geq \beta^{\min} \tag{5.2}$$

where "S" denotes the system reliability index and is its minimal admissible value. According to a specific model, elemental and system constraints can be mixed. Reliability constraints can alternatively be expressed in terms of failure probabilities.

If failure occurs at a random time, the reliability-based optimization problem is formulated within the framework of time-variant reliability. Then, bounds on reliability indices have to be replaced by bounds on failure rates, for example

$$h(\mathbf{p}) \leq h(t \to \infty)^{\max} \tag{5.3}$$

with obvious generalization to series systems. It is important to use asymptotic failure rates as in Eq. (3.14) implying a exponential distribution of failure times. Any other assumption leads inevitably to an inversion of the Laplace transform of the renewal intensity which is numerically a notoriously difficult problem.

Additionally, the set of constraints assuring the mathematical and physical admissibility of the design parameter vector and simple lower and upper bounds for the transformed basic variable vector and the design vector should be observed:

$$\begin{aligned} h_\ell(\mathbf{p}) &\leq 0, && \ell = 1, \ldots, q, \\ u_{\min} \leq u_j &\leq u_{\max}, && j = 1, \ldots, n, \\ p_{\min,k} \leq p_k &\leq p_{\max,k}, && k = 1, \ldots, m, \end{aligned} \tag{5.4}$$

where $h_\ell(\mathbf{p})$ may contain inequality constraint and equality constraints for the design parameters. Simple lower and upper bounds on stochastic variables resp. design parameters usually are also introduced.

5.2. Constraints based on societal criteria for risk acceptance [43]

Recently, interesting concepts have been proposed for the assessment of public risk acceptance [33, 41, 31, 34, 43, 42]. Those considerations are valid for the acceptance of involuntary risks to human life and limb from technical installations or the natural environment by an anonymous member of society. In essence, they set out from a composite social indicator, the *societal life quality index*, also to be interpreted as a utility function which encompasses three important indicators of life quality, that is life expectancy, consumption (income net of taxes) and the time necessary to raise the total income by paid work, i.e. the time not available for leisure. In [43] the following version has been derived

$$L_{\bar{E}} = \frac{g^q}{q}\int_0^{a_u} e_d(a,,\zeta,\rho,n)h(a,n)da = \frac{g^q}{q}\bar{E} \tag{5.5}$$

with

$$\bar{E} = \int_0^{a_u} e_d(a,\zeta,\rho,n)h(a,n)da, \tag{5.6}$$

$$e_d(a,\zeta,\rho,n) = \frac{\exp((\rho+\zeta-n)a)}{\ell(a)}\int_a^{a_u} \exp\left[-\int_0^t (\mu(\tau)+(\rho+\zeta-n))d\tau\right] dt, \tag{5.7}$$

$$h(a,n) = \frac{\exp\left[-na\right]\ell(a)}{\int_0^{a_u}\exp\left[-na\right]\ell(a)da}. \tag{5.8}$$

In these formula $g \approx 0.6\,GDP$ is the part of the GDP available for risk reduction interventions (approximately the part available for private use), $q = \frac{w}{1-w}$ a risk aversion parameter with w the life working time as a fraction of life expectancy at birth $e(0) = \int_0^{a_u}\ell(a)da$ with survival probability $\ell(a) = \exp[-\int_0^a \mu(t)dt]$ at age a and $\mu(t)$ the age dependent mortality obtainable from life tables, $e_d(a,\zeta,\rho,n)$ the "discounted" remaining life expectancy given that a person has survived until age a, ρ the socalled time preference rate, n the population growth rate, ζ the rate of economic growth and $h(a,n)$ the density of the distribution of ages in a (stable) population. Dividing Eq. (5.5) by the marginal utility $u'(g) = g^{q-1}$ gives the so-called *societal value of a statistical life*

$$\overline{SVSL} = \frac{g}{q}\bar{E}. \tag{5.9}$$

Using Eq. (5.5) a small relative change in the societal life quality index can be assessed as

$$\frac{dL_{\bar{E}}}{L_{\bar{E}}} = \frac{dg}{g} + \frac{1}{q}\frac{d\bar{E}}{\bar{E}},$$

so that the requirement $dL_{\bar{E}} \geq 0$ leads to a general acceptance criterion

$$\frac{dg}{g} + \frac{1}{q}\frac{d\bar{E}}{\bar{E}} \geq 0. \tag{5.10}$$

The change in age-averaged, discounted life expectancy can be expressed in terms of a change in (crude) mortality as

$$\frac{d\bar{E}}{\bar{E}} \approx \frac{\frac{d}{dx}\bar{E}(x)\mid_{x=0} x}{\bar{E}} = -\frac{c_{x\bar{E}}(\zeta,\rho,n)}{m}dm. \tag{5.11}$$

The *societal willingness to pay* is finally defined as

$$dC_Y = -dg = g\frac{1}{q}\frac{c_{x\bar{E}}(\zeta,\rho,n)}{m}dm = G_{x\bar{E}}(\zeta,\rho,n)dm. \tag{5.12}$$

The demographic constant $c_{x\bar{E}}(\zeta,\rho,n)$ depends on the mortality reduction scheme x of a particular intervention, for example whether the intervention reduces mortality proportional to age-dependent mortality or simply as a constant at all ages. In the following only constant mortality changes denoted by scheme Δ will be considered.

Application to technical objects requires that the mortality change is expressed in terms of changes in the failure rate. Let dm be proportional to the increment in the mean failure rate $dh(p)$, i.e. it is assumed that the process of failures and renewals is already in a stationary state that is for $t \to \infty$ (see Eq. (3.14)). Rearrangement and introducing the incremental cost and the failure rate as a function of a (scalar) parameter p yields

$$\frac{dC_Y(p)}{dh(p)} \geq -k\frac{c_{\Delta\bar{E}}(\zeta,\rho,n)}{m}g\frac{1}{q} = -kG_{\Delta\bar{E}}(\zeta,\rho,n) \tag{5.13}$$

where

$$dm = kdh(p), \quad 0 < k \leq 1, \tag{5.14}$$

the proportionality constant k relating the changes in mortality to changes in the failure rate. Note that for any reasonable risk reducing intervention there is necessarily $dh(p)/dp < 0$. k $(0 \leq k \leq 1)$ must be determined by careful failure consequence analysis.

The life saving cost (LSC) or *implied cost of averting a fatality* ($ICAF$) can be obtained from the equality of Eq. (5.10) after replacing $\bar{E}$ by $e = e(0)$, separation and integration from g to $g + \Delta g$ and e to $e + \Delta e$, i.e. the cost $\Delta C = -\Delta g$ per year to extend a person's life by Δe is:

$$\Delta C = -\Delta g = g\left[1 - \left(1 + \frac{\Delta e}{e}\right)^{-\frac{1}{q}}\right].$$

Because ΔC is a yearly cost and the (undiscounted) LSC has to be spent for safety related investments into technical projects at the decision point $t = 0$, one should multiply by $e_r = \Delta e$ and

$$LSC(e_r) = g\left[1 - \left(1 + \frac{e_r}{e}\right)^{-\frac{1}{q}}\right]e_r \tag{5.15}$$

follows. The societal equality principle prohibits to differentiate with respect to special ages within a group. The conditional (remaining) life expectancy given that the person has survived up to age a is:

$$e(a) = \int_a^{a_u} \frac{\ell(t)}{\ell(a)} dt = \frac{1}{\ell(a)} \int_a^{a_u} \exp\left[-\int_0^t \mu(\tau)d\tau\right] dt. \tag{5.16}$$

Therefore, averaging the remaining life expectancy over the age distribution leads to the societal life saving cost ($SLSC$):

$$SLSC = \int_0^{a_u} LSC(e(a))h(a,n)da \tag{5.17}$$

where $h(a,n)$ is the density of the age distribution of the population with n its population growth rate.

The criterion Eq. (5.13) is derived for safety-related regulations for a larger group in a society or the entire society. For a specific project it makes sense to apply criterion (5.13) to the specific group exposed. Therefore, the "life saving cost" of a technical project with N_{PE} potential endangered persons is:

$$H_F = SLSC\ kN_{PE}. \tag{5.18}$$

The monetary losses in case of failure are decomposed into $H = H_M + H_F$ in formulations of the type Eq. (4.8) with H_M all losses not related to human life and limb.

Criterion (5.13) changes accordingly into:

$$\frac{dC_Y(p)}{dh(p)} \geq -G_{\Delta\bar{E}}(\zeta,\rho,n)kN_{PE}. \tag{5.19}$$

All quantities in Eq. (5.19) are related to one year. For a particular technical project all design and construction cost, denoted by $dC(p)$, must be raised at the decision point $t = 0$. The yearly cost must be replaced by the erection cost $dC(p)$ at $t = 0$ on the left hand side of Eq. (5.19) and discounting is necessary. The method of discounting is the same as for discharging an annuity. If the public is involved $dC_Y(p)$ may be interpreted as cost of societal financing of $dC(p)$ such that $dC_Y(p) = dC(p)\frac{\gamma\exp[\gamma t_s]}{\exp[\gamma t_s]-1}$. The (real) interest rate to be used must then be a societal interest rate. Otherwise the interest rate is the market rate. g in $G_{\Delta\bar{E}}(\zeta,\rho,n)$ also grows in the long run approximately exponentially with rate ζ, the rate of economic growth in a country (see [32] for an empirical verification). It can be taken into account by discounting. The acceptability criterion for individual technical projects then is (discount factor for discounted erection cost moved to the right hand side):

$$\frac{dC(p)}{dh(p)} \geq -\frac{\exp[\gamma t_s]-1}{\gamma\exp[\gamma t_s]} G_{\Delta\bar{E}}(\zeta,\rho,n)\frac{\zeta\exp[\zeta t_s]}{\exp[\zeta t_s]-1}kN_{PE} \\ \rightarrow_{t_s\rightarrow\infty} -G_{\Delta\bar{E}}(\zeta,\rho,n)kN_{PE}\frac{\zeta}{\gamma} \tag{5.20}$$

where t_s is service time. For $\zeta \rightarrow 0$ as well as $\gamma \rightarrow 0$ we have the interesting limiting result for arbitrary t_s:

$$\frac{dC(p)}{dh(p)} \geq \rightarrow_{\zeta\rightarrow 0,\gamma\rightarrow 0} -G_{\bar{E}}(\zeta,\rho,n)kN_{PE}. \tag{5.21}$$

