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LECTURE NOTES 11

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Energy Methods in Non-Linear Mechanics

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Preface

This volume contains the text of 7 lectures given during the month of June 2002 at the Centre of Excellence for Advanced Materials and Structures in the Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw.

It begins by summarizing the framework of a thermodynamical description of continua, including a description of the kinematics of deformation, and a summary of the equations governing the motion.

Determination of the evolution of system is studied through the definition of functionals presented in the case of non-linear dynamics. After a short description of the motion of the system and of mechanical interactions, the first part is devoted to the Lagrangian and to Hamiltonian functionals of the system. Quasistatic characterization is then deduced.

The case of evolution of elastoplastic system is investigated. The formulation of the rate boundary value problem for the system evolution is given in term of variational inequality. Conditions of uniqueness are discussed.

In the some manner, the framework is extended to fracture mechanics. The condition of propagation of cracks is analysed and a generalisation of Griffith's law based on energy criterion is proposed. The formulation of the rate boundary value problem has the same form as in elastoplasticity provided that the internal variable for the structure is reduced to the crack length.

For damage material and wear, a similar energetical approach is proposed. It is based on the dissipation associated with moving surfaces along which the mechanical fields experience discontinuities. Such discontinuities produce entropy and dissipation. If we consider that the moving surface is governed by an energy criterion of the some type as in classical rupture mechanics, the formulation of the rate boundary value problem of propagation of damage front is given.

Finally, the energy approach is applied to homogeneisation in non-linear mechanics for locally plastic or damaged material. Some properties are given in connection with the analysis of stability and bifurcation of equilibrium path.

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Warsaw, June 2003 *Claude Stolz*

Chapter 1

Introduction

Determination of the evolution of a system is studied through the definition of functionals presented here in the case of non-linear dynamics. After a short account of the necessary notions for the description of the motion and of mechanical interactions, this introduction is devoted to the Lagrangian and Hamiltonian functionals of the system.

1.1. Some general features

In order to explain and to predict the motion and the equilibrium of bodies or structures subjected to various physical interactions, a kinematical description of the motion is first performed. In the case of continuum, such a description must ensure the continuity of the body during its motion. Usually one looks for the motion of a material point $\underline{X}$ from a reference configuration by describing its displacement $\underline{u}(\underline{X}, t)$.

After the kinematical description of the body, one has to deal with the mechanical interactions. Many statements permit the description of these interactions, for example the virtual power statement can be used. This shows the manner to describe the mechanical interaction between each material point of the body with respect to a given loading distribution. For sake of simplicity and conciseness of this presentation, a thermodynamical description of interaction is chosen.

1.2. Description of the motion

Let a body Ω be submitted to external forces described by vector fields $\underline{f}$ in Ω and vector fields $\underline{T}$ along the boundary $\partial\Omega$. The external forces are generally functions of time.

Under these loadings the body is transformed and deformed. The actual position $\underline{x}$ of a material point is a function $\underline{\Phi}$ of it's initial position $\underline{X}$ and of the time. The displacement $\underline{u}$ is then defined by:

$$\underline{x}(\underline{X},t) = \underline{\Phi}(\underline{X},t) = \underline{X} + \underline{u}(\underline{X},t). \tag{1.1}$$

Consider now two material points $\underline{X}$ and $\underline{X} + \mathrm{d}\underline{X}$, then we have:

$$\underline{x}(\underline{X} + \mathrm{d}\underline{x}, t) = \underline{\Phi}(\underline{X} + \mathrm{d}\underline{X}, t) = \underline{\Phi}(\underline{X},t) + \frac{\partial \underline{\Phi}}{\partial \underline{X}}.\mathrm{d}\underline{X} + o(\mathrm{d}\underline{X}) \tag{1.2}$$

Hence, a material element $\mathrm{d}\underline{X}$ is convected by the motion in the actual material element $\mathrm{d}\underline{x}$, the corresponding transformation is the linear application associated with the gradient $\mathbf{F}$:

$$\mathrm{d}\underline{x} = \frac{\partial \underline{x}}{\partial \underline{X}}.\mathrm{d}\underline{X} = \mathbf{F}.\mathrm{d}\underline{X}. \tag{1.3}$$

The actual length of the material element is given by:

$$\mathrm{d}\underline{x}.\mathrm{d}\underline{x} = \mathrm{d}\underline{X}.\mathbf{F}^T.\mathbf{F}.\mathrm{d}\underline{X} = \mathrm{d}\underline{X}.\boldsymbol{C}.\mathrm{d}\underline{X}. \tag{1.4}$$

The variations of the local geometry, stretching of fibers and gliding, are determined by the Cauchy-Green tensor $\boldsymbol{C} = \mathbf{F}^T.\mathbf{F}$.

In small perturbations the gradient of the displacement is small and the strain is reduced to the linearized strain $\boldsymbol{\varepsilon}(\underline{u})$:

$$2\,\boldsymbol{\varepsilon}(\underline{u}) = \mathrm{grad}\,\underline{u} + \mathrm{grad}^T\,\underline{u}. \tag{1.5}$$

For the sake of simplicity, all subsequent studies will be made within the framework small perturbations.

1.3. The mobility and the interactions

The body Ω is considered as a continuous set of elements, positions of which are denoted by $\underline{x}$. This material element of volume $\mathrm{d}\Omega$ has an elementary mass $\mathrm{d}m = \rho\,\mathrm{d}\Omega$, where ρ is the mass density.

The mobility of the body is defined by its set of virtual motions associated with any vector field $\tilde{\underline{v}}^*$ which can be interpreted as a virtual velocity field.

Forces are defined by a linear form: $\tilde{\underline{v}}^* \to \mathcal{P}(\tilde{\underline{v}}^*)$ where $\mathcal{P}$ is a real number, named the virtual power of the forces developed by the virtual motion $\tilde{\underline{v}}^*$.

In classical continuum mechanics, the external forces applied to the body are given by vector field $\underline{f}$ defined over the volume Ω and by vector field $\underline{\mathrm{T}}$ defined over the boundary $\partial\Omega$. In this case the virtual power of external forces is expressed by:

$$\mathcal{P}_e(\tilde{\underline{v}}^*) = \int_\Omega \underline{f}.\tilde{\underline{v}}^* \, \mathrm{d}\Omega + \int_{\partial\Omega} \underline{\mathrm{T}}.\tilde{\underline{v}}^* \, \mathrm{d}S. \tag{1.6}$$

The power of internal interactions is given by a field of second-order tensors $\boldsymbol{\sigma}(\underline{x})$ such that:

$$\mathcal{P}_i(\tilde{\underline{v}}^*) = -\int_\Omega \boldsymbol{\sigma} : \operatorname{grad} \underline{v}^* \, \mathrm{d}\Omega, \tag{1.7}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor. This expression is the simplest form for a local description of internal interactions compatible with external loading defined only in terms of vector fields.

The axiom of objectivity states that the power of local interaction $\mathcal{P}_i$ is equal to zero for any rigid body motion:

$$\mathcal{P}_i(\tilde{\underline{v}}^*) = 0, \quad \forall\, \tilde{\underline{v}}^* \in \mathcal{R}.\mathcal{B}.\mathcal{M}. \tag{1.8}$$

The set $\mathcal{R}.\mathcal{B}.\mathcal{M}$ is the set of rigid body motions:

$$\mathcal{R}.\mathcal{B}.\mathcal{M} = \{\tilde{\underline{v}} \mid \tilde{\underline{v}} = \underline{v}_o + \boldsymbol{\omega}_o.\underline{X}\}, \tag{1.9}$$

with $\boldsymbol{\omega}_o^T = -\boldsymbol{\omega}_o$. By application of (1.8) to any domain Ω, we conclude that the Cauchy stress $\boldsymbol{\sigma}$ is a symmetric second-order tensor.

If we assume that there is no jump for the velocity, the virtual power of acceleration is given by:

$$\mathcal{A}(\tilde{\underline{v}}^*) = \int_\Omega \rho\underline{\gamma}.\tilde{\underline{v}}^* \, \mathrm{d}\Omega, \quad \underline{\gamma} = \dot{\underline{v}}. \tag{1.10}$$

Virtual Power. *The fundamental statement of dynamics is written in terms of virtual power: the sum of the virtual power of internal interactions and of*

the virtual power of external forces is equal to the virtual power of acceleration quantities:

$$\mathcal{P}_i(\tilde{\underline{v}}^*) + \mathcal{P}_e(\tilde{\underline{v}}^*) = \mathcal{A}(\tilde{\underline{v}}^*). \tag{1.11}$$

for any virtual motion $\tilde{\underline{v}}^*$.

Using this equality for any virtual motion and any subdomain Ω^*, the local form of the balance of momentum is obtained:

$$\operatorname{div} \boldsymbol{\sigma} + \rho \underline{f} = \rho \underline{\gamma}, \quad \boldsymbol{\sigma}^T = \boldsymbol{\sigma}, \quad \boldsymbol{\sigma}.\underline{n} = \underline{\mathrm{T}}. \tag{1.12}$$

By the same reasoning the principle of action and reaction is recovered. Along an elementary internal surface $\mathrm{d}S$ with normal $\underline{n}$ the stress vector is continuous, if there is no jump of velocity:

$$\boldsymbol{\sigma}^+.\underline{n} = \boldsymbol{\sigma}^-.\underline{n}\,. \tag{1.13}$$

These equations of conservation are not sufficient to determine the internal state, some complementary information are needed.

A thermodynamical point of view is chosen. The body is a thermodynamical system formed by a collection of small elements defined as material points. Each small element has a density ρ and the local state is characterized by a set of state variables.

1.4. Conservation of energy

The internal state is described by the actual value of a set of state variables. To described effectively the behaviour of the material, we must measure a great number of mechanical quantities; to gain in efficiency the concept of internal parameters is adopted. The choice of these parameters is governed by the observation and the ability of the modelisation to describe the studied phenomenon with sufficient accuracy. The state variables are the strain $\boldsymbol{\varepsilon}$, the temperature T and a set of internal variables α. Attached with these parameters are the internal energy density e and entropy density s. Then the internal energy and the entropy of the body Ω are given by integration over the body:

$$\mathrm{E} = \int_\Omega \rho\, \mathrm{e}\, \mathrm{d}\Omega, \quad S = \int_\Omega \rho\, s\, \mathrm{d}\Omega. \tag{1.14}$$

Conservation of energy

The conservation of energy is written as:

$$\dot{\mathcal{K}} + \dot{\mathrm{E}} = \mathcal{P}_{cal} + \mathcal{P}_e. \tag{1.15}$$

Assuming that the calorific power is due to conduction:

$$\mathcal{P}_{cal} = -\int_{\partial\Omega} \underline{q}.\underline{n}\, \mathrm{d}S = -\int_{\Omega} \operatorname{div} \underline{q}\, \mathrm{d}\Omega, \tag{1.16}$$

the local expression of the energy conservation is deduced from (1.11), (1.15), (1.16) applied to any volume Ω:

$$\rho\, \dot{\mathrm{e}} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \operatorname{div} \underline{q}. \tag{1.17}$$

Entropy production

The second law for the whole system is written as:

$$\dot{S} + \int_{\partial\Omega} \frac{\underline{q}.\underline{n}}{T}\, \mathrm{d}S \geq 0. \tag{1.18}$$

After integration by parts we obtain:

$$\int_{\Omega} \left(\rho \dot{s} + \frac{1}{T} \operatorname{div} \underline{q}\right) \mathrm{d}\Omega - \int_{\Omega} \frac{\underline{q}.\operatorname{grad} T}{T^2}\, \mathrm{d}\Omega \geq 0. \tag{1.19}$$

The two terms have different origin, the first one is due to internal mechanical irreversibility, the second one is due to conduction.

In non-linear mechanics the internal state is generally associated with irreversibility. Then the fundamental inequality of thermodynamics implies that the internal production of entropy must be non-negative.

In the total dissipation we distinguish the part due to the conduction and the part due to internal forces. We assume that the choice of state parameters is a normal set of variables. In this case, a variation of temperature does not induce variation of kinetic energy, so the two dissipations are individually positive:

$$D_m = \rho \dot{s} + \frac{1}{T} \operatorname{div} \underline{q} \geq 0, \tag{1.20}$$

$$D_{th} = -\frac{\underline{q}.\operatorname{grad} T}{T^2} \geq 0. \tag{1.21}$$

By introducing the conservation of energy in the first equation, we can use the free energy w instead of internal energy $e = w + sT$. Then, the intrinsic dissipation D_m is rewritten as:

$$D_m = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \rho(\dot{w} + s\dot{T}) \geq 0. \tag{1.22}$$

This inequality must be satisfied by any real evolution of the body, from the state defined by the actual values of the state variables $\boldsymbol{\varepsilon}, \alpha, T$.

1.5. The linear thermoelasticity

When the constitutive behaviour is elastic, all thermodynamical quantities are functions only of the actual state $\boldsymbol{\varepsilon}, T$.

For linear thermoelasticity and small perturbations around a natural state at the temperature T_o, the free energy has the following form ($\tau = T - T_o$):

$$\rho\, w = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon} + \boldsymbol{k}.\boldsymbol{\varepsilon}\tau - \frac{1}{2}C\tau^2, \tag{1.23}$$

The positivity of entropy production is satisfied by any real variations of the state near a thermodynamical equilibrium state (i.e. mechanical equilibrium under uniform temperature), then we deduce the equations of state:

$$\boldsymbol{\sigma} = \rho\frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \quad s = -\frac{\partial w}{\partial T}. \tag{1.24}$$

In this case the stress $\boldsymbol{\sigma}$ must satisfy the balance of momentum. Therefore the elastic behaviour is essentially reversible.

1.6. More general cases

In general, the intrinsic dissipation D_m has a form driven by the choice of the free energy w, which depends on the strain $\boldsymbol{\varepsilon}$, internal parameters α and temperature T. The entropy production is rewritten as:

$$D_m = (\boldsymbol{\sigma} - \rho\frac{\partial w}{\partial \boldsymbol{\varepsilon}}) : \dot{\boldsymbol{\varepsilon}} - \rho(\frac{\partial w}{\partial T} + s)\dot{T} - \rho\frac{\partial w}{\partial \alpha}\dot{\alpha} \geq 0. \tag{1.25}$$

Defining the thermodynamical forces associated with the state variables by the state equations:

$$\boldsymbol{\sigma}_r = \rho\frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \quad \mathrm{A} = -\rho\frac{\partial w}{\partial \alpha}, \quad s = -\frac{\partial w}{\partial T}, \tag{1.26}$$

the dissipation takes the form:

$$D_m = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_r) : \dot{\boldsymbol{\varepsilon}} + \mathrm{A}\dot{\alpha} = \boldsymbol{\sigma}_{ir} : \dot{\boldsymbol{\varepsilon}} + \mathrm{A}.\dot{\alpha} \geq 0. \tag{1.27}$$

There exists two sources of entropy production, namely one due to the variations of internal parameters and the second due to the strain rates.

The equations of state don't provide the full constitutive equations, some complementary laws are needed to describe the evolution of the irreversibility. Such laws are determined by observations and experimentations. First we must define the domain of reversibility, we must discuss the influence of the strain rates and finally we must determine constitutive relations between the rates $\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}$ and the thermodynamical forces $\boldsymbol{\sigma}_{ir}, \mathrm{A}$.

Generalized standard materials.

A powerful method is to consider the existence of potential for the dissipation. Let us assume that the behaviour belongs to the class of the generalized standard materials, (B. Halphen and Q.S. Nguyen [1975]). This ensures the existence of dissipation potentials and the evolution of the internal state satisfies normality law:

$$\dot{\alpha} = \frac{\partial \mathrm{d}^*}{\partial \mathrm{A}}, \ \dot{\boldsymbol{\varepsilon}} = \frac{\partial \mathrm{d}^*}{\partial \boldsymbol{\sigma}_{ir}}, \quad \text{or} \quad \mathrm{A} = \frac{\partial \mathrm{d}}{\partial \dot{\alpha}}, \ \boldsymbol{\sigma}_{ir} = \frac{\partial \mathrm{d}}{\partial \dot{\boldsymbol{\varepsilon}}}. \tag{1.28}$$

The potentials d^* and d are convex functions of the variables, with a minimum value at the origin.

Cases of linear visco-elasticity

For example, the potential of dissipation is:

$$\mathrm{d}(\dot{\boldsymbol{\varepsilon}}) = \frac{1}{2}\dot{\boldsymbol{\varepsilon}} : \eta : \dot{\boldsymbol{\varepsilon}}, \tag{1.29}$$

with η a positive definite operator. The complementary law gives:

$$\boldsymbol{\sigma}_{ir} = \frac{\partial \mathrm{d}}{\partial \dot{\boldsymbol{\varepsilon}}} = \eta : \dot{\boldsymbol{\varepsilon}}. \tag{1.30}$$

The stresses $\boldsymbol{\sigma}$ used in the balance of momentum is decomposed in two terms $\boldsymbol{\sigma} = \boldsymbol{\sigma}_r + \boldsymbol{\sigma}_{ir}$.

- For the model of Kelvin-Voigt of linear viscoelasticity, the two potentials take the form:

$$\rho\, w(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon}\ ; \quad \mathrm{d}(\dot{\boldsymbol{\varepsilon}}) = \frac{1}{2}\dot{\boldsymbol{\varepsilon}} : \eta : \dot{\boldsymbol{\varepsilon}}, \tag{1.31}$$

and then the constitutive behaviour implies that:

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} + \eta : \dot{\boldsymbol{\varepsilon}}. \tag{1.32}$$

- The Maxwell description is obtained by choosing the thermodynamical potential in the following form:

$$\rho\, w(\boldsymbol{\varepsilon}, \alpha) = \frac{1}{2}(\boldsymbol{\varepsilon} - \alpha) : \boldsymbol{C} : (\boldsymbol{\varepsilon} - \alpha), \tag{1.33}$$

and the pseudo-potential of dissipation in a quadratic manner:

$$\mathrm{d} = \frac{1}{2}\dot{\alpha} : \eta : \dot{\alpha}. \tag{1.34}$$

Then $\boldsymbol{\sigma}_{ir} = 0, \mathrm{A} = \boldsymbol{C} : (\boldsymbol{\varepsilon} - \alpha)$ and the complementary law gives the relation:

$$\mathrm{A} = \eta : \dot{\alpha}. \tag{1.35}$$

Then the constitutive behaviour is given by:

$$\boldsymbol{\sigma} = \boldsymbol{C} : (\boldsymbol{\varepsilon} - \alpha), \quad \mathrm{A} = \boldsymbol{C} : (\boldsymbol{\varepsilon} - \alpha) = \eta : \dot{\alpha}. \tag{1.36}$$

Normality rule

In the case of a regular and differentiable function, the convexity of the potential of dissipation gives us the characterization of internal state evolution by the equalities:

$$\boldsymbol{\sigma}_{ir} = \frac{\partial \mathrm{d}}{\partial \dot{\boldsymbol{\varepsilon}}}, \quad \mathrm{A} = \frac{\partial \mathrm{d}}{\partial \dot{\alpha}}. \tag{1.37}$$

More generally the definition of the gradient is replaced by the notion of subgradient [1].

Normality rule. *The internal state satisfies the evolution laws given by the normality rule*

$$(\boldsymbol{\sigma}_{ir}, \mathrm{A}) \in \partial \mathrm{d}(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}). \tag{1.38}$$

The set $\partial \mathrm{d}$ *is the set of thermodynamical forces whose satisfy the inequality:*

$$\mathrm{d}(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}) + \boldsymbol{\sigma}_{ir} : (\boldsymbol{\varepsilon}^* - \dot{\boldsymbol{\varepsilon}}) + \mathrm{A} : (\alpha^* - \dot{\alpha}) \leq \mathrm{d}(\boldsymbol{\varepsilon}^*, \alpha^*), \tag{1.39}$$

for all admissible values $\boldsymbol{\varepsilon}^*, \alpha^*$.

We observe that the existence of a potential for the dissipation ensures the positivity of the entropy production:

$$\boldsymbol{\sigma}_{ir} : \dot{\boldsymbol{\varepsilon}} + \mathrm{A} : \dot{\alpha} = < \partial \mathrm{d}(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}), (\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}) > \; \geq \mathrm{d}(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}) - \mathrm{d}(0, 0) \geq 0. \tag{1.40}$$

[1] Elements of convex analysis are synthesized in J.J. Telega (2002).

1.7. The quasistatic evolution

Consider a body Ω submitted to some prescribed boundary conditions. The boundary is decomposed into $\partial\Omega_u$ where the displacement is imposed and $\partial\Omega_T$ where the stress vector is prescribed, $\partial\Omega = \partial\Omega_u \cup \partial\Omega_T$ and $\emptyset = \partial\Omega_u \cap \partial\Omega_T$. A solution $(\tilde{\underline{u}}, \tilde{\alpha}, \tilde{T})$ of the problem of quasistatic evolution satisfies:

- the compatibility equations for strains and displacements: the strain field $\tilde{\boldsymbol{\varepsilon}}$ is associated with the displacement $\tilde{\underline{u}}$, $\boldsymbol{\varepsilon}(\underline{u}) = (\text{grad}\,\underline{u} + \text{grad}^T\,\underline{u})/2$, the displacement satisfies the boundary condition $\underline{u} = \underline{u}^d$ on $\partial\Omega_u$,
- the state equations:

$$\boldsymbol{\sigma}_r = \rho\frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \quad \mathrm{A} = -\rho\frac{\partial w}{\partial \alpha}, \quad s = -\frac{\partial w}{\partial T}, \tag{1.41}$$

- the equations of evolution for the state variables:

$$(\boldsymbol{\sigma}_{ir}, \mathrm{A}) \in \partial \mathrm{d}(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}), \tag{1.42}$$

- the constitutive law:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_r + \boldsymbol{\sigma}_{ir}, \tag{1.43}$$

- the conservation of the momentum and boundary conditions:

$$\text{div}\ \boldsymbol{\sigma} = 0, \ \text{on}\ \Omega, \quad \boldsymbol{\sigma}.\underline{n} = \underline{\mathrm{T}}^d \ \text{on}\ \partial\Omega_T. \tag{1.44}$$

For the overall system the rule of the free energy is replaced by the global free energy:

$$W(\tilde{\boldsymbol{\varepsilon}}, \tilde{\alpha}, \tilde{T}) = \int_\Omega \rho\, w(\boldsymbol{\varepsilon}, \alpha, T)\, \mathrm{d}\Omega. \tag{1.45}$$

We recall the definition of the Gâteaux differential:

$$\frac{\partial F}{\partial \tilde{q}} \cdot \tilde{q}^* = \lim_{\eta \to 0} \frac{F(\tilde{q} + \eta\tilde{q}^*) - F(\tilde{q})}{\eta}, \tag{1.46}$$

and then we get for our particular case:

$$\frac{\partial W}{\partial \tilde{q}} \cdot \tilde{q}^* = \int_\Omega \rho \frac{\partial w}{\partial \tilde{q}}\, \tilde{q}^*\, \mathrm{d}\Omega. \tag{1.47}$$

The equations of state are now relations between fields:

$$\tilde{\boldsymbol{\sigma}}_r = \frac{\partial W}{\partial \tilde{\boldsymbol{\varepsilon}}}, \quad \tilde{\mathrm{A}} = -\frac{\partial W}{\partial \tilde{\alpha}}, \quad \tilde{s} = -\frac{\partial W}{\partial \tilde{T}}. \tag{1.48}$$

In a global description the equations of state have the same form as in the local one, and the state of the system is defined by fields of state variables.

Dissipative function

By integration over the body of the dissipation density we define the total dissipation function:

$$\mathrm{D}(\tilde{\dot{\varepsilon}}, \tilde{\dot{\alpha}}) = \int_\Omega \mathrm{d}(\dot{\varepsilon}, \dot{\alpha})\, \mathrm{d}\Omega. \tag{1.49}$$

By integration of the local inequality (1.39), we get:

$$\mathrm{D}(\tilde{\dot{\varepsilon}}, \tilde{\dot{\alpha}}) - \mathrm{D}(\tilde{\varepsilon}^*, \tilde{\alpha}^*) + \tilde{\sigma}_{ir} \cdot (\tilde{\varepsilon}^* - \tilde{\dot{\varepsilon}}) + \tilde{\mathrm{A}} \cdot (\tilde{\alpha}^* - \tilde{\dot{\alpha}}) \leq 0, \tag{1.50}$$

for all admissible fields $(\tilde{\varepsilon}^*, \tilde{\alpha}^*)$. This shows that the evolution of internal state satisfies the normality rule, rewritten in terms of fields:

$$(\tilde{\sigma}_{ir}, \tilde{\mathrm{A}}) \in \partial \mathrm{D}(\tilde{\dot{\varepsilon}}, \tilde{\dot{\alpha}}). \tag{1.51}$$

For example, in the case of regular dissipation function we have:

$$\frac{\partial \mathrm{D}}{\partial \tilde{\dot{\varepsilon}}} \cdot \tilde{\varepsilon}(\delta u) = \int_\Omega \boldsymbol{\sigma}_{ir} : \boldsymbol{\varepsilon}(\delta \underline{u})\, \mathrm{d}\Omega,$$

$$\frac{\partial \mathrm{D}}{\partial \tilde{\dot{\alpha}}} \cdot \delta\tilde{\alpha} = \int_\Omega \mathrm{A}.\delta\alpha\, \mathrm{d}\Omega.$$

These equations can be rewritten as:

$$\tilde{\sigma}_{ir} = \frac{\partial \mathrm{D}}{\partial \tilde{\dot{\varepsilon}}}, \quad \tilde{\mathrm{A}} = \frac{\partial \mathrm{D}}{\partial \tilde{\dot{\alpha}}}. \tag{1.52}$$

The isothermal boundary value problem

Consider now, for the sake of simplicity, an isothermal process. Let us assume that the external loading derives from a potential given in terms of traction $\underline{\mathrm{T}}^d$ applied on the external surface $\partial\Omega_\mathrm{T}$ of the body. Then, the global free energy W is replaced by the potential energy $\mathcal{E}$ of the system:

$$\mathcal{E}(\tilde{\underline{u}}, \tilde{\alpha}, \tilde{\underline{\mathrm{T}}}^d) = \int_\Omega \rho\, w(\boldsymbol{\varepsilon}(\underline{u}), \alpha)\, \mathrm{d}\Omega - \int_{\partial\Omega_\mathrm{T}} \underline{\mathrm{T}}^d.\underline{u}\, \mathrm{d}S. \tag{1.53}$$

By combining all the equations in terms of fields of the state variables, the quasistatic evolution is then given in a global manner by the variational system:

$$0 = \frac{\partial \mathcal{E}}{\partial \tilde{\underline{u}}} \cdot \delta\tilde{\underline{u}} + \frac{\partial \mathrm{D}}{\partial \tilde{\dot{\varepsilon}}} \cdot \tilde{\varepsilon}(\delta \underline{u}), \tag{1.54}$$

$$0 = \frac{\partial \mathcal{E}}{\partial \tilde{\alpha}} \cdot \delta\tilde{\alpha} + \frac{\partial \mathrm{D}}{\partial \tilde{\dot{\alpha}}} \cdot \delta\tilde{\alpha}. \tag{1.55}$$

These equations are defined on a set of admissible fields, the displacement is submitted to the boundary condition $\underline{u} = \underline{u}^d$ over $\partial\Omega_u$. Then the fields $\delta\tilde{\underline{u}}$ must satisfy $\delta\underline{u} = 0$ over $\partial\Omega_u$. The $\delta\tilde{\alpha}$ can have some constraints depending on the nature of irreversibility.

