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Semi-inverse solutions and Almansi's problem for viscoelastic cylinder

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WITHIN THE SET of Almansi's solutions for a viscoelastic cylinder a subset is outlined by reformulating the three-dimensional quasi-static equilibrium equations as a formal integro-differential operator for the cross-section domain. Any solution of the subset is characterized by the property that it can be treated as a plane strain in the cross-section of the cylinder, the axial variable taking the role of a parameter. In that subset, some classes of semi-inverse solutions are pointed out and they are used in order to obtain a solution for the relaxed Almansi's problem.

1. Introduction

THE ALMANSI'S problem [1, 2] is that of a homogeneous isotropic cylinder which — in the absence of body forces — is in elastic equilibrium under the action of forces distributed along the lateral surface and over its plane ends. The loadings on the lateral surface are assumed in the form of a polynomial in the axial coordinate. In the relaxed formulation of the problem the detailed assignment of the terminal tractions is abandoned in favor of prescribing merely the appropriate stress resultants. A modern treatment of the relaxed problem was given recently by IEŞAN [3].

In this paper we formulate the Almansi's problem for a viscoelastic cylinder made of an anisotropic and inhomogeneous material. It is one aim of the paper to obtain some classes of semi-inverse solutions to the Almansi's problem which are relevant to treatment of the associated relaxed problem. The main idea consists in the reformulation of the three-dimensional quasi-static equilibrium equations as a formal integro-differential operator for the plane cross-section domain, the axial variable playing the role of a parameter. Then the conditions upon the solution of Almansi's problem are determined in order to enable us to treat it as a plane cross-section strain. These conditions indicate a set of classes of semi-inverse solutions for the Almansi's problem.

In this paper, the basic equations, assumptions and a formulation of Almansi's problem are contained in Section 2, with the following section devoted to a statement of the generalized plane strain problems. Four auxiliary generalized plane strain problems are defined for the subsequent analysis. By reformulating the three-dimensional quasi-static equilibrium equations for the plane cross-section domain, in Section 4 we deduce the conditions upon the solution of Almansi's problem in order to treat it as a plane strain. In addition, Section 4 also ulitises these results to obtain some classes of semi-inverse solutions for the Almansi's problem. The primary solution class is characterized by the property that the partial derivative of a solution with respect to the axial coordinate gives rise to a rigid displacement. Any solution of this class is expressed in terms of four canonical displacements (that can be treated as plane displacements in the plane cross-section of cylinder) and depends on four arbitrary continuous functions depending only on time variable.

The other classes of semi-inverse solutions are characterized by the fact that the partial derivatives with respect to axial coordinate of a solution lead to a primary solution.

The final section is devoted to determining the solution of the relaxed Almansi's problem.

2. Statement of the problem

We consider a prismatic cylinder B whose ends are plane, and select a rectangular system of Cartesian coordinates such that the base of the cylinder lies in the (x_1, x_2)-coordinate plane and contains the origin, while the generators of the cylinder are parallel to the positive x_3-axis. We suppose that the length of the cylinder is L and that $D(x_3) \subset R^2$ represents the bounded cross-section at distance x_3 from the plane end containing the origin. The boundary ∂D of each cross-section is assumed to be sufficiently smooth to admit application of the divergence theorem in the plane of the cross-section. The lateral boundary of the cylinder is $\pi = \partial D \times [0, L]$. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers $(1, 2)$, whereas Latin subscripts — unless otherwise specified — are confined to the range $(1, 2, 3)$; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate; superposed dot denotes differentiation with respect to the time variable; where no confusion may occur, we disregard the dependence upon the spatial variables.

We assume that the body occupying B is a linearly viscoelastic material that is at rest at all times $t < 0$. Let u_i be the components of displacement field over B. Then

$$e_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.1}$$

are the components of the strain field associated with $\mathbf{u}$. The stress-strain relation has the form [4]

$$S_{ij}(\mathbf{u}) = G_{ijrs}(0)e_{rs} + \int_0^t \dot{G}_{ijrs}(t-z)e_{rs}(z)\,dz\,. \tag{2.2}$$

Here $S_{ij}(\mathbf{u})$ are the components of the stress field associated with $\mathbf{u}$, while G_{ijrs} stands for the components of the relaxation tensor. We assume that G_{ijrs} is symmetric, and functions

$$G_{ijrs} = G_{ijrs}(x_1, x_2; t) \tag{2.3}$$

are smooth functions on $\overline{B} \times [0, \infty)$. Moreover, we assume that $G_{ijrs}(0)$ is positive definite in B.

Let $s_i(\mathbf{u})$ be the components of the surface traction at regular points of the boundary ∂B of B, corresponding to the stress field $S_{ij}(\mathbf{u})$, defined by

$$s_i(\mathbf{u}) = S_{ij}(\mathbf{u})n_j\,, \tag{2.4}$$

where n_j are the components of the outward unit normal to ∂B.

We call a vector field $\mathbf{u}$ a quasi-static equilibrium displacement field for B corresponding to the body force $\mathbf{f}$, if for each time $t \in [0, T)$, we have $\mathbf{u} \in C^1(\overline{B}) \cap C^2(B)$, $\mathbf{u}$ is continuous with respect to t on $[0, T)$ and

$$S_{ij,j}(\mathbf{u}) + f_i = 0, \tag{2.5}$$

holds on B.