Here, a slight inconsistency is encountered because there is double discounting with respect to g and $G_{\Delta\bar{E}}(\zeta,\rho,n)$ by ζ. Alternatively, discounting can be performed with the same rate in Eq. (5.20) so that the effect of discounting cancels. Generalizing now to a vectorial parameter $\mathbf{p}$ we have

$$\nabla_p C(\mathbf{p}) + G_{\Delta\bar{E}}(\zeta,\rho,n)kN_{PE}\frac{\zeta}{\gamma}\nabla_p h(\mathbf{p}) \geq 0 \tag{5.22}$$

which is easily seen to be equivalent to the solution of the following optimization task:

$$\textbf{Minimize:}\quad Z'(\mathbf{p}) = C(\mathbf{p}) + G_{\Delta\bar{E}}(\zeta,\rho,n)kN_{PE}\frac{\zeta}{\gamma}h(\mathbf{p}). \tag{5.23}$$

Equation (5.22) is seen to be the optimality condition $\nabla_p Z'(\mathbf{p}) = 0$ of the (unconstrained) optimization problem Eq. (5.23). Eq. (5.23) allows solving for vectorial parameter $\mathbf{p}$. A solution to Eq. (5.22) or (5.23) can always be found because $\nabla_p C(\mathbf{p})$ usually grows approximately linearly in $\mathbf{p}$ whereas $\nabla_p h(\mathbf{p})$ decays exponentially.

Some numerical values for the various economic and demographic quantities entering Eq. (5.22) or Eq. (5.23) are given in Table 3 [43]. According to [43] the discount rate is computed from

$$\gamma = \rho + \epsilon\zeta > 0, \tag{5.24}$$

and the rate of time preference ρ is bounded to the below by

$$\rho \geq n + \zeta(1-\epsilon). \tag{5.25}$$

Here $\epsilon > 0$ is the elasticity of marginal consumption (income) and is determined to be $\epsilon = 1 - q$. It should be observed that a complex interaction

TABLE 3. Social indicators for some countries.

Country	GDP[1], g[2]	ζ[3]	m[4]	n[5]	e	q[6]	ρ	γ	$SLSC$[7,8]	$G_{\Delta\bar{E}}$,[8]	$\overline{SVSL}$
USA	34260, 22030	1.8	0.87	0.90	77	0.22	1.3	2.3	$8.7 \cdot 10^5$	$4.8 \cdot 10^6$	$2.1 \cdot 10^6$
Germany	25010, 14460	1.9	1.04	0.27	78	0.17	0.6	1.9	$5.7 \cdot 10^5$	$3.7 \cdot 10^6$	$2.1 \cdot 10^6$
Poland	9030, 5630	1.6	1.00	-0.03	73	0.19	0.2	1.3	$1.9 \cdot 10^5$	$1.4 \cdot 10^6$	$7.2 \cdot 10^5$
Switzerland	29000, 17700	1.9	0.88	0.27	79	0.17	0.6	1.8	$7.0 \cdot 10^5$	$5.3 \cdot 10^6$	$2.5 \cdot 10^6$
UK	23500, 15140	1.3	1.07	0.23	78	0.19	0.5	1.3	$5.7 \cdot 10^5$	$3.4 \cdot 10^6$	$2.3 \cdot 10^6$
Japan	26460, 15960	2.7	0.83	0.17	80	0.20	0.7	2.3	$6.0 \cdot 10^5$	$4.1 \cdot 10^6$	$1.6 \cdot 10^6$
Australia	25370, 15750	1.2	0.72	0.99	78	0.21	1.2	1.9	$6.9 \cdot 10^5$	$5.2 \cdot 10^6$	$2.4 \cdot 10^6$

[1] in PPPUS$, [2] private consumption in PPPUS$ according to [56], [3] average yearly economic growth in % for 1870-1992, after [32], [4] crude mortality (2000) in % [6], [5] population growth (2000) in % [6], [6] estimates based on [15] including 1 hour travel time per working day and a life working time of 45 years, [7] SLSC computed with g and age-averaged life expectancies, [8] computed from recent period life tables, Δ indicates constant additive mortality changes.

between economic, work-leisure time aspects and demographic factors determines the given values of $SLSC$, $G_{\Delta\bar{E}}$ and $\overline{SVSL}$. It may also be mentioned that mortality reduction scheme corresponding to $G_{\Delta\bar{E}}$ is most appropriate for technical facilities or natural hazards.

It should be clear that criteria like Eq. (5.22) are gradient constraints which, depending on the solution algorithm, impose rather strong differentiability requirements on $C(\mathbf{p})$ and $h(\mathbf{p})$.

6. Numerical techniques

6.1. Principles of a one-level approach

Let $\mathbf{p}$ be a parameter vector which enters in both the cost function and the limit state function $g(\mathbf{u},\mathbf{p}) = 0$. Benefit, construction and damage function as well as the limit state function(s) are differentiable in $\mathbf{p}$ and $\mathbf{u}$. The conditions for the application of FORM/SORM hold. In the so-called β−point $\mathbf{u}^*$ the optimality conditions (Kuhn–Tucker conditions) are [26]:

$$\begin{gathered} g(\mathbf{u},\mathbf{p}) = 0, \\ \frac{\mathbf{u}}{\|\mathbf{u}\|} = -\frac{\nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p})}{\|\nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p})\|}. \end{gathered} \tag{6.1}$$

The geometrical meaning of (6.1) is that the gradient of $g(\mathbf{u},\mathbf{p}) = 0$ is perpendicular to the vector of direction cosines of $\mathbf{u}^*$. The basic idea mentioned first in [17] and elaborated in [26] now is to use these conditions as constraints in the cost optimization problem thus avoiding a bi-level optimization. It will turn out that this concept is crucial for further numerical analysis as described below.

It is important to reduce the set of the gradient conditions in the Kuhn–Tucker conditions by one. Otherwise the system of Kuhn–Tucker conditions is overdetermined. It is also important that the remaining Kuhn–Tucker conditions are retained under all circumstances, for example, if one or more gradient Kuhn–Tucker conditions become co-linear with one or more of the other constraints possibly included in the cost-benefit optimization task. Otherwise the so-called β-point conditions are not fulfilled.

6.2. Formulations for time-variant problems

In the simplest stationary, one-component case we have:

$$Z(\mathbf{p}) = B - C(\mathbf{p}) - (C(\mathbf{p}) + H) \cdot \frac{\nu^+(\mathbf{p})}{\gamma} \tag{6.2}$$

subject to:

$$\begin{aligned}
&g(\mathbf{u},\mathbf{p}) = 0,\\
&u_i\|\nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p})\| + \nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p})_i\|\mathbf{u}\| = 0, \quad i = 1,\dots,n-1,\\
&h_k(\mathbf{p}) \le 0, k = 1,\dots,q,\\
&\nu^+(\mathbf{p}) \le \nu^+_{\text{admissible}}\\
&\text{or}\quad \nabla_p C(\mathbf{p}) + G_{x\bar{E}}(\zeta,\rho,n)kN_{PE}\frac{\zeta}{\gamma}\nabla_p \nu^+(\mathbf{p}) \ge 0,
\end{aligned}$$

depending on whether a reliability constraint is imposed exogeneously or criterion (5.22) is used.

If there are multiple failure modes one replaces $\frac{\nu^+(\mathbf{p})}{\gamma}$ by $\frac{\sum_{i=1}^{s}\nu^+(\mathbf{p})}{\gamma}$, for example. In this case generalizing ideas in [28] (see also [55])

$$Z(\mathbf{p}) = B - C(\mathbf{p}) - (C(\mathbf{p}) + H)\cdot\frac{\sum_{k=1}^{s}\nu_k^+(\mathbf{p})}{\gamma} \tag{6.3}$$

subject to:

$$\begin{aligned}
&g_k(\mathbf{u}_k,\mathbf{p}) = 0, \quad k = 1,\dots,s,\\
&u_{i,k}\|\nabla_{\mathbf{u}} g_k(\mathbf{u}_k,\mathbf{p})\| + \nabla_{\mathbf{u}} g_k(\mathbf{u}_k,\mathbf{p})_i\|\mathbf{u}_k\| = 0,\\
&\qquad\qquad i = 1,\dots,n_k-1, \quad k = 1,\dots,s,\\
&h_\ell(\mathbf{p}) \le 0, \quad \ell = 1,\dots,q,\\
&\sum_{k=1}^{s}\nu_k^+(\mathbf{p}) \le \nu^+_{\text{admissible}}, \quad k = 1,\dots,s,\\
&\text{or}\quad \nabla_p C(\mathbf{p}) + G_{x\bar{E}}(\zeta,\rho,n)kN_{PE}\frac{\zeta}{\gamma}\nabla_p\sum_{k=1}^{s}\nu_k^+(\mathbf{p}) \ge 0,
\end{aligned}$$

where the Kuhn–Tucker conditions have to be fulfilled separately for each failure mode. Note that there are s distinct independent vectors $\mathbf{u}_k$.