The preceding equations are general. They contain the essential structure of a problem of quasistatic evolution. The first equation of this system explains the conservation of the momentum taking into account the constitutive law:

$$\operatorname{div}\boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_r + \boldsymbol{\sigma}_{ir}, \quad \boldsymbol{\sigma}.\underline{n} = \underline{T}^d \text{ on } \partial\Omega_T, \tag{1.56}$$

the second one provides the complementary law as a relation between the forces A and the internal parameters:

$$\mathrm{A}\cdot\delta\tilde{\alpha} = -\frac{\partial\mathcal{E}}{\partial\tilde{\alpha}}\cdot\delta\tilde{\alpha} = \frac{\partial \mathrm{D}}{\partial\dot{\tilde{\alpha}}}\cdot\delta\tilde{\alpha}. \tag{1.57}$$

1.8. The Lagrangian and the dynamical case

By definition, the Lagrangian is the difference between the kinetic energy and the potential of interaction applied to the system. For all kinematically admissible fields, the potential of interaction is the potential energy:

$$\mathcal{E}(\tilde{\underline{u}},\tilde{\alpha},\tilde{T},\tilde{\underline{T}}^d) = \int_\Omega w(\boldsymbol{\varepsilon},\alpha,T)\rho\,\mathrm{d}\Omega - \int_{\partial\Omega_T}\underline{T}^d.\underline{u}\,\mathrm{d}S. \tag{1.58}$$

The kinetic energy is defined as:

$$\mathcal{K}(\tilde{\underline{v}}) = \int_\Omega \frac{1}{2}\rho\underline{v}^2\,\mathrm{d}\Omega, \tag{1.59}$$

and the Lagrange's functional is then:

$$\mathcal{L}(\tilde{\underline{u}},\tilde{\underline{v}},\tilde{T},\tilde{\underline{T}}^d) = \mathcal{K}(\tilde{\underline{v}}) - \mathcal{E}(\tilde{\underline{u}},\tilde{\alpha},\tilde{T},\tilde{\underline{T}}^d). \tag{1.60}$$

The acceleration is denoted by $\underline{\gamma}$, $\underline{\gamma} = \dot{\underline{v}}$. Some variations of the Lagrangian are useful:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\tilde{\underline{u}}}\cdot\delta\tilde{\underline{u}} &= -\int_\Omega \boldsymbol{\sigma}_r : \delta\boldsymbol{\varepsilon}\,\mathrm{d}\Omega + \int_{\partial\Omega_T}\underline{T}^d.\delta\underline{u}\,\mathrm{d}S,\\
\frac{\partial\mathcal{L}}{\partial\tilde{\underline{v}}}\cdot\delta\tilde{\underline{v}} &= \int_\Omega \rho\underline{v}.\delta\underline{v}\,\mathrm{d}\Omega,\\
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\mathcal{L}}{\partial\tilde{\underline{v}}}\cdot\delta\tilde{\underline{u}}\Big) &= \frac{\partial\mathcal{L}}{\partial\tilde{\underline{v}}}\cdot\delta\tilde{\underline{v}} + \Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\tilde{\underline{v}}}\Big)\cdot\delta\tilde{\underline{u}}\\
&= \int_\Omega(\rho\underline{v}.\delta\underline{v} + \rho\underline{\gamma}.\delta\underline{u})\,\mathrm{d}\Omega + \int_\Gamma m[\underline{v}]_\Gamma.\delta\underline{u}\,\mathrm{d}S.
\end{aligned}$$

m is the mass flux through the moving surface Γ along which the velocity v has discontinuities $[v]_\Gamma$. The equations of the motion are given by the conservation of the momentum:

$$\operatorname{div} \boldsymbol{\sigma} = \rho \underline{\gamma} \text{ on } \Omega, \tag{1.61}$$

$$[\boldsymbol{\sigma}]_\Gamma . \underline{n} = m[\underline{v}]_\Gamma \text{ on } \Gamma. \tag{1.62}$$

Then a variational form for the conservation of momentum is easily deduced:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \underline{u}) \, \mathrm{d}\Omega = \int_{\partial\Omega} \underline{n}.\boldsymbol{\sigma}.\delta \underline{u} \, \mathrm{d}S + \int_\Gamma \underline{n}.[\boldsymbol{\sigma}]_\Gamma.\delta \underline{u} \, \mathrm{d}S + \int_\Omega \rho \, \underline{\gamma}.\delta \underline{u} \, \mathrm{d}\Omega. \tag{1.63}$$

The stress $\boldsymbol{\sigma}$ is decomposed as previously as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_r + \boldsymbol{\sigma}_{ir}$ taking the constitutive law into account. Γ is a moving surface, where the velocities have discontinuities. Taking all these relations into account, the evolution of the system is governed by the generalized Lagrange's equations:

$$-\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \underline{\tilde{v}}} - \frac{\partial \mathcal{L}}{\partial \underline{\tilde{u}}}\right) \cdot \delta \underline{\tilde{u}} = \frac{\partial \mathrm{D}}{\partial \dot{\tilde{\boldsymbol{\varepsilon}}}} \cdot \tilde{\boldsymbol{\varepsilon}}(\delta \underline{\tilde{u}}), \tag{1.64}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{\alpha}} \cdot \delta \tilde{\alpha} = \frac{\partial \mathrm{D}}{\partial \dot{\tilde{\alpha}}} \cdot \delta \tilde{\alpha}, \tag{1.65}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{T}} \cdot \delta \tilde{T} = \int_\Omega \rho s \delta T \, \mathrm{d}\Omega. \tag{1.66}$$

These equations are a generalisation to non-linear dynamics (C. Stolz [1988]) of the classical Lagrange's formulation, they have the same form as the expression given by Biot (M.A. Biot [1978]) in viscoelasticity. In this formulation, we have defined as previously the dissipative function by:

$$\mathrm{D}(\dot{\tilde{\boldsymbol{\varepsilon}}}, \dot{\tilde{\alpha}}) = \int_\Omega \mathrm{d}(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}) \, \mathrm{d}\Omega. \tag{1.67}$$

The first equation is the equation of motion, the second one the evolution law for the internal state, the last one defines the local entropy. To this set of equations we must append a conduction law for completeness.

1.9. The Hamiltonian

The Hamiltonian is a Legendre transformation of the Lagrangian, with respect to the velocity and temperature (C. Stolz [1988]):

$$\mathcal{H}(\underline{\tilde{u}}, \underline{\tilde{p}}, \tilde{\alpha}, \tilde{s}, \tilde{\underline{\mathrm{T}}}^d) = \int_\Omega (\underline{p}.\underline{u} + T \rho s) \, \mathrm{d}\Omega - \mathcal{L}(\underline{\tilde{u}}, \underline{\tilde{v}}, \tilde{\alpha}, \tilde{T}, \tilde{\underline{\mathrm{T}}}^d). \tag{1.68}$$

Then we have:

$$\mathcal{H}(\tilde{\underline{u}}, \tilde{\underline{p}}, \tilde{\alpha}, \tilde{s}, \tilde{\underline{T}}^d) = \int_\Omega \frac{1}{2}\underline{p}^2/\rho \, \mathrm{d}\Omega + \int_\Omega \rho \mathrm{e}(\boldsymbol{\varepsilon}(\underline{u}), \alpha, T) \, \mathrm{d}\Omega - \int_{\partial\Omega} \underline{T}^d.\underline{u} \, \mathrm{d}S.$$

where $\underline{p}$ is the momentum $\rho\underline{v}$. In this expression the density of internal energy: $\mathrm{e} = w + T\,s$ appears. In a global formulation, we obtain successively:

$$\frac{\partial \mathcal{H}}{\partial \tilde{\underline{p}}} \cdot \tilde{\underline{p}}^* = \tilde{\underline{v}} \cdot \tilde{\underline{p}}^* = \frac{\mathrm{d}\tilde{\underline{u}}}{\mathrm{d}t} \cdot \tilde{\underline{p}}^*,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{\underline{u}}} \cdot \delta\tilde{\underline{u}} = \int_\Omega \rho \frac{\partial w}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}(\delta\underline{u}) \, \mathrm{d}\Omega - \int_{\partial\Omega_\mathrm{T}} \underline{T}^d.\delta\underline{u} \, \mathrm{d}S$$

$$= \int_\Omega \boldsymbol{\sigma}_r : \boldsymbol{\varepsilon}(\delta\underline{u}) \, \mathrm{d}\Omega - \int_{\partial\Omega_\mathrm{T}} \underline{T}^d.\delta\underline{u} \, \mathrm{d}S.$$

Taking account of the conservation of: the momentum, decomposition of the stress into reversible and irreversible parts, of the boundary conditions and of jump conditions, the expressions are then modified to:

$$\frac{\partial \mathcal{H}}{\partial \tilde{\underline{u}}} \cdot \delta\tilde{\underline{u}} = -\int_\Omega \boldsymbol{\sigma}_{ir} : \boldsymbol{\varepsilon}(\delta\underline{u}) \, \mathrm{d}\Omega - \int_\Omega \rho\underline{\gamma}.\delta\underline{u} \, \mathrm{d}\Omega + \int_\Gamma \underline{n}.[\boldsymbol{\sigma}]_\Gamma.\delta\underline{u} \, \mathrm{d}S.$$

Recall that $\tilde{\boldsymbol{\sigma}_{ir}} = \partial \mathrm{D}/\partial\tilde{\dot{\boldsymbol{\varepsilon}}}$ and consider the relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \underline{p}.\delta\underline{u} \, \mathrm{d}\Omega = \int_\Omega \rho\underline{\gamma}.\delta\underline{u} \, \mathrm{d}\Omega + \int_\Gamma m[\underline{v}]_\Gamma.\delta\underline{u} \, \mathrm{d}S. \tag{1.69}$$

We then obtain the conservation of the momentum in the Hamiltonian's form:

$$\frac{\partial \mathcal{H}}{\partial \tilde{\underline{u}}} \cdot \delta\tilde{\underline{u}} = -\frac{\partial \mathrm{D}}{\partial \tilde{\dot{\boldsymbol{\varepsilon}}}} \cdot \tilde{\boldsymbol{\varepsilon}}(\delta\underline{u}) - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \underline{p}.\delta\underline{u} \, \mathrm{d}\Omega. \tag{1.70}$$

Finally, the Hamiltonian formulation of the evolution problem is obtained:

$$\frac{\partial \mathcal{H}}{\partial \tilde{\underline{p}}} \cdot \tilde{\underline{p}}^* = \tilde{\underline{v}} \cdot \tilde{\underline{p}}^* = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\underline{u}} \cdot \tilde{\underline{p}}^*,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{\underline{u}}} \cdot \tilde{\underline{u}}^* = -\frac{\partial \mathrm{D}}{\partial \tilde{\boldsymbol{\varepsilon}}(\underline{v})} \cdot \tilde{\boldsymbol{\varepsilon}}(\tilde{\underline{u}}^*) - \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\underline{p}} \cdot \tilde{\underline{u}}^*),$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{\alpha}} \cdot \tilde{\alpha}^* = -\frac{\partial \mathrm{D}}{\partial \tilde{\dot{\alpha}}} \cdot \tilde{\alpha}^*,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{s}} \cdot \tilde{s}^* = \tilde{T} \cdot \tilde{\rho s}.$$

As previously a conduction law must be specified and the positivity of the entropy production must be verified to determine the evolution of the system.

1.10. Some properties

Generalisation

The definitions of the Lagrangian and Hamiltonian can be extended to structures like beams or plates. The proposed description is performed when the behaviour of the system is described by two potentials: a global free energy and a dissipative function. If some particular internal constraints exist, the preceding description must be modified.

Conservation of energy

For the real motion, the value of the Hamiltonian is the sum of: the kinetic energy, internal energy and potential energy of the external loading. Then the conservation of the energy of the system can be easily rewritten as:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} - \frac{\partial \mathcal{H}}{\partial \tilde{\underline{\mathrm{T}}}^d} \cdot \frac{\mathrm{d}\tilde{\underline{\mathrm{T}}}^d}{\mathrm{d}t} = \mathcal{P}_{cal}. \tag{1.71}$$

When the external loading is time independent $\frac{\mathrm{d}\underline{\mathrm{T}}^d}{\mathrm{d}t} = 0$, the exchange of energy is only due to the heat rate supply $\mathcal{P}_{cal}$. Generally this quantity has the form:

$$\mathcal{P}_{cal} = -\int_{\partial\Omega} \underline{q}.\underline{n}\, \mathrm{d}S, \tag{1.72}$$

where $\underline{q}$ is the heat flux. This result is useful in fracture mechanics to quantify the heat generated by the propagation of crack as presented in the following section.

Conservation law

In the case of conservative system, in an adiabatic evolution ($\mathcal{P}_{cal} = 0$), the Hamiltonian is constant, i.e.:

$$\mathcal{H}(t) = \mathcal{H}(0). \tag{1.73}$$

This property can be rewritten in terms of the Lagrangian:

$$\mathcal{L} - \tilde{\underline{v}} \cdot \frac{\partial \mathcal{L}}{\partial \tilde{\underline{v}}} = \mathcal{H}(0). \tag{1.74}$$

Property of stationarity

The Lagrangian has the property of stationarity in elasticity or viscoelasticity: let us consider the variation of the Lagrangian in isothermal evolution:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \tilde{\underline{v}}} \cdot \delta\tilde{\underline{v}} + \frac{\partial \mathcal{L}}{\partial \tilde{\underline{u}}} \cdot \delta\tilde{\underline{u}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \tilde{\underline{v}}} \cdot \delta\tilde{\underline{u}}\right) + \frac{\partial \mathrm{D}}{\partial \tilde{\dot{\boldsymbol{\varepsilon}}}} \cdot \tilde{\boldsymbol{\varepsilon}}(\delta\underline{u}), \tag{1.75}$$

then

$$\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = \delta \mathrm{D}, \tag{1.76}$$

where δD is the total viscous dissipation during the variation.

Finally, let us note that the above results may be adapted to the case of other type of boundary conditions.

Chapter 2

Elasto-plasticity

For almost any metal and at rather small rates of loading, we obtain a mean curve $\sigma(\varepsilon)$ during a tensile experiment: this is a static curve. The whole static curve may be analysed with the help of several successive mathematical relations, which are models of the elasto-plastic behaviour.

2.1. Introduction

Consider the tension curve on a test specimen. The behaviour is linear below a limit value σ_o, whilst above this value some permanent strains ε_p are present after a total unloading. The reversible part of the strain $\varepsilon - \varepsilon_p$ is the elastic strain. The form of linear elastic behaviour is preserved:

$$\boldsymbol{\sigma} = \boldsymbol{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) \tag{2.1}$$

At the microscale metals are crystalline solids, this means that they consist of atoms arranged in a pattern which is periodically repeated. The whole system is build of unit cells or lattice. To each lattice we can associated a triad of vectors.

The plastic transformation of monocrystal is generally described by slip lines contained in crystallographic planes: the slip-plane of normal vector $\underline{n}$. The two parts of the crystal at both sides of this plane were submitted to a relative small displacement along a crystallographic direction, the slip

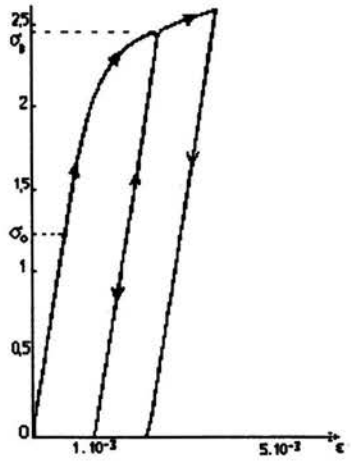

FIGURE 2.1. A quasi-static tensile curve.

direction $\underline{m}$. This mode of deformation is the slip process. The set consisting of the slip plane and slip direction characterizes the slip system $(\underline{n}, \underline{m})$. The activation of this transformation is reached when the resolved shear stress on the slip system attains the limit value, that is the Schmid law:

$$f(\boldsymbol{\sigma}) = \underline{n}.\boldsymbol{\sigma}.\underline{m} - \tau_o \leq 0. \tag{2.2}$$

Those sets depend upon the crystallographic class of crystal. Metals are polycristalline media, composed of an aggregate of different shape and orientation. The modes of deformation of polycrystals stay the same as those of monocrystals but the grains boundary introduce supplementary modes of transformation. It is rather difficult to modelise directly the behaviour of elasto-plasticity of polycrystal, we adopted some simplified modelisation of such a complex reality.

Classical triaxial test are performed on cylindrical samples. These tests show that the pressure $\mathrm{tr}\,\boldsymbol{\sigma}$ does not influence the plastic behaviour of most metals. Some other results are concerned with the determination of the domain of reversibility in stress space. If initially the shape of this domain is nearly an ellipse, this shape becomes complicated after loading history. There are three effects: an expansion, a global translation in the direction of the loading point and a local deformation near the stress point which can be very pronounced. It was also pointed out that the projection of the plastic strain increment is almost normal to the elastic boundary in the stress

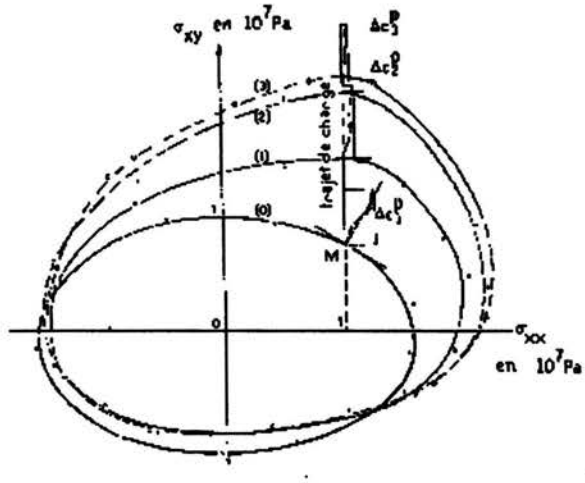

FIGURE 2.2. Experiments on copper, after H.D. Bui (1972).

space (Fig. 2.2). The test shows also that the domain of reversibility C is convex in the stress space. This is the principal argument to admit the principle of maximal plastic work.

Maximal plastic work, normality rule. *The principle of maximal plastic work defines the evolution of the plastic strain*

$$\text{Let } \boldsymbol{\sigma} \in C; \quad \text{for all } \boldsymbol{\sigma}^* \in C, \ (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) : \dot{\boldsymbol{\varepsilon}}_p \geq 0.$$

The plastic strain is normal to the convex set C.

2.2. The domain of elasticity

The domain of elasticity is modelised by a convex function f on the stress space which depends on internal parameters $\mathcal{E}$; these parameters are often chosen to represent also possible change of shape of the domain in the stress space:

$$C = \{\boldsymbol{\sigma} \mid f(\boldsymbol{\sigma}, \mathcal{E}) \leq 0\}. \tag{2.3}$$

Taking into account the principle of maximal plastic work we deduce the property:

$$\begin{cases} f(\boldsymbol{\sigma}) < 0, & \dot{\boldsymbol{\varepsilon}}_p = 0, \\ f(\boldsymbol{\sigma}) = 0, & \dot{\boldsymbol{\varepsilon}}_p = \mu \dfrac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \mu \geq 0. \end{cases} \tag{2.4}$$

Generally, the function f is taken as invariant by translation of $\operatorname{tr}\boldsymbol{\sigma}$, then f depends upon the deviatoric part of $\boldsymbol{\sigma}$, i.e. $\boldsymbol{s} = \boldsymbol{\sigma} - \frac{\operatorname{tr}\boldsymbol{\sigma}}{3}\boldsymbol{I}$. The normality rule induces then that the plastic strain is isochoric.

The function f must be invariant under the group of material symmetry.

Consider now an isotropic material; then the function f depends only upon the invariant of the stress:

$$\mathrm{I}_1 = \operatorname{tr}\boldsymbol{\sigma}, \quad \mathrm{I}_2 = \frac{1}{2}\operatorname{tr}\boldsymbol{\sigma}.\boldsymbol{\sigma}, \quad \mathrm{I}_3 = \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}.\boldsymbol{\sigma}.\boldsymbol{\sigma}, \tag{2.5}$$

or on the principal stresses:

$$\sigma_1, \quad \sigma_2, \quad \sigma_3. \tag{2.6}$$

Generally the function f is not a polynomial of the invariants. Some typical yielding functions are the Tresca criterion and the Huber–von Mises function.

The Tresca criterion: The domain of elasticity is defined by the function

$$f(\boldsymbol{\sigma}) = \sup_{i,j\in\{1,2,3\}} \{\sigma_i - \sigma_j - \sigma_o\} \leq 0, \tag{2.7}$$

where the value σ_o has the following interpretation:

- σ_o is the limit of elasticity in tension,
- $\frac{1}{2}\sigma_o$ is the limit of elasticity in shear.

This function depends only on the deviatoric part of the stress $\boldsymbol{s}$.

The Huber–von Mises criterion: The domain of elasticity is defined by the function

$$f(\boldsymbol{\sigma}) = \sqrt{J_2} - k \leq 0, \tag{2.8}$$

where $J_2 = \frac{1}{2}\operatorname{tr}\boldsymbol{s}.\boldsymbol{s}$, and k is interpreted as follows:

- k is the limit in shear,
- $k\sqrt{3}$ is the limit in tension.

To modelise the expansion of the domain of elasticity, the value k or σ_o is a function of some hardening parameters $\mathcal{E}$ and the translation in the stress space is obtained by replacing $\boldsymbol{\sigma}$ by $\boldsymbol{A} = \boldsymbol{\sigma} - \boldsymbol{\alpha}(\mathcal{E})$.

2.3. The evolution of internal state

The domain of elasticity being defined by such a function f, the evolution of the internal parameters $\boldsymbol{\varepsilon}_p, \alpha, \mathcal{E}$ satisfies the complementarity conditions:

$$f \leq 0, \quad \mu \geq 0, \quad \mu f = 0. \tag{2.9}$$

By derivation with respect to time of the last equation, we can deduced that: for a state of stress on the boundary of the domain of elasticity $f = 0$, we have obviously $\dot{f} \leq 0$. Then a necessary condition to obtain $\mu \geq 0$ is the consistency condition $\dot{f} = 0$, whose expression is:

$$\dot{f} = \frac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f^{\mathrm{T}}}{\partial \mathcal{E}} \dot{\mathcal{E}} = 0. \tag{2.10}$$

To determine μ effectively, one must give complementary laws for the hardening:

$$\dot{\mathcal{E}} = \mu \ g(\boldsymbol{\sigma}, \alpha, \boldsymbol{\varepsilon}_p, \mathcal{E}), \tag{2.11}$$

where g is a given function of the local state. Considering now the constitutive law for the stress:

$$\boldsymbol{\sigma} = \boldsymbol{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p), \tag{2.12}$$

we determine the value of μ [1)]

$$\mu = \frac{< \dfrac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \dot{\boldsymbol{\varepsilon}} >}{\dfrac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}} - \dfrac{\partial f^{\mathrm{T}}}{\partial \mathcal{E}} g}, \tag{2.13}$$

if the hardening modulus $H = \dfrac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}} - \dfrac{\partial f^{\mathrm{T}}}{\partial \mathcal{E}} g$ is positive. Therefore, the evolution of the stress satisfies:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{C} : \dot{\boldsymbol{\varepsilon}} - \frac{< \dfrac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \dot{\boldsymbol{\varepsilon}} >}{\dfrac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}} - \dfrac{\partial f^{\mathrm{T}}}{\partial \mathcal{E}} g} \boldsymbol{C} : \frac{\partial f}{\partial \boldsymbol{\sigma}}, \tag{2.14}$$

and the local behaviour is determined by the potential $U(\dot{\boldsymbol{\varepsilon}})$, (Hill [1958]):

$$U(\dot{\boldsymbol{\varepsilon}}) = \frac{1}{2} \dot{\boldsymbol{\varepsilon}} : \boldsymbol{C} : \dot{\boldsymbol{\varepsilon}} - \frac{1}{2} \frac{< \dfrac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \dot{\boldsymbol{\varepsilon}} >^2}{\dfrac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}} - \dfrac{\partial f^{\mathrm{T}}}{\partial \mathcal{E}} g}, \tag{2.15}$$

$$\dot{\boldsymbol{\sigma}} = \frac{\partial U}{\partial \dot{\boldsymbol{\varepsilon}}}. \tag{2.16}$$

Combining these properties, the equations of equilibrium $\operatorname{div} \dot{\boldsymbol{\sigma}} = 0$ in Ω, the boundary condition $\dot{\underline{T}} = \dot{\boldsymbol{\sigma}} . \underline{n}$ over $\partial\Omega_T$ and the boundary condition $\underline{v} = \underline{v}^d$

[1)] The positive part of any quantities F is $< F >= \frac{1}{2}(F + |F|)$.

on the complementary part $\partial\Omega_u$, the solution $\underline{v}$ of the rate boundary value problem is a stationary point of the functional:

$$\mathcal{U}(\tilde{\underline{v}}^*, \tilde{\underline{\dot{T}}}^d) = \int_\Omega U(\boldsymbol{\varepsilon}(\underline{v}^*))\,\mathrm{d}\Omega - \int_{\partial\Omega_T} \underline{\dot{T}}^d.\underline{v}^*\,\mathrm{d}A, \tag{2.17}$$

among the set of admissible $\underline{v}^*$ satisfying the condition $\underline{v}^* = \underline{v}^d$ on $\partial\Omega_u$.

Proof. Consider the variation of the functional $\mathcal{U}$:

$$\frac{\partial \mathcal{U}}{\partial \tilde{\underline{v}}^*}(\tilde{\underline{v}}, \tilde{\underline{\dot{T}}}^d) \cdot \delta\tilde{\underline{v}}^* = \int_\Omega \frac{\partial U}{\partial \dot{\boldsymbol{\varepsilon}}}(\boldsymbol{\varepsilon}(\underline{v})) : \boldsymbol{\varepsilon}(\delta\underline{v}^*)\,\mathrm{d}\Omega - \int_{\partial\Omega_T} \underline{\dot{T}}^d.\delta\underline{v}^*\,\mathrm{d}A = 0, \tag{2.18}$$

among the set of kinematical fields $\delta\underline{v}^*$ satisfying $\delta\underline{v}^* = 0$ on $\partial\Omega_u$. After integration by part, it is obvious that $\dot{\boldsymbol{\sigma}} = \dfrac{\partial U}{\partial \dot{\boldsymbol{\varepsilon}}}$ satisfies the equilibrium equation and the boundary condition over $\partial\Omega_T$.

2.4. A model of perfect plasticity

The model of perfect plasticity is illustrated by the tensile curve (Fig. 2.3). In this case the domain of reversibility is constant. The convex function f depends only upon the stress $\boldsymbol{\sigma}$. The relation between stress and strain is given by:

$$\boldsymbol{\sigma} = \boldsymbol{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p). \tag{2.19}$$

The free energy is reduced to:

$$\rho w = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) : \boldsymbol{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p). \tag{2.20}$$

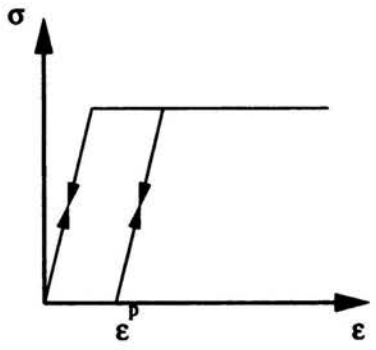

FIGURE 2.3. Elastic-perfectly plastic tensile curve

The increment of plastic strain is normal to the domain of reversibility: at a point σ such $f = 0$,

$$\dot{\varepsilon}_p = \mu \frac{\partial f}{\partial \sigma}, \quad \mu \geq 0. \tag{2.21}$$

Obviously, for all states of stress, we have

$$\mu f = 0, \quad \mu \geq 0, \quad f \leq 0. \tag{2.22}$$

The evolution satisfies a consistency condition, which is obtained by derivation of the equation (2.22) with respect to time:

$$\dot{\mu} f + \mu \dot{f} = 0. \tag{2.23}$$

The only way to have $\mu \geq 0$ implies that $\dot{f} = 0$. This is the consistency condition, which defines the value of μ:

$$\frac{\partial f^{\mathrm{T}}}{\partial \sigma} : \boldsymbol{C} : (\dot{\varepsilon} - \dot{\varepsilon}_p) = 0. \tag{2.24}$$

Hence we deduce that

$$\mu = \frac{< \dfrac{\partial f^{\mathrm{T}}}{\partial \sigma} : \boldsymbol{C} : \dot{\varepsilon} >}{\dfrac{\partial f^{\mathrm{T}}}{\partial \sigma} : \boldsymbol{C} : \dfrac{\partial f}{\partial \sigma}}. \tag{2.25}$$

2.5. The rate boundary value problem

In the case of generalized standard materials, the free energy is a function of the strain ε and of internal parameter α. We assume that there is no viscosity and we consider isothermal problem. The equations of state are:

$$\sigma = \rho \frac{\partial w}{\partial \varepsilon}, \quad \mathrm{A} = -\rho \frac{\partial w}{\partial \alpha}. \tag{2.26}$$

The evolution of the internal state is defined by the normality rule associated with a convex domain $\mathcal{C}$ in the A-space:

$$\mathcal{C} = \{\mathrm{A} \mid f(\mathrm{A}) \leq 0\}. \tag{2.27}$$

At each time the body Ω is decomposed in two domains, one where the evolution is reversible and the other one where the evolution can be irreversible Ω_p:

$$\Omega_p = \{\underline{X} \mid f(\mathrm{A}(\underline{X})) = 0\}. \tag{2.28}$$

Due to the definition of the evolution, the rate of the internal parameter is $\dot{\alpha}(\underline{X}) = \mu(\underline{X}) \dfrac{\partial f}{\partial \mathrm{A}}$ where $\mu(\underline{X}) \geq 0$ if and only if $\underline{X}$ is in Ω_p, $\mu(\underline{X}) = 0$ otherwise.

Characterization of equilibrium

A state of equilibrium is defined by a field of displacement $\underline{u}$ and a field of internal parameter α satisfying the set of local equations:

- the displacement is compatible with the boundary condition prescribed on $\partial\Omega_u$: $\underline{u} = \underline{u}^d$, and the strain is $\boldsymbol{\varepsilon}(\underline{u}) = (\text{grad}\,\underline{u} + \text{grad}^T\,\underline{u})/2$,
- the equations of state define the internal forces $\boldsymbol{\sigma}$, A:

$$\boldsymbol{\sigma} = \rho\frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \quad A = -\rho\frac{\partial w}{\partial \alpha}, \tag{2.29}$$

- the stresses $\boldsymbol{\sigma}$ are statically admissible with the boundary conditions prescribed over $\partial\Omega_T$,

$$\text{div}\,\boldsymbol{\sigma} = 0, \text{ on } \Omega \quad \boldsymbol{\sigma}.\underline{n} = \underline{T}^d \text{ over } \partial\Omega_T, \tag{2.30}$$

- the forces A are plastically admissible, that is $f(A) \leq 0$ everywhere.

For this state of equilibrium, the plastic domain Ω_p is known.

For a given distribution of the internal parameter α, a solution of equilibrium is obtained also by the property of stationarity of the potential energy. This energy is defined by the internal potential of strain and the potential of external loading, so that:

$$\mathcal{E}(\tilde{\underline{u}}, \tilde{\alpha}, \tilde{\underline{T}}^d) = \int_\Omega \rho w(\boldsymbol{\varepsilon}(\underline{u}), \alpha)\,d\Omega - \int_{\partial\Omega_T} \underline{T}^d.\underline{u}\,dS. \tag{2.31}$$

For any internal state $\tilde{\alpha}$, a state of equilibrium satisfies

$$\frac{\partial \mathcal{E}}{\partial \underline{u}} \cdot \delta\underline{u} = 0. \tag{2.32}$$

It is necessary to verify that the field $A(\tilde{\alpha}, \tilde{\underline{u}})$ is also plastically admissible. The expression (2.32) is differentiated with respect to time to obtain the rate equilibrium equations. The variation of the total potential energy is given by:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{E}}{\partial \underline{u}}\right) \cdot \delta\underline{u} = \int_\Omega \dot{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}(\delta\underline{u})\,d\Omega - \int_{\partial\Omega_T} \dot{\underline{T}}^d.\delta\underline{u}\,dS = 0. \tag{2.33}$$

The stress rate $\dot{\boldsymbol{\sigma}}$ verifies the constitutive law:

$$\dot{\boldsymbol{\sigma}} = \rho\frac{\partial^2 w}{\partial \boldsymbol{\varepsilon}\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \rho\frac{\partial^2 w}{\partial \boldsymbol{\varepsilon}\partial \alpha}\dot{\alpha}. \tag{2.34}$$

The rate of internal parameters satisfies the normality rule:

$$\dot{\alpha} = \lambda\frac{\partial f}{\partial A}, \quad \lambda \geq 0, \quad \lambda f = 0. \tag{2.35}$$

The consistency condition must be taken into account:

- if $f(\mathrm{A}) = 0$ and $\dot{f} = 0$, then $\lambda \geq 0$,
- if $f(\mathrm{A}) = 0$ and $\dot{f} < 0$, then $\lambda = 0$.