On the lateral surface we assume the boundary conditions

$$s_i(\mathbf{u}) = p_i \quad \text{on } \pi\,, \tag{2.6}$$

and at the ends we have

$$s_i(\mathbf{u}) = s_i^{(1)} \quad \text{on} \quad D(0),\ s_i(\mathbf{u}) = s_i^{(2)} \quad \text{on } D(L)\,, \tag{2.7}$$

for each time $t \in [0,T)$. Here, $p_i, s_i^{(1)}$ and $s_i^{(2)}$ are functions prescribed on π, $D(0)$ and $D(L)$, respectively, for each time $t \in [0,T)$.

The necessary and sufficient conditions for the existence of the solution to this problem are given by

$$\begin{gathered}\int\limits_B f_i\,dV + \int\limits_\pi p_i d\sigma + \int\limits_{D(0)} s_i^{(1)}\,da + \int\limits_{D(L)} s_i^{(2)}\,da = 0\,,\\ \int\limits_B \varepsilon_{ijk}x_j f_k\,dV + \int\limits_\pi \varepsilon_{ijk}x_j p_k\,d\sigma + \int\limits_{D(0)} \varepsilon_{ijk}x_j s_k^{(1)}\,da + \int\limits_{D(L)} \varepsilon_{ijk}x_j s_k^{(2)}\,da = 0\,,\end{gathered} \tag{2.8}$$

where ε_{ijk} is the alternating symbol.

Under suitable smoothness hypotheses on π, and on the given forces, the solution of the above problem exists and is continuous with respect to time on $[0,T)$ (cf. FICHERA [5]).

If we introduce the relations (2.1) and (2.2) into (2.4) and (2.5), it follows that the solution $\mathbf{u}$ of the above problem satisfies the boundary-value problem $(\mathcal{A})$ defined by the equations

$$\mathcal{A}_i(\mathbf{u}) \equiv (G_{ijrs}(0)u_{r,s})_{,j} + \int\limits_0^t (\dot{G}_{ijrs}(t-z)u_{r,s}(z))_{,j}\,dz = -f_i \qquad \text{in} \quad B \equiv D \times (0,L)\,, \tag{2.9}$$

the lateral boundary conditions

$$\mathcal{B}_i(\mathbf{u}) \equiv [G_{i\alpha rs}(0)u_{r,s} + \int\limits_0^t \dot{G}_{i\alpha rs}(t-z)u_{r,s}(z)\,dz]n_\alpha = p_i \qquad \text{on} \quad \pi \equiv \partial D \times (0,L)\,, \tag{2.10}$$

and the end boundary conditions

$$S_{3i}(\mathbf{u}) = -s_i^{(1)} \quad \text{on } D(0), \quad S_{3i}(\mathbf{u}) = s_i^{(2)} \quad \text{on } D(L)\,, \tag{2.11}$$

for each $t \in [0,T)$.

In this paper we assume that the body force $\mathbf{f}$ and the surface force $\mathbf{p}$ are polynomials of degree n in the axial coordinate, i.e.

$$f_i = \sum_{k=0}^{n} \frac{1}{k!} f_{ik}(x_1,x_2;t)x_3^k, \qquad p_i = \sum_{k=0}^{n} \frac{1}{k!} p_{ik}(x_1,x_2;t)x_3^k\,, \tag{2.12}$$

where f_{ik} and p_{ik} ($i = 1,2,3$; $k = 0,1,\ldots,n$) are given functions, continuous on $[0,T)$.

Almansi's problem $(\mathcal{A}_n)$ consists in finding a quasi-static equilibrium displacement field on B that corresponds to the body force $\mathbf{f}$ and satisfies the boundary conditions (2.6) and (2.7), when $\mathbf{f}$ and $\mathbf{p}$ are given by relation (2.12). We denote by $\mathcal{A}$ the set of solutions of the Almansi's problem.

3. Generalized plane strain state

The state of generalized plane strain for the plane domain $D \subset R^2$ is characterized by the relation

$$v_i = v_i(x_1, x_2; t), (x_1, x_2) \in D,\ t \in [0, T)\,. \tag{3.1}$$

Such a displacement vector, in conjunction with the stress-strain-displacement relations, imply that the components of the stress field are functions of x_1 and x_2 and t, i.e. $T_{ij} = T_{ij}(x_1, x_2; t)$. Moreover, we have

$$T_{ij}(\mathbf{v}) = G_{ijk\beta}(0)v_{k,\beta} + \int_0^t \dot{G}_{ijk\beta}(t-z)v_{k,\beta}(z)\,dz\,. \tag{3.2}$$

A vector field $\mathbf{v}$ is an admissible displacement field provided $\mathbf{v}$ is continuous with respect to time variable on $[0, T)$ and, moreover, for each $t \in [0, T)$,

(i) $\mathbf{v}$ is independent of x_3;
(ii) $\mathbf{v} \in C^1(\overline{D}) \cap C^2(D)$.