If the problem is non-stationary it is sufficient to determine the asymptotic renewal intensity or the mean value of time between renewals in many cases (see Eq. (3.14)). Several but not always successful methods have been studied in [44]. However, for (locally) non-stationary problems, especially aging problems and for problems with non-Poissonian failures, a rather general, numerical solution can be proposed. More precisely, the Laplace transform is taken numerically and each value of the failure density is computed by FORM/SORM. A first model makes use of the asymptotic result in Eq. (3.14), i.e. requires the computation of the mean failure time. A better failure model certainly is a model where mean and standard deviation of the failure times are determined. As mentioned this is also an asymptotic approximation for arbitrary failure models being identical to the Gaussian model. Both models may be used as approximations. The first two moments of an arbitrary failure model then need to be computed from Eq. (2.3). The integrals are represented as sums of equi-distant values of the integrand:

$$I_k(\mathbf{p}) = \Delta \sum_{j=0}^{m} w_j i_k(t_j, \mathbf{p}) \tag{6.4}$$

where w_j are the weights (for example according to Simpson or Newton) and $i(t_j)$ are the values of the integrands, that is $\Phi(\beta(t_j, \mathbf{p}))$ and $t_j\Phi(\beta(t_j, \mathbf{p}))$ according to Eq. (2.3), respectively, for $k = 1$ and $k = 2$ (SORM-factor neglected). The integrand function $\exp\left[-\gamma t\right] f_T(t, \mathbf{p})$ is bell-shaped. Any suitable integration schemes can be used alternatively. Then, with $f^*(\gamma, \mathbf{p}) = \exp\left[\frac{1}{2}\gamma\left(\gamma\sigma(\mathbf{p})^2 - 2m(\mathbf{p})\right)\right]$ and $m(\mathbf{p}) = E[T(\mathbf{p})] = \Delta\sum_{j=0}^{n} w_j i_1(t_j, \mathbf{p})$ as well as $\sigma(\mathbf{p})^2 = \Delta\sum_{j=0}^{n} w_j i_2(t_j, \mathbf{p}) - m(\mathbf{p})^2$ the Kuhn–Tucker-conditions must be fulfilled at each t_j and one can write similar to the procedure for series systems

$$Z(\mathbf{p}) \approx B - C(\mathbf{p}) - (C(\mathbf{p}) + H) \cdot \frac{\exp\left[\frac{1}{2}\gamma\left(\gamma\sigma(\mathbf{p})^2 - 2m(\mathbf{p})\right)\right]}{1 - \exp\left[\frac{1}{2}\gamma\left(\gamma\sigma(\mathbf{p})^2 - 2m(\mathbf{p})\right)\right]} \tag{6.5}$$

subject to:

$$g(\mathbf{u}_j, \mathbf{p}, t_j) = 0 \quad \text{for} \quad j = 0, 1, \ldots, m,$$

$$u_{i,j}\|\nabla_{\mathbf{u}} g(\mathbf{u}_j, \mathbf{p}, t_j)\| + \nabla_{\mathbf{u}} g(\mathbf{u}_j, \mathbf{p}, t_j)_i \|\mathbf{u}_j\| = 0, \qquad i = 1, \ldots, n-1, \quad j = 0, \ldots, m,$$

$$h_k(\mathbf{p}) \le 0, \quad k = 1, \ldots, q,$$

$$\frac{1}{m(\mathbf{p})} \le h_{\text{admissible}}$$

$$\text{or} \quad \nabla_p C(\mathbf{p}) + G_{x\bar{E}}(\zeta, \rho, n) k N_{PE} \frac{\zeta}{\gamma} \nabla_p\left(\frac{1}{m(\mathbf{p})}\right) \ge 0,$$

where $\beta(t_j, \mathbf{p}) = \|\mathbf{u}_j^*\|$. The vectors $\mathbf{u}_j$, $j = 0, 1, \ldots, m$, are mutually independent. Therefore, the size of the optimization problem grows as $n \times m$.

The same scheme, however, applies to the full Laplace transform of non-stationary problems.

$$Z(\mathbf{p}) \approx B - C(\mathbf{p}) - (C(\mathbf{p}) + H) \cdot \frac{f^*(\gamma, \mathbf{p})}{1 - f^*(\gamma, \mathbf{p})}, \tag{6.6}$$

subject to:

$$g(\mathbf{u}_j, \mathbf{p}, t_j) = 0 \quad \text{for} \quad j = 0, 1, \ldots, m,$$

$$u_{i,j} \|\nabla_{\mathbf{u}} g(\mathbf{u}_j, \mathbf{p}, t_j)\| + \nabla_{\mathbf{u}} g(\mathbf{u}_j, \mathbf{p}, t_j)_i \|\mathbf{u}_j\| = 0, \qquad i = 1, \ldots, n-1, \quad j = 0, \ldots, m,$$

$$h_\ell(\mathbf{p}) \le 0, \quad \ell = 1, \ldots, q,$$

$$\frac{1}{m(\mathbf{p})} \le h_{\text{admissible}}$$

$$\text{or} \quad \nabla_p C(\mathbf{p}) + G_{x\bar{E}}(\zeta, \rho, n) k N_{PE} \frac{\zeta}{\gamma} \nabla_p \left(\frac{1}{m(\mathbf{p})}\right) \ge 0,$$

where

$$f^*(\gamma, \mathbf{p}) \approx \Delta \sum_{j=0}^{m} w_j \exp\left[-\gamma t_j\right] f_T(t_i, \mathbf{p}). \tag{6.7}$$

For the case in Eq. (2.2) it is

$$f^*(\gamma, \mathbf{p}) \approx \Delta \sum_{j=0}^{m} w_j \exp\left[-\gamma t_j\right] \left(-\varphi(\beta(t_j, \mathbf{p}))\right) \frac{d\beta(t_j, \mathbf{p})}{dt} \tag{6.8}$$

with $\frac{d\beta(t_j,\mathbf{p})}{dt} = \frac{\frac{\partial}{\partial t} g(\mathbf{u}_j, t_j, \mathbf{p})}{\|\nabla_{\mathbf{u}} g(\mathbf{u}_j, t_j, \mathbf{p})\|}$ [23] and in the case (2.11):

$$f^*(\gamma, \mathbf{p}) \approx \Delta \sum_{j=0}^{m} w_j \exp\left[-\gamma t_j\right] \cdot \left(P_f(0)\delta(0) + \left(\sum_{i=1}^{n_J} \lambda_i \Phi_2(\beta(t_j, \mathbf{p}), -\beta(t_j, \mathbf{p}); \rho_i(t_j, \mathbf{p})) + \omega_0 \frac{\varphi(\beta(t_j, \mathbf{p}))}{\sqrt{2\pi}} \right) \right). \tag{6.9}$$

If $f_T(t, \mathbf{p})$ depends on a random parameter $\mathbf{R}$ one has to use the approximation $E_{\mathbf{R}}\left[\frac{f^*(\gamma,\mathbf{p},\mathbf{R})}{1-f^*(\gamma,\mathbf{p},\mathbf{R})}\right] \approx \frac{f^*(\gamma,\mathbf{p},E[\mathbf{R}])}{1-f^*(\gamma,\mathbf{p},E[\mathbf{R}])}$. It can be shown that usually the approximation $E_{\mathbf{R}}\left[\frac{f^*(\gamma,\mathbf{p},\mathbf{R})}{1-f^*(\gamma,\mathbf{p},\mathbf{R})}\right] \approx \frac{E_R[f^*(\gamma,\mathbf{p},\mathbf{R})]}{1-f^*(\gamma,\mathbf{p},E[\mathbf{R}])}$ is found to be closest if $f^*(\gamma, \mathbf{p}, \mathbf{R}) \ll 1$. A similar computation scheme can, of course, be used if obsolescence and/or inspections and repairs are included.

Finally, the case of multiple failure mode system is given for arbitrary failure models:

$$
\begin{aligned}
Z(\mathbf{p}) &= B - C(\mathbf{p}) - \sum_{k=1}^{s}(C_k(\mathbf{p}) + H_k) \cdot \frac{f_{1,k}^{**}(\gamma, \mathbf{p})}{1 - \sum_{k=1}^{s} f_{1,k}^{**}(\gamma, \mathbf{p})} \\
&\leq B - C(\mathbf{p}) - \sum_{k=1}^{s}(C_k(\mathbf{p}) + H_k) \cdot \frac{f_k^{*}(\gamma, \mathbf{p})}{1 - \sum_{k=1}^{s} f_k^{*}(\gamma, \mathbf{p})}
\end{aligned}
\tag{6.10}
$$

subject to:

$$g_k(\mathbf{u}_{j,k}, \mathbf{p}, t_j) = 0 \quad \text{for} \quad k = 1, \ldots, s, \quad j = 0, 1, \ldots, m,$$

$$u_{i,j,k} \|\nabla_{\mathbf{u}} g_k(\mathbf{u}_{j,k}, \mathbf{p}, t_j)\| + \nabla_{\mathbf{u}} g_k(\mathbf{u}_{j,k}, \mathbf{p}, t_j)_i \|\mathbf{u}_{j,k}\| = 0,$$
$$i = 1, \ldots, n_k - 1, \quad j = 0, \ldots, m, \quad k = 1, \ldots, s,$$

$$h_\ell(\mathbf{p}) \leq 0, \quad \ell = 1, \ldots, q,$$

$$\sum_{k=1}^{s} \frac{1}{m(\mathbf{p})} \leq h_{\text{admissible}}$$

$$\text{or} \quad \nabla_p C(\mathbf{p}) + G_{x\bar{E}}(\zeta, \rho, n) k N_{PE} \frac{\zeta}{\gamma} \nabla_p \sum_{k=1}^{s} (\frac{1}{m_k(\mathbf{p})}) \geq 0,$$

where, for example, for the case in Eq. (4.47)

$$
\begin{aligned}
f_s^{**}(\gamma) &= \int_0^\infty \exp[-\gamma t] f_s(t) dt = \int_0^\infty \exp[-\gamma t] \sum_{k=1}^{s} f_{1,k}(t) dt = \sum_{k=1}^{s} f_{1,k}^{**}(t) \\
&\approx \sum_{k=1}^{s} \sum_{j=0}^{m} w_j \exp[-\gamma t_j] \varphi_1(\beta_k(t_j)) \Phi_{s-1}(\hat{\mathbf{c}}_k; \hat{\mathbf{R}}_k) \left(\frac{-\frac{\partial}{\partial t} g_k(\mathbf{u}_j^*, t_j)}{\|\nabla_{\mathbf{u}} g_k(\mathbf{u}_j^*, t_j)\|} \right).
\end{aligned}
\tag{6.11}
$$

The problem now can be rather large, i.e. there are $s \times m$ independent random vectors of length n_k. Clearly, the most difficult part in such calculations is the assessment of t_m and m. However, the exponent term in Eq. (6.7) usually lets the integrand decay sufficiently fast. If reliability restrictions are imposed it is necessary in all practical cases to use Eq. (3.14) because the inversion of the Laplace transform of the renewal density is numerically extremely difficult. It is further noted that the scheme proposed above can also be used when the benefit is non-constant as in Eq. (4.33) or the damage term is non-constant as in Eq. (4.35).