This condition is rewritten equivalently as the inequality:

$$(\lambda - \lambda^*)\dot{f} \geq 0, \tag{2.36}$$

among the set of admissible fields λ^*:

$$\mathcal{K} = \{\lambda^* \mid \lambda^*(x) \geq 0, \text{ if } f(\mathrm{A}(x)) = 0 \text{ and } \lambda^*(x) = 0 \text{ otherwise}\}. \tag{2.37}$$

Taking the constitutive law into account, $(\mathrm{A} = -\rho\dfrac{\partial w}{\partial \alpha})$, the consistency condition is rewritten as:

$$(\lambda - \lambda^*)\rho\left(\frac{\partial f^{\mathrm{T}}}{\partial \mathrm{A}}\frac{\partial^2 w}{\partial \alpha \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial f^{\mathrm{T}}}{\partial \mathrm{A}}\frac{\partial^2 w}{\partial \alpha \partial \alpha}\lambda\frac{\partial f}{\partial \mathrm{A}}\right) \leq 0. \tag{2.38}$$

The average of this inequality over the whole volume gives an inequality defined for the system.

The solution of the boundary value problem satisfies the equilibrium and the normality rule, these conditions are rewritten in a global manner using sets of compatible fields $(\delta\underline{\tilde{u}}, \tilde{\lambda}^*)$:

$$0 = \int_\Omega \boldsymbol{\varepsilon}(\delta\underline{u}) : \left(\frac{\partial^2 w}{\partial \boldsymbol{\varepsilon}\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}(\underline{v}) + \frac{\partial^2 w}{\partial \boldsymbol{\varepsilon}\partial \alpha}\lambda\frac{\partial f}{\partial \mathrm{A}}\right)\rho \,\mathrm{d}\Omega - \int_{\partial\Omega_{\mathrm{T}}} \dot{\underline{\mathrm{T}}}^d . \delta\underline{u} \,\mathrm{d}S,$$

$$0 \geq \int_\Omega (\lambda - \lambda^*)\left(\frac{\partial f^{\mathrm{T}}}{\partial \mathrm{A}}\frac{\partial^2 w}{\partial \alpha \partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}(\underline{v}) + \frac{\partial f^{\mathrm{T}}}{\partial \mathrm{A}}\frac{\partial^2 w}{\partial \alpha \partial \alpha}\lambda\frac{\partial f}{\partial \mathrm{A}}\right)\rho \,\mathrm{d}\Omega.$$

The potential of the rate. *For any kinematically admissible fields* $(\underline{v}, \lambda)$

$$(\underline{v}, \lambda) \in \left\{(\underline{\tilde{v}}, \tilde{\lambda}) \mid \underline{v} = \underline{v}^d, \text{ on } \partial\Omega_{\mathrm{u}}, \tilde{\lambda} \in \mathcal{K}\right\} \tag{2.39}$$

we define the functional

$$\mathcal{F}(\underline{v}, \lambda) = \int_\Omega \left(\frac{1}{2}\boldsymbol{\varepsilon}(\underline{v}) : \frac{\partial^2 w}{\partial \boldsymbol{\varepsilon}\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}(\underline{v}) + \boldsymbol{\varepsilon}(\underline{v}) : \frac{\partial^2 w}{\partial \boldsymbol{\varepsilon}\partial \alpha}\lambda\frac{\partial f}{\partial \mathrm{A}}\right)\rho \,\mathrm{d}\Omega$$

$$\int_\Omega \frac{1}{2}\lambda^2\frac{\partial f^{\mathrm{T}}}{\partial \mathrm{A}}\frac{\partial^2 w}{\partial \alpha \partial \alpha}\frac{\partial f}{\partial \mathrm{A}}\rho \,\mathrm{d}\Omega - \int_{\partial\Omega_{\mathrm{T}}} \dot{\underline{\mathrm{T}}}^d . \underline{v} \,\mathrm{d}S. \tag{2.40}$$

The rate boundary value problem. *The solution* $(\underline{v}, \lambda) \in \mathcal{K}.\mathcal{A}.$ *of the rate boundary value problem satisfies the variational inequality:*

$$\frac{\partial \mathcal{F}}{\partial \underline{v}} \cdot (\underline{v} - \underline{v}^*) + \frac{\partial \mathcal{F}}{\partial \lambda}(\lambda - \lambda^*) \leq 0, \tag{2.41}$$

among the set of admissible fields:

$$(\underline{v}^*, \lambda^*) \in \mathcal{K}.\mathcal{A}. = \left\{ (\tilde{\underline{v}}, \tilde{\lambda}) \mid \underline{v} = \underline{v}^d, \text{ on } \partial\Omega_{\mathbf{u}}, \tilde{\lambda} \in \mathcal{K} \right\}. \tag{2.42}$$

Chapter 3

Fracture Mechanics

We consider the problem of the propagation of a crack in a continuous medium. First we study the case of classical linear fracture mechanics, we investigate the asymptotic singular stress field and we propose a global approach to rupture. A variational formulation is given to solve the rate boundary value problem. Some extensions to finite strain and dynamics are presented.

3.1. Introduction

Consider a body Ω with a crack, represented by a straight line. Around the crack tip we distinguish three domains determined by the distance from the tip (see Fig. 3.1):

- Zone I, the nearest zone, is the domain where all physical processes of rupture occur, that is the process zone.
- Zone II, where the mechanical fields are represented by singularities outside the process zone.
- Zone III, where the mechanical fields satisfy all matching conditions with given conditions at infinity.

The crack is represented at our scale by a line oriented by $\underline{e}_x$. The normal is $\underline{e}_y$ in the plane, and $\underline{e}_z$ normal to the plane.

If the singularities of mechanical fields govern the propagation of the crack, it is not necessary to take into account the process of rupture. This

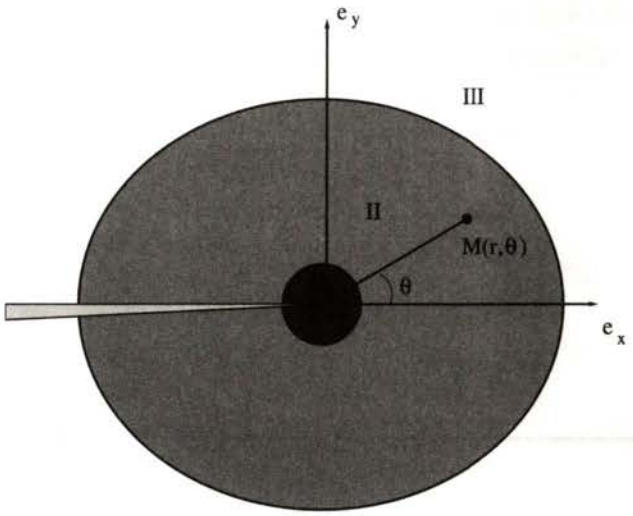

FIGURE 3.1. The description of the behaviour near the crack tip.

is an approximation which leads to a global approach of rupture. Such a description is powerful and constitutes the key point for describing classical fracture mechanics. In this case the singularities characterize the loading applied to the process zone.

3.2. Case of linear elasticity

The global approach of rupture is based on the study of the singularities of the mechanical fields.

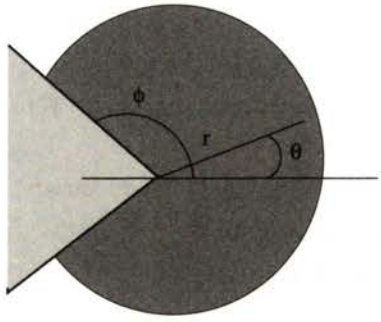

FIGURE 3.2. The geometry to study the singularities in a corner

The asymptotic stress fields near a crack tip in a linearly elastic material are given by solving a classical homogeneous boundary value problem with vanishing loading condition at infinity. The local displacement $\underline{u}$ is the solution of the set of local equations given by:

- the constitutive law: $\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}$ over Ω,
- the compatibility: $2\boldsymbol{\varepsilon} = \operatorname{grad} \underline{u} + \operatorname{grad}^{\mathrm{T}} \underline{u}$ in Ω,
- the equilibrium, with null body forces: $\operatorname{div} \boldsymbol{\sigma} = 0$ in Ω,
- the boundary conditions: fixed displacement $\underline{u} = 0$ or traction-free conditions on crack faces $\boldsymbol{\sigma}.\underline{n} = 0$, or mixed conditions, fixed displacement on one face and free-traction on the other face,
- vanishing conditions at infinity: $\boldsymbol{\sigma} \to 0$ as $r \to \infty$.

The solution of this problem is given by a displacement of the form:

$$\underline{u} = r^{\alpha} \underline{f}(\theta). \tag{3.1}$$

In general, the power α is a complex function which depends on the local behaviour. For isotropic linear elastic homogeneous body, the value depends on the Poisson ratio ν, angle ϕ and boundary conditions along the crack surface:

$$\alpha = \alpha(\nu, B.C, \phi). \tag{3.2}$$

However, for traction-free boundary conditions along a line crack, $\phi = \pi$, we obtain:

$$\alpha = \frac{1}{2}. \tag{3.3}$$

Associated with this power, we can distinguish three eigenfunctions for the problem of singularities, which correspond to three modes for opening the crack:

- mode I:

$$[\underline{u}]_{\Gamma} . \underline{e}_y \geq 0, \tag{3.4}$$

- mode II:

$$[\underline{u}]_{\Gamma} . \underline{e}_x \neq 0, \tag{3.5}$$

- mode III:

$$[\underline{u}]_{\Gamma} . \underline{e}_z \neq 0. \tag{3.6}$$

Then with these properties the stresses have singularities:

$$\boldsymbol{\sigma} = K r^{-\frac{1}{2}} \boldsymbol{f}(\theta), \tag{3.7}$$

where K denotes the stress intensity factors $(K_{\mathrm{I}}, K_{\mathrm{II}}, K_{\mathrm{III}})$ and the eigenfunctions $\boldsymbol{f}$ are known functions for mode I and II:

$$\sigma_{11} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right),$$

$$\sigma_{12} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} - \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right),$$

$$\sigma_{22} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2},$$

$$[\underline{u}]_{\Gamma}\cdot\underline{e}_y = \frac{K_{\mathrm{I}}(\eta+1)}{\mu}\sqrt{\frac{r}{2\pi}}, \qquad [\underline{u}]_{\Gamma}\cdot\underline{e}_x = \frac{K_{\mathrm{II}}(\eta+1)}{\mu}\sqrt{\frac{r}{2\pi}},$$

where $\eta = 3 - 4\nu$ in plane strain. The antiplane solution (mode III) satisfies:

$$\sigma_{13} = -\frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \sin\frac{\theta}{2},$$

$$\sigma_{23} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \cos\frac{\theta}{2},$$

$$[\underline{u}]_{\Gamma}\cdot\underline{e}_z = \frac{4K_{\mathrm{III}}}{\mu}\sqrt{\frac{r}{2\pi}}.$$

The mechanical fields being determined, we study now the propagation of the crack. We assume that the propagation is rectilinear.

3.3. Characterization of the propagation

The essential difficulty of the problem of propagation is the dependence of Ω on the crack length and the presence of moving singularities accompanying the crack.

One possibility has been investigated by Destuynder and Djaoua [1981]; by introducing a geometrical Lagrangian description. We propose to apply to our description the concept of singularity transport (Q.S. Nguyen [1980], C. Stolz [1998]). Inside a moving frame in translation with the crack tip, the nature of the singularity is conserved. The crack singularity is surrounded by a curve Γ delimiting a domain V_{Γ}. This domain translates with the tip of the crack position given by a function $l(t)$. All mechanical quantities are expressed in terms of the classical fixed coordinates outside V_{Γ} and in terms

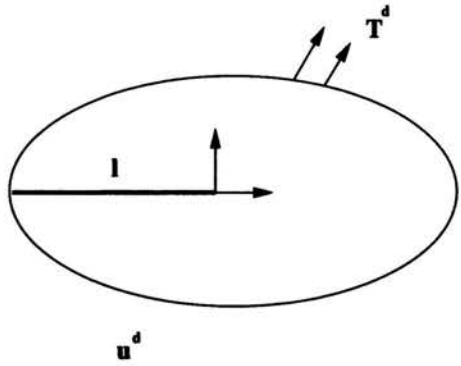

FIGURE 3.3. The boundary value problem.

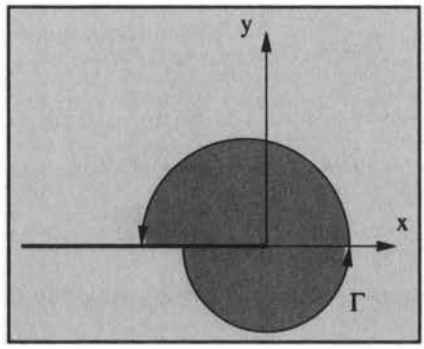

FIGURE 3.4. Decomposition of Ω in $\Omega_\Gamma \cup V_\Gamma$.

of moving coordinates inside V_Γ:

$$x = X - l(t),$$
$$y = Y.$$

All mechanical quantities F possesses time derivative given by f^*, which represents the variation of f in the moving frame:

$$F(X,Y,t) = f(x,y,t), \qquad \dot{f} = \frac{\partial F}{\partial t} = -\dot{l}\frac{\partial f}{\partial X} + f^*. \tag{3.8}$$

Applications of these definitions to average quantities on the whole domains are used to separate the contribution of the crack tip in the expression of the

dissipation:

$$\mathcal{F} = \int_{\Omega} f\rho \, \mathrm{d}\Omega = \int_{\Omega_\Gamma} f\rho \, \mathrm{d}\Omega + \int_{V_\Gamma} f\rho \, \mathrm{d}\Omega,$$

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_\Gamma} f\rho \, \mathrm{d}\Omega\right) + \int_{V_\Gamma} f^*\rho \, \mathrm{d}\Omega,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_\Gamma} f\rho \, \mathrm{d}\Omega\right) = \int_{\Omega_\Gamma} \dot{f}\rho \, \mathrm{d}\Omega - \int_{\Gamma} f\rho \dot{l} n_x \, \mathrm{d}S.$$

Let us introduce the notations: $f_x = f.\underline{e}_x, \nabla_x f = \nabla f.\underline{e}_x$.

Dissipation

The dissipation of the whole system can be rewritten as

$$D_m = \mathcal{P}_e - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} w\rho \, \mathrm{d}\Omega \geq 0, \tag{3.9}$$

where the power of external forces is given in term of local stresses, taking into account the conservation of the momentum:

$$\mathcal{P}_e = \int_{\partial\Omega} \underline{n}.\boldsymbol{\sigma}.\underline{v} \, \mathrm{d}S. \tag{3.10}$$

This quantity is decomposed in two terms using the divergence theorem:

$$\int_{\partial\Omega} \underline{n}.\boldsymbol{\sigma}.\underline{v} \, \mathrm{d}S = \int_{\Omega_\Gamma} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) \, \mathrm{d}\Omega + \int_{\Gamma} \underline{n}.\boldsymbol{\sigma}.\underline{v} \, \mathrm{d}S. \tag{3.11}$$

Using now the decomposition of the overall volume ($\Omega = \Omega_\Gamma \cup V_\Gamma$) we first obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} w\rho \, \mathrm{d}\Omega = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_\Gamma} w\rho \, \mathrm{d}\Omega\right) + \int_{V_\Gamma} w^*\rho \, \mathrm{d}\Omega. \tag{3.12}$$

Next, by application of general relations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\Gamma} w\rho \, \mathrm{d}\Omega = \int_{\Omega_\Gamma} \dot{w}\rho \, \mathrm{d}\Omega - \int_{\Gamma} \rho w n_x \, \mathrm{d}S \, \dot{l}, \tag{3.13}$$

$$\int_{V_\Gamma} w^*\rho \, \mathrm{d}\Omega = \int_{\Gamma} \underline{n}.\boldsymbol{\sigma}.\underline{v}^* \, \mathrm{d}S, \tag{3.14}$$

where the traction-free boundary condition $\boldsymbol{\sigma}.\underline{n} = 0$ along the crack has been taken into account. The dissipation takes finally the form:

$$D_m = \int_{\Omega_\Gamma} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \rho\dot{w}) \, \mathrm{d}\Omega + \int_{\Gamma} (\underline{n}.\boldsymbol{\sigma}.(\underline{v} - \underline{v}^*) + \rho w \underline{n}.\underline{e}_x \, \dot{l}) \, \mathrm{d}S \geq 0. \tag{3.15}$$

The displacement $\underline{u}$ is continuous along the curve Γ. The condition of compatibility implies Hadamard relations on the rates:

$$[\underline{u}]_\Gamma = 0 \Rightarrow \underline{v} = \underline{v}^* - \nabla_x \underline{u}\, \dot{l}. \tag{3.16}$$

Then the dissipation is decomposed in two terms: a volume part due to irreversibility of the body and a surface term associated with the propagation of the crack:

$$D_m = \int_{\Omega_\Gamma} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \rho \dot{w})\, \mathrm{d}\Omega + \int_\Gamma (-\underline{n}.\boldsymbol{\sigma}.\nabla_x \underline{u} + \rho w n_x)\, \mathrm{d}S\ \dot{l} \geq 0. \tag{3.17}$$

If the constitutive law is linearly elastic, the local behaviour is reversible and $\boldsymbol{\sigma} = \rho \dfrac{\partial w}{\partial \boldsymbol{\varepsilon}}$, then there is no dissipation in the volume:

$$\int_{\Omega_\Gamma} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \rho \dot{w})\, \mathrm{d}\Omega = 0. \tag{3.18}$$

When Γ is reduced to the crack tip, the result is conserved:

$$\lim_{\Gamma \to 0} \int_{\Omega_\Gamma} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \rho \dot{w})\, \mathrm{d}\Omega = 0. \tag{3.19}$$

The global dissipation contains only the contribution of the crack:

$$D_m = \lim_{\Gamma \to 0} \int_\Gamma (-\underline{n}.\boldsymbol{\sigma}.\nabla_x \underline{u} + \rho w n_x)\, \mathrm{d}S\ \dot{l}. \tag{3.20}$$

The thermodynamical force associated with the propagation is the free energy release rate $\mathcal{G}$ defined by:

$$\mathcal{G} = \lim_{\Gamma \to 0} \int_\Gamma (-\underline{n}.\boldsymbol{\sigma}.\nabla_x \underline{u} + \rho w n_x)\, \mathrm{d}S. \tag{3.21}$$

3.4. Energy interpretation

The total potential energy for the system is given by:

$$\mathcal{E}(\underline{u}, l, \underline{\mathrm{T}}^d) = \int_\Omega \rho\, w(\boldsymbol{\varepsilon}(\underline{u}))\, \mathrm{d}\Omega - \int_{\partial\Omega_\mathrm{T}} \underline{\mathrm{T}}^d.\underline{u}\, \mathrm{d}S, \tag{3.22}$$

whilst the dissipation is rewritten as:

$$D_m = \mathcal{P}_e - \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E} + \int_{\partial\Omega_\mathrm{T}} \underline{\mathrm{T}}^d.\underline{u}\, \mathrm{d}S) \geq 0. \tag{3.23}$$

By derivation of the potential energy with respect to its arguments we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}) = \frac{\partial \mathcal{E}}{\partial \tilde{\underline{u}}}.\tilde{\underline{v}} + \frac{\partial \mathcal{E}}{\partial l}\dot{l} + \frac{\partial \mathcal{E}}{\partial \tilde{\underline{\mathrm{T}}}^d}.\tilde{\underline{\mathrm{T}}}^d. \tag{3.24}$$

Since

$$\frac{\partial \mathcal{E}}{\partial \tilde{\underline{u}}}.\tilde{\underline{v}} = \int_{\partial\Omega} \underline{n}.\boldsymbol{\sigma}.\underline{v} - \int_{\partial\Omega_\Gamma} \underline{\mathrm{T}}^d.\underline{v}\,\mathrm{d}S, \tag{3.25}$$

hence

$$D_m = -\frac{\partial \mathcal{E}}{\partial l}\dot{l} \geq 0. \tag{3.26}$$

Release of the rate of energy. *For the overall system, the only internal parameter is the length of the crack. The thermodynamical force associated with the propagation is the release rate of energy $\mathcal{G}$ obtained by the global state equation:*

$$\mathcal{G} = -\frac{\partial \mathcal{E}}{\partial l}. \tag{3.27}$$

Remark. We must emphasize that this result is due to:

- the homogeneity of the constitutive material,
- the absence of discontinuity of velocity inside V_Γ,
- some hypothesis due to stationarity and self-similarity of processes which are related to the regularity of f^*.

3.5. Invariance and J-integral

Consider now a closed loop S inside a domain Ω, over which the body forces are null. For an homogeneous linearly elastic material the density ρ is uniform. The stresses satisfy both the equation of state and the conservation of the momentum:

$$\boldsymbol{\sigma} = \rho\frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \quad \mathrm{div}\,\boldsymbol{\sigma} = 0 \text{ on } \Omega. \tag{3.28}$$

Consider the integral C

$$C = \int_S (\rho w\, n_k - \sigma_{ij}u_{i,k}n_j)\,\mathrm{d}S, \tag{3.29}$$

then by divergence theorem the integral is equal to

$$\int_{\Omega_S} \frac{\partial(\rho w)}{\partial x_k} - \frac{\partial(\sigma_{ij}u_{i,k})}{\partial x_j}\,\mathrm{d}\Omega = \int_{\Omega_S} (\sigma_{ij}\varepsilon_{ij,k} - \boldsymbol{\sigma}_{ij}u_{i,jk} - \frac{\partial \sigma_{ij}}{\partial x_j}u_{i,k})\,\mathrm{d}\Omega. \tag{3.30}$$

Using now the conservation of momentum, the integral C vanishes. This is the expression of the law of conservation of energy.

>From this result it follows that the integral J_Γ:

$$J_\Gamma = \int_\Gamma (\rho w\, n_k - \sigma_{ij} u_{i,k} n_j)\, \mathrm{d}S, \tag{3.31}$$

is independent of the choíce of the loop Γ, provided that of the free-traction condition along the faces of the crack is taken onto account. Hence

$$J_\Gamma = \lim_{\Gamma \to 0} J_\Gamma = \mathcal{G}. \tag{3.32}$$

Invariance in non-linear elasticity. *Consider now a closed loop S inside a body Ω. Let a non-linear elastic behaviour be defined by the free energy function of the expansion tensor $\boldsymbol{C}$. The nominal stress $\boldsymbol{\theta}$ satisfies the equation of state and the conservation of momentum:*

$$\boldsymbol{\theta} = \rho \frac{\partial w}{\partial \mathbf{F}}, \quad \mathrm{Div}\, \boldsymbol{\theta}^{\mathrm{T}} = 0, \ \textit{in}\ \Omega. \tag{3.33}$$

By the same reasoning we deduce that:

$$\int_S (\rho w n_k - \theta_{ij} u_{i,k} n_j)\, \mathrm{d}S = 0, \tag{3.34}$$

i.e.,

$$\int_{\Omega_S} \Big(\frac{\partial \rho w}{\partial X_k} - \frac{\partial (\theta_{ij} u_{i,k})}{\partial X_j} \Big)\, \mathrm{d}\Omega = \int_{\Omega_S} (\theta_{ij} F_{ij,k} - \theta_{ij} u_{i,jk})\, \mathrm{d}\Omega = 0. \tag{3.35}$$

(c.f. Knowles and Sternberg [1981]). The invariance is obtained with the same properties as in small perturbations.

3.6. On the rate boundary value problem

To solve the problem of evolution of a crack inside a body Ω a law of propagation is needed.

Griffith's law

We consider a propagation law of Griffith's form.

$$\begin{cases} \mathcal{G} < G_c, & \dot{l} = 0, \\ \mathcal{G} = G_c, & \dot{l} \geq 0. \end{cases} \tag{3.36}$$

Similarly to plasticity, the evolution satisfies the following property:

$$\mathcal{G} \leq G_c, \quad \dot{l} \geq 0, \quad \dot{l}(\mathcal{G} - G_c) = 0. \tag{3.37}$$

By derivation with respect to time we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{l}(\mathcal{G} - G_c)) = 0 \;\Rightarrow\; \dot{l}\dot{\mathcal{G}} = 0. \tag{3.38}$$

At an equilibrium state such that $\mathcal{G} = G_c$, the propagation satisfies the inequality:

$$\forall \beta \geq 0, \quad (\dot{l} - \beta)\dot{\mathcal{G}} \geq 0. \tag{3.39}$$

This formulation characterizes the evolution of the crack.

The equations of the rate boundary value problem

The solution of the rate boundary value problem in terms of rate of displacement and rate of propagation satisfies the local set of equations:

- the conservation of momentum: $\operatorname{div} \dot{\boldsymbol{\sigma}} = 0$ on Ω, $\dot{\boldsymbol{\sigma}}.\underline{n} = \dot{\underline{T}}^d$ on $\partial\Omega_T$,
- the compatibility conditions: $\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}(\underline{v})$ on Ω, $\underline{v} = \underline{v}^d$ on $\partial\Omega_u$,
- the constitutive law: $\dot{\boldsymbol{\sigma}} = \dfrac{\partial^2 w}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \boldsymbol{C} : \dot{\boldsymbol{\varepsilon}}$ in Ω,
- the propagation law.

Choice of representation

Introducing now the decomposition of the domain $\Omega = \Omega_\Gamma \cup V_\Gamma$ to take into account the presence of the singularity and the rate $\hat{f}$ of any field f:

$$\begin{cases} \underline{x} \in \Omega_\Gamma, & \hat{f} = \dot{f}, \\ \underline{x} \in V_\Gamma, & \hat{f} = f^*, \end{cases} \tag{3.40}$$

then $\hat{f}$ is discontinuous on Γ:

$$0 = [f]_\Gamma + \dot{l}\, \nabla_x f \tag{3.41}$$

where $[\hat{f}]_\Gamma = \dot{f} - f^*$.

Therefore for the mechanical fields we obtain:

$$\begin{cases} \underline{x} \in \Omega_\Gamma, & \hat{\underline{v}} = \underline{v}, \quad \hat{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}, \\ \underline{x} \in V_\Gamma, & \hat{\underline{v}} = \underline{v}^*, \quad \hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^*. \end{cases} \tag{3.42}$$

and

$$0 = [\hat{\underline{v}}]_\Gamma + \dot{l}\, \nabla_x \underline{u}, \tag{3.43}$$

$$0 = \underline{n}.[\hat{\boldsymbol{\sigma}}]_\Gamma + \dot{l}\, \underline{n}.\nabla_x \boldsymbol{\sigma}. \tag{3.44}$$

Expression of $\dot{\mathcal{G}}$

$\dot{\mathcal{G}}$ is given by path independent integral

$$\dot{\mathcal{G}} = \int_\Gamma (\underline{n}.\nabla_x \boldsymbol{\sigma}.\underline{v}^* - \underline{n}.\boldsymbol{\sigma}^*.\nabla_x \underline{u})\, \mathrm{d}S. \tag{3.45}$$

The proof is obtained by time differentiation of $\mathcal{G}$ in the moving frame which first gives the expression:

$$\dot{\mathcal{G}} = \int_\Gamma \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^* n_x - \underline{n}.\boldsymbol{\sigma}.\nabla_x \underline{v}^* - \underline{n}.\boldsymbol{\sigma}^*.\nabla_x \underline{u}\, \mathrm{d}S. \tag{3.46}$$

The result follows by taking into account the following property:

$$\int_\Gamma \left((\boldsymbol{\sigma} : \boldsymbol{\varepsilon}^*) n_x - \underline{n}.\boldsymbol{\sigma}.\nabla_x \underline{v}^* - \underline{n}.\nabla_x \boldsymbol{\sigma}.\nabla_x \underline{v}^* \right) \mathrm{d}S = 0. \tag{3.47}$$

The path independence is naturally deduced from the definition of $\mathcal{G}$. A direct proof is given using the Green's formula applied to the fields:

$$\nabla_x \boldsymbol{\sigma} = \boldsymbol{C} : \nabla_x \boldsymbol{\varepsilon}, \ \operatorname{div} \nabla_x \boldsymbol{\sigma} = 0, \tag{3.48}$$

$$\boldsymbol{\sigma}^* = \boldsymbol{C} : \boldsymbol{\varepsilon}^*, \ \operatorname{div} \boldsymbol{\sigma}^* = 0. \tag{3.49}$$

Then we get the relation

$$\int_{V_S} \left(\operatorname{div}(\nabla_x \boldsymbol{\sigma}.\underline{v}^*) + \nabla_x \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^* - \boldsymbol{\sigma}^* : \nabla_x \boldsymbol{\varepsilon} - \operatorname{div}(\boldsymbol{\sigma}^*.\nabla_x \underline{u}) \right) \mathrm{d}S = 0, \tag{3.50}$$

which ensures that for a closed loop S we have:

$$\int_S (\underline{n}.\boldsymbol{\sigma}.\nabla_x \underline{v}^* - \underline{n}.\boldsymbol{\sigma}^*.\nabla_x \underline{u})\, \mathrm{d}S = 0. \tag{3.51}$$

Global formulation of the rate boundary value problem

The necessity to take into account the singularities leads to the formulation of the rate boundary value problem in terms of rate fields $\hat{\underline{v}}, \hat{\boldsymbol{\sigma}}$. The formulation of the rate boundary value problem with this representation is given by following equations:

- the conservation of the momentum:

$$\operatorname{div} \hat{\boldsymbol{\sigma}} = 0 \text{ on } \Omega, \quad \hat{\boldsymbol{\sigma}}.\underline{n} = \dot{\underline{T}}^d \text{ on } \partial\Omega_T, \tag{3.52}$$

$$[\hat{\boldsymbol{\sigma}}]_\Gamma.\underline{n} + \dot{l}\, \nabla_x \boldsymbol{\sigma}.\underline{n} = 0 \text{ on } \Gamma; \tag{3.53}$$

- the compatibility:

$$\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}(\hat{\underline{v}}) \text{ in } \Omega, \quad \hat{\underline{v}} = \underline{v}^d \text{ on } \partial\Omega_u, \tag{3.54}$$

$$[\hat{\underline{v}}]_\Gamma + \dot{l}\, \nabla_x \underline{u} = 0, \text{ on } \Gamma; \tag{3.55}$$

- the constitutive law: $\hat{\boldsymbol{\sigma}} = \boldsymbol{C} : \hat{\boldsymbol{\varepsilon}}$, in Ω;
- the propagation of the crack:

$$(\dot{l} - \beta) \quad \dot{\mathcal{G}} \geq 0, \quad \mathcal{G} \leq G_c, \quad \forall \beta \geq 0, \dot{l} \geq 0.$$

If the propagation is known, the velocity $\underline{v}$ is a solution of a non-classical problem of linear elasticity. Indeed, $\hat{\underline{v}}$ is not continuous on Γ and surface forces $\dot{l}\, \nabla_i \boldsymbol{\sigma}.\underline{n}$ are applied on Γ.