Under given body forces $\mathbf{F}(x_1, x_2; t)$ on $D \times [0, T)$ and boundary forces $\mathbf{P}(x_1, x_2; t)$ on $\partial D \times [0, T)$, the generalized plane strain problem for $D \cup \partial D$ consists in finding an admissible displacement field $\mathbf{v}$ which satisfies the quasi-static equilibrium equations

$$T_{\alpha i,\alpha}(\mathbf{v}) + F_i = 0 \quad \text{in } D\,, \tag{3.3}$$

and the boundary conditions

$$T_{\alpha i}(\mathbf{v})n_\alpha = P_i \quad \text{on } \partial D\,, \tag{3.4}$$

for each $t \in [0, T)$. If we substitute the relation (3.2) into (3.3) and (3.4), we obtain the displacement plane boundary value problem $(\mathcal{P})$ for $D \cup \partial D$, defined by

$$\mathcal{P}_i(\mathbf{v}) \equiv \Big(G_{i\alpha k\beta}(0)v_{k,\beta}\Big)_{,\alpha} + \int_0^t (\dot{G}_{i\alpha k\beta}(t-z)v_{k,\beta}(z))_{,\alpha}dz = -F_i \quad \text{in } D\,, \tag{3.5}$$

$$\mathcal{T}_i(\mathbf{v}) \equiv \Big[G_{i\alpha k\beta}(0)v_{k,\beta} + \int_0^t \dot{G}_{i\alpha k\beta}(t-z)v_{k,\beta}(z)\,dz\Big]n_\alpha = P_i \quad \text{on } \partial D\,, \tag{3.6}$$

for each $t \in [0, T)$.

The necessary and sufficient conditions for the existence of the solution $\mathbf{v}$ of the plane boundary value problem $(\mathcal{P})$ associated with $D \cup \partial D$, are given by

$$\int_D F_i da + \int_{\partial D} P_i\, ds = 0\,, \tag{3.7}$$

$$\int_D \varepsilon_{3\alpha\beta}x_\alpha F_\beta\, da + \int_{\partial D} \varepsilon_{3\alpha\beta}x_\alpha P_\beta\, ds = 0\,. \tag{3.8}$$

Under suitable smoothness hypotheses on ∂D and on the forces given, a solution of the generalized plane strain problem $(\mathcal{P})$ exists for each $t \in [0, T)$ (cf. FICHERA [5]). In what follows, we denote by $\mathcal{P}$ the set of plane displacement solutions associated with the cross-section of the cylinder.

We will have the opportunity to use four special problems $\mathcal{P}^{(s)}$ $(s = 1, 2, 3, 4)$ of generalized plane strain. We denote by $\mathbf{w}^{(s)}$ $(s = 1, 2, 3, 4)$ the solution of the problem $P^{(s)}$ $(s = 1, 2, 3, 4)$. The problems $P^{(s)}$ $(s = 1, 2, 3, 4)$ are characterized by the equations [6]

$$
\begin{aligned}
\mathcal{P}_i(\mathbf{w}^{(\beta)}) + (G_{i\alpha 33}(t)x_\beta)_{,\alpha} &= 0 \quad (\beta = 1, 2), \\
\mathcal{P}_i(\mathbf{w}^{(3)}) + G_{i\alpha 33,\alpha}(t) &= 0, \\
\mathcal{P}_i(\mathbf{w}^{(4)}) - \varepsilon_{3\rho\beta}(G_{i\alpha\rho 3}(t)x_\beta)_{,\alpha} &= 0 \quad \text{in } D,
\end{aligned}
\tag{3.9}
$$

and the boundary conditions

$$
\begin{aligned}
\mathcal{T}_i(\mathbf{w}^{(\beta)}) + G_{i\alpha 33}(t)x_\beta n_\alpha &= 0 \quad (\beta = 1, 2), \\
\mathcal{T}_i(\mathbf{w}^{(3)}) + G_{i\alpha 33}(t)n_\alpha &= 0, \\
\mathcal{T}_i(\mathbf{w}^{(4)}) - \varepsilon_{3\rho\beta}G_{i\alpha\rho 3}(t)x_\beta n_\alpha &= 0 \quad \text{on } \partial D.
\end{aligned}
\tag{3.10}
$$

The necessary and sufficient conditions for the existence of the solution are satisfied for each boundary value problem $P^{(s)}$ $(s = 1, 2, 3, 4)$. Therefore, we can assume that the fields $\mathbf{w}^{(s)}$ $(s = 1, 2, 3, 4)$ are known.

4. Semi-inverse solutions by plane cross-section displacements

We have now completed all the necessary preliminaries required to analyse the three-dimensional problem $(\mathcal{A})$ by means of the generalized plane strain problem associated with the cross-section of the cylinder. Obviously, the last problem is more tractable.

Therefore, we consider the system (2.9) and the boundary conditions (2.10) on the cross-section $D \cup \partial D$ so that we have the plane boundary-value problem

$$\mathcal{A}_i(\mathbf{u}) = -f_i \quad \text{in } D, \tag{4.1}$$

$$\mathcal{B}_i(\mathbf{u}) = p_i \quad \text{on } \partial D, \tag{4.2}$$

where $x_3 \in (0, L)$ and $t \in [0, T)$ are viewed as parameters. In this connection we pose the following question: When the solution $\mathbf{u} \in \mathcal{A}$ belongs to the set $\mathcal{P}$ of the plane displacements associated with the cross-section $D \cup \partial D$ of the cylinder?