6.3. Solution algorithm

In order to solve the optimization problem a suitable optimization algorithm is required. Based on sequential quadratic programming methods a new optimization algorithm JOINT5 has been developed from an earlier algorithm proposed by Enevoldsen/Sorensen [14]. This turned out necessary because the tasks in (6.2), (6.3), (6.5), (6.6) and (6.10) require special precautions which are not necessarily available in most of the off-shelf algorithms. For example, the algorithm includes a reliable and robust slow down strategy to improve stability of the algorithm instead of an exact (or approximate) line search which too often is the reason for non-convergence [35]. A special 'extended' equation system is solved in case of failure in the quadratic subalgorithm, e.g., due to linear dependence of the linearized constraints. In addition, the algorithm contains a careful active set strategy (for further details see [53]).

Gradient-based methods need first derivatives of the objective and all active constraints. In case of cost optimization under reliability constraints first order Kuhn–Tucker optimality conditions for a design point are restrictions to the optimization problem. These equations are given in terms of the first derivatives of the limit state function. The gradients of these conditions involve second derivatives. Thus, the solution of the quadratic subproblem needs second derivatives, i.e. the complete Hessian of $g(\mathbf{u}, \mathbf{p})$. The determination of the Hessian in each iteration step is laborious and can be numerically inexact. In order to avoid this, an approximation by iteration is proposed. The Hessian is first preset with zeros. Note that linear limit state functions always have a zero Hessian matrix. This implies loss of efficiency, but the overall numerical effort needs not to rise, because calculation of the Hessian is no more necessary. In order to improve the results in case of nonlinear limit state functions, it is possible to evaluate the Hessian after the first optimization run and restart the algorithm. The solution is the new starting point and the Hessian matrix is fixed for the whole run. This iterative improvement with subsequent restarts continues until the results differ only with respect to a given precision which is usually after very few steps. The results can be simultaneously improved by including second-order corrections during reiteration (see [30]). Any other more exact improvement can be taken into account in a similar manner.

All in all, the techniques proposed enable the solution of quite general problems. They are still based on a one-level optimization but rather strong requirements on differentiability of the objective, limit state functions and other restrictions must be made. Also, a possibly substantial increase of the problem dimension must be expected in extreme cases and, hence, much computing time will be necessary.

It must be mentioned that there are alternative solution algorithms such as a bi-level optimization [52] or algorithms based on semi-infinite programming [11]. Also, very little is known how to solve problems if objectives and constraints do not fulfill certain differentiability requirements.

7. Illustrating examples

7.1. Example 1: Random capacity and random shock load [43]

As a first example from the structures area we take a rather simple case of a single-mode system where failure is defined if a random resistance or capacity is exceeded by a random demand. The demand is modelled as a one-dimensional, stationary marked Poissonian renewal process of disturbances (earthquakes, wind storms, explosions, etc.) with stationary renewal rate λ and random, independent sizes of the disturbances $S_i, i = 1, 2, \ldots$. The resistance is log-normally distributed with mean p and a coefficient of variation V_R. The disturbances are independently log-normally distributed with mean equal to unity and coefficient of variation V_S so that p can be interpreted as central safety factor. A disturbance causes failure with probability:

$$P_f(p) = \Phi\left(-\frac{\ln\left\{p\sqrt{\frac{1+V_S^2}{1+V_R^2}}\right\}}{\sqrt{\ln\left((1+V_R^2)(1+V_S^2)\right)}}\right). \tag{7.1}$$

An appropriate objective function then is with $b = b(\mathbf{p})$:

$$Z(p) = \frac{b}{C_0\gamma} - \left(1 + \frac{C_1}{C_0}p^a\right) - \left(1 + \frac{C_1}{C_0}p^a + \frac{H_M}{C_0} + \frac{H_F}{C_0}\right)\frac{\lambda P_f(p)}{\gamma}. \tag{7.2}$$

The criterion (5.20) has the form:

$$\frac{d}{dp}\left(C_0 + C_1 p^a\right) \geq -G_{x\bar{E}}(\rho, n)kN_{PE}\frac{\zeta}{\gamma}\frac{d}{dp}\left(\lambda P_f(p)\right). \tag{7.3}$$

Some more or less realistic, typical parameter assumptions are: $C_0 = 10^6$, $C_1 = 10^4$, $a = 1.25$, $H_M = 3 \cdot C_0$, $V_R = 0.2$, $V_S = 0.3$, and $\lambda = 1$ [1/year]. The LQI-data is $e = 77$, $GDP = 25000$, $g = 15000$, $m = 0.01$, $c_{1\bar{E}} = 0.25$ or $c_{2\bar{E}} = 0.75$, $w = 0.15, N_{PE} = 100, k = 0.1$ so that $H_F = SLSC\ kN_{PE} = 8.4 \cdot 10^6$, $G_{1\bar{E}}(\rho, n)kN_{PE} = 2.1 \cdot 10^7$ and $G_{2\bar{E}}(\rho, n)kN_{PE} = 6.2 \cdot 10^7$. The value of N_{PE} is chosen relatively large for demonstration purposes. Monetary values are in US$. Optimization is performed for the public and for the owner separately.

For the public $b_S = \beta C_0$ with $\beta = 0.02$ from Table 3 and $\gamma_S = 0.0185$ determined from Eq. (4.56) are chosen. Also, we take $\frac{\zeta}{\gamma_S} = 1$ for simplicity. In particular, benefit and discount rate are chosen such that the public does not make direct profit from an economic activity of its members. Optimization including the cost H_F gives $p_S^* = 4.35$, the corresponding failure rate is $1.2 \cdot 10^{-5}$. Criterion (5.20) is already fulfilled for $p_l = 3.34$ and $p_u = 3.68$, respectively, corresponding to yearly failure rates of $2.5 \cdot 10^{-4}$ and $9.1 \cdot 10^{-5}$, respectively, but $Z_S(p_l)/C_0$ and $Z_S(p_u)/C_0$ being already negative. It is notable that although the two demographic constants $C_{\pi \bar{E}}$ differ by a factor of three the acceptability limits are close together. It is also interesting to see that in this case the public can do better in adopting the optimal solution rather than just realizing the facility at its acceptability limit.

The owner uses some typical values of $b_O = 0.07C_0$ and $\gamma_O = 0.05$ and does or does not include societal life saving cost. If he includes life saving cost the objective function is shifted to the right (dotted line). The calculations yield $p_O^* = 3.76$ and $p_O^* = 4.03$, respectively, and the corresponding failure rates are $7.1 \cdot 10^{-5}$ and $3.2 \cdot 10^{-5}$. The $SLQI$-based acceptability criterion limits the owner's region for reasonable designs. Inclusion of life saving cost has relatively little influence on the position of the optimum.

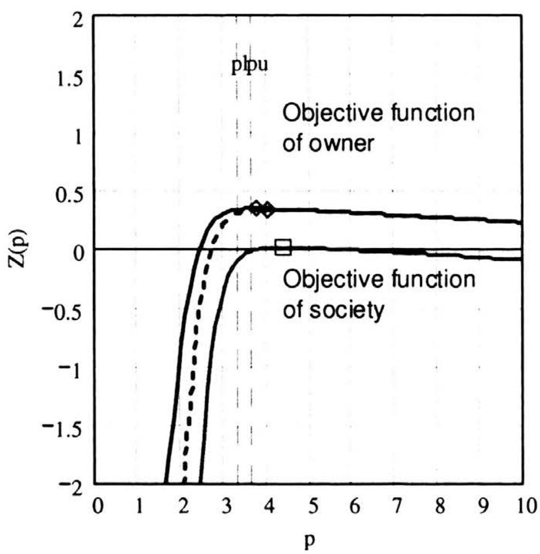

FIGURE 3. Objective function for society and owner (with and without life saving cost).

It is noted that the stochastic model and the variability of capacity and demand also play an important role for the magnitude and location of the optimum as well as the acceptability limit. The specific marginal cost (rate of change) of a safety measure and its effect on a reduction of the failure rate are equally important.

This example also allows to derive risk-consequence curves by varying the number of fatalities in an event. With the same data as before but $SLSC = 7 \cdot 10^5$ and $G_{x\bar{E}}(\rho, n) = 4 \cdot 10^6$ for $N_F = 1$ we first vary the cost effectiveness of the safety measure (see Fig. 4). Here, only the ratio C_1/C_0 is changed. The upper bounds (solid lines) are derived from Eq. (5.13) and the lower bounds (dashed lines) corresponds to the societal optimum according to Eq. (4.8) ($b_S = 0.02C_0$, $\gamma_S = 0.0185$).

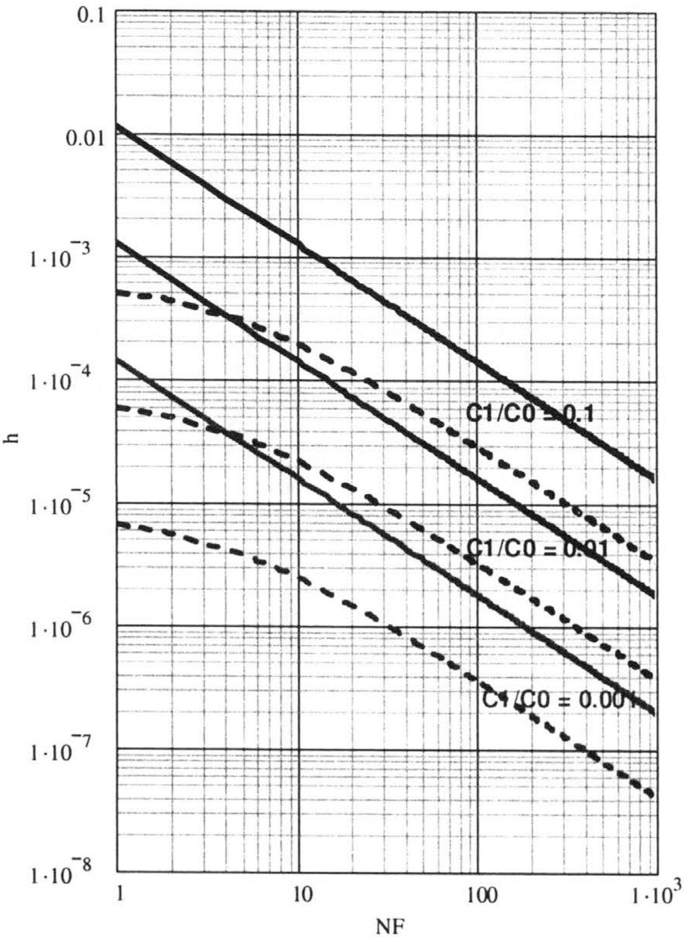

FIGURE 4. Acceptable failure rate over number of fatalities for different C_1/C_0. Dashed lines correspond to optimal solution for the public.