The rate boundary value problem. *The solution* $(\hat{\underline{v}}, \dot{l})$ *satisfies the inequality*

$$\frac{\partial \mathcal{F}}{\partial \hat{v}} \cdot (\hat{\underline{v}} - \tilde{\underline{v}}) + \frac{\partial \mathcal{F}}{\partial \dot{l}} (\dot{l} - \beta) \geq 0,$$

among the set of kinematically admissible fields

$$\mathcal{K}.\mathcal{A} = \left\{ (\hat{\underline{v}}, \beta) \mid \hat{\underline{v}} = \underline{v}^d \text{ on } \partial\Omega_u,\ \beta \in \mathcal{K},\ [\hat{\underline{v}}]_\Gamma + \beta \nabla_x \underline{u} = 0, \text{ on } \Gamma \right\}.$$

Here $\mathcal{F}$ *is the functional given by*

$$\mathcal{F}(\hat{\underline{v}}, \dot{l}) = \int_\Omega \frac{1}{2} \hat{\boldsymbol{\varepsilon}} : \boldsymbol{C} : \hat{\boldsymbol{\varepsilon}}\, \mathrm{d}\Omega - \int_\Gamma \dot{l} \underline{n}.\nabla_x \boldsymbol{\sigma}.\underline{v}^*\, \mathrm{d}S + \frac{1}{2} \dot{l}^2 \int_\Gamma \underline{n}.\nabla_x \boldsymbol{\sigma}.\nabla_x \underline{u}\, \mathrm{d}S - \int_{\partial\Omega_T} \dot{\underline{T}}^d.\underline{v}\, \mathrm{d}S.$$

Proof. The proof follows immediately from the properties of the fields. The variation of the functional $\mathcal{F}$ is given by

$$\delta\mathcal{F} = \delta\int_\Omega \frac{1}{2}\hat{\boldsymbol{\varepsilon}} : \boldsymbol{C} : \hat{\boldsymbol{\varepsilon}}\,\mathrm{d}\Omega - \int_\Gamma \delta l\, \underline{n}.\nabla_x\boldsymbol{\sigma}.\underline{v}^*\,\mathrm{d}S - \int_\Gamma \dot{l}\,\underline{n}.\nabla_x\boldsymbol{\sigma}.\delta\underline{v}^*\,\mathrm{d}S$$
$$+ \dot{l}\delta l\int_\Gamma \underline{n}.\nabla_x\boldsymbol{\sigma}.\nabla_x\underline{u}\,\mathrm{d}S - \int_{\partial\Omega_T}\dot{\underline{T}}^d.\delta\underline{v}\,\mathrm{d}S,$$

where

$$\delta\int_\Omega \frac{1}{2}\hat{\boldsymbol{\varepsilon}} : \boldsymbol{C} : \hat{\boldsymbol{\varepsilon}}\,\mathrm{d}\Omega = \delta\int_{\Omega_T}\frac{1}{2}\dot{\boldsymbol{\varepsilon}} : \boldsymbol{C} : \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}\Omega + \delta\int_{V_T}\frac{1}{2}\boldsymbol{\varepsilon}^* : \boldsymbol{C} : \boldsymbol{\varepsilon}^*\,\mathrm{d}\Omega,$$

and then

$$\delta\int_\Omega \frac{1}{2}\hat{\boldsymbol{\varepsilon}} : \boldsymbol{C} : \hat{\boldsymbol{\varepsilon}}\,\mathrm{d}\Omega - \int_{\partial\Omega_T}\dot{\underline{T}}^d.\delta\underline{v}\,\mathrm{d}S = \int_\Gamma\Big(-\underline{n}.\dot{\boldsymbol{\sigma}}.\delta\underline{v} + \underline{n}.\boldsymbol{\sigma}^*.\delta\underline{v}^*\Big)\,\mathrm{d}S.$$

Taking into account the compatibility of the fields

$$\delta\underline{v} = \delta\underline{v}^* - \delta l\,\nabla_x\underline{u}, \tag{3.56}$$

the last term is rewritten as

$$\int_\Gamma\Big(-\underline{n}.(\dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^*).\delta\underline{v}^* + \delta l\,\underline{n}.\dot{\boldsymbol{\sigma}}.\nabla_x\underline{u}\Big)\,\mathrm{d}S. \tag{3.57}$$

Substituting all the terms in the functional we obtain:

$$\delta\mathcal{F} = \int_\Gamma \underline{n}.(-\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^*).\delta\underline{v}^*\,\mathrm{d}S - \int_\Gamma \dot{l}\,\underline{n}.\nabla_x\boldsymbol{\sigma}.\delta\underline{v}^*\,\mathrm{d}S + \int_\Gamma \delta l\,\underline{n}.\dot{\boldsymbol{\sigma}}.\nabla_x\underline{u}\,\mathrm{d}S$$
$$- \int_\Gamma \delta l\,\underline{n}.\nabla_x\boldsymbol{\sigma}.\underline{v}^*\,\mathrm{d}S + \dot{l}\delta l\int_\Gamma \underline{n}.\nabla_x\boldsymbol{\sigma}.\nabla_x\underline{u}\,\mathrm{d}S,$$

and finally for any variation we have

$$\delta\mathcal{F} = \int_\Gamma\Big(\underline{n}.(-\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^*) - \dot{l}\,\underline{n}.\nabla_x\boldsymbol{\sigma}\Big).\delta\underline{v}^*\,\mathrm{d}S$$
$$+ \int_\Gamma \delta l\Big((\underline{n}.\dot{\boldsymbol{\sigma}} + \dot{l}\,\underline{n}.\nabla_x\boldsymbol{\sigma}).\nabla_x\underline{u} - \underline{n}.\nabla_x\boldsymbol{\sigma}.\underline{v}^*\Big)\,\mathrm{d}S. \tag{3.58}$$

In the last term we recognize $(\underline{n}.\dot{\boldsymbol{\sigma}} + \dot{l}\,\underline{n}.\nabla_x\boldsymbol{\sigma}).\nabla_x\underline{u} = \underline{n}.\boldsymbol{\sigma}^*.\nabla_x\underline{u}$. The terms correspond respectively to the jump of the stress vector and to the propagation law.

3.7. Interaction of cracks

For a body with many cracks, we can apply the same reasoning for each tip of crack, $i = 1, \ldots, n$.

The rate boundary value problem for many cracks. *The solution* $(\underline{\hat{v}}, \dot{l}_i)$ *satisfies*

$$\frac{\partial \mathcal{F}}{\partial \hat{v}} \cdot (\underline{\hat{v}} - \underline{\tilde{v}}) + \sum_i \frac{\partial \mathcal{F}}{\partial \dot{l}_i}(\dot{l}_i - \beta_i) \geq 0,$$

among the set of admissible fields $(\mathcal{K}.\mathcal{A})$*; moreover, the functional* $\mathcal{F}$ *is*

$$\mathcal{F}(\underline{\hat{v}}, \dot{l}_i) = \int_\Omega \frac{1}{2}\hat{\boldsymbol{\varepsilon}} : \boldsymbol{C} : \hat{\boldsymbol{\varepsilon}} \, \mathrm{d}\Omega - \sum_i \int_\Gamma \dot{l}_i \underline{n}.\nabla_i \boldsymbol{\sigma}.\underline{v}^* \, \mathrm{d}S$$

$$+ \sum_i \frac{1}{2}\dot{l}_i^2 \int_\Gamma \underline{n}.\nabla_i \boldsymbol{\sigma}.\nabla_i \underline{u} \, \mathrm{d}S - \int_{\partial\Omega_T} \dot{\underline{T}}^d.\underline{v} \, \mathrm{d}S,$$

$$\mathcal{K}.\mathcal{A} = \left\{ (\underline{\hat{v}}, \underline{\beta}) \mid \underline{\hat{v}} = \underline{v}^d \text{ on } \partial\Omega_u, \ \underline{\beta} \in \mathcal{K}, \ [\underline{\hat{v}}]_\Gamma + \beta_i \nabla_i \underline{u} = 0 \text{ on } \Gamma_i \right\},$$

$$\mathcal{K} = \left\{ \underline{\beta} \mid \beta_i \geq 0 \text{ if } \mathcal{G}_i = G_c, \ \beta_i = 0 \text{ otherwise} \right\}.$$

It is easy to prove in an analogous manner as previously that a solution of the inequality satisfies the set of the classical local equations given below:

- the compatibility of the velocity

$$\boldsymbol{\varepsilon}(\underline{\hat{v}}) = \frac{1}{2}(\nabla \underline{\hat{v}} + \nabla^{\mathrm{T}} \underline{\hat{v}}) \text{ in } \Omega, \quad \underline{\hat{v}} = \underline{v}^d, \text{ on } \partial\Omega_u$$

$$[\underline{\hat{v}}]_\Gamma + \dot{l}_i \nabla_i \underline{u} = 0 \text{ on each } \Gamma_i,$$

- the conservation of the momentum:

$$\operatorname{div} \hat{\boldsymbol{\sigma}} = 0 \text{ in } \Omega, \quad \hat{\boldsymbol{\sigma}}.\underline{n} = \dot{\underline{T}}^d \text{ on } \partial\Omega_T,$$

$$\underline{n}.[\hat{\boldsymbol{\sigma}}]_\Gamma + \dot{l}_i \underline{n}.\nabla_i \boldsymbol{\sigma} = 0 \text{ on each } \Gamma_i,$$

- the constitutive law: $\hat{\boldsymbol{\sigma}} = \boldsymbol{C} : \boldsymbol{\varepsilon}(\underline{\hat{v}})$ in Ω,
- the propagation law:

$$\mathcal{K} = \left\{ \underline{\beta} / \beta_i \geq 0, \text{ if } \mathcal{G}_i = G_c, \quad \beta_i = 0 \text{ otherwise} \right\}, \tag{3.59}$$

$$\forall \underline{\beta} \in \mathcal{K}, \quad \sum_i (\dot{l}_i - \beta_i) \, \dot{\mathcal{G}}_i \geq 0. \tag{3.60}$$

These equations are classical equations of an elasticity problem with non-classical boundary conditions along each Γ_i. The solution $\hat{v}$ is a linear function of $v^d(S)$ and $\dot{l}_i$:

$$\hat{\underline{v}}(x) = \int_{\partial\Omega_u} \boldsymbol{v}(x,S).v^d(S)\,\mathrm{d}S + \int_{\partial\Omega_T} \boldsymbol{L}(x,S).\underline{T}^d(S)\,\mathrm{d}S + \sum_i \boldsymbol{U}_i(x)\dot{l}_i. \quad (3.61)$$

By substitution of $\hat{\underline{v}}$ in terms of the propagation of cracks we obtain a reduced functional $\mathcal{F}^*(\dot{l}_i)$:

$$\mathcal{F}^* = \frac{1}{2}\underline{\dot{l}}.\boldsymbol{B}.\underline{\dot{l}} - \bar{Q}.\underline{\dot{l}}. \quad (3.62)$$

The existence of a solution is ensured by the positivity condition:

$$\underline{\beta}.\boldsymbol{B}.\underline{\beta} > 0, \quad \forall \underline{\beta} \neq \underline{0} \in \mathcal{K}, \quad (3.63)$$

where

$$\mathcal{K} = \{\underline{\beta} \mid \beta_i = 0, \mathcal{G}_i < G_c,\ \beta_i \geq 0,\ \text{if } \mathcal{G}_i = G_c\}.$$

The condition of uniqueness is given by

$$\begin{gathered} \underline{\beta}.\boldsymbol{B}.\underline{\beta} \geq 0, \\ \forall \underline{\beta} \in \{\underline{\beta} \mid \beta_i = 0,\ \mathcal{G}_i < G_c,\ \text{and } \beta_i \text{ anything otherwise}\}, \end{gathered} \quad (3.64)$$

The global formulation of the rate boundary value problem give us criterion for study the stability and the uniqueness of crack growth.

3.8. Case of hyperelasticity

The case of propagation of cracks in non-linear mechanics is more complex and only few results exist for specific classes of behaviour. The case of propagation of crack under antiplane shear have been studied by some authors for the class of Knowles and Sternberg materials:

$$w(I_1) = \frac{\mu}{2b}\Big((1 + \frac{b}{n}(I_1 - 3))^n - 1\Big), \quad I_1 = \mathrm{tr}(\mathbf{F}^{\mathrm{T}}.\mathbf{F}). \quad (3.65)$$

The shear curve is plotted in Fig. 3.5.

Due to the presence of non-linearities the equations of motion have particular properties, which are summarized as follows:

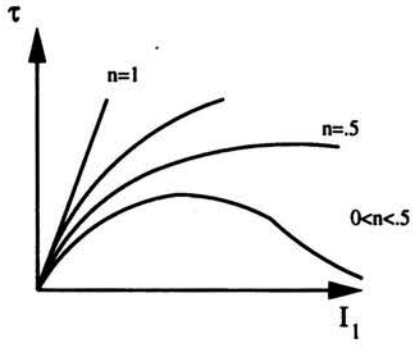

FIGURE 3.5. Illustration of Knowles and Sternberg materials.

- $n > \frac{1}{2}$: Equations of motion are always elliptic, this ensures the presence of singularities near the crack tip.
- $n < \frac{1}{2}$: Equations of motion are elliptic before the maximum and are hyperbolic after. Near the crack tip this induces jumps of gradient of displacement.

In the first case, the previous analysis is conserved, the existence of $\mathcal{G}$ is related to singularities. In the other cases the existence of jump of the gradient of displacement determines a shock curve along where the dissipation is distributed (see Fig. 3.6).

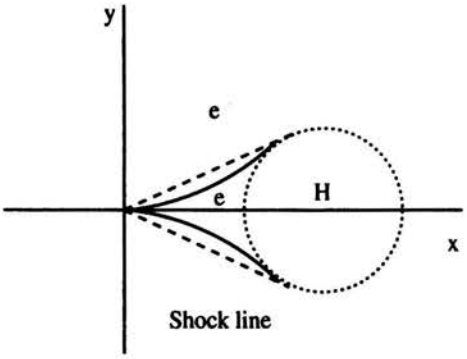

FIGURE 3.6. Shock curve.

In this case the expression of the dissipation in terms of the limit ($n_x = \underline{n}.\underline{e}_x$):

$$D_m = \int_\Gamma \left(\underline{n}.\boldsymbol{\sigma}.(\underline{v} - \underline{v}^*) + \rho w n_x \dot{l} \right) \mathrm{d}S, \tag{3.66}$$

must be reconsidered in the presence of discontinuities inside V_Γ:

$$[\underline{v}^*]_S + \dot{l}[\nabla_x \underline{u}]_S = 0. \tag{3.67}$$

Then we get

$$D_m = \lim_{\Gamma \to S} \int_\Gamma (-\underline{n}.\boldsymbol{\sigma}.\nabla_x \underline{u} + \rho w n_x)\, \mathrm{d}S\ \dot{l} \geq 0. \tag{3.68}$$

Hence the energy release rate takes the form:

$$\mathcal{G} = \int_S (-\underline{n}.\boldsymbol{\sigma}.[\nabla_x \underline{u}]_S + \rho [w]_S n_x)\, \mathrm{d}S. \tag{3.69}$$

The dissipation is distributed along the shock curve and the singularities disappear. The process of dissipation has been changed.

3.9. Case of dynamics

For the dynamical case we use the notion of Hamiltonian. We decompose the volume Ω into $\Omega = \Omega_\Gamma \cup V_\Gamma$; then the Hamiltonians of each domain are:

$$H_{\Omega_\Gamma} = \int_{\Omega_\Gamma} (\frac{p^2}{2\rho} + \rho(w + sT))\, \mathrm{d}\Omega - \int_{\partial\Omega_\Gamma} \underline{\mathrm{T}}^d.\underline{u}\, \mathrm{d}S, \tag{3.70}$$

$$H_{V_\Gamma} = \int_{V_\Gamma} (\frac{p^2}{2\rho} + \rho(w + sT))\, \mathrm{d}\Omega. \tag{3.71}$$

Since the conservation of energy can be expressed by:

$$\frac{\mathrm{d}}{\mathrm{d}t} H - \frac{\partial H}{\partial \underline{\mathrm{T}}^d} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \underline{\mathrm{T}}^d = - \int_{\partial\Omega} \underline{q}.\underline{n}\, \mathrm{d}S, \tag{3.72}$$

we deduce a relation between variations of the Hamiltonian and power heat supply:

$$\frac{\partial H}{\partial l}\, \dot{l} = - \lim_{\Gamma \to 0} \int_\Gamma \underline{q}.\underline{n}\, \mathrm{d}S, \tag{3.73}$$

where the variation of the Hamiltonian is:

$$-\frac{\partial H}{\partial l}\, \dot{l} = - \lim_{\Gamma \to 0} \int_\Gamma (-\underline{n}.\boldsymbol{\sigma}.\nabla_x \underline{u} + \frac{1}{2}\rho \underline{v}^2 n_x + \rho n_x (w + sT))\, \mathrm{d}S\, \dot{l}. \tag{3.74}$$

Consider now the entropy production:

$$\dot{S} = \lim_{\Gamma \to 0} \int_\Gamma \frac{(-\underline{n}.\boldsymbol{\sigma}.\nabla_x \underline{u} + \frac{1}{2}\rho \underline{v}^2 n_x + \rho n_x w)}{T}\, \mathrm{d}S\, \dot{l}, \tag{3.75}$$

then two different energy release rates are distinguished: one for heat source and one for entropy source:

$$G_e = -\lim_{\Gamma\to 0}\int_\Gamma (-\underline{n}.\boldsymbol{\sigma}.\nabla_x\underline{u} + \frac{1}{2}\rho n_x \underline{v}^2 + \rho n_x(w+sT))\,\mathrm{d}S, \tag{3.76}$$

$$G_s = \lim_{\Gamma\to 0}\int_\Gamma \frac{(-\underline{n}.\boldsymbol{\sigma}.\nabla_x\underline{u} + \frac{1}{2}\rho n_x \underline{v}^2 + \rho n_x w)}{T}\,\mathrm{d}S. \tag{3.77}$$

When strong discontinuities are present into the volume V_Γ, the limit to the crack tip is replaced by the limit to the line of discontinuities.

If we denote by $\bar{f}$ the mean value of f along the line S of discontinuities, we have the property

$$\frac{1}{2}[\underline{v}^2]_S = [\underline{v}]_S.\bar{\underline{v}},$$

and the conservation of the momentum implies

$$[\boldsymbol{\sigma}]_S.\underline{\nu} = m[\underline{v}]_S, \quad m = \rho \dot{l}\nu_x.$$

Taking into account the Hadamard relations along the line S, the jump of velocity is given by

$$[\underline{v}]_S + \dot{l}[\nabla\underline{u}]_S.\underline{e}_x = 0.$$

Finally we get

$$[-\underline{\nu}.\boldsymbol{\sigma}.\underline{v} + \frac{1}{2}\rho\nu_x\underline{v}^2\dot{l}]_S = -\underline{\nu}.\bar{\boldsymbol{\sigma}}.[\nabla\underline{u}]_S.\underline{e}_x\dot{l}.$$

The velocity has disappeared, the relation is objective, and the two energy release rates take the following form:

$$\begin{aligned} G_e\dot{l} &= -\lim_{\Gamma\to S}\int_\Gamma (-\underline{n}.\boldsymbol{\sigma}.\underline{v} + \frac{1}{2}\rho\underline{v}^2 n_x\dot{l} + \rho n_x(w+sT)\dot{l})\,\mathrm{d}S, \\ G_s\dot{l} &= \lim_{\Gamma\to S}\int_\Gamma \frac{(-\underline{n}.\boldsymbol{\sigma}.\underline{v} + \frac{1}{2}\rho n_x\underline{v}^2\dot{l} + \rho n_x w\dot{l})}{T}\,\mathrm{d}S. \end{aligned} \tag{3.78}$$

Assuming that the temperature is continuous along the line S we finally obtain:

$$\begin{aligned} G_e &= -\int_S (-\underline{\nu}.\bar{\boldsymbol{\sigma}}.[\nabla\underline{u}]_S.\underline{e}_x + [\rho(w+sT)]_S\underline{\nu}.\underline{e}_x\,\mathrm{d}S, \\ G_s &= \int_S \frac{(-\underline{\nu}.\bar{\boldsymbol{\sigma}}.[\nabla\underline{u}]_S.\underline{e}_x + [\rho w]_S\underline{\nu}.\underline{e}_x)}{T}\,\mathrm{d}S. \end{aligned} \tag{3.79}$$

We can notice that for isothermal evolution only one energy release rate is needed.

Chapter 4

Moving Discontinuities

The propagation of moving surface inside a body is analysed within the framework of thermomechanical couplings, when the moving surface is associated with an irreversible change of mechanical properties. The moving surface is a surface of heat sources and entropy production, intensities of which are related to particular energy release rates defined in terms of Hamiltonian gradients. As example, we analyse the evolution of partial damage in a composite sphere.

4.1. Introduction

The propagation of damage has been usually studied in connection with fracture mechanics and different approaches based on macroscopic and microscopic descriptions of degradation of mechanical properties have been proposed.

During a loading history damage in continuum mechanics can be induced by the initiation and growth of micro-cracks and micro-cavities. These descriptions, which are based on the evolution of the microscopic properties, propose to take into account the growth of pores or micro-cracks, through the idea that when some threshold value is reached, the material can not support further tensile loading.

Variational formulations were performed to describe the evolution of the surface between the sound and damaged material (Bui *et al.* [1981];

Pradeilles-Duval and Stolz [1995]). In the framework of thermomechanical coupling, similarly to fracture mechanics the analysis defines two different energy release rates associated with the heat and entropy production (Stolz [1995]; Stolz and Pradeilles-Duval [1996]).

This Chapter is mostly concerned with the description of damage on the evolution of a moving interface along which mechanical transformation occurs. Some connections can be made with the notion of configurational forces, (Gurtin [1995]; Maugin [1995]; Truskinovsky [1987]; Grinfeld [1980, 1991]).

4.2. General features

The domain Ω is composed of two distinct volumes Ω_1, Ω_2 of two materials with different mechanical characteristics. The contact between the two phases is perfect and the interface is denoted by Γ, ($\Gamma = \partial\Omega_1 \cap \partial\Omega_2$). The external surface $\partial\Omega$ is decomposed in two parts $\partial\Omega_u$ and $\partial\Omega_T$ on which the displacement $\underline{u}^d$ and the loading $\underline{T}^d$ are prescribed, respectively.

The material 1 changes into material 2 along the interface Γ by an irreversible process. Hence Γ moves with the normal velocity $\underline{c} = \phi\underline{\nu}$ in the reference state, $\underline{\nu}$ is the outward normal to Ω_2; then ϕ is positive.

When the surface Γ is moving, any mechanical quantity f can experience a jump denoted by $[f]_\Gamma = f_1 - f_2$, and any volume average has a rate defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(\Gamma)} f \,\mathrm{d}\Omega = \int_{\Omega(\Gamma)} \dot{f} \,\mathrm{d}\Omega - \int_\Gamma [f]_\Gamma \, c.\underline{\nu} \,\mathrm{d}S \tag{4.1}$$

The state of the system is characterized by the displacement field $\underline{u}$, from which the strain field $\boldsymbol{\varepsilon}$ is derived. The other parameters are the temperature T and spatial distribution of the two phases given by the position of the boundary Γ.

We analyse quasistatic evolution of Γ under given loading prescribed on the boundary $\partial\Omega$.

The behaviour of the phase i is defined by the free energy density w_i, function of the strain $\boldsymbol{\varepsilon}$ and temperature T. The mass density ρ of the two phases is the same. The state equations of each phase are

$$\boldsymbol{\sigma} = \rho\frac{\partial w_i}{\partial \boldsymbol{\varepsilon}}, \quad s = -\frac{\partial w_i}{\partial T}, \tag{4.2}$$

where $\boldsymbol{\sigma}$ is the reversible stress and s the entropy. If the materials have no viscosity then $\boldsymbol{\sigma}$ is the stress satisfying the momentum equation.

Assume now that the two phases are linear elastic materials.

The two phases are linear elastic

The potential energy $\mathcal{E}$ of the structure Ω ($\Omega_1 \cup \Omega_2$) has the following form

$$\mathcal{E}(\tilde{\underline{u}}, \Gamma, \tilde{\underline{\mathrm{T}}}^d) = \sum_{i=1,2} \int_{\Omega_i} \rho\, w_i(\boldsymbol{\varepsilon}(u), T)\, \mathrm{d}\Omega - \int_{\partial\Omega_{\mathrm{T}}} \underline{\mathrm{T}}^d . \underline{u}\, \mathrm{d}S.$$

The potential energy plays the role of the global free energy in a thermodynamical description; we can notice that the position of the interface Γ becomes an internal parameter for the global system. The characterization of an equilibrium state is given by the stationarity of the potential energy

$$\frac{\partial \mathcal{E}}{\partial \underline{u}} \cdot \delta\underline{u} = \sum_{i=1,2} \int_{\Omega_i} \rho \frac{\partial w_i}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}(\delta\underline{u})\, \mathrm{d}\Omega - \int_{\partial\Omega_{\mathrm{T}}} \underline{\mathrm{T}}^d . \delta\underline{u}\, \mathrm{d}S = 0, \tag{4.3}$$

for all $\delta\underline{u}$ kinematically admissible field satisfying $\delta\underline{u} = 0$ over $\partial\Omega_{\mathrm{u}}$. This formulation is equivalent to the set of local equations:

- local constitutive relations:

$$\boldsymbol{\sigma} = \rho \frac{\partial w_i}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{C}^i : \boldsymbol{\varepsilon} \text{ on } \Omega_i, \tag{4.4}$$

- momentum equations:

$$\operatorname{div} \boldsymbol{\sigma} = 0, \text{ on } \Omega, \quad [\boldsymbol{\sigma}]_\Gamma . \underline{\nu} = 0 \text{ on } \Gamma, \quad \boldsymbol{\sigma} . \underline{n} = \underline{\mathrm{T}}^d \text{ on } \partial\Omega_{\mathrm{T}}, \tag{4.5}$$

- compatibility relations:

$$2\boldsymbol{\varepsilon} = \nabla\underline{u} + \nabla^t\underline{u}, [\underline{u}]_\Gamma = 0 \text{ on } \Gamma, \quad \underline{u} = \underline{u}^d \text{ on } \partial\Omega_{\mathrm{u}}. \tag{4.6}$$

They are equations of a problem of heterogeneous elasticity. The solution is denoted by $\underline{u}^{sol}$, this field depends upon the quantities $(u^d, \underline{\mathrm{T}}^d, \Gamma)$. For an equilibrium state

$$\mathcal{E}(\underline{u}^{sol}, \underline{\mathrm{T}}^d, \Gamma) = W(\underline{u}^d, \underline{\mathrm{T}}^d, \Gamma). \tag{4.7}$$

This equation expresses the fact that the position of the interface Γ plays the role of internal parameters.

At a given state of equilibrium, for a given value of the prescribed loading $(\underline{u}^d, \underline{\mathrm{T}}^d)$, the position of the interface Γ is known. At this time a variation of

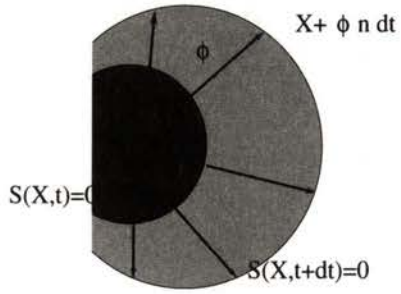

FIGURE 4.1. Propagation of the interface.

the loading is imposed, the mechanical quantities evolve and propagation of the interface can occur according to a given evolution law. For a prescribed history of the loading, we must determine the rate of all mechanical fields and the normal propagation ϕ to characterize the position of the interface Γ at each time. Along the interface Γ perfect bonding is preserved at each time. Let us introduce the notion of convected derivative.