The answer to this question will be given in the terms of the vector-valued linear functionals $\mathcal{R}$ and $\mathcal{M}$, whose components are defined by

$$\mathcal{R}_i(\mathbf{u}) = \int_D S_{3i}(\mathbf{u})\, da, \qquad \mathcal{M}_i(\mathbf{u}) = \int_D \varepsilon_{ijk} x_j S_{3k}(\mathbf{u})\, da, \tag{4.3}$$

and which represent the resultant force and the resultant moment about 0 of the tractions acting on the cross-section D of the cylinder. Let us note that

$$\mathcal{M}_\alpha(\mathbf{u}) = \varepsilon_{3\alpha\beta} \int_D x_\beta S_{33}(\mathbf{u})\, da - x_3 \varepsilon_{3\alpha\beta} \mathcal{R}_\beta(\mathbf{u}), \tag{4.4}$$

$$\mathcal{M}_3(\mathbf{u}) = \varepsilon_{3\alpha\beta} \int_D x_\alpha S_{3\beta}(\mathbf{u})\, da. \tag{4.5}$$

In order to answer to the above question, we write the plane boundary-value problem

(4.1) and (4.2) in the form

$$\text{(4.6)}\quad \mathcal{P}_i(\mathbf{u}) + \left[G_{i\alpha k3}(0)u_{k,3} + \int_0^t \dot{G}_{i\alpha k3}(t-z)u_{k,3}(z)\,dz\right]_{,\alpha} + S_{3i,3}(\mathbf{u}) + f_i = 0 \quad \text{in } D,$$

$$\text{(4.7)}\quad \mathcal{T}_i(\mathbf{u}) = -\left[G_{i\alpha k3}(0)u_{k,3} + \int_0^t \dot{G}_{i\alpha k3}(t-z)u_{k,3}(z)\,dz\right]n_\alpha + p_i \quad \text{on } \partial D.$$

Therefore, we can imagine the boundary value problem defined by (4.6) and (4.7) as a generalized plane strain boundary-value problem of the above section, with

$$\text{(4.8)}\qquad F_i = \left[G_{i\alpha k3}(0)u_{k,3} + \int_0^t \dot{G}_{i\alpha k3}(t-z)u_{k,3}(z)\,dz\right]_{,\alpha} + S_{3i,3}(\mathbf{u}) + f_i\,,$$

$$\text{(4.9)}\qquad P_i = -\left[G_{i\alpha k3}(0)u_{k,3} + \int_0^t \dot{G}_{i\alpha k3}(t-z)u_{k,3}(z)\,dz\right]n_\alpha + p_i\,.$$

From the necessary and sufficient conditions (3.7) and (3.8) it results that $\mathbf{u} \in \mathcal{P}$ if

$$\text{(4.10)}\qquad \int_D S_{3i,3}(\mathbf{u})da = -\int_D f_i\,da - \int_{\partial D} p_i\,ds\,,$$

$$\text{(4.11)}\qquad \int_D \varepsilon_{3\alpha\beta}x_\alpha S_{3\beta,3}(\mathbf{u})\,da = -\int_D \varepsilon_{3\alpha\beta}x_\alpha f_\beta\,da - \int_{\partial D} \varepsilon_{3\alpha\beta}x_\alpha p_\beta\,ds\,.$$

Under the hypothesis (2.3), it is easy to observe that the relation (2.2) gives

$$\text{(4.12)}\qquad \frac{\partial^k}{\partial x_3^k}S_{3i}(\mathbf{u}) = S_{3i}\left(\frac{\partial^k \mathbf{u}}{\partial x_3^k}\right), \qquad k = 0, 1, \ldots,$$

so that, by means of the relation (2.12), the relations (4.10) and (4.11) give

$$\text{(4.13)}\qquad \int_D S_{3i}\left(\frac{\partial^{n+2}\mathbf{u}}{\partial x_3^{n+2}}\right)da = 0\,,$$

$$\text{(4.14)}\qquad \int_D \varepsilon_{3\alpha\beta}x_\alpha S_{3\beta}\left(\frac{\partial^{n+2}\mathbf{u}}{\partial x_3^{n+2}}\right)da = 0\,.$$

On the other hand, the relations (4.10) and (4.11) prove, by means of the relations (4.3) and (4.5), that

$$\text{(4.15)}\qquad \mathcal{R}_i(\mathbf{u}) = \mathcal{R}_i^0 - \sum_{k=0}^{n}\frac{1}{(k+1)!}R_{ik}(\mathbf{f}, \mathbf{p})x_3^{k+1}\,,$$

$$\text{(4.16)}\qquad \mathcal{M}_3(\mathbf{u}) = \mathcal{M}_3^0 - \sum_{k=0}^{n}\frac{1}{(k+1)!}M_{3k}(\mathbf{f}, \mathbf{p})x_3^{k+1}\,,$$

where

$$R_{ik}(\mathbf{f},\mathbf{p}) = \int_D f_{ik}(x_1,x_2;t)\,da + \int_{\partial D} p_{ik}(x_1,x_2;t)\,ds,$$

(4.17)
$$M_{3k}(\mathbf{f},\mathbf{p}) = \int_D \varepsilon_{3\alpha\beta}x_\alpha f_{\beta k}(x_1,x_2;t)\,da + \int_{\partial D} \varepsilon_{3\alpha\beta}x_\alpha p_{\beta k}(x_1,x_2;t)\,ds, \qquad k=0,1,\ldots,n,$$

and $\mathcal{R}_i^0$ and $\mathcal{M}_3^0$ are arbitrary functions of t, continuous on $[0,T)$.