Most realistic is probably a ratio of $C_1/C_0 = 0.001$. The failure rate of approximately 10^{-4} per year corresponds well with the controllable crude mortality of the same magnitude as mentioned earlier. In Fig. 5 the mortality reduction regimes are varied indicating that this is of only moderate influence. In this figure the region between the upper bound(s) and the lower curve derived from the societal optimum may be interpreted as ALARP-region (ALARP = **A**s **L**ow **A**s **R**easonably **P**racticable).

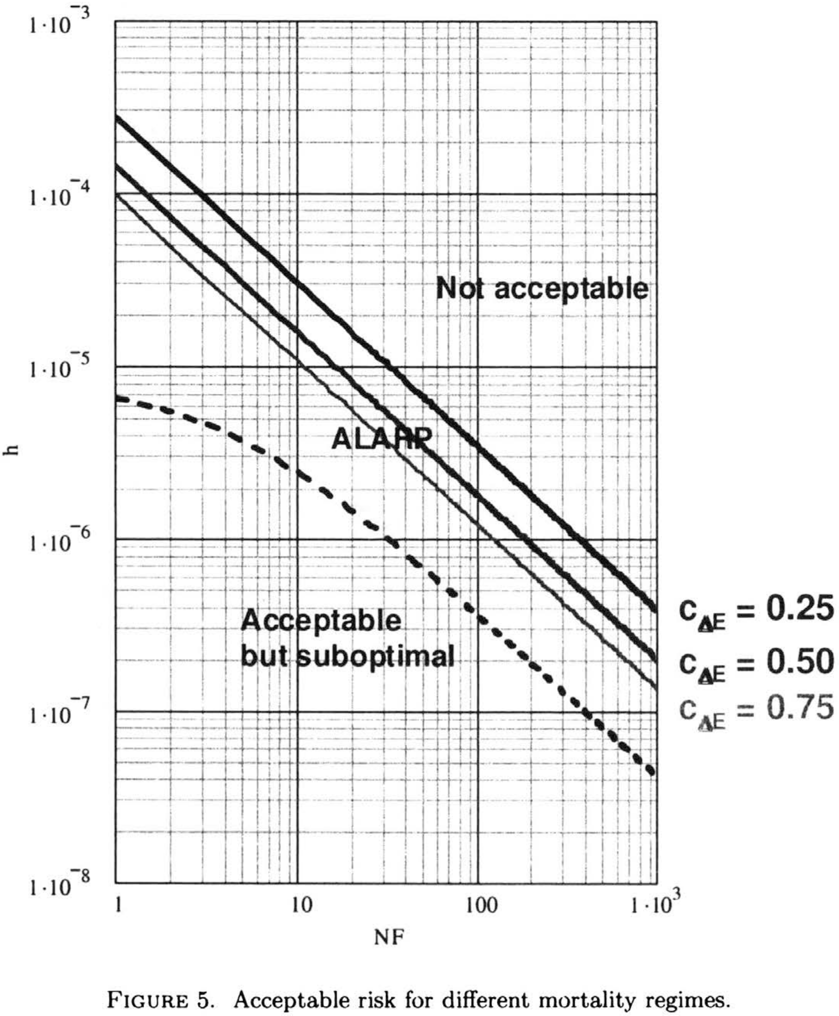

FIGURE 5. Acceptable risk for different mortality regimes.

Note that in these figures the failure rate is given by λP_f and the number of fatalities is given by $N_F = kN_{PE}$. Therefore, these figures cover the full range of λ and P_f and k and N_{PE}, respectively. However, it is to be emphasized that in both figures the precise location and slope of the acceptability curve depend on the specific physical and stochastic model.

7.2. Example 2: Earthquake resistant design

A technically more involved example tries to establish a basis for the codification of design values for earthquake resistant design. It introduces so-called risk integrals. We follow closely the considerations in [47] but use slightly different data. In a seismic region Poissonian earthquakes occur with rate $\lambda = 2.9$ [1/year]. Magnitudes between $m_u = 4.0$ and $m_o = 7.5$ are considered. A truncated Weibull distribution (for maxima) has been found to model the data adequately

$$F_M(m) = \frac{\exp\left[-\left(\frac{m_o-m}{m_o-w}\right)^k\right] - \exp\left[-\left(\frac{m_o-m_u}{m_o-w}\right)^k\right]}{1-\exp\left[-\left(\frac{m_o-m_u}{m_o-w}\right)^k\right]} \tag{7.4}$$

with $w = 4.35$ and $k = 8.11$. These data are characteristic for an area with medium to high seismicity. With the attenuation law:

$$a = h(m,r) = b_1 \exp(b_2 m)(r^2 + 7.3^2)^{-1/2} \exp(-b_3 r) = \exp(b_2 m) b(r) \tag{7.5}$$

where $b_1 = 0.0955g, b_2 = 0.573, b_3 = 0.00587$, one determines the density of peak ground acceleration as

$$f_A(a,r) = \frac{k\frac{(m_o-h^{-1}(a,r))^{k-1}}{(m_o-w)^k} \exp\left[-\left(\frac{(m_o-h^{-1}(a,r))}{m_o-w}\right)^k\right] \frac{dh^{-1}(a,r)}{da}}{1-\exp\left[-\left(\frac{m_o-m_u}{m_o-w}\right)^k\right]} \tag{7.6}$$

with $m_u \leq h^{-1}(a,r) = \frac{1}{b_2}\ln(\frac{a}{b(r)}) \leq m_o$. Possible epicentra are uniformly distributed around the site in a radius of $r_{\max} = 200\,\text{km}$. Hence, the density of peak ground acceleration is:

$$f_A(a) = \int_0^{r_{\max}} f_A(a,r) \frac{2r}{r_{\max}^2} dr. \tag{7.7}$$

Peak ground acceleration then varies with a coefficient of variation of $V_A = 1.55$. The maximum responses given peak ground acceleration vary log-normally with coefficient of variation of $V_S = 0.60$. A simplified limit state function then is

$$g(\mathbf{X}) = R - KSAE \leq 0 \tag{7.8}$$

for shear resistance versus shear demand. Herein, R is a log-normal resistance with $V_R \approx 0.2$, K contains all system-specific properties and is, without loss of generality, assumed to equal untity, S is the log-normal variability in the (elastic) spectral enhancement factor with mean $m_S = 1$ and A is peak ground acceleration. The systematic frequency-dependent part of S must be taken into account in K. E is the log-normal error in relation (7.5) with mean 1 and coefficient of variation $V_E \approx 0.6$. The conditional failure probability (fragility curve) is

$$P_f(p \mid a) = \Phi\left(-\frac{\ln\left\{\frac{p}{K\cdot m_S\cdot a\cdot m_E}\sqrt{\frac{(1+V_S^2)(1+V_E^2)}{1+V_R^2}}\right\}}{\sqrt{\ln\left((1+V_R^2)(1+V_S^2)(1+V_E^2)\right)}}\right) \tag{7.9}$$

with $p = m_R$ the design parameter because $m_S m_E = 1.0$. The objective function (without benefit term) for systematic reconstruction is

$$Z(p) = C(p)f(N_{PE}) + E_A\left[\left(C_R(p,a)(1 - P_f(p \mid a))f(N_{PE})\frac{\lambda}{\gamma}\right)\right.$$
$$\left. + \left((C(p) + H_0 + H_M(a))f(N_{PE}) + H_F(a)\right)\frac{\lambda P_f(p \mid a)}{\gamma}\right]. \tag{7.10}$$

The acceptability criterion Eq. (5.19) to be used as a constraint for Eq. (7.10) correspondingly reads:

$$\frac{d}{dp}\left(C_0 + C_1 p^\delta\right)$$
$$\geq -E_A\left[G_{\Delta\bar{E}}(\rho,n)\frac{1}{2}(1 - \exp(-0.25a))N_{PE}\frac{\zeta}{\gamma}\frac{d}{dp}\left(\lambda P_f(p,a)\right)\right]. \tag{7.11}$$

The following widely verified relationship between peak ground acceleration and MSK-intensity $\log(a) = 0.31MSK - 2.5$ is assumed. We distinguish between normal damage to the building during an earthquake and building collapse. Construction, retrofitting, loss of business and physical damage cost are slightly underproportional to the occupation rate of a unit in a residential building so that we choose $f(N_{PE}) = (N_{PE}/3)^{0.8}$. $C(p) = \left(C_0 + C_1 p^\delta\right)$is the construction cost, $C_R(p,a) = C(p)(1-\exp(-0.25a))$ is the cost of retrofitting taking account of the fact that retrofitting cost approach the cost of complete reconstruction for larger a, $H_M(a) = H_M a^{0.4}$ is the physical damage cost. The physical damage term includes infrastructure losses for large accelerations. Indirect cost such as loss of business is approximated by $H_0 = \ell C_0$. The estimation of human losses is difficult. They also depend on a as people are increasingly trapped at collapse. Immediate death then has probability 0.3 to 0.5 or more but some 10 to 20% die later in hospitals. This leads to $H_F(a) = SLSC\ \frac{1}{2}(1 - \exp(-0.25a))N_{PE}$ for the compensation cost for loss of human life. Note that the factor $\frac{1}{2}(1 - \exp(-0.25a))$ replaces the constant k in Eq. (5.19). The constant 0.25 in these relationships appears to vary with building type and material. The concavity of the function with respect to a implies convexity with respect to MSK in agreement with estimates in the literature. It is certainly only a rough approximation. Furthermore, we have $C_0 = 10^6$, $C_1 = 3 \cdot 10^4$, $\delta = 1.1$, $H_M = 5 \cdot 10^5$, $\gamma = 0.02$, $\frac{\zeta}{\gamma} = 1$, $N_{PE} = 3$. It is beyond the scope of this paper to discuss all these special choices in detail but they are essentially in line with the findings in [7] and other sources in the literature. The damage term in Eq. (7.10) is conditional on a. The expectation operation removes the condition. The damage term is also called risk integral. Table 4 collects typical data for three different socio-economic levels.