Convected derivation. *The convected derivative* $\mathcal{D}_\phi$ *of any function* $f(\underline{X}_\Gamma, t)$ *is*

$$\mathcal{D}_\phi f = \lim_{\tau \to 0} \frac{f(\underline{X}_\Gamma + \phi \underline{\nu}\tau, t+\tau) - f(\underline{X}_\Gamma, t)}{\tau}. \tag{4.8}$$

With this definition, we can express the transport of the normal vector at point $\underline{x}$

$$\mathcal{D}_\phi \underline{\nu} = -\nabla\phi . \underline{e}_\alpha \, \underline{e}_\alpha, \tag{4.9}$$

where $\underline{e}_1, \underline{e}_2$ is a basis of the plane tangent to Γ at point $\underline{x}$. We can notice that the equation of the surface Γ, $S(\underline{X}, t) = 0$ satisfies immediately

$$\mathcal{D}_\phi S = \frac{\partial S}{\partial \underline{X}} . \dot{\underline{X}} + \frac{\partial S}{\partial t} = 0, \tag{4.10}$$

which defines the normal velocity $\underline{c}$ of Γ:

$$\underline{c} = \phi \underline{\nu}, \quad \underline{\nu} = \frac{\partial S}{\partial \underline{X}} / \parallel \frac{\partial S}{\partial \underline{X}} \parallel, \tag{4.11}$$

and finally for any differentiable fields f the convected derivative takes the form

$$\mathcal{D}_\phi f = \frac{\partial f}{\partial t} + \phi \, \nabla f . \underline{\nu} \, . \tag{4.12}$$

Hadamard's relations

On account of perfect bonding between the phases, the displacement and stress vectors are continuous along Γ. Their rates have discontinuities according to the general compatibility conditions of Hadamard, rewritten in term of the convected derivative:

- continuity of displacement

$$[\underline{u}]_\Gamma = 0 \Rightarrow \mathcal{D}_\phi([\underline{u}]_\Gamma) = [\underline{v}]_\Gamma + \phi[\nabla \underline{u}]_\Gamma.\underline{\nu} = 0, \tag{4.13}$$

- continuity of the stress vector

$$[\boldsymbol{\sigma}]_\Gamma.\underline{\nu} = 0 \Rightarrow \mathcal{D}_\phi([\boldsymbol{\sigma}]_\Gamma.\underline{\nu}) = [\dot{\boldsymbol{\sigma}}]_\Gamma.\underline{\nu} - \mathrm{div}_\Gamma([\boldsymbol{\sigma}]_\Gamma\ \phi) = 0. \tag{4.14}$$

The last equation is obtained by taking into account the equilibrium equation. Indeed, we have

$$\mathcal{D}_\phi([\boldsymbol{\sigma}]_\Gamma.\underline{\nu}) = \mathcal{D}_\phi[\boldsymbol{\sigma}]_\Gamma.\underline{\nu} + [\boldsymbol{\sigma}]_\Gamma.\mathcal{D}_\phi\underline{\nu}, \tag{4.15}$$

where

$$\mathcal{D}_\phi[\boldsymbol{\sigma}]_\Gamma = [\dot{\boldsymbol{\sigma}}]_\Gamma + \phi\underline{\nu}.[\nabla\boldsymbol{\sigma}]_\Gamma.\underline{\nu}. \tag{4.16}$$

Using the conservation of momentum

$$\underline{e}_\alpha.\nabla\boldsymbol{\sigma}.\underline{e}_\alpha + \underline{\nu}.\nabla\boldsymbol{\sigma}.\underline{\nu} = 0, \tag{4.17}$$

and the expression of the surface divergence given by

$$\mathrm{div}_\Gamma F = \mathrm{div}\, F - \underline{\nu}.\nabla F.\underline{\nu}, \tag{4.18}$$

the required result is obtained.

Orthogonality property for discontinuities

Since the displacement is continuous along the interface:

$$[\underline{u}]_\Gamma = 0, \Rightarrow [\nabla\underline{u}]_\Gamma.\underline{e}_\alpha = 0, \tag{4.19}$$

the discontinuities of the gradient must satisfy

$$[\nabla\underline{u}]_\Gamma = \underline{U}(x) \otimes \underline{\nu}. \tag{4.20}$$

Since the stress vector is continuous on Γ:

$$[\boldsymbol{\sigma}]_\Gamma.\underline{\nu} = 0, \tag{4.21}$$

the discontinuities of $\boldsymbol{\sigma}$ and of $\nabla\underline{u}$ have the property of orthogonality as pointed out by Hill [1986]:

$$[\boldsymbol{\sigma}]_\Gamma : [\nabla\underline{u}]_\Gamma = 0. \tag{4.22}$$

4.3. Dissipation analysis

The mass conservation leads to the continuity of the mass flux $m = \rho\phi$. The first and second laws of thermodynamics imply the local equations inside the volume and along the moving boundary Γ:

$$\rho\dot{\mathrm{e}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \operatorname{div} \underline{q}, \text{ in } \Omega, \tag{4.23}$$

$$0 = m[\mathrm{e}]_\Gamma - \underline{\nu}.\boldsymbol{\sigma}.[\underline{v}]_\Gamma + \underline{\nu}.[\underline{q}]_\Gamma, \text{ on } \Gamma. \tag{4.24}$$

Here e is the internal energy density ($\mathrm{e} = w + T\ s$), and $\underline{q}$ is the heat flux associated to the heat conduction.

Due to Hadamard compatibility equations, the heat power supply is given in terms of the release rate of internal energy $\mathcal{G}_{th}$ as an objective quantity defined along Γ

$$-\underline{\nu}.[\underline{q}]_\Gamma = \mathcal{G}_{th}\phi, \text{ with } \mathcal{G}_{th} = \rho[\mathrm{e}]_\Gamma - \boldsymbol{\sigma} : [\boldsymbol{\varepsilon}]_\Gamma. \tag{4.25}$$

The value of $\mathcal{G}_{th}$ is obtained considering the orthogonality condition on the discontinuities. When $\phi = 0$ in the reference state the interface Γ does not move, and the normal heat flux is continuous. When the transformation occurs, the moving interface is a surface of heat sources intensities given by $\mathcal{G}_{th}\phi$.

The total internal energy of the structure is equal to

$$\mathrm{E}(\tilde{\underline{u}}, \Gamma, T, T^d) = \int_{\Omega(\Gamma)} \rho\, \mathrm{e}\, \mathrm{d}\Omega - \int_{\partial\Omega_T} \underline{T}^d.\underline{u}\, \mathrm{d}S = \mathcal{E} + \int_{\Omega(\Gamma)} \rho\, sT\, \mathrm{d}\Omega. \tag{4.26}$$

For quasistatic evolution, the first law of thermodynamics is written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E} - \frac{\partial \mathrm{E}}{\partial \underline{T}^d}.\dot{\underline{T}}^d = -\int_{\partial\Omega} \underline{q}.\underline{n}\, \mathrm{d}S. \tag{4.27}$$

Taking into account the momentum conservation we have

$$\frac{\partial \mathrm{E}}{\partial \Gamma}.\dot{\Gamma} = \int_\Gamma [\underline{q}]_\Gamma.\underline{\nu}\, \mathrm{d}S = -\int_\Gamma \mathcal{G}_{th}\phi\, \mathrm{d}S. \tag{4.28}$$

Then the derivative of the total energy relatively to the position of the interface determine the source of heat due to the irreversible process with the intensity is governed by the internal energy release rate:

$$\mathcal{G}_{th} = -\frac{\partial \mathrm{E}}{\partial \Gamma}. \tag{4.29}$$

The entropy production is given by

$$\int_\Omega (\rho \dot{s} + \frac{\operatorname{div} \underline{q}}{T} - \underline{q}.\frac{\nabla T}{T^2})\, \mathrm{d}\Omega + \int_\Gamma (-m[s]_\Gamma - \underline{\nu}.[\frac{\underline{q}}{T}]_\Gamma)\, \mathrm{d}S \geq 0. \qquad (4.30)$$

Under the assumption of separability of the two dissipations, the volume term is reduced to the conduction, whilst the term along the surface is then

$$D_\Gamma = \frac{\rho[w]_\Gamma - \boldsymbol{\sigma} : [\boldsymbol{\varepsilon}]_\Gamma}{T}\, \phi = \frac{\mathcal{G}_s}{T}\, \phi \geq 0, \qquad (4.31)$$

where $\mathcal{G}_s$ is the release rate of free energy.

This quantity has a form similar to the driving traction force acting on a surface of strain discontinuity proposed by Abeyaratne and Knowles [1990]. The criteria governing the evolution of the interface may be written as function of this quantity.

In a thermomechanical coupling, two different release rates must be distinguished. One, defined in terms of variation of the total internal energy, gives rise to the heat source associated with the moving surface; the second one implies the production of entropy.

In the case of isothermal evolution the total dissipation is given in terms of the derivative of the potential energy relatively to the position of the interface

$$\frac{\partial \mathcal{E}}{\partial \Gamma}.\dot{\Gamma} = -\int_\Gamma \mathcal{G}_s \phi\, \mathrm{d}S, \text{ or } \mathcal{G}_s(\underline{x}) = -\frac{\partial \mathcal{E}}{\partial \Gamma}(\underline{x}). \qquad (4.32)$$

with $\mathcal{G}_s = \rho[w]_\Gamma - \boldsymbol{\sigma} : [\boldsymbol{\varepsilon}]_\Gamma$.

In this case, there is only one energy release rate characterizing the propagation, it gives the sources of entropy production and the dissipation.

These relations can be generalized to the dynamical case by replacing the internal energy of the system by its Hamiltonian.An extension to the case of running cracks, and to more general behaviour and structures is also possible (Stolz [1995], Stolz and Pradeilles-Duval [1996, 1997]).

4.4. Quasistatic evolution

In isothermal evolution, to describe the irreversibility we must specify complementary relations. An energy criterion is chosen as a generalized form of the well-known theory of Griffith. Then, we assume

$$\phi \geq 0, \text{ if } \mathcal{G}_s = G_c \text{ on } \Gamma, \text{ or } \phi = 0, \text{ otherwise.} \qquad (4.33)$$

This is a local energy criterion. At each equilibrium state, the interface Γ is decomposed into two subsets where the propagation is either possible or not. Let us denote by Γ^+ the subset of Γ where the critical value G_c is reached. The evolution of the interface is governed by the consistency condition. If at the geometrical point $\underline{x}_\Gamma(t)$ the criterion is reached:

$$\mathcal{G}_s(\underline{x}_\Gamma(t), t) = G_c, \tag{4.34}$$

then the derivative of $\mathcal{G}_s$ following the moving surface vanishes $\mathcal{D}_\phi \mathcal{G}_s = 0$. This leads to the consistency condition written for all points belonging to Γ^+

$$(\phi - \phi^*)\mathcal{D}_\phi \mathcal{G}_s \geq 0, \forall \phi^* \geq 0, \text{ on } \Gamma^+, \tag{4.35}$$

otherwise $\phi = 0$.

Evaluation of $\mathcal{D}_\phi \mathcal{G}_s$

Along the interface the displacement is continuous; then the velocities satisfy the Hadamard relation:

$$\underline{v}_2 + \phi \nabla \underline{u}_2 . \underline{\nu} = \underline{v}_1 + \phi \nabla \underline{u}_1 . \underline{\nu}. \tag{4.36}$$

To calculate $\mathcal{D}_\phi \mathcal{G}_s$, we derive term by term, the first of them is the jump of the free energy

$$\mathcal{D}_\phi(\rho[w]_\Gamma) = -\boldsymbol{\sigma}_2 : (\nabla \underline{v}_2 + \phi \nabla \nabla \underline{u}_2 . \underline{\nu}) + \boldsymbol{\sigma}_1 : (\nabla \underline{v}_1 + \phi \nabla \nabla \underline{u}_1 . \underline{\nu}). \tag{4.37}$$

Then we get

$$\begin{aligned} \mathcal{D}_\phi \mathcal{G} &= \mathcal{D}_\phi [w]_\Gamma - \mathcal{D}_\phi \boldsymbol{\sigma}_2 : [\nabla \underline{u}]_\Gamma - \boldsymbol{\sigma}_2 : [\mathcal{D}_\phi \nabla \underline{u}]_\Gamma \\ &= [\boldsymbol{\sigma}]_\Gamma : (\nabla \underline{v}_1 + \phi \nabla \nabla \underline{u} . \underline{\nu}) - (\dot{\boldsymbol{\sigma}}_2 + \phi \nabla \boldsymbol{\sigma}_2 . \underline{\nu}) : [\nabla \underline{u}]_\Gamma. \end{aligned} \tag{4.38}$$

Hence, after rearrangement of terms we obtain

$$\mathcal{D}_\phi \mathcal{G} = [\boldsymbol{\sigma}]_\Gamma : \nabla \underline{v}_1 - \dot{\boldsymbol{\sigma}}_2 : [\nabla \underline{u}]_\Gamma - \phi G_n, \tag{4.39}$$

where

$$G_n = -[\boldsymbol{\sigma}]_\Gamma : (\nabla \nabla \underline{u}_1 . \underline{\nu}) + \nabla \boldsymbol{\sigma}_2 . \underline{\nu} : [\nabla \underline{u}]_\Gamma. \tag{4.40}$$

4.5. The rate boundary value problem

The solution $(\tilde{\underline{v}}, \tilde{\phi})$ must satisfy:

- the constitutive law: $\dot{\boldsymbol{\sigma}} = \boldsymbol{C}_i : \dot{\boldsymbol{\varepsilon}}$ in Ω,
- the strain-displacement relation: $\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}(\nabla \underline{v} + \nabla^{\mathrm{T}} \underline{v})$ in Ω, and the boundary condition: $\underline{v} = \underline{v}^d$ on $\partial\Omega_{\mathrm{u}}$,
- the conservation of the momentum: $\operatorname{div} \dot{\boldsymbol{\sigma}} = 0$ in Ω, and $\dot{\boldsymbol{\sigma}}.\underline{n} = \dot{\underline{\mathrm{T}}}^d$ on $\partial\Omega_{\mathrm{T}}$,
- the compatibility conditions on the moving perfect interface: $[\mathcal{D}_\phi \underline{v}]_\Gamma = 0$, $[\mathcal{D}_\phi(\boldsymbol{\sigma}.\underline{\nu})]_\Gamma = 0$,
- the propagation law: $\forall \beta \in \mathcal{K}$, $(\beta - \phi)\mathcal{D}_\phi \mathcal{G} \geq 0$.

This system is now written in a global form.

The rate boundary value problem. *The evolution is determined by the functional*

$$\mathcal{F}(\underline{v}, \phi, \dot{\underline{\mathrm{T}}}^d) = \int_\Omega \frac{1}{2}\boldsymbol{\varepsilon}(\underline{v}) : \boldsymbol{C} : \boldsymbol{\varepsilon}(\underline{v}) \, \mathrm{d}\Omega - \int_{\partial\omega_{\mathrm{T}}} \dot{\underline{\mathrm{T}}}^d.\underline{v} \, \mathrm{d}S - \int_\Gamma \phi[\boldsymbol{\sigma}]_\Gamma : \nabla \underline{v}_1 \, \mathrm{d}S + \int_\Gamma \frac{1}{2}\phi^2 G_n \, \mathrm{d}S.$$

The solution satisfies the inequality

$$0 \leq \frac{\partial \mathcal{F}}{\partial \underline{v}}(\underline{v} - \underline{v}^*) + \frac{\partial \mathcal{F}}{\partial \phi}(\beta - \phi), \tag{4.41}$$

among the set $\mathcal{K}.\mathcal{A}$ of admissible fields $(\underline{v}^*, \phi^*)$:

$$\mathcal{K}.\mathcal{A} = \Big\{ (\underline{v}, \phi \mid \underline{v} = \underline{v}^d \text{ on } \partial\Omega_{\mathrm{u}}, [\underline{v}]_\Gamma + \phi[\nabla \underline{u}]_\Gamma = 0, \ \phi \in \mathcal{K} \Big\}, \tag{4.42}$$

$$\mathcal{K} = \{\beta \mid \beta \geq 0 \text{ on } \Gamma^+, \ \beta = 0 \text{ otherwise}\}. \tag{4.43}$$

Proof. The variation of the functional is given by:

$$\delta\mathcal{F} = \int_\Omega \boldsymbol{\varepsilon}(\underline{v}) : \boldsymbol{C} : \boldsymbol{\varepsilon}(\delta \underline{v}) \, \mathrm{d}\Omega - \int_{\partial\Omega_{\mathrm{T}}} \dot{\underline{\mathrm{T}}}^d.\delta\underline{v} \, \mathrm{d}S - \int_\Gamma \delta\phi[\boldsymbol{\sigma}]_\Gamma : \nabla \underline{v}_1 \, \mathrm{d}S + \int_\Gamma \phi\delta\phi G_n \, \mathrm{d}S - \int_\Gamma \phi[\boldsymbol{\sigma}]_\Gamma : \nabla \delta\underline{v}_1 \, \mathrm{d}S.$$

After integration by part we obtain:

$$\delta\mathcal{F} = \int_\Gamma \underline{n}.[\dot{\boldsymbol{\sigma}}.\delta\underline{v}]_\Gamma \, \mathrm{d}S + \int_{\partial\Omega_T} (\dot{\boldsymbol{\sigma}}.\underline{n} - \dot{\underline{T}}^d).\delta\underline{v} \, \mathrm{d}S - \int_\Gamma \phi[\boldsymbol{\sigma}]_\Gamma : \nabla\delta\underline{v}_1 \, \mathrm{d}S$$
$$- \int_\Gamma \delta\phi[\boldsymbol{\sigma}]_\Gamma : \nabla\underline{v}_1 \, \mathrm{d}S + \int_\Gamma \phi\delta\phi G_n \, \mathrm{d}S.$$

Using now the compatibility conditions for the variation:

$$\delta\underline{v} = 0 \text{ on } \partial\Omega_u, \quad \delta\underline{v}_2 + \delta\phi\nabla\underline{u}_2.\underline{\nu} = \delta\underline{v}_1 + \delta\phi\nabla\underline{u}_1.\underline{\nu} \text{ on } \Gamma, \tag{4.44}$$

we finally obtain:

$$\delta\mathcal{F} = \int_\Gamma \left(\underline{\nu}.[\dot{\boldsymbol{\sigma}}]_\Gamma - \mathrm{div}_\Gamma(\phi[\boldsymbol{\sigma}]_\Gamma)\right).\delta\underline{v}_1 \, \mathrm{d}S + \int_{\partial\Omega_T} (\underline{n}.\dot{\boldsymbol{\sigma}} - \dot{\underline{T}}^d).\delta\underline{v} \, \mathrm{d}S$$
$$- \int_\Gamma \delta\phi([\boldsymbol{\sigma}]_\Gamma : \nabla\underline{v}_1 - \dot{\boldsymbol{\sigma}}_2.[\nabla\underline{u}]_\Gamma - \phi \, G_n) \, \mathrm{d}S.$$

Hence, we recover the conservation of the momentum and the propagation law.

Stability and bifurcation

The discussion of the stability and bifurcation during an evolution process can be investigated as in Pradeilles-Duval and Stolz [1995].

Consider the velocity $\underline{v}$, a solution of the rate boundary value problem for any given velocity ϕ. The field $\underline{v}$ satisfies:

$$\mathrm{div}\,\dot{\boldsymbol{\sigma}} = 0, \quad \dot{\boldsymbol{\sigma}} = \rho\frac{\partial^2 w}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}(\underline{v}) \text{ in } \Omega,$$

$$\underline{v} = \underline{v}^d \text{ on } \partial\Omega_u, \quad \dot{\boldsymbol{\sigma}}.\underline{n} = \dot{\underline{T}}^d \text{ on } \partial\Omega_T,$$

and non-classical boundary conditions on Γ:

$$\mathcal{D}_\phi([\boldsymbol{\sigma}]_\Gamma.\underline{\nu}) = 0, \quad \mathcal{D}_\phi[\underline{u}]_\Gamma = 0.$$

Consider the value W of $\mathcal{F}$ for this solution, i.e., for $\underline{v}(\phi, \underline{v}^d, \dot{\underline{T}}^d)$:

$$W(\phi, \underline{v}^d, \dot{\underline{T}}^d) = \mathcal{F}(\underline{v}(\phi, \underline{v}^d, \dot{\underline{T}}^d), \phi, \dot{\underline{T}}^d). \tag{4.45}$$

The stability of the actual state is determined by the condition of the existence of a solution

$$\delta\phi\frac{\partial^2 W}{\partial\phi^2}\delta\phi \geq 0, \quad \delta\phi \geq 0 \text{ on } \Gamma^+, \quad \delta\phi \neq 0, \tag{4.46}$$

and the uniqueness and non-bifurcation is characterized by

$$\delta\phi \frac{\partial^2 W}{\partial\phi\partial\phi}\delta\phi \geq 0, \quad \delta\phi \neq 0 \text{ on } \Gamma^+. \tag{4.47}$$

The functional W has a complicated form and can be written as follows:

$$W = \int_\Gamma \int_\Gamma \frac{1}{2}\phi(s).\boldsymbol{B}(s,s')\phi(s') \, \mathrm{d}S \, \mathrm{d}S' - \int_\Gamma \bar{Q}.\phi(s) \, \mathrm{d}S, \tag{4.48}$$

where $\boldsymbol{B}(s,s')$ is an integral operator.

4.6. An example

Consider a composite sphere with kernel and shell composed from linear elastic materials with different moduli, see Fig. 4.2. The sphere is submitted to an isotropic loading, the radial displacement is prescribed on its external boundary ($r = R_e$).

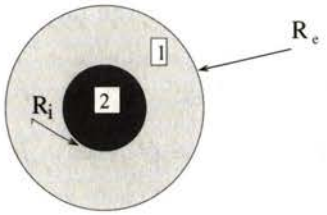

FIGURE 4.2. The composite sphere.

The solution of the elasticity problem is given considering a radial displacement

$$\underline{u} = u_i(r)\underline{e}_r, \quad u_i(r) = A_i r + \frac{B_i}{r^2}. \tag{4.49}$$

The imposed boundary conditions are:

$$u_1(R_e) = E \, R_e, \quad u_2(0) = 0. \tag{4.50}$$

For a given history of E, the traction on the external surface is radial:

$$\boldsymbol{\sigma}_1(R_e).\underline{e}_r = T\underline{e}_r. \tag{4.51}$$

On the interface Γ the energy release rate has the form

$$\mathcal{G}(R_i, E) = \frac{9E^2}{D^2(c)}(\kappa_1 - \kappa_2)(3\kappa_2 + 6\mu_1)(3\kappa_1 + 4\mu_1), \tag{4.52}$$

where

$$D(c) = 3\kappa_2 + 4\mu_1 + 3c(\kappa_1 - \kappa_2), \quad c = \frac{R_i^3}{R_e^3}.$$

The loading parameter E is increasing. Initially, the kernel does'nt evolve, the critical value G_c is not reached. At a certain instant the critical value is reached and the radius of the kernel increases. The actual value of R_i is determined by the implicit equation

$$\mathcal{G}(R_i(t), E(t)) = G_c. \tag{4.53}$$

This is the consistency condition.

If all the sphere is not deformed, the decreasing of the loading ensures that $\mathcal{G}(R_i(t), E(t)) < G_c$, then the composite sphere behaves as elastic heterogeneous medium with new concentration $c = R_i^3/R_e^3$. The global bulk modulus decreases with the deformation.

With the given propagation law for the interface we have, successively,

$$\begin{cases} E < E_c, & \mathcal{G}(R_i, E) < G_c, & R_i(t) = R_i(0), \\ E > E_c, & \mathcal{G}(R_i(t), E(t)) = G_c, \ \Rightarrow & R_i(t) < R, \\ E_T, & \mathcal{G}(R_e, E_T) = G_c, & R_i(T) = R_e, \\ E > E_T, & & R_i(t) = R_e, \end{cases} \tag{4.54}$$

and the response can be plotted as in Fig. 4.3.

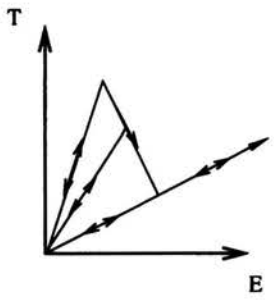

FIGURE 4.3. The response of the composite sphere.

4.7. Dissipation analysis in dynamical case

Now we take into account the inertia effects. Thus the two thermodynamical principles must be rewritten. The mass conservation leads to the continuity of the mass flux $m = \rho\phi$, where ρ denotes the mass density. The first and second laws of thermodynamics lead to local equations in the volume and on the moving surface Γ:

$$\rho\dot{e} = \boldsymbol{\sigma} : \dot{\varepsilon} - \operatorname{div} q, \text{ in } \Omega,$$

$$0 = m[\mathrm{e} + \frac{\underline{v}^2}{2}]_\Gamma - \underline{\nu}.[\boldsymbol{\sigma}.\underline{v}]_\Gamma + \underline{\nu}.[\underline{q}]_\Gamma \text{ on } \Gamma.$$

Then taking into account the conservation of the momentum and the continuity of the displacement:

$$[\underline{u}]_\Gamma = 0, \quad [\boldsymbol{\sigma}]_\Gamma.\underline{\nu} = m[\underline{v}]_\Gamma, \tag{4.55}$$

we obtain the heat power supply defined by the internal energy release rate $\mathcal{G}_{th}$ ($\bar{\boldsymbol{\sigma}} = \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$):

$$-\underline{\nu}.[\underline{q}]_\Gamma = \mathcal{G}_{th}\phi, \quad \mathcal{G}_{th} = \rho[\mathrm{e}]_\Gamma - \underline{\nu}.\bar{\boldsymbol{\sigma}}.[\nabla\underline{u}]_\Gamma.\underline{\nu}.$$

The Hamiltonian of the structure is the sum of the kinetic energy and the total internal energy; the potential energy is defined as above:

$$\mathcal{H} = \int_\Omega \frac{1}{2}\underline{p}^2/\rho \, \mathrm{d}\Omega + \mathcal{E} + \int_\Omega \rho s T \, \mathrm{d}\Omega \tag{4.56}$$

The momentum conservation is then given by the set of equations:

$$\frac{\partial\mathcal{H}}{\partial\underline{p}} \bullet \delta\underline{p} = \int_\Omega \underline{v}.\delta\underline{p} \, \mathrm{d}\Omega, \tag{4.57}$$

$$\frac{\partial\mathcal{H}}{\partial\underline{u}} \circ \delta\underline{u} = -\frac{d}{dt}\int_\Omega \underline{p}.\delta\underline{u} \, \mathrm{d}\Omega, \tag{4.58}$$

where $\underline{p}$ is the momentum. These equations lead to the classical equation of motion. The first law of thermodynamics is rewritten as follows:

$$\frac{d\mathcal{H}}{dt} - \frac{\partial\mathcal{H}}{\partial\underline{\Gamma}^d}.\dot{\underline{\Gamma}}^d = \int_{\partial\Omega} -\underline{q}.\underline{n} \, \mathrm{d}S. \tag{4.59}$$

Now taking into account the momentum conservation, we have

$$\frac{\partial\mathcal{H}}{\partial\Gamma}.\dot{\Gamma} = \int_\Gamma [\underline{q}]_\Gamma.\underline{\nu} \, \mathrm{d}S = -\int_\Gamma \mathcal{G}_{th}\phi \, \mathrm{d}S. \tag{4.60}$$

The second law has the same form as previously. The interface is perfect at each time. Under the assumption of separability of the two dissipations the volume term is reduced to the conduction whilst the surface term is then: $D_\Gamma = \frac{\mathcal{G}_s}{T}\phi$ where $\mathcal{G}_s$ has also the form of the energy release rate:

$$\mathcal{G}_s = \rho[w]_\Gamma - \underline{\nu}.\bar{\boldsymbol{\sigma}}.[\nabla \underline{u}]_\Gamma.\underline{\nu}. \tag{4.61}$$

In case of a thermomechanical coupling, two different release rates must be distinguished; one defined in terms of variation of the Hamiltonian leads to the heat source associated with the moving surface whilst the second one describes the production of entropy.

In the case of isothermal evolution, we can define another Hamiltonian

$$\mathcal{H} = \int_\Omega \frac{1}{2}\underline{p}^2/\rho \, \mathrm{d}\Omega + \mathcal{E}, \tag{4.62}$$

and the total dissipation is then given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} - \frac{\partial \mathcal{H}}{\partial \underline{\tilde{\Gamma}}^d}.\underline{\dot{\tilde{\Gamma}}}^d = \frac{\partial \mathcal{H}}{\partial \Gamma}.\dot{\Gamma} = -\int_\Gamma \mathcal{G}_{dyn}\phi \, \mathrm{d}S, \tag{4.63}$$

where $\mathcal{G}_{dyn} = \rho[w]_\Gamma - \underline{\nu}.\bar{\boldsymbol{\sigma}}.[\nabla \underline{u}]_\Gamma.\underline{\nu}$.

4.8. Connection with fracture

In this section we briefly investigate some connections with fracture.

Consider a crack in mode III in an infinite medium. The displacement has the form

$$\underline{u} = w(x, y)\underline{e}_z,$$

and the stress field is

$$\boldsymbol{\sigma}_{i3} \to \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}}(-\sin(\frac{1}{2}\theta), \cos(\frac{1}{2}\theta)).$$

Consider now that the crack is a layer of depth h with a continuous boundary in which the criterion $\mathcal{G} = G_c$ is reached. The applied loading are the stresses obtained by asymptotic expansion in mode III, at infinity. That is a matching condition between the classical crack and the quasicrack.

The solution exists and we found that Γ is a cycloid (H.D. Bui [1978]).

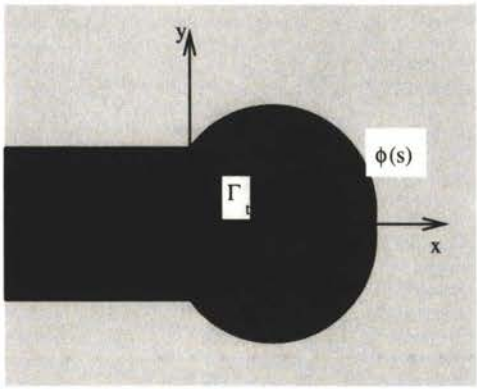

FIGURE 4.4. A quasicrack.

If we compare the dissipation obtained by classical fracture mechanics for a steel and the dissipation obtained by damage modelisation, the depth of the damaged zone can be evaluated as follows:

$$G_c h = K_{c\mathrm{III}}^2 \quad \rightarrow \quad h : \ 50\mu - 100\mu. \tag{4.64}$$

This value is evaluated by assuming that the critical value G_c is the elastic energy at a strain of the order of percent and K_c has the value typical for classical steel.

More general results are given by Bui and Ehrlacher [1980].

Chapter 5

Delamination of laminates

We propose another application of moving surfaces. The previous description depends on only two potentials, where the first is the free energy and the second the potential of dissipation. The change of mechanical characteristics along moving front is used for the study of degradation of laminates. The propagation of delamination front is analysed when the laminate is considered as an assemblage of beams or plates.