We are thus led the following result.

PROPOSITION 1. Let $\mathbf{u}$ be a solution to the Almansi's problem $(\mathcal{A}_n)$. Then $\mathbf{u}$ can be treated as a generalized plane strain solution of the plane boundary-value problem defined by relations (4.6) and (4.7) if and only if the relations (4.15) and (4.16) hold true.

COROLLARY 1. Let $\mathbf{u}$ be a solution of the Almansi's problem $(\mathcal{A}_n)$ for which the relations (4.15) and (4.16) hold true. Then

(4.18)
$$\mathcal{M}_\alpha(\mathbf{u}) = \mathcal{M}_\alpha^0 - \sum_{k=0}^{n}\frac{1}{(k+1)!}M_{\alpha k}(\mathbf{f},\mathbf{p})x_3^{k+1} + \sum_{k=0}^{n}\frac{1}{(k+2)k!}\varepsilon_{3\alpha\beta}R_{\beta k}(\mathbf{f},\mathbf{p})x_3^{k+2},$$

where

(4.19)
$$M_{\alpha k}(\mathbf{f},\mathbf{p}) = \varepsilon_{3\alpha\beta}\left[\int_D x_\beta f_{3k}(x_1,x_2;t)\,da + \int_{\partial D} x_\beta p_{3k}(x_1,x_2;t)\,ds\right],$$

and $\mathcal{M}_\alpha^0$ are arbitrary functions of t continuous on $[0,T)$.

Proof. In view of the quasi-static equilibrium equations (2.5) and the boundary conditions (2.6), we get

(4.20)
$$\begin{aligned}\int_D x_\alpha S_{33}(\mathbf{u}_{,3})\,da &= \int_D x_\alpha S_{33,3}(\mathbf{u})\,da = -\int_D x_\alpha S_{\rho 3,\rho}(\mathbf{u})\,da - \int_D x_\alpha f_3\,da \\ &= -\int_D (x_\alpha S_{\rho 3}(\mathbf{u}))_{,\rho}\,da + \int_D S_{3\alpha}(\mathbf{u})\,da - \int_D x_\alpha f_3\,da \\ &= -\int_{\partial D} x_\alpha p_3\,ds - \int_D x_\alpha f_3\,da + \mathcal{R}_\alpha(\mathbf{u}).\end{aligned}$$

On the basis of the relations (4.4), (4.10) and (4.20), we deduce

(4.21)
$$(\mathcal{M}_\alpha(\mathbf{u}))_{,3} = -\sum_{k=0}^{n}\frac{1}{k!}M_{\alpha k}(\mathbf{f},\mathbf{p})x_3^k + \sum_{k=0}^{n}\frac{1}{k!}\varepsilon_{3\alpha\beta}R_{\beta k}(\mathbf{f},\mathbf{p})x_3^{k+1},$$

which leads to the relation (4.18) and the proof is complete.

REMARK 1. The relations (4.15) and (4.20) yield

(4.22)
$$\int_D x_\alpha S_{33}\left(\frac{\partial^{n+3}\mathbf{u}}{\partial x_3^{n+3}}\right)da = 0.$$

REMARK 2. The relations (4.13), (4.14) and (4.22) allow us to point out some classes of semi-inverse solutions to the Almansi's problem that can be treated as plane displacements in the cross-section considering x_3 as a parameter. In what follows, we describe the set of classes of semi-inverse solutions.

THE CLASS C_0 (the primary solution class). For the vanishing body force and lateral boundary force, the relations (4.10) and (4.11) take the form

$$\int_D S_{3i}(\mathbf{u}_{,3})\,da = 0, \qquad \int_D \varepsilon_{3\alpha\beta} x_\alpha S_{3\beta}(\mathbf{u}_{,3})\,da = 0, \tag{4.23}$$

and this leads to a class of semi-inverse solutions for Saint-Venant's problem. Thus, we are led to introduce the set C_0 consisting of the solutions of the Saint-Venant's problem for which

$$\mathbf{u}_{,3} \tag{4.24}$$

is a rigid displacement.

If $\mathbf{u}^0 \in C_0$, then it can be expressed in the form [6]

$$\mathbf{u}^0 = \sum_{s=1}^{4} a_s \otimes \mathbf{u}^{(s)}, \tag{4.25}$$

where the canonical displacements $\mathbf{u}^{(s)}$ $(s = 1, 2, 3, 4)$ (solutions of the plane boundary value problem (4.6) and (4.7)) are expressed in terms of the auxiliary generalized plane displacements $\mathbf{w}^{(s)}$ $(s = 1, 2, 3, 4)$ by

$$\begin{aligned} &u_\alpha^{(\beta)} = -\frac{1}{2}x_3^2\delta_{\alpha\beta} + w_\alpha^{(\beta)}, \qquad u_3^{(\beta)} = x_\beta x_3 + w_3^{(\beta)} \qquad (\beta = 1, 2), \\ &u_\alpha^{(3)} = w_\alpha^{(3)}, \qquad u_3^{(3)} = x_3 + w_3^{(3)}, \\ &u_\alpha^{(4)} = \varepsilon_{3\beta\alpha} x_\beta x_3 + w_\alpha^{(4)}, \qquad u_3^{(4)} = w_3^{(4)}, \end{aligned} \tag{4.26}$$