TABLE 4.

Socio-economic level	*High*	*Medium*	*Low*
GDP	23500	6500	1500
w	0.145	0.17	0.20
e	77	65	55
C_Δ	35	50	65
ℓ	3.62	1.0	0.23

An additional FORM/SORM-analysis can then determine the design values a^* corresponding to p^* and some other results of interest in the following results table (Table 5).

TABLE 5.

Socio-economic level	*High*	*Medium*	*Low*
p^*	4.80	4.00	3.54
$h(p^*) = \lambda P_f(p^*)$ (FORM)	$3.7{\cdot}10^{-4}$	$6.3{\cdot}10^{-4}$	$8.8{\cdot}10^{-4}$
Return period $\frac{1}{h(p^*)}$	2700	1600	1100
a^*	1.03	0.97	0.84
$a^* s^* \varepsilon^*$	3.11	2.76	2.43
$C(p^*)/C_0$	1.17	1.14	1.12

The design values of the accelerations a^* have a return period of about 120 years. Roughly the same design accelerations for all socio-economic levels indicate that the quantity $E_A[\lambda P_f(p \mid a)]$ decays very slowly with a. The values of $a^*s^*\varepsilon^* < p^*$ are also given. Surprisingly, the acceptance criterion Eq. (7.11) is not active and produces values $p_{\lim}$ of 1.5, 1.0 and 0.5, respectively. These values imply an order of magnitude larger failure rates than the optimal solution. This example is somewhat extreme because the loading side varies very much. Large changes in the values of p^* result in rather small changes in the failure rates. This explains why the differences between the different socio-economic climates are relatively small. Also, in contrary to the previous example, both construction cost and damage cost have been referred to a residential unit and are roughly proportional to N_F. Therefore, the effect of varying N_F is insignificant and the failure rates in the second line of the results table are the individual risks due to earthquakes in that region.

7.3. Example 3: Rigid plastic two-bay frame [53]

In this simple example a double-bay frame as shown in Fig. 6 using rigid-plastic theory with random horizontal and vertical loading and random plastic moments at nodes 1 to 10 will be optimized under reliability constraints.

The structure can fail in eight different failure modes as shown in Fig. 7. The first three failure events are elementary mechanisms, the others combined mechanisms.

A limit state function for each failure mode is available using the energy theorem:

$$G_1(x,\mathbf{p}) = X_2 + 2X_3 + X_4 - X_{12}\cdot\frac{h}{2},$$

$$G_2(x,\mathbf{p}) = X_6 + 2X_7 + X_8 - X_{13}\cdot\frac{h}{2},$$

$$G_3(x,\mathbf{p}) = X_1 + X_2 + X_5 + X_8 + X_9 + X_{10} - X_{11}\cdot h,$$

$$G_4(x,\mathbf{p}) = X_2 + 2X_3 + X_4 + X_6 + 2X_7 + X_8 - X_{12}\cdot\frac{h}{2} - X_{13}\cdot\frac{h}{2},$$

$$G_5(x,\mathbf{p}) = X_1 + X_2 + X_5 + X_6 + 2X_7 + 2X_8 + X_9 + X_{10} - X_{11}\cdot h - X_{13}\cdot\frac{h}{2},$$

$$G_6(x,\mathbf{p}) = X_1 + X_2 + X_4 + 2X_7 + 2X_8 + X_9 + X_{10} - X_{11}\cdot h - X_{13}\cdot\frac{h}{2},$$

$$G_7(x,\mathbf{p}) = X_1 + 2X_3 + X_4 + X_5 + X_8 + X_9 + X_{10} - X_{11}\cdot h - X_{12}\cdot\frac{h}{2},$$

$$G_8(x,\mathbf{p}) = X_1 + 2X_3 + 2X_4 + 2X_7 + 2X_8 + X_9 + X_{10} - X_{11}\cdot h - X_{12}\cdot\frac{h}{2} - X_{13}\cdot\frac{h}{2},$$

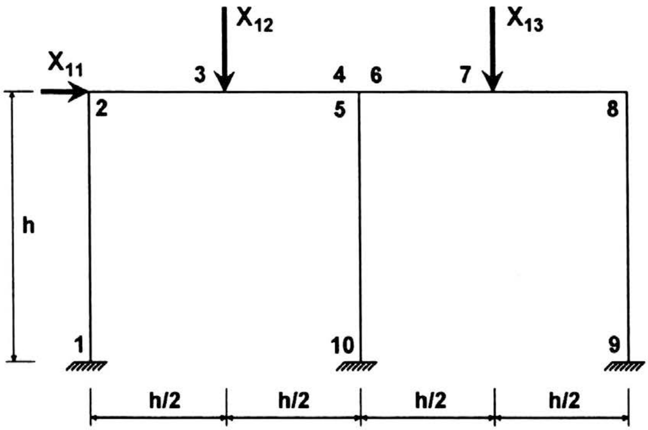

FIGURE 6. Loads and system geometry of the frame.

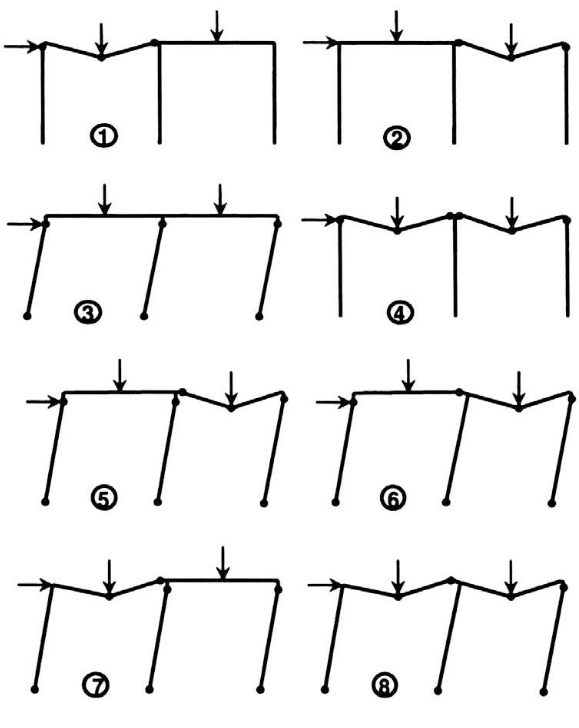

FIGURE 7. Failure modes of the frame.

where X_i, $i = 1, \ldots, 10$ are the plastic moments of the frame at node i. X_{11}, X_{12} and X_{13} are stochastic loads at node 2, 3 and 7. The stochastic properties of the random variables X_i are given in the following table. The random plastic moments at node X_i, $i = 1, \ldots, 10$ are assumed to be independently and lognormally distributed, the stochastic loads X_{11}, X_{12} and X_{13} are normally distributed (see Table 6).

TABLE 6.

Stochastic variable	[Unit]	*Mean/St.deviat.*
Plastic moment at node $1,2,5,8,9,10$	X_i [kNm]	$p_1/0.1 \cdot p_1$
Plastic moment at node $3,4,6,7$	X_i [kNm]	$p_2/0.1 \cdot p_2$
Load at node 2	X_{11} [kN]	2/0.6
Load at node 3	X_{12} [kN]	4/1.2
Load at node 7	X_{13} [kN]	6/1.8

The assumption of log-normal resistance variables makes the reliability problem slightly non-linear in the standard space. The loads at node 3 and 7 are modeled as stationary rectangular wave renewal processes with jump rates $\lambda_{12} = \lambda_{13} = 0.5$ [1/year]. The load at node 2 is modeled as stationary differentiable Gaussian process with autocorrelation function $\rho_{ij}(\tau) = \exp(-\tau^2)$. It follows that formula (4.46) applies for $n_J = 2$ and $n_D = 1$. The design parameters p_1 and p_2 are the mean values of the appropriate stochastic variables. The bounds for p_1 and p_2 are as follows: $p_1 \in [5.0; 80.0]$ kNm, $p_2 \in [5.0; 80.0]$ kNm. The objective function, which will be minimized in the optimization program, is defined as construction cost depending on the mean values of the plastic moments at nodes $1, \ldots, 10$ as $C(\mathbf{p}) = p_1 + 2.0 \cdot p_2$. The failure cost are $H = 900$ and the interest rate is $\gamma = 0.02$. The optimization problem contains of 106 optimization variables. The objective function to be minimized then is:

$$
\begin{aligned}
Z(\mathbf{p}) &= C(\mathbf{p}) + E_{\mathbf{R}}\left[\sum_{k=1}^{s} \left(C(\mathbf{p})_k + H_k\right) \frac{\nu^+(\mathbf{p}, R)_k}{\gamma}\right] \\
&\leq C(\mathbf{p}) + \sum_{k=1}^{s} \left(C(\mathbf{p})_k + H_k\right) \frac{E_{\mathbf{R}}\left[\nu^+(\mathbf{p}, R)_k\right]}{\gamma}
\end{aligned}
\tag{7.12}
$$

with $\mathbf{R} = (X_1, \ldots, X_2)$ collecting the random vector of resistances. Note that first line in Eq. (7.12) is exact but the Poissonian nature of outcrossings is lost if the different modes had also different damage cost. In this case, however, the damage cost are assumed to be equal for each failure mode and, therefore, the second line of Eq. (7.12) is also exact. The optimal cost parameter for time- variant cost optimization under reliability constraints of this series system with $h = 20\,\mathrm{m}$ and a time interval of one year are

$$p_1^* = 16.95, \qquad p_2^* = 38.45,$$

and the optimal cost are $C_{tot}(\mathbf{p}) = 93.85$ $[CU]$. The time-variant failure probability in each mode is computed as: $(P_{f,1}(\mathbf{p}^*)$, $P_{f,2}(\mathbf{p}^*)$, $P_{f,3}(\mathbf{p}^*)$, $P_{f,4}(\mathbf{p}^*)$, $P_{f,5}(\mathbf{p}^*)$, $P_{f,6}(\mathbf{p}^*)$, $P_{f,7}(\mathbf{p}^*)$, $P_{f,8}(\mathbf{p}^*)) = (9.76 \cdot 10^{-11}$, $2.21 \cdot 10^{-4}$, $2.53 \cdot 10^{-6}$, $2.37 \cdot 10^{-11}$, $4.13 \cdot 10^{-8}$, $1.82 \cdot 10^{-6}$, $9.66 \cdot 10^{-10}$, $1.38 \cdot 10^{-9})$. The system failure probability is $2.25 \cdot 10^{-4}$ with corresponding equivalent reliability index 3.51.