5.1. Introduction

To study the delamination of laminates we assume that a sound laminate (domain 0) with known characteristics is transformed into two laminates (1 and 2), separated by the crack of delamination as shown in Fig. 5.1. The laminate i ($i = 0, 2$) is described by an homogeneous plate or by an homogeneous beam.

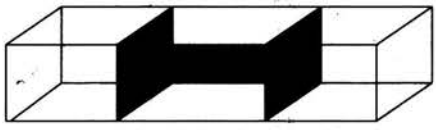

FIGURE 5.1. Delamination of a plate.

One has to choose, at different scales or levels, kinematical properties for describing the system. The kinematic modelling of beams and plates

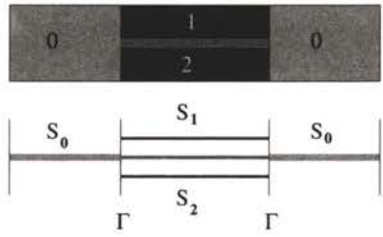

FIGURE 5.2. Modelisation with beams.

has a great influence on the behaviour of the delamination as well as on the modelling of the continuity relations of displacements along the delamination front, thus inducing specific value for the energy release rate.

The choice of the free energy allows us to study different regime of transformation by taking into account particular non-linearities. We can describe delamination within the framework of small strain or finite strain and then we can study the interaction between buckling and delamination.

5.2. The kinematic of the plates

The motion of a material point of the plate is described by the motion of the middle surface and by the rotation of the section.

A point of the middle surface S has curvilinear coordinates (x_1, x_2). The normal to this surface is denoted by $\underline{e}_3$. A point on this normal vector is referred to the coordinate x_3. We consider a kinematical description defined

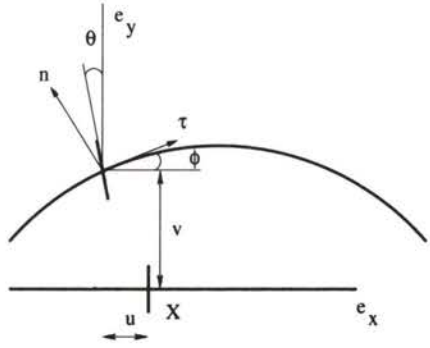

FIGURE 5.3. Kinematics of plates.

by a function of position on the surface and coordinate along the normal at this point.

The displacement of the point (x_1, x_2, x_3) is defined by:

$$\underline{\xi} = \underline{u}(x_1, x_2) + w(x_1, x_2)\underline{e}_3 + \underline{\theta}(x_1, x_2)x_3, \quad (x_1, x_2) \in S, \tag{5.1}$$

where $\underline{u}$ is the plane displacement, w is the normal displacement and θ is the local rotation of the normal vector of the middle surface. With these fields, the strain inside the plate has the following form:

$$\boldsymbol{\varepsilon}(\underline{\xi}) = \boldsymbol{\varepsilon}(\underline{u}) + \frac{1}{2}(\underline{e}_3 \otimes \underline{\gamma} + \underline{\gamma} \otimes \underline{e}_3) - x_3\boldsymbol{\kappa}, \tag{5.2}$$

where the distortion is defined by

$$\underline{\gamma} = \nabla w - \underline{\theta}, \tag{5.3}$$

and the local rotation has the form

$$\boldsymbol{\kappa} = \frac{1}{2}(\nabla\underline{\theta} + \nabla^{\mathrm{T}}\underline{\theta}). \tag{5.4}$$

We observe that the membrane strain $\boldsymbol{\varepsilon}(\underline{u})$ satisfies

$$\boldsymbol{\varepsilon}(\underline{u}) = \frac{1}{2}(\nabla\underline{u} + \nabla^{\mathrm{T}}\underline{u}). \tag{5.5}$$

The free energy

The free energy of the plate is chosen naturally as a function of the generalized strains:

$$W = W(\boldsymbol{\varepsilon}(\underline{u}), \boldsymbol{\kappa}, \underline{\gamma}). \tag{5.6}$$

The generalized stresses associated to the generalized strains are given by the local state equations:

$$\boldsymbol{N} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}, \quad \boldsymbol{M} = \frac{\partial W}{\partial \boldsymbol{\kappa}}, \quad \underline{Q} = \frac{\partial W}{\partial \underline{\gamma}}. \tag{5.7}$$

The assemblage of beams or plates

It is useful to introduce global notations:

- the set of generalized parameters q_i:

$$\underline{q}_i = (\underline{u}_i, w_i, \underline{\theta}_i), \tag{5.8}$$

- the associated generalized strains:

$$\nabla \underline{q}_i = (\nabla \underline{u}_i, \nabla w_i, \nabla \underline{\theta}_i), \tag{5.9}$$

The free energy takes the form:

$$W_i(\boldsymbol{\varepsilon}(\underline{u}_i), \boldsymbol{\kappa}_i, \boldsymbol{\gamma}_i) = F_i(\underline{q}_i, \nabla \underline{q}_i), \tag{5.10}$$

from which the generalized stresses are derived:

$$\boldsymbol{\sigma}_i = \frac{\partial F_i}{\partial \nabla \underline{q}_i}, \quad \underline{T}_i = \frac{\partial F_i}{\partial \underline{q}_i}. \tag{5.11}$$

Kinematical aspects of assemblage

Along the front, the section of the sound plate ($i = 0$) imposes its motion to the two others plates ($i = 1, 2$):

$$\underline{u}_i = \underline{u}_o + h_i \underline{\theta}_o, \tag{5.12}$$

$$w_i = w_o, \tag{5.13}$$

$$\underline{\theta}_i = \underline{\theta}_o. \tag{5.14}$$

These continuity conditions are easily rewritten by using the set of generalized parameters:

$$\underline{q}_i = \boldsymbol{l}_i . \underline{q}_o, \quad \underline{x} \in \Gamma. \tag{5.15}$$

Then the associated Hadamard condition takes the form:

$$\mathcal{D}_\phi \underline{q}_i = \dot{\underline{q}}_i + \phi \nabla \underline{q}_i . \underline{\nu} = \boldsymbol{l}_i . \mathcal{D}_\phi \underline{q}_o. \tag{5.16}$$

Therefore, the derivation of the continuity relations with respect to tangent vector $\underline{\tau}$ yields:

$$\nabla \underline{u}_i . \underline{\tau} = \nabla \underline{u}_o . \underline{\tau} + h_i \nabla \theta_o . \underline{\tau}, \tag{5.17}$$

$$\nabla w_i . \underline{\tau} = \nabla w_o . \underline{\tau} \tag{5.18}$$

$$\nabla \underline{\theta}_i . \underline{\tau} = \nabla \underline{\theta}_o . \underline{\tau}. \tag{5.19}$$

Hence, in terms of generalized relations we have

$$\nabla \underline{q}_i . \tau = \boldsymbol{l}_i . (\nabla \underline{q}_o . \tau), \quad \underline{x} \in \Gamma. \tag{5.20}$$

5.3. Conservation of the momentum

By using the method of the virtual power the equilibrium equations are derived. We assume that the power of internal state of stresses for each plate is defined by

$$\mathcal{P}_i(\tilde{\underline{v}}^*) = -\int_{S_i} \boldsymbol{N}_i : \boldsymbol{\varepsilon}(\underline{u}_i^*) + \boldsymbol{M}_i : \boldsymbol{\kappa}_i(\underline{\theta}_i^*) + \underline{Q}_i.(\nabla w_i^* - \underline{\theta}_i^*)\,\mathrm{d}S. \tag{5.21}$$

The external forces are reduced to given quantities acting on external middle surface:

$$\mathcal{P}_e = \int_{\partial S} (\underline{T}.\underline{u}^* + T_3 w^* + \underline{C}.\underline{\theta}^*)\,\mathrm{d}S. \tag{5.22}$$

The equilibrium state satisfies the set of field equations:

$$0 = \operatorname{div} \boldsymbol{N}^{\mathrm{i}}, \tag{5.23}$$

$$0 = \operatorname{div} \boldsymbol{M}^{\mathrm{i}} + \underline{Q}^{\mathrm{i}}, \tag{5.24}$$

$$0 = \operatorname{div} \underline{Q}^{\mathrm{i}}, \tag{5.25}$$

and the boundary conditions on ∂S

$$\boldsymbol{N}_o.\underline{n} = \underline{T}, \quad \boldsymbol{M}_o.\underline{n} = \underline{C}, \quad \underline{Q}_o.\underline{n} = T_3. \tag{5.26}$$

The virtual velocity field are chosen such as the continuity relations are satisfied. Consequently the continuity conditions on the stress vector on Γ are obtained in the form:

$$0 = (\boldsymbol{N}^1 + \boldsymbol{N}^2 - \boldsymbol{N}_o).\underline{\nu},$$

$$0 = (\boldsymbol{M}^1 + h_1\boldsymbol{N}^1 + \boldsymbol{M}^2 + h_2\boldsymbol{N}^2 - \boldsymbol{M}_o).\underline{\nu},$$

$$0 = (\underline{Q}^1 + \underline{Q}^2 - \underline{Q}_o).\underline{\nu}\,.$$

Characterization of equilibrium

The state of equilibrium in terms of general field $\tilde{\underline{q}}$ satisfies:

- the state equations associated with the free energy:

$$W = F_i(\underline{q}_i, \nabla\,\underline{q}_i), \quad \boldsymbol{\sigma}_i = \frac{\partial F_i}{\partial \nabla\,\underline{q}_i}, \quad \underline{T}_i = \frac{\partial F_i}{\partial\,\underline{q}_i}, \tag{5.27}$$

- the continuity of displacement along the front:

$$\underline{q}_i = \boldsymbol{l}_i.\underline{q}_o, \tag{5.28}$$

- the equilibrium equation:

$$0 = \frac{\partial F_i}{\partial \underline{q}_i} - \operatorname{div} \boldsymbol{\sigma}^i = \underline{T}_i - \operatorname{div} \boldsymbol{\sigma}_i \text{ on } S_i, \tag{5.29}$$

$$0 = \underline{\nu}.(-\sum_{i=1,2} \boldsymbol{\sigma}^i.\boldsymbol{l}_i + \boldsymbol{\sigma}_o) = \underline{\nu}.[|\boldsymbol{\sigma}|]_\Gamma \text{ on } \Gamma, \tag{5.30}$$

$$\underline{\mathrm{T}}^d = \boldsymbol{\sigma}_o.\underline{n} \text{ on } \partial S_T. \tag{5.31}$$

Here the tractions $\underline{\mathrm{T}}^d$ are imposed on ∂S_T and $\underline{q} = \underline{q}^d$ on the complementary part ∂S_q.

We have introduced the useful notation $[|f|]_\Gamma = f_o - \sum_{i=1,2} f_i l_i$.

The solution of the problem of equilibrium is obtained also by searching stationary points of the potential energy $\mathcal{E}$. Consider the potential energy of the system:

$$\mathcal{E}(\tilde{\underline{q}}) = \sum_{i=0,2} \int_{S_i} F_i \, \mathrm{d}S - \int_{\partial S_T} \underline{\mathrm{T}}^d.\underline{q} \, \mathrm{d}s, \tag{5.32}$$

defined for any kinematically admissible fields $\tilde{\underline{q}}$:

$$\tilde{\underline{q}} \in \mathcal{K}.\mathcal{A}. = \{\tilde{\underline{q}}^* \mid \underline{q}^* = \underline{q}^d \text{ on } \partial S_q, \ \underline{q}_i^* = \boldsymbol{l}_i.\underline{q}_o^* \text{ on } \Gamma\}.$$

A kinematically admissible variation $\delta \underline{q}$ satisfies:

$$\delta \underline{q} = 0 \text{ on } \partial S_q, \quad \delta \underline{q}_i = \boldsymbol{l}_i.\delta \underline{q}_o \text{ on } \Gamma. \tag{5.33}$$

Then the variation of the potential energy takes the form:

$$\frac{\partial \mathcal{E}}{\partial \tilde{\underline{q}}} \cdot \delta \tilde{\underline{q}} = \sum_{i=0,2} \int_{S_i} \frac{\partial F_i}{\partial \underline{q}_i}.\delta \underline{q}_i + \frac{\partial F_i}{\partial \nabla \underline{q}_i} : \delta \nabla \underline{q}_i \, \mathrm{d}S - \int_{\partial S_T} \underline{\mathrm{T}}^d.\delta \underline{q} \, \mathrm{d}s. \tag{5.34}$$

Using the global notation and taking account the definitions of $\boldsymbol{\sigma}$ and $\underline{T}$ we write:

$$\frac{\partial \mathcal{E}}{\partial \tilde{\underline{q}}} \cdot \delta \tilde{\underline{q}} = \sum_{i=0,2} \int_{S_i} (\underline{T}_i.\delta \underline{q}_i + \boldsymbol{\sigma}_i : \delta \nabla \underline{q}_i) \, \mathrm{d}S - \int_{\partial S_T} \underline{\mathrm{T}}^d.\delta \underline{q} \, \mathrm{d}s. \tag{5.35}$$

Integrating by parts, the condition of stationarity $\dfrac{\partial \mathcal{E}}{\partial \tilde{\underline{q}}} \cdot \delta \tilde{\underline{q}} = 0$ yields:

$$0 = \sum_{i=0,2} \int_{S_i} (-\operatorname{div} \boldsymbol{\sigma}_i + \underline{T}_i).\delta \underline{q}_i \, \mathrm{d}S + \int_{\partial S_i} \underline{\nu}.\boldsymbol{\sigma}_i.\delta \underline{q}_i \, \mathrm{d}s - \int_{\partial S_T} \underline{\mathrm{T}}^d.\delta \underline{q} \, \mathrm{d}s.$$

Taking into account the admissibility of $\delta \underline{q}$, the equilibrium conditions are recovered.

5.4. Dissipation analysis

The dissipation is given by the balance of the power of external loading and the reversible stored energy:

$$D_m = \mathcal{P}_e - \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=0,3}\int_{S_i} F_i \,\mathrm{d}S = \mathcal{P}_e - \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E} + \int_{\partial S_T} \underline{\mathrm{T}}^d.\underline{q}\,\mathrm{d}s) \geq 0. \quad (5.36)$$

The definition of the energy release rate is recovered

$$D_m = \int_\Gamma \mathcal{G}\phi \,\mathrm{d}S = -\frac{\partial \mathcal{E}}{\partial \Gamma}.\dot{\Gamma} \geq 0, \quad (5.37)$$

where the thermodynamical force $\mathcal{G}$ is the field:

$$\mathcal{G}(s) = -\frac{\partial \mathcal{E}}{\partial \Gamma}(s). \quad (5.38)$$

The free energy release rate is expressed locally as:

$$\mathcal{G} = F_o - \underline{\nu}.\boldsymbol{\sigma}_o.\nabla \underline{q}_o.\underline{\nu} - \sum_{i=1,2}(F_i - \underline{\nu}.\boldsymbol{\sigma}^i.\nabla \underline{q}_i.\underline{\nu}). \quad (5.39)$$

By using the continuity conditions, the following expression is obtained:

$$\mathcal{G} = F_o - F_1 - F_2 - \sum_{i=1,2}\boldsymbol{\sigma}_i : (\boldsymbol{l}_i.\nabla \underline{q}_o - \nabla \underline{q}_i). \quad (5.40)$$

Propagation law

A normality rule is considered. The propagation law is an extension of Griffith's law:

$$\begin{cases} \mathcal{G}(s,t) < G_c, & \phi = 0, \\ \mathcal{G}(s,t) = G_c, & \phi \geq 0. \end{cases} \quad (5.41)$$

The propagation is possible when the critical value is reached. The solution of the normal propagation of delamination is defined on the set:

$$\mathcal{K} = \{\phi^* \mid \phi^*(s) \geq 0, \text{ if } \mathcal{G}(s) = G_c, \quad \phi^* = 0, \text{ otherwise}\}. \quad (5.42)$$

5.5. The rate boundary value problem

The rate boundary value problem is written in terms of rate of displacement $\underline{v}$ and normal propagation ϕ. The solution $(\dot{\underline{q}}, \phi)$ of the rate boundary value problem satisfies:

- the constitutive law:

$$\dot{\boldsymbol{\sigma}}_i = \frac{\partial^2 F_i}{\partial \underline{q} \partial \nabla \underline{q}} . \dot{\underline{q}}_i + \frac{\partial^2 F_i}{\partial \nabla \underline{q} \partial \nabla \underline{q}} . \nabla \dot{\underline{q}}_i \text{ on } S_i, \tag{5.43}$$

- the compatibility relations for the velocity along the front Γ:

$$\dot{\underline{q}}_i + \phi \nabla \underline{q}_i . \underline{\nu} = \boldsymbol{l}_i . (\dot{\underline{q}}_o + \phi \nabla \underline{q}_o . \underline{\nu}), \quad i = 1 \text{ or } i = 2, \tag{5.44}$$

- the compatibility relations for the stress vector on Γ:

$$\mathcal{D}_\phi (\underline{\nu} . (\sum_{i=1,2} \boldsymbol{\sigma}^i . \boldsymbol{l}_i - \boldsymbol{\sigma}_o)) = \mathcal{D}_\phi (\underline{\nu} . [\![\boldsymbol{\sigma}]\!]_\Gamma) = 0, \tag{5.45}$$

- the propagation law on Γ:

$$(\phi - \phi^*)\, \mathcal{D}_\phi \mathcal{G} \geq 0, \quad \phi \in \mathcal{K}, \quad \forall \phi^* \in \mathcal{K}. \tag{5.46}$$

The effective expression of the consistency condition gives a relation between the rate of the displacement and the velocity of propagation.

Derivation of the energy release rate

By using the definition of the convective derivation, we determine successively the contribution of each term of $\mathcal{D}_\phi \mathcal{G}$:

- for the free energy:

$$\begin{aligned} \mathcal{D}_\phi F_i &= \frac{\partial F_i}{\partial \nabla \underline{q}_i} : (\nabla \dot{\underline{q}}_i + \phi \nabla \nabla \underline{q}_i . \underline{\nu}) + \frac{\partial F_i}{\partial \underline{q}_i} . (\dot{q}_i + \phi \nabla \underline{q}_i . \underline{\nu}) \\ &= \boldsymbol{\sigma}^i : (\nabla \dot{\underline{q}}_i + \phi \nabla \nabla \underline{q}_i . \underline{\nu}) + \frac{\partial F_i}{\partial \underline{q}_i} . (\dot{\underline{q}}_i + \phi \nabla \underline{q}_i . \underline{\nu}), \end{aligned} \tag{5.47}$$

- for the stresses:

$$\mathcal{D}_\phi \boldsymbol{\sigma}^i = \dot{\boldsymbol{\sigma}}^i + \phi \nabla \boldsymbol{\sigma}^i . \underline{\nu}, \tag{5.48}$$

- for the normal vector:

$$\mathcal{D}_\phi \underline{\nu} = -\nabla \phi . \underline{e}_\alpha \quad \underline{e}_\alpha . \tag{5.49}$$

The boundary conditions on the stress vector is evaluated as

$$0 = \mathcal{D}_\phi (\underline{\nu} . (\sum_{i=1,2} \boldsymbol{\sigma}^i . \boldsymbol{l}_i - \boldsymbol{\sigma}_o)) = \underline{\nu} . (\sum_{i=1,2} \dot{\boldsymbol{\sigma}}^i . \boldsymbol{l}_i - \dot{\boldsymbol{\sigma}}_o) - \operatorname{div}_\Gamma (\phi (\sum_{i=1,2} \boldsymbol{\sigma}^i . \boldsymbol{l}_i - \boldsymbol{\sigma}_o)) + \phi (\frac{\partial F_i}{\partial \underline{q}_i} \boldsymbol{l}_i - \frac{\partial F_o}{\partial \underline{q}_o}).$$

Finally we get

$$\mathcal{D}_\phi(\underline{\nu}.[|\boldsymbol{\sigma}|]_\Gamma) = \underline{\nu}.[|\dot{\boldsymbol{\sigma}}|]_\Gamma - \mathrm{div}_\Gamma(\phi[|\boldsymbol{\sigma}|]_\Gamma) + \phi[|\underline{T}|]_\Gamma. \tag{5.50}$$

Then, using the equilibrium equations on the surface, the variation of $\mathcal{G}$ is given by:

$$\mathcal{D}_\phi\mathcal{G} = -(\sum_{i=1,2} \boldsymbol{\sigma}^i.\boldsymbol{l}_i - \boldsymbol{\sigma}_o) : \mathcal{D}_\phi(\nabla \underline{q}_o) - (\frac{\partial F_i}{\partial q_i}\boldsymbol{l}_i - \frac{\partial F_o}{\partial q_o}).\mathcal{D}_\phi q_o + \sum_{i=1,2} \mathcal{D}_\phi(\boldsymbol{\sigma}_i).(\nabla \underline{q}_i - \boldsymbol{l}_i.\nabla \underline{q}_o).$$

Hence

$$\mathcal{D}_\phi\mathcal{G} = [|\boldsymbol{\sigma}|]_\Gamma : \mathcal{D}_\phi(\nabla \underline{q}_o) + [|\underline{T}|]_\Gamma : \mathcal{D}_\phi \underline{q}_o + \sum_{i=1,2} \mathcal{D}_\phi(\boldsymbol{\sigma}_i).(\nabla \underline{q}_i - \boldsymbol{l}_i.\nabla \underline{q}_o)$$

and

$$\mathcal{D}_\phi\mathcal{G} = [|\boldsymbol{\sigma}|]_\Gamma : \nabla \dot{\underline{q}}_o + [|\underline{T}|]_\Gamma : \dot{\underline{q}}_o + \sum_{i=1,2} \dot{\boldsymbol{\sigma}}_i.(\nabla \underline{q}_i - \boldsymbol{l}_i.\underline{q}_o) - \phi G_n,$$

$$G_n = -[|\boldsymbol{\sigma}|]_\Gamma : \nabla\nabla \underline{q}_o.\underline{\nu} - [|\underline{T}|]_\Gamma : \nabla \underline{q}_o.\underline{\nu} + \sum_{i=1,2} (\nabla \boldsymbol{\sigma}_i.\underline{\nu}).(\nabla \underline{q}_i - \boldsymbol{l}_i.\underline{q}_o).$$

5.6. The variational formulation

Consider the potential $\mathcal{F}(\dot{\underline{q}}, \phi)$ depending on the rate quantities:

$$\begin{aligned}\mathcal{F}(\dot{\underline{q}}, \phi) = &\sum_{i=0,2} \int_{S_i} \frac{1}{2}(\dot{\underline{q}}_i.\frac{\partial^2 F_i}{\partial \underline{q}_i \partial \underline{q}_i}.\dot{\underline{q}}_i + \dot{\underline{q}}_i.\frac{\partial^2 F_i}{\partial \underline{q}_i \partial \nabla \underline{q}_i}.\nabla \dot{\underline{q}}_i)\, \mathrm{d}S \\ &+ \sum_{i=0,2} \int_{S_i} \frac{1}{2}(\nabla \dot{\underline{q}}_i.\frac{\partial^2 F_i}{\partial \nabla \underline{q}_i \partial \underline{q}_i}.\dot{\underline{q}}_i + \nabla \dot{\underline{q}}_i.\frac{\partial^2 F_i}{\partial \nabla \underline{q}_i \partial \nabla \underline{q}_i}.\nabla \dot{\underline{q}}_i)\, \mathrm{d}S \\ &- \int_\Gamma \Big([|\boldsymbol{\sigma}|]_\Gamma : \nabla \dot{\underline{q}}_o - [|\underline{T}|]_\Gamma : \dot{\underline{q}}_o\Big)\phi\, \mathrm{d}S + \int_\Gamma \frac{1}{2}\phi^2 G_n\, \mathrm{d}S \\ &- \int_{\partial S} \dot{\underline{T}}^d.\dot{\underline{q}}\, \mathrm{d}s.\end{aligned}$$

Then the solution of the rate boundary value problem is characterized by the following property:

The rate boundary value problem. *The solution of the rate boundary value problem* $(\dot{\underline{q}}, \dot{\phi}) \in \mathcal{K}.\mathcal{A}$ *is a solution of the variational inequality*

$$\frac{\partial \mathcal{F}}{\partial \dot{\underline{q}}}.(\dot{\underline{q}} - \dot{\underline{q}}^*) + \frac{\partial \mathcal{F}}{\partial \dot{\phi}}.(\dot{\phi} - \dot{\phi}^*) \geq 0, \tag{5.51}$$

among the set $(\dot{\underline{q}}^*, \dot{\phi}^*) \in \mathcal{K}.\mathcal{A}$.

$$\mathcal{K}.\mathcal{A}. = \Big\{(\dot{\underline{q}}, \dot{\phi}) \;\Big|\; \dot{\underline{q}}_i + \phi \nabla \underline{q}_i.\underline{\nu} = \underline{l}_i.(\dot{\underline{q}}_o + \phi \nabla \underline{q}_o.\underline{\nu}) \text{ on } \Gamma,$$
$$\underline{q}_o = \underline{q}^d \text{ on } \partial S, \quad \phi \in \mathcal{K}\Big\}.$$

Proof. The variations $\delta \dot{\underline{q}}_i, \delta \dot{\underline{q}}_o, \delta \dot{\phi}$ are constrained by the Hadamard relations on Γ:

$$\delta \underline{q}_i + \delta\phi \nabla \underline{q}_i.\underline{\nu} = \underline{l}_i.(\delta \underline{q}_o + \delta\phi \nabla \underline{q}_o.\underline{\nu}). \tag{5.52}$$

The variation of $\mathcal{F}$ is given by

$$\delta \mathcal{F} = \frac{\partial \mathcal{F}}{\partial \dot{\underline{q}}} \cdot \delta \dot{\underline{q}} + \frac{\partial \mathcal{F}}{\partial \dot{\phi}} \cdot \delta \dot{\phi}.$$

Then we obtain

$$\delta \mathcal{F} = \sum_{i=0,2} \int_{S_i} \Big(\dot{\underline{q}}_i . \frac{\partial^2 F_i}{\partial \underline{q}_i \partial \underline{q}_i} + \nabla \dot{\underline{q}}_i . \frac{\partial^2 F_i}{\partial \underline{q}_i \partial \nabla \underline{q}_i} \Big) . \delta \underline{q}_i \, \mathrm{d}S$$
$$+ \sum_{i=0,2} \int_{S_i} \Big(\dot{\underline{q}}_i . \frac{\partial^2 F_i}{\partial \underline{q}_i \partial \nabla \underline{q}_i} + \nabla \dot{\underline{q}}_i . \frac{\partial^2 F_i}{\partial \nabla \underline{q}_i \partial \nabla \underline{q}_i} \Big) . \nabla \delta \underline{q}_i \, \mathrm{d}S$$
$$- \int_\Gamma \phi \Big([\![\boldsymbol{\sigma}]\!]_\Gamma : \nabla \delta \underline{q}_o + \phi [\![\underline{T}]\!]_\Gamma \delta \underline{q}_o \Big) \, \mathrm{d}S - \int_{\partial S_T} \dot{\underline{T}}^d . \delta \underline{q} \, \mathrm{d}S$$
$$- \int_\Gamma \delta\phi \Big([\![\boldsymbol{\sigma}]\!]_\Gamma : \nabla \dot{\underline{q}}_o + [\![\underline{T}]\!]_\Gamma \dot{\underline{q}}_o - \phi G_n \Big) \, \mathrm{d}S.$$

Let us introduce the notation:

$$\dot{\boldsymbol{\sigma}}_i = \frac{\partial^2 F_i}{\partial \underline{q}_i \partial \nabla \underline{q}_i} . \dot{\underline{q}}_i + \frac{\partial^2 F_i}{\partial \nabla \underline{q}_i \partial \nabla \underline{q}_i} . \nabla \dot{\underline{q}}_i,$$

$$\dot{\underline{T}}_i = \frac{\partial^2 F_i}{\partial \underline{q}_i \partial \underline{q}_i} . \dot{\underline{q}}_i + \frac{\partial^2 F_i}{\partial \underline{q}_i \partial \nabla \underline{q}_i} : \nabla \dot{\underline{q}}_i .$$

Then the variation of $\mathcal{F}$ are rewritten as follows:

$$\begin{aligned}\delta\mathcal{F} = &\sum_{i=0,2}\int_{S_i}(\dot{\underline{T}}_i.\delta\underline{q}_i + \dot{\boldsymbol{\sigma}}_i.\nabla\delta\underline{q}_i)\,\mathrm{d}S\\ &-\int_\Gamma \phi\,[|\underline{T}|]_\Gamma\delta\underline{q}_o\,\mathrm{d}S - \int_\Gamma \phi\,[|\boldsymbol{\sigma}|]_\Gamma : \nabla\delta\underline{q}_o\,\mathrm{d}S - \int_{\partial S_T}\dot{\underline{T}}^d.\delta\underline{q}\,\mathrm{d}S\\ &-\int_\Gamma \delta\phi([|\boldsymbol{\sigma}|]_\Gamma : \nabla\dot{\underline{q}}_o + [|\underline{T}|]_\Gamma\,\dot{\underline{q}}_o - \phi G_n)\,\mathrm{d}S.\end{aligned}$$

Using now the Hadamard relations and integration by part, we obtain:

$$\begin{aligned}\delta\mathcal{F} = &\sum_{i=1,2}\int_S(-\operatorname{div}\dot{\boldsymbol{\sigma}}_i + \dot{\underline{T}}_i).\delta\underline{q}_i\,\mathrm{d}S\\ &+\int_\Gamma\Big(\sum_{i=1,2}\underline{\nu}.\dot{\boldsymbol{\sigma}}_i.\delta\underline{q}_i - \underline{\nu}.\dot{\boldsymbol{\sigma}}_o.\delta\underline{q}_o\Big)\,\mathrm{d}S - \int_\Gamma\phi\,[|\underline{T}|]_\Gamma\delta\underline{q}_o\,\mathrm{d}S\\ &-\int_\Gamma\phi\,[|\boldsymbol{\sigma}|]_\Gamma : \nabla\delta\underline{q}_o\,\mathrm{d}S - \int_{\partial S_T}\dot{\underline{T}}^d.\delta\underline{q}\,\mathrm{d}S\\ &-\int_\Gamma\delta\phi\Big([|\boldsymbol{\sigma}|]_\Gamma : \nabla\dot{\underline{q}}_o + [|\underline{T}|]_\Gamma\cdot\dot{\underline{q}}_o - \phi G_n\Big)\,\mathrm{d}S.\end{aligned}$$

Finally, we recover:

- the equations of equilibrium in each beam:

$$-\operatorname{div}\dot{\boldsymbol{\sigma}}_i + \dot{\underline{T}}_i = 0, \tag{5.53}$$

- the boundary conditions on the interface:

$$-\underline{\nu}.[|\dot{\boldsymbol{\sigma}}|]_\Gamma + \operatorname{div}_\Gamma([|\boldsymbol{\sigma}|]_\Gamma\phi) - [|\underline{T}|]_\Gamma\phi = 0, \tag{5.54}$$

- the expression for $\mathcal{D}_\phi\mathcal{G}$:

$$\mathcal{D}_\phi\mathcal{G} = [|\boldsymbol{\sigma}|]_\Gamma : \nabla\dot{\underline{q}}_o + [|\underline{T}|]_\Gamma\,\dot{\underline{q}}_o - \phi G_n + \sum_{i=1,2}\dot{\boldsymbol{\sigma}}_i : (\nabla\underline{q}_i - \boldsymbol{l}_i.\nabla\underline{q}_o). \tag{5.55}$$

This framework can be extended to alternative models of beams or plates and to dynamics using kinetic energy and hamiltonian formalism (Stolz [1995]).