and a_s $(s = 1, 2, 3, 4)$ are arbitrary functions of t continuous on $[0, T)$. Moreover, in relation (4.25) we have used the notation

$$(g \otimes h)(t) = g(0)h(t) + \int_0^t \dot{g}(t - z)h(z)\,dz. \tag{4.27}$$

It follows from (2.2) and (4.25) that

$$S_{ij}(\mathbf{u}^0) = \sum_{s=1}^{4} a_s \otimes S_{ij}(\mathbf{u}^{(s)}), \tag{4.28}$$

where

$$\begin{aligned} &S_{ij}(\mathbf{u}^{(\alpha)}) = T_{ij}(\mathbf{w}^{(\alpha)}) + G_{ij33}(t)x_\alpha, \qquad S_{ij}(\mathbf{u}^{(3)}) = T_{ij}(\mathbf{w}^{(3)}) + G_{ij33}(t), \\ &S_{ij}(\mathbf{u}^{(4)}) = T_{ij}(\mathbf{w}^{(4)}) - \varepsilon_{3\rho\beta} G_{ij\rho 3}(t) x_\beta. \end{aligned} \tag{4.29}$$

Obviously, the relations (3.9), (3.10) and (4.29) give

$$S_{\alpha i,\alpha}(\mathbf{u}^{(s)}) = 0 \quad \text{in } D, \tag{4.30}$$

$$S_{\alpha i}(\mathbf{u}^{(s)}) n_\alpha = 0 \quad \text{on } \partial D, \qquad s = 1, 2, 3, 4. \tag{4.31}$$

These relations imply

$$\int_D S_{3\alpha}(\mathbf{u}^{(s)})\,da = 0, \qquad s = 1,\,2,\,3,\,4. \tag{4.32}$$

We note that $S_{ij}(\mathbf{u}^{(s)})$ $(s = 1, 2, 3, 4)$ at $t = 0$ coincide with the components of the stress tensor in the auxiliary strain problems of the classical elasticity corresponding to an elastic material with the positive definite elasticity tensor $G_{ijrs}(0)$.

Further, we deduce that for $\mathbf{u}^0 \in C_0$, we have

$$\begin{aligned} &\mathcal{R}_\alpha(\mathbf{u}^0) = 0, \qquad \mathcal{R}_3(\mathbf{u}^0) = \sum_{s=1}^{4} D_{3s} \otimes a_s\,, \\ &\mathcal{M}_\alpha(\mathbf{u}^0) = \sum_{s=1}^{4} \varepsilon_{3\alpha\beta} D_{\beta s} \otimes a_s, \qquad \mathcal{M}_3(\mathbf{u}^0) = \sum_{s=1}^{4} D_{4s} \otimes a_s\,, \end{aligned} \tag{4.33}$$

where

$$\begin{aligned} &D_{\beta s} = \int_D x_\beta S_{33}(\mathbf{u}^{(s)})\,da, \qquad D_{3s} = \int_D S_{33}(\mathbf{u}^{(s)})\,da\,, \\ &D_{4s} = \int_D \varepsilon_{3\alpha\beta} x_\alpha S_{3\beta}(\mathbf{u}^{(s)})\,da, \qquad s = 1,\,2,\,3,\,4. \end{aligned} \tag{4.34}$$

We denote by $\mathcal{D}(t)$ the 4×4-matrix whose components are $D_{rs}(t)$, $(r, s = 1, 2, 3, 4)$. It should be pointed out that $\mathcal{D}(0)$ coincides with the corresponding matrix [3] for an elastic medium with the elastic coefficients $G_{ijrs}(0)$. It was shown that the matrix $\mathcal{D}(0)$ is invertible [3].

Finally, the corresponding body force $\mathcal{F}(\mathbf{u}^0)$ and boundary force $\mathbf{s}(\mathbf{u}^0)$ are given by

$$\begin{aligned} &\mathcal{F}_i(\mathbf{u}^0) \equiv -S_{ji,j}(\mathbf{u}^0) = 0 \quad \text{in } B, \qquad s_i(\mathbf{u}^0) \equiv S_{\alpha i}(\mathbf{u}^0) n_\alpha = 0 \quad \text{on } \pi\,, \\ &s_i(\mathbf{u}^0) = -\sum_{s=1}^{4} S_{3i}(\mathbf{u}^{(s)}) \otimes a_s \quad \text{on } D(0)\,, \\ &s_i(\mathbf{u}^0) = \sum_{s=1}^{4} S_{3i}(\mathbf{u}^{(s)}) \otimes a_s \quad \text{on } D(L)\,. \end{aligned} \tag{4.35}$$

Let $\widehat{a}(t)$ be the four-dimensional vector field $(a_1(t), a_2(t), a_3(t), a_4(t))$. We shall write $\mathbf{u}^0\{\widehat{a}\}$ for the displacement vector $\mathbf{u}^0$ defined by relation (4.25), indicating thus its dependence on the functions $a_1(t), a_2(t), a_3(t)$ and $a_4(t)$.