7.4. Example 4: Optimal replacement of a reinforced concrete structure subject to chloride corrosion in warm sea water [55]

Following [44] a simplified failure criterion for chloride corrosion in the splash zone in warm sea water is:

$$C_{cr} - C_s(1 - \operatorname{erf}\left(\frac{c}{2\sqrt{Dt}}\right)) \leq 0,$$

where C_{cr} = critical chloride content, C_s = surface chloride content, c = concrete cover and D = diffusion parameter. The stochastic model is presented in Table 7.

TABLE 7.

Variable	*Distr. function*	*Parameters*
C_{cr}	Uniform	0.125, 0.175
C_s	Uniform	0.2, 0.4
c	Log-normal	$m_c, 1$
D	Uniform	0.1, 0.315

The uniform distributions reflect the large uncertainty in the variables. The units are chosen such that t is in years. Inspection are performed at regular intervals a. They are followed by renewals (repairs) with probability $P_R(a) = 1 - \exp\left[-a_R a^2\right]$. The optimization variables are the mean concrete cover m_c and the length a of the inspection interval. Erection cost are $C(m_c) = C_0 + C_1 m_c^2$, inspection cost are $I_0 = 0.1C_0$, repair cost are $I_1 = 0.5C_0$ and we have $C_0 = 10^6$, $C_1 = 10^4$, $H = 10C_0$, $b = 015C_0$, $\gamma = 0.03$ and $a_R = 0.01$. the solution is $a^* = 66$ and $m_c^* = 6.5$. It turns out that preventive repairs should be performed every 66 years which saves up to 30% of the cost. These results comply well with practical experience with such structures. The contributions to the total damage cost are shown in Fig. 8.

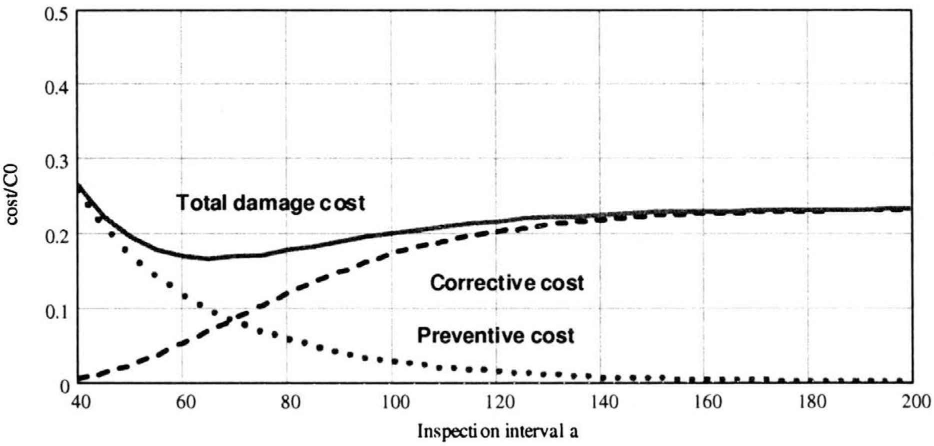

FIGURE 8. Total cost for regular inspections and renewals.

Relatively small variations in the repair model or in the cost factors will, however, result in cases where it is better not to inspect and repair but just wait for failure. It is noted that for the given failure model no mean time to failure exists.

7.5. Example 5: Series system of corroding expansion joints

A long multi-span bridge has s expansion joints which are exposed to corrosion due to heavy winter salting. For illustration purposes the state function is taken as $g(\mathbf{X}) = R(1 - C\sqrt{t}) - (S_1 + S_2)$ where $R \sim LN(m_R, 2)$, $C \sim UN(0.085, 0.115)$, $S_1 \sim N(1, 0.3)$ and $S_2 \sim GU(0, 0.2)$. If any of the expansion joints fails the bridge must be closed off. We investigate the quality of various computation schemes for series systems. The optimization variable is taken as the mean of resistance m_R and it is assumed that there are $s = 10$ joints. The objective function can be written as

$$Z(p) = \frac{b}{\gamma} - C(m_R) - (C(m_R) + H)h_s^*(m_R, \gamma),$$

$$h_s^*(m_R, \gamma) = \frac{f_S^*(m_R, \gamma)}{1 - f_S^*(m_R, \gamma)}, \qquad f_S^*(m_R, \gamma) = \int_0^\infty \exp\left[-\gamma t\right] f_s(t) dt,$$

$$f_s(t) = \frac{\partial}{\partial t} F_S(t) = \frac{\partial}{\partial t} P(\bigcup_{k=1}^{s} \left\{R(1 - C\sqrt{t}) - (S_{1,k} + S_{2,k}) \leq 0\right\}).$$

It is seen that R and C are common to all spots while the other variables are assumed to be independent from spot to spot and, therefore, $\rho_{ij}(t) = \alpha_R^2(t) + \alpha_C^2(t) \geq 0$. Here, we only compute the "discount factor" $h_s^*(m_R, \gamma)$ for various interest rates but different computation schemes for $m_R = 8$. Figure 9 shows that the upper bound solution appears to be rather conservative for larger s while the exact solution (with component correlation) and the solution for independent components are very close together. It can also be shown that consideration of the time variations in the correlations can as well be neglected. We also see that the upper bound solution breaks down for $\gamma < 0.04$.

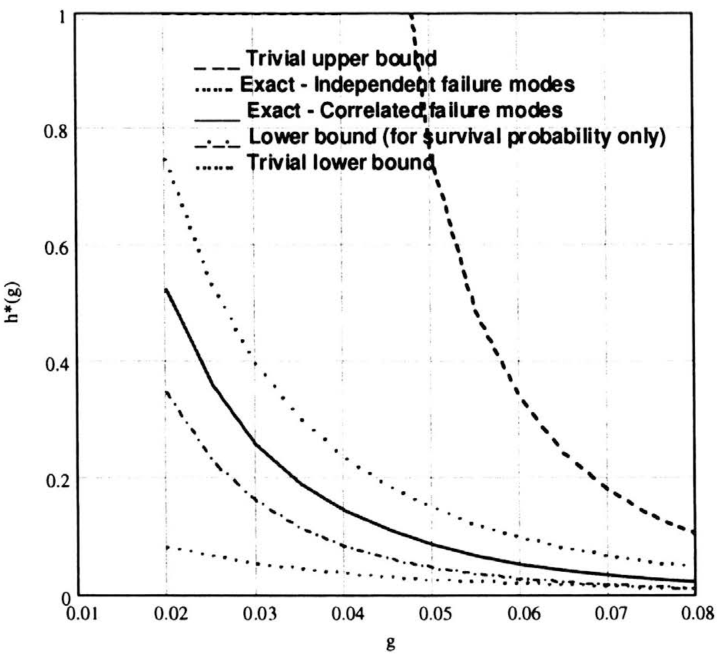

FIGURE 9. Discount factor for different solution schemes over discount rate ($s = 10$).

8. Summary and conclusions

Objective functions for cost-benefit optimization based on a continuous renewal model for a series of cases frequently met in practice are formulated. In particular, they include failures by outcrossings of loading processes and by random disturbances, non-constant benefit and damage functions, finite renewal times, repeated reconstructions and inspection and repair. Modifications to account for serviceability losses are proposed. A method for reliability-oriented time-variant structural optimization of dependent series systems using first order reliability methods (FORM) in standard space is developed generalizing theories proposed earlier for component problems and time-invariant series system problems in a special one-level approach. Certain improvements by taking account of dependencies among failure modes are also proposed. Approximations for time-variant failure probabilities are computed via the outcrossing method for locally stationary rectangular wave renewal and differentiable Gaussian processes. Numerical Laplace transforms are proposed for the treatment of aging components.

The optimization problem is solved by the newly developed gradient-based, locally convergent algorithm JOINT5. It requires second derivatives of the limit state functions. This can be avoided by iteration. In the first iteration the Hessian is approximated by a zero matrix corresponding to linear limit state functions. In the second iteration the Hessian is determined once and kept fixed. The results can, thus, be improved by reiteration of the complete optimization task. The same reiteration loop can be used to update the results by SORM or any other suitable method. More details about the technical aspects of reliability-based optimization are contained in [53].

Several examples illustrate theory and numerical aspects.

References

1. Barlow, R.E., Proschan, F., (1965), *Mathematical Theory of Reliability*, Wiley, New York.

2. Bhat, U.N., (1972), *Elements of Applied Stochastic Processes*, Wiley, New York.

3. Breitung, K., (1984), Asymptotic approximations for multinormal integrals, *Journal of Engineering Mechanics*, Vol.110, No.3, pp.357-366.

4. Breitung, K., (1988), Asymptotic approximations for the out-crossing rates of stationary vector processes, *Stochastic Processes and their Applications*, Vol.29, pp.195-207.

5. Breitung, K., Rackwitz, R., (1982), Nonlinear combination of load processes, *J. Struct. Mech.*, Vol.10, No.2, pp.145-166.

6. *CIA-factbook 2001*, `www.cia.gov/cia/publications/factbook/`

7. Coburn, A., Spence, R., (1992), *Earthquake Protection*, J. Wiley & Sons, Chichester.

8. Cox, D.R., (1962), *Renewal Theory*, Methuen, London.

9. Cramer, H., Leadbetter, M.R., (1967), *Stationary and Related Processes*, Wiley & Sons, New-York.

10. Der Kiureghian, A., Liu, P.-L., (1986), Structural reliability under incomplete probability information, *J. Eng. Mech.*, ASCE, Vol.112, No.1, pp.85-104.

11. ROYSET, J.O., DER KIUREGHIAN, A., POLAK, E., (2003), Successive approximations for the solution of optimal design problems with probabilistic objective and constraints, *Proc. ICASP'03*, A. Der Kiureghian (Ed.), Millpress, Rotterdam, pp.1049-1057.