Chapter 6

A thermodynamical approach to contact wear

Wear phenomena due to contact and relative motion between two solids depend on the loading conditions and material mechanical properties. Friction between contacting bodies induces damage of materials, producing surface and subsurface cracks. Particles are detached from sound solids when some local criteria are satisfied at the boundary. Wear leads to geometrical changes and modification of contact conditions. Wear debris induce a specific layer with particular properties. Then the interface between the bodies is a complex medium made of detached particles, eventually a lubricant fluid, and damaged zones. We propose to describe the evolution of the interface using the framework developed before for inducing the general form of a wear-law.

6.1. Introduction

The system consists of two sliding contacting bodies Ω_1, Ω_2 separated by a contact interface Ω_3. We assume the properties of Ω_3 are known, and we attempt to characterize the behaviour and evolution of the interface, taking into account modelisation of wear phenomenon. Such an interface Ω_3 must be considered at a macroscopic level as an homogeneous body obtained by some averaging process through the thickness H of Ω_3. This thickness is so small compared to the size of the contact zone and tribological system that the condition of homogeneity is acceptable.

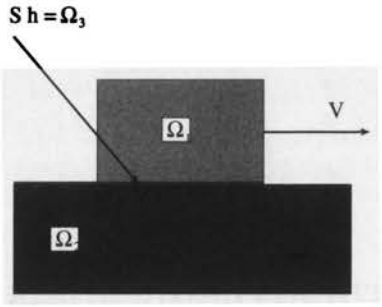

FIGURE 6.1. Macroscopic description of contact.

6.2. The energy approach

The behaviour of each zone is defined through the free energy w_i and dissipation potential. As a result of wear, the boundary Γ_i moves. Along each front Γ_i, the normal $\underline{\nu}_i$ is oriented toward the sound solid. We denote by $\phi_i \underline{\nu}_i$ the normal velocity of the surface Γ_i. Along these boundaries, mechanical quantities with jump $[f]_{\Gamma_i} = f_i^+ - f_i^-$. By expressing the conservation laws in Ω_i and across each Γ_i, we get a set of local equations for the characterisation of the rate quantities:

- mass conservation

$$\begin{cases} \text{on } \Gamma_i, & m_i = \rho_i \phi_i \underline{\nu}_i, \\ \text{in } \Omega_i, & \dot{\rho}_i + \operatorname{div} \rho_i \underline{v} = 0, \end{cases} \tag{6.1}$$

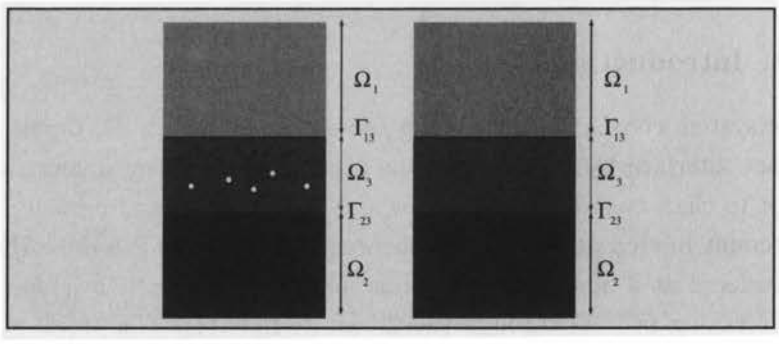

FIGURE 6.2. The moving boundaries and the interface medium.

- momentum conservation

$$\begin{cases} \text{on } \Gamma_i, & \boldsymbol{\sigma}.\underline{\nu}_i = 0, \\ \text{in } \Omega_i, & \operatorname{div}\boldsymbol{\sigma} = 0, \end{cases} \tag{6.2}$$

- energy balance equation

$$\begin{cases} \text{on } \Gamma_i, & m_i[w + sT]_{\Gamma_i} - \underline{\nu}_i\boldsymbol{\sigma}.[\underline{v}]_{\Gamma_i} + \underline{\nu}_i.[\underline{q}]_{\Gamma_i} = 0, \\ \text{in } \Omega_i, & \rho\dot{e} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \operatorname{div}\underline{q}. \end{cases} \tag{6.3}$$

- continuity of the displacement along each Γ_i

$$[\underline{u}]_{\Gamma_i} = 0 \quad \Rightarrow [\underline{v}]_{\Gamma_i} + \phi_i[\nabla\underline{u}]_{\Gamma_i}.\underline{\nu}_i = 0. \tag{6.4}$$

6.3. The dissipation

The internal entropy production is positive and is decomposed in three contributions:

- the volume thermal conduction

$$D_{th} = -\underline{q}.\frac{\nabla T}{T^2}, \tag{6.5}$$

- the volume term due to intrinsic mechanical irreversibility

$$D_m = \frac{1}{T}\Big(\boldsymbol{\sigma} : \operatorname{grad}\underline{v} - \rho(\dot{w} + s\dot{T})\Big), \tag{6.6}$$

- the surface term due to mechanical discontinuities

$$D_\Gamma = \frac{1}{T}(m_i[w]_{\Gamma_i} - \underline{\nu}_i.\boldsymbol{\sigma}.[\underline{v}]_{\Gamma_i}). \tag{6.7}$$

We observe that if the mass flux m_i is zero, the velocity ϕ_i is zero too. Then the velocity jump verifying the Hadamard relation is zero, and no dissipation on Γ_i occurs. Thus the dissipation D_Γ is a characteristic feature of loss of matter and consequently of the phenomenon of wear.

Description of the interface

The interface Ω_3 is described by its middle surface Γ with the equation $S(X,t) = 0$, normal vector $\underline{n}(X,t)$ and thickness $H(X,t) = 2h(X,t)$.

The two boundaries Γ_{13}, Γ_{23} are then defined by

$$x_1 = X + h(X,t)\underline{n}, \quad x_2 = X - h(X,t)\underline{n}. \tag{6.8}$$

Denoting ϕ the velocity of Γ, it is obvious that

$$\mathcal{D}_\phi(x_1) = \phi_1 \underline{\nu}_1, \quad \mathcal{D}_\phi(x_2) = \phi_2 \underline{\nu}_2. \tag{6.9}$$

The continuity of the displacement is then rewritten as

$$[\underline{u}(X \pm h(X,t)\underline{n}, t)]_{\Gamma_i} = 0. \tag{6.10}$$

The dissipation by unit area of Γ is then equal to

$$D_m = D_{\Gamma_{13}} j_1 + D_{\Gamma_{23}} j_2 + \int_H d_m \, j(z) \, dz, \tag{6.11}$$

where $j_1 = j(h)$, $j_2 = j(-h)$, $j(z) = \det(\boldsymbol{I} - z\boldsymbol{b})$ and $\boldsymbol{b}$ is the curvature tensor.

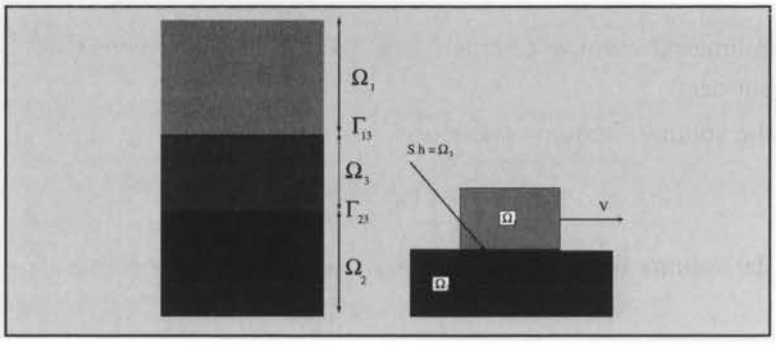

FIGURE 6.3. The mesoscopic description.

Interpretation

The contributions to the dissipation have different nature. The dissipation on the interface Γ_{13}, Γ_{23} are characteristics of the loss of sound material. The last term in (6.11) is the dissipation due to the irreversibility inside the interface. For example, if Ω_3 is a viscous fluid, this term is due to the shear stresses, and consequently we have the possibility to describe some resistance to slip. The term

$$D_3 = \int_h d_m \, j(z) \, dz = \int_H (\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{w}) \, j(z) \, dz, \tag{6.12}$$

describes the friction associated with the relative motion of solids.

For a given mechanical behaviour of the interface, the resolution of the evolution equation must be completed by evolution laws allowing to determine the propagation of the damage material. For example, we may consider the Griffith criterion:

$$\begin{cases} \mathcal{G}(X,t) < G_c, & \phi = 0, \\ \mathcal{G}(X,t) = G_c, & \phi \geq 0. \end{cases} \tag{6.13}$$

Main difficulty consists in choosing an appropriate modelisation to describe the behaviour in the interface and a procedure for its identification.

6.4. An example

Consider now an example of such a behaviour, to emphasize the possibility of the combined description of the friction as well as the wear phenomena. Consider a rigid punch Ω_1 moving on an elastic half-plane. The interface is composed of a viscous fluid with particles in suspension, the global behaviour of this fluid is defined by the bulk modulus κ and viscous modulus η which are functions of the concentration of particles. This particular case has been studied by M. Dragon-Louiset [2002] using integral equations.

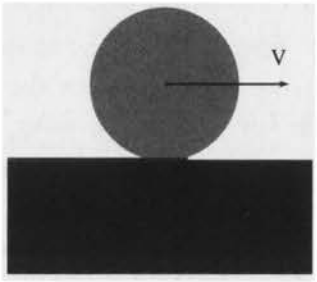

FIGURE 6.4. Moving punch on the elastic half-space.

We assume that the shear is essentially due to the viscosity and the elastic behaviour has only uniaxial effect in the direction $\underline{e}_y$. The motion inside Ω_3 is approximated by a linear profile for the velocity and displacement with respect to the normal coordinate of the middle surface S; then $\dot{\boldsymbol{\varepsilon}}$ is associated with the jump of the velocity.

The constitutive law inside Ω_3 is given by:

$$\sigma_{xy} = m(c)(\dot{u}_x^1 - \dot{u}_x^2), \quad \sigma_{yy} = k(c)(u_y^1 - u_y^2). \tag{6.14}$$

These expressions are compatible with experimental observations. We are now interested in the steady-state solution, then

$$\dot{u}_x^1 - \dot{u}_x^2 = -V(u_{x,x}^1 - u_{x,x}^2). \tag{6.15}$$

The half-plane is linear elastic; the displacement on the interface Γ_{23} is given by solving the Galin's equations:

$$co_1\, u_{x,x}(x) = co_2\, \sigma_{yy}(x) + Vp\frac{1}{\pi}\int_{-a}^{a} \frac{\sigma_{xy}(s)}{s-x}\, \mathrm{d}s, \tag{6.16}$$

$$co_1\, u_{y,x}(x) = -co_2\, \sigma_{xy}(x) + Vp\frac{1}{\pi}\int_{-a}^{a} \frac{\sigma_{yy}(s)}{s-x}\, \mathrm{d}s, \tag{6.17}$$

where the coefficients are:

$$co_1 = \frac{E}{2(1-\nu^2)}, \quad co_2 = \frac{(1-2\nu)}{2(1-\nu)}. \tag{6.18}$$

As usual, E is the Young modulus, ν the Poisson ratio, and Vpf denotes the principal value of f in the sense of Cauchy.

The solution is found by the method of perturbation, using an asymptotic expansion with respect to the concentration c of particles. Finally we get the following results:

- at the 0 order, the Hertz contact solution is recovered,
- at the first order, a dependance on the concentration is obtained. Obviously, this dependance is influenced by the criterion of wear. For sake of simplicity, a linear law is chosen: $\phi = \lambda\sigma_{yy}^2$.

Due to the presence of viscous fluid, the maximum pressure under the punch is not at the center of the contact area, as in the case of contact with friction, (Bui *et al.* [1970]). This example shows the pertinence of the modelisation and possibility to study the problem of thin layer interface for describing the interaction between wear and friction.

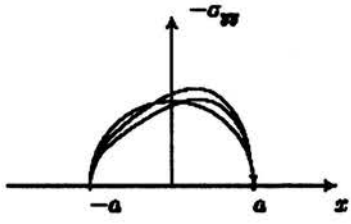

FIGURE 6.5. Pressure in the contact area.

Global approach to the interface

With the same hypotheses, we take the average value of the free energy to define the global free energy of the interface for the unit of contact area:

$$\psi_S(\underline{u}_1, \underline{u}_2, \alpha) = \frac{1}{\rho_S} \int_H \rho(x,z) w_3(\boldsymbol{\varepsilon}(x + z\underline{n}), \alpha) dz, \tag{6.19}$$

where the mass density is defined by:

$$\rho_S = \int_H \rho(x,z) dz. \tag{6.20}$$

In the same way, we can define the potential of dissipation

$$\mathrm{D}_S(\underline{v}_1, \underline{v}_2, \dot{\alpha}) = \frac{1}{\rho_S} \int_H \rho(x,z) \mathrm{d}_3(\dot{\boldsymbol{\varepsilon}}(x + z\underline{n}), \dot{\alpha}) dz. \tag{6.21}$$

As pointed out, the equilibrium on the interface is defined by the equations obtained by variations of the potential energy with respect to the fields $\underline{u}_1, \underline{u}_2$. Then we obtain:

$$\boldsymbol{\sigma}.\underline{n}_i = \rho_S \left(\frac{\partial \psi_S}{\partial \underline{u}_i} + \frac{\partial \mathrm{D}_S}{\partial \underline{v}_i} \right). \tag{6.22}$$

This result suggests a more general study of the behaviour of the interface by asymptotic expansion of the displacement:

$$\underline{u}_3 = \underline{u}_o + z\underline{u}_1 + z^2\underline{u}_2 + \ldots \tag{6.23}$$

with respect to the normal coordinate inside the interface Ω_3. Then specific functions for the constitutive behaviour should be deduced in a similar manner.

Chapter 7

On relationships between micro-macro quantities

This Chapter is concerned with the description of the general relationships between microscopic and macroscopic mechanical quantities in non-linear mechanics. Many studies havedealt with relations between mechanical average quantities as stresses or strains in small or finite transformation Hill [1972], Mandel [1964, 1980], and Rice [1971, 1975].

Our purpose is to reformulate these relations in the framework of a thermodynamical point of view as proposed by Germain et al. *[1983]. Some extensions of classical relations to non-linear mechanics are proposed.*

To determine the overall behaviour of a body, whose local properties are known, we must solve a complicated boundary value problem. The thermodynamical point of view is used to determine the partition between reversibility and irreversibility of the global response.

7.1. Introduction

Consider a small volume element; two scales are distinguished in this volume. The microscopic one, where the properties vary from point to point as in a highly heterogeneous body, and the macroscopic one, where the properties are those of a homogeneous continuum.

In order to determine the overall behaviour with accuracy, it is essential to define the so-called representative volume element (RVE), which must be small enough to allow us to distinguish the microscopic heterogeneities and sufficiently large to be representative for the overall behaviour. The scale of the RVE is chosen with respect to the scale of the heterogeneities and their interactions. A discussion can be found in Drugan and Willis [1996] how to specify the condition of the existence of such a RVE.

The local behaviour is determined by two thermodynamical potentials: the local free energy w to define the equilibrium state and reversibility, and the potential of dissipation d which governs the evolution of the irreversibility.

To characterize the overall behaviour in the thermodynamical sense, it is only necessary to know the corresponding two macroscopic potentials (macroscopic free energy W and macroscopic potential of dissipation D).

The macroscopic free energy W is related to the equilibrium state and the reversible part of the evolution; the potential of dissipation characterizes the irreversibility.

For sake of simplicity, we consider only isothermal processes or in a more general case we assume that the variation of temperature τ in the RVE is uniform. This condition is a necessary condition to determine the global free energy W of the body, because this quantity is defined only for a thermodynamical state of equilibrium. The thermodynamical state of equilibrium is a mechanical equilibrium under uniform temperature.

When the two potentials are determined, the quasistatic evolution of the system can be studied.

We propose to establish the relations between potentials at the microlevel and at the macro one, and to characterize some macroscopic state variables. Denoting the volume of the RVE by Ω, with any microscopic quantity f, we can associate its macroscopic value F by an averaging process on the RVE:

$$F = \frac{1}{V}\int_{\Omega} f \, \mathrm{d}\omega = < f > . \tag{7.1}$$

In this way a unique macrostate quantity is defined for each microstate. The macroscopic free energy at a given state is the total free energy at an equilibrium state, given by the solution of boundary value problem, with particular boundary conditions. To be efficient, these conditions must verify some properties, summarized in the concept of concentration process or localization process (Francfort *et al.* [1983]; Germain *et al.* [1983], Stolz [1995]). This concentration process is defined as a specific boundary value problem.

In particular, the bonding conditions between phases are taken into account. The interface between phases is assumed to be perfect.

We analyse successively the mode of localization in the case of small perturbations and present applications to linear thermoelasticity, plasticity and partially damaged materials as defined in Pradeilles-Duval and Stolz [1995].

7.2. Mode and process of localization

The mode of localization is defined by suitable boundary conditions and properties for the characterization of the bonding between phases.

We denote by $\underline{n}$ the unit normal to the boundary $\partial\Omega$ of Ω and we assume that $\partial\Omega = \partial\Omega_T \cup \partial\Omega_u$ where $\partial\Omega_T$ and $\partial\Omega_u$ are disjoint parts of $\partial\Omega$, on which the stress vector and the displacement vector are prescribed, respectively. The boundary conditions on $\partial\Omega$ must be chosen such that all equations of continuum mechanics are verified in a compatible manner with the averaging process. The local stresses $\boldsymbol{\sigma}$ satisfy:

- the equations of equilibrium

$$\operatorname{div}\boldsymbol{\sigma} = 0 \text{ in } \Omega, \tag{7.2}$$

- the boundary condition

$$\boldsymbol{\sigma}.\underline{n} = \underline{T}^d \text{ on } \partial\Omega_T. \tag{7.3}$$

In the heterogeneous media the interface between phases is perfect, so that the stress vector is continuous along each interface Γ:

$$[\boldsymbol{\sigma}]_\Gamma.\underline{n} = 0 \text{ on } \Gamma. \tag{7.4}$$

All stress fields $\boldsymbol{\sigma}$ satisfying these conditions will be called statically admissible (S.A.) with $\boldsymbol{\Sigma} = <\boldsymbol{\sigma}>$ in the mode of localization. The boundary conditions must be compatible with the averaging process

$$\boldsymbol{\Sigma} = <\boldsymbol{\sigma}> = \frac{1}{V}\int_{\partial\Omega} \{\boldsymbol{\sigma}.\underline{n} \otimes x\}_s \, \mathrm{d}S. \tag{7.5}$$

The local displacement $\underline{u}$ satisfies the boundary conditions $\underline{u} = \underline{U}^d$ on $\partial\Omega_u$. The strain $\boldsymbol{\varepsilon}$ associated with this displacement is defined by

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\underline{u} + \nabla^T\underline{u}), \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{7.6}$$

in Ω, and the macroscopic strain is then deduced to be given by

$$\boldsymbol{E} = <\boldsymbol{\varepsilon}> = \frac{1}{V}\int_{\partial\Omega}(\underline{u}\otimes\underline{n}+\underline{n}\otimes\underline{u})\,\mathrm{d}S. \tag{7.7}$$

The displacement is continuous along all interfaces between phases

$$[\underline{u}]_\Gamma = 0 \text{ on } \Gamma. \tag{7.8}$$

All strain fields $\boldsymbol{\varepsilon}$ satisfying these conditions will be said to be kinematically admissible (K.A.) with $\boldsymbol{E}$ in the mode of localization.

Finally, the boundary conditions $(\underline{T}^d, \underline{U}^d)$ must satisfy the hypothesis of macrohomogeneity in the sense of Hill–Mandel: for any stress field $\boldsymbol{\sigma}^*$ S.A. with $\boldsymbol{\Sigma}^* = <\boldsymbol{\sigma}^*>$ in the mode and any field $\boldsymbol{\varepsilon}'$ K.A. with $\boldsymbol{E}' = <\boldsymbol{\varepsilon}'>$ in the mode, we have:

$$\boldsymbol{\Sigma}^* : \boldsymbol{E}' = <\boldsymbol{\sigma}^* : \boldsymbol{\varepsilon}'> . \tag{7.9}$$

Since the local constitutive law is known, we can study the evolution of the system for a given history of the prescribed boundary conditions. However, the determination of the macroscopic behaviour requires that the process of localization, defined by the mode of localization and local constitutive law, ensures existence and uniqueness of the microscopic fields. In such a case we can deduce the form of the macroscopic constitutive law in the following way. For given macroscopic quantities we solve the boundary value problem associated with the process of localization and then the local fields are determined. Finally, using the averaging process we find the unknown macroscopic quantities.

There exist three particular well-known modes of localization for which the boundary conditions, the average process and the Hill–Mandel conditions are simultaneously verified. The first one is the concentration process under macrohomogeneous stresses $\underline{T}^d = \boldsymbol{\Sigma}.\underline{n}$ on $\partial\Omega$, where $\boldsymbol{\Sigma}$ is a second order symmetric tensor. Then for all $\boldsymbol{\sigma}^*$ S.A. in the mode, $\boldsymbol{\Sigma}^*$ must be equal to $\boldsymbol{\Sigma}$. The displacement $\underline{u}'$ is close to $\underline{U}' = \boldsymbol{E}'.x$ on $\partial\Omega$

$$\int_{\partial\Omega}(\underline{u}' - \underline{U}')\otimes\underline{n}\,\mathrm{d}S = 0. \tag{7.10}$$

This is obtained by taking $\boldsymbol{\sigma}^* = \boldsymbol{\Sigma}$ in the Hill–Mandel condition, and then we have $\boldsymbol{E}' = <\boldsymbol{\varepsilon}(\underline{u}')>$.

Secondly, the concentration process under macrohomogeneous strain $\underline{U}^d = \boldsymbol{E}.x$ on $\partial\Omega$ can be chosen. All kinematical fields $\underline{u}'$ verify automatically the average condition on strains $\boldsymbol{E} = <\boldsymbol{\varepsilon}(\underline{u}')>$, and for any statically

admissible field $\boldsymbol{\sigma}^*$ we obtain from the macrohomogenous condition the average condition on stresses: $\boldsymbol{\Sigma}^* = <\boldsymbol{\sigma}^*>$.

The third mode is the periodic description. The RVE is reduced to the geometry of the elementary cell.

The traction $\underline{T} = \boldsymbol{\sigma}.\underline{n}$ are opposite on opposite sides of $\partial\Omega$, and the displacement is defined as $\underline{u} = \boldsymbol{E}.\underline{x} + \underline{v}$, where $\underline{v}$ is a periodic field, on $\partial\Omega_u$. The average condition on stresses is due the equilibrium equation, the average condition on strains is deduced by the compatibility of the local strain, and the Hill–Mandel macrohomogeneous condition is deduced from the periodicity (Sanchez-Palencia [1980], Suquet [1982, 1987]).

7.3. Potentials and general properties

The local behaviour is defined by the local free energy $w(\boldsymbol{\varepsilon}, \alpha, \tau)$, where $\boldsymbol{\varepsilon}$ is the strain, α represents a set of internal variables and τ is the variation of temperature. The state equations are given by

$$\boldsymbol{\sigma}_R = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \quad \mathrm{A} = -\frac{\partial w}{\partial \alpha}, \quad s = -\frac{\partial w}{\partial \tau}. \tag{7.11}$$

$\boldsymbol{\sigma}_R$ is the reversible stress, A is the thermodynamical force associated with the evolution of α and s is the entropy. The Clausius–Duhem inequality of entropy production is reduced to

$$D = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - (\dot{w} + s\dot{\tau}) \geq 0, \tag{7.12}$$

where the stresses $\boldsymbol{\sigma}$ are in equilibrium inside the body. Then

$$D = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_R) : \dot{\boldsymbol{\varepsilon}} + \mathrm{A}\,\dot{\alpha} \geq 0. \tag{7.13}$$

We have two sources of dissipation, one due to viscosity with the thermodynamical force $\boldsymbol{\sigma}_{ir} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_R$, another one is associated with the evolution of internal variables.

To solve the problem of evolution, we assume that a complementary law is given in the form of a potential of dissipation $d(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha})$, which is a convex function of its arguments; the thermodynamical forces $(\boldsymbol{\sigma}_{ir}, \mathrm{A})$ satisfy the normality rule:

$$(\boldsymbol{\sigma}_{ir}, \mathrm{A}) \in \partial d(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}), \tag{7.14}$$

where the set $\partial d(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha})$ is defined by the property:

$$\forall(\dot{\boldsymbol{\varepsilon}}^*, \dot{\alpha}^*), \quad d(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}) + \boldsymbol{\sigma}_{ir} : (\boldsymbol{\varepsilon}^* - \dot{\boldsymbol{\varepsilon}}) + \mathrm{A}\,(\dot{\alpha}^* - \dot{\alpha}) \leq d(\dot{\boldsymbol{\varepsilon}}^*, \dot{\alpha}^*) \tag{7.15}$$

We assume henceforth that the local behaviour has no viscosity, then $\boldsymbol{\sigma}_{ir}$ vanishes and the reversible stress verifies the conservation of momentum.

The boundary value problem

We prescribe a macroscopic strain $\boldsymbol{E}$ and a uniform variation of temperature τ for a given distribution of internal parameters α. We denote by $\underline{\alpha}$ the field of internal parameters. We must find the local fields $\underline{u}, \boldsymbol{\sigma}$ as functions of $\boldsymbol{E}$ and α by solving the boundary value problem:

- the local stresses are statically admissible

$$\operatorname{div} \boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma}.\underline{n} = \underline{T}^d \text{ on } \partial\Omega_T, \tag{7.16}$$

- the strain $\boldsymbol{\varepsilon}$ is kinematically admissible in the mode:

$$\boldsymbol{E} = < \boldsymbol{\varepsilon}(\underline{u}) >, \quad \underline{u} = \underline{U}^d \text{ on } \partial\Omega_u, \tag{7.17}$$

- the internal interfaces are perfect, the stress vector and displacement are continuous:

$$[\boldsymbol{\sigma}]_\Gamma.\underline{n} = 0, [\underline{u}]_\Gamma = 0, \tag{7.18}$$

- the stress and strain are related by the constitutive law

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\underline{u}), \alpha, \tau). \tag{7.19}$$

The global free energy

The macroscopic free energy W is defined by

$$W(\boldsymbol{E}, \tilde{\alpha}, \tau) = < w(\boldsymbol{\varepsilon}(\underline{u}), \alpha, \tau) >, \tag{7.20}$$

where $\underline{u}$ is the solution of the boundary value problem of localization, α and τ being given at the equilibrium state. Then, from the Hill–Mandel macrohomogeneity condition we deduce the macroequation of state:

$$\frac{\partial W}{\partial \boldsymbol{E}} = < \frac{\partial w}{\partial \boldsymbol{\varepsilon}} : \frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{E}} > = < \boldsymbol{\sigma} : (\boldsymbol{I} + \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{E}}) > = < \boldsymbol{\sigma} > : < (\boldsymbol{I} + \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{E}}) > . \tag{7.21}$$

Noting that $\boldsymbol{\varepsilon}$ is written as $\boldsymbol{E} + \boldsymbol{\eta}$, with $\boldsymbol{\eta}$ being a kinematically admissible strain such that $< \boldsymbol{\eta} >$, then we have $\dfrac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{E}} = 0$ and hence the macroscopic stress $\boldsymbol{\sigma}$ is related to the macroscopic strain by the state equation

$$\frac{\partial W}{\partial \boldsymbol{E}} = < \boldsymbol{\sigma} > = \boldsymbol{\Sigma}. \tag{7.22}$$

The macrostress at equilibrium is defined in the same way as the microstress, owing to the definition of the macroscopic thermodynamical potential W. For a perturbation of temperature $\delta\tau$, the variation of energy is

$$-\frac{\partial W}{\partial \tau}\,\delta\tau = -<\frac{\partial w}{\partial \tau}\,\delta\tau> = <s>\,\delta\tau = S\,\delta\tau. \tag{7.23}$$

Then the global entropy S, the average of the local one, is related to the variation of the global free energy similarly to the micro level.

The other state equations are expressed as follows:

$$\underline{\mathrm{A}} \bullet \delta\underline{\alpha} = -\int_{\Omega} \frac{\partial w}{\partial \alpha}\,\delta\alpha\,\mathrm{d}\Omega = -\frac{\partial W}{\partial \underline{\alpha}} \bullet \delta\underline{\alpha}. \tag{7.24}$$

The internal state in a global description for the system is defined by the value of $\alpha(x)$ at each point of Ω, so the internal state is defined by a field of internal variables. This interpretation is emphasized by considering the potential of dissipation.