THE CLASS C_r, $r \geq 1$. We denote by C_r the class of solutions of the Almansi's problem for which

$$\frac{\partial^r \mathbf{u}}{\partial x_3^r} \in C_0\,. \tag{4.36}$$

For $\mathbf{u}^* \in C_r$, it follows that

$$\frac{\partial^r \mathbf{u}^*}{\partial x_3^r} = \mathbf{u}^0\{\widehat{a}^{(r)}\}\,, \tag{4.37}$$

so that, by means of the relations (4.25) and (4.26), we get

$$(4.38)\quad \begin{aligned} u_\alpha^* &= -\sum_{k=0}^{r}\frac{1}{(k+2)!}a_\alpha^{(k)}(t)x_3^{k+2} - \sum_{k=0}^{r}\frac{1}{(k+1)!}a_4^{(k)}(t)\varepsilon_{3\alpha\beta}x_\beta x_3^{k+1} \\ &\quad + \sum_{k=0}^{r}\frac{1}{k!}\sum_{s=1}^{4}a_s^{(k)}\otimes w_\alpha^{(s)}x_3^k + \sum_{k=0}^{r-1}\frac{1}{k!}v_\alpha^{(k)}x_3^k\,, \\ u_3^* &= \sum_{k=0}^{r}\frac{1}{(k+1)!}[a_\rho^{(k)}(t)x_\rho + a_3^{(k)}(t)]x_3^{k+1} + \sum_{k=0}^{r}\frac{1}{k!}\sum_{s=1}^{4}a_s^{(k)}\otimes w_3^{(s)}x_3^k \\ &\quad + \sum_{k=0}^{r-1}\frac{1}{k!}v_3^{(k)}x_3^k\,, \end{aligned}$$

where $\mathbf{v}^{(k)}(x_1,x_2;t)(k=0,1,\ldots,r-1)$ are plane displacements in $\mathcal{P}$, and $\widehat{a}^{(k)}(k=0,1,\ldots,r)$ are arbitrary four-dimensional vector fields depending only on the time t continuous on $[0,T)$.

The components of the stress field corresponding to the displacement $\mathbf{u}^*$ defined by (4.38) have the form

$$(4.39)\quad \begin{aligned} S_{ij}(\mathbf{u}^*) &= \sum_{k=0}^{r}\frac{1}{k!}\sum_{s=1}^{4}a_s^{(k)}\otimes S_{ij}(\mathbf{u}^{(s)})x_3^k + \sum_{k=0}^{r-1}\frac{1}{k!}K_{ij}^{(k)}x_3^k \\ &\quad + \sum_{k=0}^{r-2}\frac{1}{k!}L_{ij}^{(k)}x_3^k + \sum_{k=0}^{r-1}\frac{1}{k!}T_{ij}(\mathbf{v}^{(k)})x_3^k\,, \end{aligned}$$

where

$$(4.40)\quad K_{ij}^{(k)} = \sum_{s=1}^{4}a_s^{(k+1)}\otimes G_{ijm3}\otimes w_m^{(s)},\qquad L_{ij}^{(k)} = G_{ijm3}\otimes v_m^{(k+1)}.$$

Further, the corresponding body force $\mathcal{F}(\mathbf{u}^*)$ is given by

$$(4.41)\quad \begin{aligned} \mathcal{F}_i(\mathbf{u}^*) \equiv -S_{ji,j}(\mathbf{u}^*) &= -\sum_{k=0}^{r-1}\frac{1}{k!}x_3^k\Big[T_{\alpha i,\alpha}(\mathbf{v}^{(k)}) + K_{\alpha i,\alpha}^{(k)} \\ &\quad + \sum_{s=1}^{4}a_s^{(k+1)}\otimes S_{3i}(\mathbf{u}^{(s)})\Big] - \sum_{k=0}^{r-2}\frac{1}{k!}x_3^k\Big[L_{\alpha i,\alpha}^{(k)} + K_{3i}^{(k+1)} \\ &\quad + T_{3i}(\mathbf{v}^{(k+1)})\Big] - \sum_{k=0}^{r-3}\frac{1}{k!}L_{3i}^{(k+1)}x_3^k\,, \end{aligned}$$

and the lateral boundary force $\mathbf{s}(\mathbf{u}^*)$ is

$$(4.42)\quad \begin{aligned} s_i(\mathbf{u}^*) \equiv S_{\alpha i}(\mathbf{u}^*)n_\alpha &= \sum_{k=0}^{r-1}\frac{1}{k!}x_3^k[T_{\alpha i}(\mathbf{v}^{(k)})n_\alpha + K_{\alpha i}^{(k)}n_\alpha] \\ &\quad + \sum_{k=0}^{r-2}\frac{1}{k!}x_3^k L_{\alpha i}^{(k)}n_\alpha \quad \text{on } \pi\,, \end{aligned}$$

and the end boundary forces

$$s_i(\mathbf{u}^*) \equiv -S_{3i}(\mathbf{u}^*) = -\Big[\sum_{s=1}^{4} a_s^{(0)} \otimes S_{3i}(\mathbf{u}^{(s)}) + K_{3i}^{(0)} + L_{3i}^{(0)} + T_{3i}(\mathbf{v}^{(0)})\Big] \text{ on } D(0),$$

$$\begin{aligned}(4.43)\quad s_i(\mathbf{u}^*) \equiv S_{3i}(\mathbf{u}^*) &= \sum_{k=0}^{r} \frac{1}{k!} \sum_{s=1}^{4} a_s^{(k)} \otimes S_{3i}(\mathbf{u}^{(s)}) L^k \\ &+ \sum_{k=0}^{r-1} \frac{1}{k!}[K_{3i}^{(k)} + T_{3i}(\mathbf{v}^{(k)})] L^k + \sum_{k=0}^{r-2} \frac{1}{k!} L_{3i}^{(k)} L^k \text{ on } D(L).\end{aligned}$$