12. DITLEVSEN, O., (1979), Narrow reliability bounds for structural systems, *Journal of Struct. Mech.*, Vol.7, No.4, pp.405-435.

13. DITLEVSEN, O., MADSEN, H.O., (1996), *Structural Reliability Methods*, Wiley, Chichester.

14. ENEVOLDSEN, I., SORENSEN, J.D., (1992), Optimization algorithms for calculation of the joint design point in parallel systems, *Structural Optimization*, Vol.4, Springer: Berlin, Heidelberg, New York, pp.121-127.

15. EVANS, J.M., LIPPOLDT, D.C. MARIANNA, P., (2001), *Trends in Working Hours in OECD Countries*, 45, OECD, Paris.

16. FELLER, W., (1970), *An Introduction to Probability and its Applications*, I, 3rd ed., Wiley, New York.

17. FRIIS HANSEN, F., MADSEN, H.O., (1992), A Comparison of some algorithms for reliability-based structural optimization and sensitivity analysis, *Proc. 4th IFIP WG 7.5 Conf. Munich*, R. Rackwitz and P. Thoft-Christensen (Eds.), Springer Verlag Berlin, pp.443-451.

18. FOX, B., (1966), Age replacement with discounting, *Operations Research*, Vol.14, pp.533-537.

19. GOLLWITZER, S., RACKWITZ, R., (1988), An efficient numerical solution to the multinormal integral, *Probabilistic Engineering Mechanics*, Vol.3, No.2, pp.98-101.

20. HASOFER, A.M., (1974), Design for infrequent overloads, *Earthquake Eng. and Struct. Dynamics*, Vol.2, No.4, pp.387-388.

21. HASOFER, A.M., RACKWITZ , R., (2000), Time-dependent models for code optimization, *Proc. ICASP'99*, R.E. Melchers and M.G. Stewart (Eds.), Balkema, Rotterdam, Vol.1, pp.151-158.

22. HOHENBICHLER, M., RACKWITZ, R., (1981), Non-normal dependent vectors in structural safety, *J. Eng. Mech. Div.*, ASCE, Vol.107, No.6, pp.1227-1249.

23. HOHENBICHLER, M., RACKWITZ, R., (1986), Sensitivity and importance measures in structural reliability, *Civil Engineering Systems*, Vol.3, December, pp.203-209.

24. HOHENBICHLER, R., RACKWITZ, R., (1988), Improvement of second-order reliability estimates by importance sampling, *Journal of Eng. Mech.*, ASCE, Vol.114, No.12, pp.2195-2199.

25. IIZUKA, H., (1988), A statistical study on life time of bridges, *Struct. & Earthquake Eng. Journ.*, JSCE, Vol.5, No.1, pp.51-60.

26. KUSCHEL, N., RACKWITZ, R., (1997), Two basic problems in reliability-based structural optimization, *Mathematical Methods of Operations Research*, Vol.46, pp.309-333.

27. KUSCHEL, N., RACKWITZ, R., (1998), Design for optimal reliability, *Proc. ICOSSAR'97*, S. Shiraishi, M.Shinozuka, and Y.K. Wen (Eds.), A.A. Balkema, Rotterdam, pp.1077-1085.

28. KUSCHEL, N., RACKWITZ, R., (2000), Optimal design under time-variant reliability constraints, *Structural Safety*, Vol.22, No.2, pp.113-128.

29. KUSCHEL, N., RACKWITZ, R., (2000), A new approach for structural optimization of series systems, *Proc. of the ICASP8 Conference, Sydney, 12-15 Dec., 1999*, R.E. Melchers and M. G. Stewart (Eds.), Balkema, Rotterdam, Vol.2, pp.987-994.

30. KUSCHEL, N., RACKWITZ, R., (2000), Time-variant reliability-based structural optimization using soRM, *Optimization*, Vol.47, No.3-4, pp.349-368.

31. LIND, N.C., (2002), Social and economic criteria of acceptable risk, *Reliability Engineering and Systems Safety*, Vol.78, pp.21-25.

32. MADDISON, A., (1995), *Monitoring the World Economy 1820-1992*, OECD, Paris.

33. NATHWANI, J.S., LIND, N.C., PANDEY, M.D., (1997), *Affordable Safety by Choice: The Life Quality Method*, Institute for Risk Research, University of Waterloo, Waterloo, Canada.

34. PANDEY, M.D., NATHWANI, J.S., (2002), Canada Wide Standard for Particulate Matter and Ozone: Cost-benefit analysis using a Life-Quality Index, to be published in: *Journ. Risk Analysis.*

35. PSHENICHNYJ, B.N., (1994), *The Linearization Method for Constrained Optimization*, Springer Berlin.

36. RACKWITZ, R., FIESSLER, B., (1978), Structural reliability under combined random load sequences, *Comp. & Struct.*, Vol.9, pp.484-494.

37. RACKWITZ, R., (1985) Reliability of systems under renewal pulse loading, *Journal of Eng. Mech.*, ASCE, Vol.111, No.9, pp.1175-1184.

38. RACKWITZ, R., (1993), On the combination of non-stationary rectangular wave renewal processes, *Structural Safety*, Vol.13, No.1+2, pp.21-28.

39. RACKWITZ, R., (1998), Computational techniques in stationary and non-stationary load combination – A review and some extensions, *J. Struc. Eng.*, SERC, Vol.25, No.1, pp.1-20.

40. RACKWITZ, R., (2000), Optimization – the basis of code making and reliability verification, *Structural Safety*, Vol.22, No.1, pp.27-60.

41. RACKWITZ, R., (2002), Optimization and risk acceptability based on the Life Quality Index, *Structural Safety*, Vol.24, pp.297-331.

42. RACKWITZ, R., (2003), Discounting for optimal and acceptable technical facilities, *Proc. ICASP'03*, A. Der Kiureghian (Ed.), Millpress, Rotterdam, pp.725-734.

43. RACKWITZ, R., (2003), *Discounting for Optimal and Acceptable Technical Facilities Involving Risks*, to be published in HERON.

44. RACKWITZ, R., BALAJI RAO, K., (2000), Numerical computation of mean failure times for locally non-stationary failure models, *Proc. ICASP'99*, R.E. Melchers and M.G. Stewart (Eds.), Balkema, Rotterdam, Vol.1, pp.159-165.

45. ROSENBLUETH, E., MENDOZA, E., (1971), Reliability optimization in isostatic structures, *J. Eng. Mech. Div.*, ASCE, Vol.97, EM6, pp.1625-1642.

46. ROSENBLUETH, E., (1976), Optimum design for infrequent disturbances, *J. Struct. Div.*, ASCE, Vol.102, ST9, pp.1807-1825.

47. SANCHEZ SILVA, M., RACKWITZ, R., (2003), *Socio-economic Implications of the Life Quality Index in the Design of Optimum Structures to Withstand Earthquakes*, to be published.

48. SCHALL, G., GOLLWITZER, S., RACKWITZ, R., (1988), Integration of multinormal densities on surfaces, *Proc. 2nd IFIP WG-7.5 Working Conference on Reliability and Optimization of Structural Systems, London, September, 26.-28., 1988*, pp.235-248, Springer, Heidelberg.

49. SCHALL, G., FABER, M., RACKWITZ, R., (1991), The ergodicity assumption for sea states in the reliability assessment of offshore structures, *J. Offshore Mech. and Arctic Eng.*, ASME, Vol.113, No.3, pp.241-246.

50. SCHRUPP, K., RACKWITZ, R., (1985), Outcrossing rates of Gaussian vector processes for cut sets of componential failure domains, *Proc. ICOSSAR'85*, I. Konishi, A.H.-S. Ang, and M. Shinozuka (Eds.), IASSAR, Shinko Printing, Vol.III, 601-609.

51. SLEPIAN, D., (1962), The one-sided barrier problem for Gaussian noise, *Bell System Tech. Journal*, Vol.11, pp.463-501.

52. SØRENSEN, J.D., P. THOFT-CHRISTENSEN, A. SIEMASZKO, J.M.B CARDOSO and J.L.T. SANTOS, (1995), Interactive reliability-based optimal design, *Proc. 6th IFIP WG 7.5 Conf. on 'Reliability and optimization of structural systems'*, Chapman & Hall, pp.249-256.

53. STREICHER, H., RACKWITZ, R., (2002), Structural optimization – a one level approach, *Proc. RBO'02 Workshop*, IPPT, Warsaw, Poland.

54. STREICHER, H., RACKWITZ, R., (2003), Time-variant reliability-oriented structural optimization of series systems, *Proc. CSM4, Corfu*, P.D. Spanos and G. Deodatis (Eds.), Millpress Rotterdam, pp.599-608.

55. STREICHER, H., RACKWITZ, R., (2004), Renewal models for optimal life-cycle cost of aging civil infrastructures, in: *"Life-Cycle Performance of Deteriorating Structures – Assessment, Design And Management", 3rd Int. IABMAS Workshop on Life-Cycle Cost Analysis and Design of Civil Infrastructure Systems*, Lausanne, March 24-26, 2003, D.M. Frangopol, E. Brúhwiler, M.H. Faber, and B. Adey (Eds.), ASCE, pp.401-412.

56. UNITED NATIONS, *Human Development Report 2001*, `www.undp.org/hdr2001`

57. VAN NOORTWIJK, J.M., (2001), Cost-based criteria for obtaining optimal design decisions, *Proc. ICOSSAR 01, Newport Beach 32-25 June, Structural Safety and Reliability*, Corotis et al. (Eds.), Swets & Zeitlinger Publ., Lisse.

58. VENEZIANO, D., GRIGORIU, M., CORNELL C.A., (1977), Vector-process models for system reliability, *J. Eng. Mech. Div.*, ASCE, Vol.103 (EM3), pp.441-460.

59. VON NEUMANN, J., MORGENSTERN, A., (1943), *Theory of Games and Economical Behavior*, Princeton University Press.

60. WINTERSTEIN, S.R., BJERAGER, P., (1987), The use of higher moments in reliability estimation, *Proceedings ICASP 5, Int. Conf. on Appl. of Statistics and Probability in Soil and Structural Engineering*, Vol.2, pp.1027-1036.