The global dissipation function

If the evolution of the internal parameters is given by the potential of dissipation $\mathrm{d}(\dot{\alpha})$, convex function of $\dot{\alpha}$, the thermodynamical forces A are defined by the normality rule $\mathrm{A} \in \partial \mathrm{d}(\dot{\alpha})$. We define the global dissipation function as the function $\mathrm{D}(\tilde{\dot{\alpha}}) = <\mathrm{d}(\dot{\alpha})>$ of the field of internal parameters $\tilde{\dot{\alpha}}$. The expression of the normality rule is transposed in terms of fields by integration in Ω:

$$\forall \tilde{\dot{\alpha}}^*, \quad \mathrm{D}(\tilde{\dot{\alpha}}) + <\mathrm{A}(\dot{\alpha}^* - \dot{\alpha})> \leq \mathrm{D}(\tilde{\dot{\alpha}}^*). \tag{7.25}$$

It is obvious that D is a functional of $\dot{\alpha}$, and A is a linear form $<\mathrm{A}\alpha^*>$ on fields α^* defined in Ω. Then the normality rule is written in terms of fields

$$\underline{\mathrm{A}} \in \partial \mathrm{D}(\tilde{\dot{\alpha}}). \tag{7.26}$$

>From a general point of view, the governing equations for the macrostate have the same form as the governing equations for the microstate except that the set of internal variables is defined by a set of fields of internal variables.

7.4. Macrohomogeneous body and linear elasticity

For linear elasticity, the macroscopic elastic modulus has not the same value when macrohomogeneous strain or stress conditions are prescribed on

the boundary $\partial\Omega$. But when the body is macrohomogeneous in the sense of Hill [1966, 1967] and Mandel [1964] the difference between the two moduli vanishes. More details could be found in the paper of Drugan and Willis [1996] or in the book of Nemat Nasser and Horii [1993] about the relations between the definition of the RVE and the macrohomogeneity condition.

Assuming that all constituent phases are linear elastic, the local free energy density is defined by $w(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{c}(x) : \boldsymbol{\varepsilon}$, where c depends on the point x of Ω. The displacement $\underline{u}$, the solution of the boundary value problem minimizes the potential energy of the system. When one prescribes homogeneous strain condition ($\underline{u} = \boldsymbol{E}.y$ for $y \in \partial\Omega$), the potential energy is reduced to W. The displacement $\underline{u}$ depends only on the given $\boldsymbol{E}$ and spatial distribution of the mechanical phases.

The local stress $\boldsymbol{\sigma}$ is obtained as the solution of a problem of heterogeneous elasticity, and this proves the existence of concentration tensors $\boldsymbol{A}$ for stresses and $\boldsymbol{B}$ for strains and Green functions $\mathcal{L}, \mathcal{M}$ for the displacement $\underline{u}$, such that

$$\boldsymbol{\sigma} = \boldsymbol{A} : \boldsymbol{\Sigma}, \quad \boldsymbol{\varepsilon} = \boldsymbol{B} : \boldsymbol{E},$$

$$\underline{u} = \mathcal{L} : \boldsymbol{E}, \quad \boldsymbol{B} = \frac{1}{2}(\nabla\mathcal{L} + \nabla^T\mathcal{L}),$$

$$\underline{u} = \mathcal{M} : \boldsymbol{\Sigma}, \quad \mathcal{M} : \boldsymbol{C} = \mathcal{L}.$$

Properties of the concentration tensors

For fixed subscripts (p, q), $\boldsymbol{A}_{ijpq}$ satisfies the equilibrium equations and homogeneous boundary conditions

$$\boldsymbol{A}_{ijpq,j} = 0 \text{ in } \Omega,$$

$$\boldsymbol{A}_{ijpq}n_j = \frac{1}{2}(n_p\delta_{iq} + n_q\delta_{ip}) \text{ on } \partial\Omega.$$

The strain $\boldsymbol{\varepsilon}_E = \boldsymbol{s} : \boldsymbol{A} : \boldsymbol{\Sigma} = \boldsymbol{B} : \boldsymbol{E}$ satisfies the condition of compatibility ($\boldsymbol{s} = \boldsymbol{c}^{-1}$) and the relations between micro and macro scales can be defined

$$\boldsymbol{\Sigma} = \boldsymbol{C} : \boldsymbol{E}, \quad \boldsymbol{C} = < \boldsymbol{B}^T : \boldsymbol{c} : \boldsymbol{B} >,$$

$$\boldsymbol{S} = < \boldsymbol{A}^T : \boldsymbol{s} : \boldsymbol{A} > = \boldsymbol{C}^{-1}.$$

We have used the notation $(\boldsymbol{A}^T)_{ijpq} = \boldsymbol{A}_{pqij}$. Moreover, we have the set of relations:

$$\boldsymbol{s} : \boldsymbol{A} = \boldsymbol{B} : \boldsymbol{S}, \quad \boldsymbol{A} : \boldsymbol{C} = \boldsymbol{c} : \boldsymbol{B}, \quad < \boldsymbol{A} > = I, \quad < \boldsymbol{B} > = \boldsymbol{I}.$$

For a more complex behaviour, we can solve the problem of localization with fixed (α, τ); the solution associated with a variation of the macroscopic strain $d\boldsymbol{E}$ represents then an elastic response. The solution of this problem of heterogeneous elasticity is written as follows

$$d\boldsymbol{\varepsilon} = \boldsymbol{B} : d\boldsymbol{E}, \qquad d\boldsymbol{\sigma} = \boldsymbol{A} : d\boldsymbol{\Sigma} = \boldsymbol{c} : d\boldsymbol{\varepsilon}.$$

Hence the value $\boldsymbol{c}(x)$ is the local instantaneous modulus of elasticity $\dfrac{\partial^2 w}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}}$. Then the concentration tensors are associated with these reversible tangent moduli for which we can define a macroscopic tangent modulus satisfying the general relation

$$\boldsymbol{C} = < \boldsymbol{B}^T : \boldsymbol{c} : \boldsymbol{B} > . \tag{7.27}$$

7.5. On the decomposition of the macroscopic strain

Let $\boldsymbol{\Sigma}$ be the real macrostress and $\boldsymbol{\sigma}$ the corresponding microscopic one. The local solution during purely elastic behaviour is as previously: $\boldsymbol{\sigma}_E = \boldsymbol{A} : \boldsymbol{\Sigma}$. The stress field $\boldsymbol{r} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_E$ is then self-equilibrated.

In small strain, the total deformation $\boldsymbol{\varepsilon}$ is the sum of the elastic strain $\boldsymbol{\varepsilon}_e$ and some initial strain $\boldsymbol{\varepsilon}_i$. The elastic strain is related to $\boldsymbol{\sigma}$ by the constitutive law $(\boldsymbol{\varepsilon}_e = \boldsymbol{s} : \boldsymbol{\sigma})$. The initial strain $\boldsymbol{\varepsilon}_i$ induces an internal stress field $\boldsymbol{r}$ such that the local strain $\boldsymbol{\varepsilon}_{res}$ satisfies the compatibility conditions and the constitutive behaviour

$$\boldsymbol{\varepsilon}_{res} = \boldsymbol{s} : \boldsymbol{r} + \boldsymbol{\varepsilon}_i. \tag{7.28}$$

The macroscopic elastic strain $\boldsymbol{E}_E$ is the strain recovered by a purely elastic unloading, which corresponds locally to the interpretation of $\boldsymbol{\sigma}_E$. The local strains $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}_E = \boldsymbol{s} : \boldsymbol{\sigma}_E$ are kinematically admissible respectively with $\boldsymbol{E}$ and $\boldsymbol{E}_E$ in the mode of localization. From the Hill–Mandel condition applied with $\boldsymbol{A} :< \boldsymbol{\sigma} >$, which is statically admissible with $< \boldsymbol{\sigma} >$ in the mode of localization, we obtain:

$$\boldsymbol{E}_E = < \boldsymbol{A}^T : \boldsymbol{\varepsilon}_E >, \quad \boldsymbol{E} = < \boldsymbol{A}^T : \boldsymbol{\varepsilon} > . \tag{7.29}$$

Then the definition of the macroscopic modulus is recovered as $\boldsymbol{S} = < \boldsymbol{A}^T : \boldsymbol{s} : \boldsymbol{A} >$. The difference $\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_E$ is a kinematically admissible field associated with the anelastic part $\boldsymbol{E}_{res}$ of the macroscopic strain $(\boldsymbol{E}_{res} = \boldsymbol{E} - \boldsymbol{E}_E)$, and we obtain

$$\boldsymbol{E}_{res} = < \boldsymbol{\varepsilon}_{res} > = < \boldsymbol{A}^T : \boldsymbol{\varepsilon}_{res} > = < \boldsymbol{A}^T : \boldsymbol{\varepsilon}_i > . \tag{7.30}$$

Since $\boldsymbol{r}$ is a self-equilibrated stress field and $\boldsymbol{s} : \boldsymbol{A}$ is a kinematically admissible field, then $< \boldsymbol{r} : \boldsymbol{s} : \boldsymbol{A} >= 0$; this property is used to establish the second equality. The thermodynamical interpretation of $\boldsymbol{E}_{res}$ must be investigated, it depends on the local meaning of the strain $\boldsymbol{\varepsilon}_i$ and its evolution.

7.6. Moving interfaces

In is well-known that the propagation of damage has been studied in connection with fracture mechanics. Different approaches based on macroscopic and microscopic descriptions of mechanical degradation properties have been proposed.

Because of loading, damage in continuum mechanics can be induced by initiation and growth of micro-cracks and micro cavities. According to the empirical description based on different concept of effective stresses in the sense of Kachanov, many works describe the degradation of mechanical properties with macroscopic parameters, in a formalism similar to that used in plasticity.

Many papers have also dealt with the relation between microscopic and macrosocpic behaviour. Such descriptions which are based on the evolution of microscopic properties, take into account the growth of cavities or pores exploiting the idea that when a critical value in strain, energy or stress is reached, the material does not suffer further tensile loading. Variational formulations were used to describe the evolution of the interface between the sound and damaged materials.

This section proposes the description of the general relationships between microscopic and macroscopic mechanical quantities in heterogeneous media with moving interfaces, similar to that presented previously.

General features

At each time the domain V is composed of two distinct volumes Ω_1 and Ω_2 which are occupied by two materials with different mechanical characteristics. The interface between the two phases is perfect and denoted by Γ. The phase 1 changes into the phase 2 in an irreversible manner due to the mechanical loading along a moving surface Γ, defined by an equation of the form $S(x,t) = 0$. The extension of the phase 2 is related to this moving surface, the equation of the surface is obtained in an explicit manner depending on the history of the loading. In order to study the general formulation of the

relationships between microscopic mechanical fields and macroscopic quantities we do not discuss the characteristics of the evolution of the interface, and at each point of the interface we assume that the normal velocity ϕ is determined.

When the interface moves, the evolution of any macroscopic quantity F is given by:

$$\dot{F} = < \dot{f} > - \frac{1}{\Omega} \int_\Gamma [f]_\Gamma \phi \, dS, \tag{7.31}$$

where $[f]_\Gamma = f_1 - f_2$ is the jump of the quantity f at a point of Γ, $\underline{\nu}$ is the normal vector to Γ outward to phase 2. As the interface moves, the transport condition for any mechanical quantity f at a geometrical point of Γ is given by the convected derivative $D_\phi f$

$$D_\phi f = \lim_{\Delta t \to 0} \frac{f(\underline{x} + \phi \underline{\nu} \Delta t, t + \Delta t) - f(\underline{x}, t)}{\Delta t}. \tag{7.32}$$

The displacement and the stress vector are continuous on Γ, then their rates verify the compatibility equations of Hadamard:

$$[D_\phi(\boldsymbol{\sigma}.\underline{\nu})]_\Gamma = 0, \quad D_\phi[\underline{u}]_\Gamma = [v]_\Gamma + \phi[\nabla \underline{u}]_\Gamma.\underline{\nu} = 0. \tag{7.33}$$

So, we must take into account the possibility of discontinuities. The displacement is continuous on Γ, $[\underline{u}]_\Gamma = 0$, hence the gradient on Γ of the displacement $\nabla_\Gamma \underline{u}$ is continuous. The discontinuity of the gradient of the displacement has the form

$$[\nabla \underline{u}]_\Gamma = \lambda \otimes \underline{\nu}. \tag{7.34}$$

The stress vector is continuous: $[\boldsymbol{\sigma}]_\Gamma.\underline{\nu} = 0$. Combining all the properties of continuity, the discontinuities of $\boldsymbol{\sigma}$ and $\nabla \underline{u}$ are interrelated by orthogonality, as pointed out by Hill [1986]:

$$[\boldsymbol{\sigma}]_\Gamma : [\nabla \underline{u}]_\Gamma = 0. \tag{7.35}$$

7.7. Case of linear elasticity

In this section, the two phases are linear elastic media. At time t, the distribution of the phases is known and the localization process is defined by the equilibrium state of a heterogeneous elastic medium. The displacement u, at equilibrium, verifies the equations of the boundary value problem associated

with the mode of localization. At each time the tensors of concentration are defined and the macroscopic behaviour is obtained as previously by

$$\boldsymbol{E} = < \boldsymbol{A}^T : \boldsymbol{s} : \boldsymbol{A} >: \boldsymbol{\Sigma} = \boldsymbol{S} : \boldsymbol{\Sigma}. \tag{7.36}$$

Between time t and t+dt, the concentration and the shape of the phases have changed, then the concentration tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ evolve and their evolution is associated with the normal velocity ϕ of propagation of the interface. The variation of the geometry of the phases induces a variation of the elastic moduli. For a macroscopic evolution of the loading, the phase l is transformed into the phase 2 along some parts of Γ. The rate of $\boldsymbol{A}$, denoted by $\dot{\boldsymbol{A}}$, is linked to the normal velocity of propagation, the same is true for $\dot{\mathcal{M}}$. So the local response is

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{A} : \dot{\boldsymbol{\Sigma}} + \dot{\boldsymbol{A}} : \boldsymbol{\Sigma}, \quad v = \mathcal{M} : \dot{\boldsymbol{\Sigma}} + \dot{\mathcal{M}} : \boldsymbol{\Sigma}. \tag{7.37}$$

The rates of the concentration tensors satisfy Hadamard's relations on Γ:

$$[\, D_\phi(\boldsymbol{A}.\underline{\nu})\,]_\Gamma = 0, \quad [\, D_\phi \mathcal{M}\,]_\Gamma = 0. \tag{7.38}$$

The global evolution of the macroscopic quantities are then deduced, using the hypothesis of macrohomogeneity:

$$\dot{\boldsymbol{\Sigma}} = < \dot{\boldsymbol{\sigma}} > - \frac{1}{\Omega} \int_\Gamma [\boldsymbol{\sigma}]_\Gamma] \, \phi \, \mathrm{d}S, \tag{7.39}$$

$$\dot{\boldsymbol{E}} = < \boldsymbol{A}^T : \dot{\boldsymbol{\varepsilon}} > - \frac{1}{\Omega} \int_\Gamma \boldsymbol{A}^T : [\nabla u]_\Gamma \, \phi \, \mathrm{d}S. \tag{7.40}$$

In a similar way, the variation of the elastic moduli is given by

$$\dot{\boldsymbol{S}} = \frac{1}{\Omega} \int_\Gamma 2\mathrm{g}\phi \, \mathrm{d}S, \quad \mathrm{g} = \frac{1}{2}[\, \boldsymbol{A}^T : \boldsymbol{s} : \boldsymbol{A}]_\Gamma - \boldsymbol{A}^T : [\nabla \mathcal{M}]_\Gamma \tag{7.41}$$

where g is the density of the energy release rate along the interface.

Total dissipation

The total energy is given by $W = \frac{1}{2}\boldsymbol{\Sigma} : \boldsymbol{S} : \boldsymbol{\Sigma}$ whilst the macroscopic dissipation has the form

$$D_m = \frac{1}{2}\boldsymbol{\Sigma} : \dot{\boldsymbol{S}} : \boldsymbol{\Sigma} = \frac{1}{\Omega} \int_\Gamma G \, \phi \, \mathrm{d}S \geq 0,$$

$$G = \boldsymbol{\Sigma} : \mathrm{g} : \boldsymbol{\Sigma},$$

where the quantity G is the energy release rate defined on Γ. So even if the local behaviour is reversible, the propagation of a surface of discontinuity inside the body generates dissipation. The macroscopic behaviour is that of an elastic medium with variable elastic moduli.

7.8. More general cases

More generally, when both materials are elastoplastic or with initial strains, because of the existence of incompatible strains, a self-equilibrated stress field $\boldsymbol{r}$ appears, and the local stress can be decomposed as follows:

$$\boldsymbol{\sigma} = \boldsymbol{A} : \boldsymbol{\Sigma} + \boldsymbol{r} = \boldsymbol{\sigma}_E + \boldsymbol{r}. \tag{7.42}$$

The field $\boldsymbol{r}$ being self-equilibrated the following relations are obtained

$$< \boldsymbol{r} > = 0, \qquad [\boldsymbol{r}]_\Gamma . \underline{\nu} = 0,$$

$$\operatorname{div} \boldsymbol{r} = 0 \text{ in } \Omega, \qquad \boldsymbol{r} . \underline{n} = 0 \text{ on } \partial\Omega.$$

The local strain $\boldsymbol{\varepsilon}$ is related with $\boldsymbol{E}$ via the K.A. displacement $\underline{u}$. Let us denote by $\boldsymbol{\varepsilon}_e$ the elastic strain

$$\boldsymbol{\varepsilon}_e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p = \boldsymbol{s} : \boldsymbol{\sigma}. \tag{7.43}$$

Next we introduce two other displacement fields:

- the first one, $\underline{u}_E = \mathcal{M} : \boldsymbol{\Sigma}$ is K.A with $\boldsymbol{E}_E = \boldsymbol{S} : \boldsymbol{\Sigma}$ and defines the strain

$$\boldsymbol{\varepsilon}_E = \boldsymbol{\varepsilon}(\underline{u}_E) = \boldsymbol{s} : \boldsymbol{\sigma}_E, \tag{7.44}$$

- the second one, $\underline{u}_{ir}$ is K.A. with $\boldsymbol{E}_{ir} = < \boldsymbol{\varepsilon}_{ir} >$ and defines the strain

$$\boldsymbol{\varepsilon}_{ir} = \boldsymbol{\varepsilon}(\underline{u}_{ir}) = \boldsymbol{\varepsilon}_p + \boldsymbol{s} : \boldsymbol{r}. \tag{7.45}$$

On account of these definitions one obtains:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_p = \boldsymbol{s} : \boldsymbol{A} : \boldsymbol{\Sigma} + \boldsymbol{s} : \boldsymbol{r} + \boldsymbol{\varepsilon}_p = \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_{ir},$$

$$\underline{u} = \underline{u}_E + \underline{u}_{ir} = \mathcal{M} : \boldsymbol{\Sigma} + \underline{u}_{ir},$$

$$\boldsymbol{E} = \boldsymbol{E}_E + \boldsymbol{E}_{ir}.$$

For a given macroscopic evolution $\dot{\boldsymbol{\Sigma}}$, the plastic strain rate can evolve and the propagation of Γ can occur. The evolution of the state obeys to the decomposition:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_p + \boldsymbol{s} : \dot{\boldsymbol{\sigma}}, \quad \dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}_E + \dot{\boldsymbol{r}}. \tag{7.46}$$

In these relations $\dot{\boldsymbol{\sigma}}_E$ corresponds to the microscopic variation for a purely elastic behaviour characterized by the same propagation of the interface Γ. Hence we conclude that

$$\dot{\boldsymbol{\sigma}}_E = \boldsymbol{A} : \dot{\boldsymbol{\Sigma}} + \dot{\boldsymbol{A}} : \boldsymbol{\Sigma}, \tag{7.47}$$

where the localization tensor $\boldsymbol{A}$ satisfies the Hadamard's compatibility equations. The rate of each displacement is discontinuous according to the continuity compatibility equations:

$$D_\phi[\underline{u}]_\Gamma = 0, \quad D_\phi[\underline{u}_E]_\Gamma = 0, \quad D_\phi[\underline{u}_{ir}]_\Gamma = 0. \tag{7.48}$$

The application of the Hill–Mandel hypothesis to these displacement fields and to related strain fields gives a set of the following relations

$$\dot{\boldsymbol{E}} = < \boldsymbol{A}^T : \dot{\boldsymbol{\varepsilon}} > - \frac{1}{\Omega} \int_\Gamma \boldsymbol{A}^T : [\nabla \underline{u}]_\Gamma \, \phi \, \mathrm{d}S,$$

$$\dot{\boldsymbol{E}}_E = < \boldsymbol{A}^T : \dot{\boldsymbol{\varepsilon}}_E > - \frac{1}{\Omega} \int_\Gamma \boldsymbol{A}^T : [\nabla \underline{u}_E]_\Gamma \, \phi \, \mathrm{d}S = \boldsymbol{S} : \dot{\boldsymbol{\Sigma}} + \dot{\boldsymbol{S}} : \boldsymbol{\Sigma}.$$

By substraction, it allows us to define the variation of the irreversible strain:

$$\dot{\boldsymbol{E}}_{ir} = < \boldsymbol{A}^T : \dot{\boldsymbol{\varepsilon}}_{ir} > - \frac{1}{\Omega} \int_\Gamma \boldsymbol{A}^T : [\nabla \underline{u}_{ir}]_\Gamma \, \phi \, \mathrm{d}S, \tag{7.49}$$

or

$$\dot{\boldsymbol{E}}_{ir} = < \boldsymbol{A}^T : \dot{\boldsymbol{\varepsilon}}_p > + < \boldsymbol{A}^T : \boldsymbol{s} : \dot{\boldsymbol{r}} > - \frac{1}{\Omega} \int_\Gamma \boldsymbol{A}^T : [\nabla \underline{u}_{ir}]_\Gamma \, \phi \, \mathrm{d}S. \tag{7.50}$$

Since the residual stress $\boldsymbol{r}$ is a self-equilibrated field, from $< \boldsymbol{r} > = 0$, $[\boldsymbol{r}]_\Gamma \cdot \underline{\nu} = 0$, we obtain

$$0 = < \dot{\boldsymbol{r}} > - \frac{1}{\Omega} \int_\Gamma [\boldsymbol{r}]_\Gamma \, \phi \, \mathrm{d}S,$$

$$0 = [D_\phi(\boldsymbol{r} . \underline{\nu})]_\Gamma.$$

Hence, the rate of internal stresses $\dot{\boldsymbol{r}}$ is not self-equilibrated and the macroscopic irreversible strain takes the form:

$$\dot{\boldsymbol{E}}_{ir} = < \boldsymbol{A}^T : \dot{\boldsymbol{\varepsilon}}_p > - \frac{1}{\Omega} \int_\Gamma \boldsymbol{A}^T : [\nabla \underline{u}_{ir}]_\Gamma \, \phi \, \mathrm{d}S + \frac{1}{\Omega} \int_\Gamma [\boldsymbol{r}]_\Gamma : \nabla \mathcal{M} \, \phi \, \mathrm{d}S.$$

The irreversible part of the macroscopic strain is decomposed into two parts: one is due to the volume irreversibility, the other to the variation of the residual stress field, essentialy dependent on the geometry of the phases. Even if the internal strain has no evolution, there exists an irreversible macroscopic strain due to the variation of internal stresses essentially dependent on the evolution of the geometry of phases.

Dissipation

In the case of plastic behaviour, the free energy of the system takes the form $W(\boldsymbol{\Sigma}, \boldsymbol{\varepsilon}_p, \alpha) = \frac{1}{2} < \boldsymbol{\sigma} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) > + h(\alpha)$ where α is any internal variable.

The energy associated with the residual stresses $\boldsymbol{r}$ then is: $W_b = \frac{1}{2} < \boldsymbol{r} : \boldsymbol{s} : \boldsymbol{r} >$. Thus the dissipation rate is equal to:

$$D_m = \boldsymbol{\Sigma} : \dot{\boldsymbol{E}}_{ir} - \dot{W}_b + \frac{1}{2} \boldsymbol{\Sigma} : \dot{\boldsymbol{S}} : \boldsymbol{\Sigma} - \frac{\partial h}{\partial \alpha} \dot{\alpha} \geq 0. \tag{7.51}$$

The expression of D_m in terms of local quantities is given by

$$D_m = < \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p > - \frac{\partial h}{\partial \alpha} \dot{\alpha} + \frac{1}{\Omega} \int_\Gamma ([w]_\Gamma - \boldsymbol{\sigma} : [\nabla \underline{u}]_\Gamma) \phi \, \mathrm{d}S. \tag{7.52}$$

In this form two parts are distinguished; the first one is related to the plastic effects, the second one is related to the moving surface. The part of the dissipation due to plasticity and hardening in not directly related to the irreversible strain. These equations shows that the main difficulty in a macroscopic approach is to determine the relative part due to plasticity and to local rupture in macroscopic tests.

7.9. Typical examples

A composite spheres assemblage of Hashin revisited

The composite spheres assemblage of Hashin is analysed in Pradeilles-Duval and Stolz [1995]. In this paper the rate boundary value problem is

discussed, provided that a criterion of propagation of the interface is given in terms of an energy release rate.

The system is composed of a compact assemblage of spheres with external radii in order to fill the whole domain. The microscopic structure consists of composite spheres with the core made of material 2 and the shell of material 1; both materials are homogeneous and linear elastic. As in the general case, the material 1 transforms into material 2; the transformation is irreversible and the criterion is a generalized Griffith's one based on the energy release rate of the transformation. The volume fraction of the material 2 is denoted by c. Applying the same method as in Hervé and Zaoui [1991], the assemblage is assumed to be well-disordered. Using the particular three phases model of Christensen and Lo [1979], the homogeneous equivalent medium denoted by material 0 is unknown. In the phase i the local characteristics are the bulk modulus denoted by k_i and shear modulus μ_i. In what follows k_1 is assumed to be larger than k_2.

There exists only one family of composite sphere in the structure; one gets the overall bulk modulus k_o in the form:

$$k_o = k_1 + c\frac{k_2 - k_1}{1 + \frac{3(k_2-k_1)(1-c)}{(3k_1+4\mu_1)}}, \tag{7.53}$$

and on the interface the energy release rate is

$$\mathcal{G} = \frac{\theta_o^2(3k_1 + 4\mu_1)((3k_2 + 4\mu_1)(k_2 - k_1)}{2(3k_2 + 4\mu_1 + 3c(k_1 - k_2))}. \tag{7.54}$$

Here θ_o is the uniform strain applied at infinity ($\boldsymbol{\varepsilon} = \theta_o \boldsymbol{I}$). Generalized Griffith's law is considered to govern the transformation:

$$\begin{cases} \mathcal{G} < G_c, & \dot{c} = 0, \\ \mathcal{G} = G_c, & \dot{c} \geq 0. \end{cases} \tag{7.55}$$

The behaviour takes the form plotted in Fig. 7.1.

Now we consider the macroscopic behaviour of composite spheres assemblage when the two families exist in the structure; the volume fractions of the phases are denoted by c_{I} and c_{II} ($c_{\mathrm{I}} > c_{\mathrm{II}}$).

It can be shown in an analytical way that

$$(\mathcal{G}_{\mathrm{I}} - \mathcal{G}_{\mathrm{II}})(c_{\mathrm{I}} - c_{\mathrm{II}})(\mu_1 - \mu_2) > 0. \tag{7.56}$$

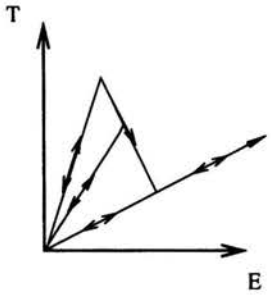

FIGURE 7.1. The response of the composite sphere.

As previously, at the beginning of the loading the macroscopic behaviour is linear elastic until the criterion of propagation is reached for one family. So we have the following cases:

- if $\mu_1 > \mu_2$, the difference between the two concentrations $(c_{\mathrm{I}} - c_{\mathrm{II}})$ increases until the larger reaches the value 1,
- if $\mu_1 < \mu_2$, the difference between the two concentration decreases, then the assemblage tends to the assemblage of only one family,
- if $\mu_1 = \mu_2$ both concentrations could increase.

So, if we consider the assemblage of the two families as a perturbation of the assemblage of one family, this study can be considered as an analysis of bifurcation for small difference $(c_{\mathrm{I}} - c_{\mathrm{II}})$. In the first case, a new well disordered family can appear along the first one. So the answer of the global behaviour in such a case is not unique.

Even, the system is composed by only one family of similar composite spheres, the local response to the loading increment is non unique. In fact, many kinds of bifurcations can exist. This shows the necessity to study stability and bifurcation of each equilibrium path in homogenization of non-linear mechanical behaviour to ensure the existence of the macroscopic law.

Case of plasticity

The case of plasticity is recovered when no transformation exist on Γ. In the dissipation two kinds of hardening are then present: the hardening due to the incompatibility of the plastic strain and the self-hardening of each constituent. The hardening is described by the energy embedded in the residual stresses and in the self-hardening energy, which emphasizes the role of the embedded energy on the hardening.

Case of damaged material

It is observed that the reduction of material stiffness is generally due to the evolution of defects such as cavities, cracks, etc. These zones cannot support tensile stresses. It is proposed to characterize damaged material only with the property that the stress vanishes in the damaged zone. It is necessary to distinguish between two different zones: the sound elastoplastic material with volume Ω and the damaged one where the stresses are identically equal to zero. The previous results of Bui *et al.* [1981] are then recovered. In particular, a relation between the global tangent modulus and local one is obtained in the form:

$$\dot{\boldsymbol{\Sigma}} : \dot{\boldsymbol{E}} = < \dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}} > - \int_{\Gamma} \nabla w . \underline{n} \phi^2 \, \mathrm{d}S. \tag{7.57}$$

This condition gives us a condition of stability in this case as pointed out by Dems and Mróz [1985].

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