In view of the relations (4.3)–(4.5), (4.20), (4.32), (4.34) and (4.39), we get

$$\begin{aligned}(4.44)\quad \mathcal{R}_\alpha(\mathbf{u}^*) &= \sum_{k=0}^{r-1} \frac{1}{k!} x_3^k \sum_{s=1}^{4} a_s^{(k+1)} \otimes D_{\alpha s} + \sum_{k=0}^{r-2} \frac{1}{k!} x_3^k \int_D x_\alpha [T_{33}(\mathbf{v}^{(k+1)}) \\ &+ K_{33}^{(k+1)}]\, da + \sum_{k=0}^{r-3} \frac{1}{k!} x_3^k \int_D x_\alpha L_{33}^{(k+1)}\, da + \int_{\partial D} x_\alpha s_3(\mathbf{u}^*)\, ds + \int_D x_\alpha \mathcal{F}_3(\mathbf{u}^*)\, da,\end{aligned}$$

$$\begin{aligned}(4.45)\quad \mathcal{R}_3(\mathbf{u}^*) &= \sum_{k=0}^{r} \frac{1}{k!} x_3^k \sum_{s=1}^{4} a_s^{(k)} \otimes D_{3s} + \sum_{k=0}^{r-1} \frac{1}{k!} x_3^k \int_D [T_{33}(\mathbf{v}^{(k)}) \\ &+ K_{33}^{(k)}]\, da + \sum_{k=0}^{r-2} \frac{1}{k!} x_3^k \int_D L_{33}^{(k)}\, da,\end{aligned}$$

$$\begin{aligned}(4.46)\quad \mathcal{M}_\alpha(\mathbf{u}^*) &= \varepsilon_{3\alpha\beta}\Bigg\{ \sum_{k=0}^{r} \frac{1}{k!} x_3^k \sum_{s=1}^{4} a_s^{(k)} \otimes D_{\beta s} + \sum_{k=0}^{r-1} \frac{1}{k!} x_3^k \int_D x_\beta [T_{33}(\mathbf{v}^{(k)}) \\ &+ K_{33}^{(k)}]\, da + \sum_{k=0}^{r-2} \frac{1}{k!} x_3^k \int_D x_\beta L_{33}^{(k)}\, da - x_3 \mathcal{R}_\beta(\mathbf{u}^*)\Bigg\},\end{aligned}$$

$$\begin{aligned}(4.47)\quad \mathcal{M}_3(\mathbf{u}^*) &= \sum_{k=0}^{r} \frac{1}{k!} x_3^k \sum_{s=1}^{4} a_s^{(k)} \otimes D_{4s} + \sum_{k=0}^{r-1} \frac{1}{k!} x_3^k \int_D \varepsilon_{3\alpha\beta} x_\alpha [T_{3\beta}(\mathbf{v}^{(k)}) \\ &+ K_{3\beta}^{(k)}]\, da + \sum_{k=0}^{r-2} \frac{1}{k!} x_3^k \int_D \varepsilon_{3\alpha\beta} x_\alpha L_{3\beta}^{(k)}\, da.\end{aligned}$$

5. The relaxed Almansi's problem

The relaxed Almansi's problem for the viscoelastic cylinder B consists in the determination of a quasi-static equilibrium displacement field $\mathbf{u}$ corresponding to the body force and lateral boundary force of the form (2.12) that satisfies global conditions at each end in the form

$$(5.1)\qquad \mathcal{R}_i(\mathbf{u}) = -R_i(t), \quad \mathcal{M}_i(\mathbf{u}) = -M_i(t) \quad \text{on } D(0),$$

where R_i and M_i are continuous functions prescribed on $[0, T)$. Similar conditions are assumed on the end located at $x_3 = L$, provided the global equilibrium of the cylinder is assured.

In this section we proceed to determine a solution of the relaxed Almansi's problem. In view of the results of the above section, by means of the relations (4.15), (4.16), (4.18) and (4.44)–(4.47), we see that a solution of the relaxed Almansi's problem has the form

$$\mathbf{u} = \mathbf{u}^*, \quad \mathbf{u}^* \in C_{n+2}. \tag{5.2}$$

Thus, the solution of the relaxed Almansi's problem is

$$\begin{aligned} u_\alpha = &-\sum_{k=0}^{n+2} \frac{1}{(k+2)!} a_\alpha^{(k)}(t) x_3^{k+2} - \sum_{k=0}^{n+2} \frac{1}{(k+1)!} a_4^{(k)}(t) \varepsilon_{3\alpha\beta} x_\beta x_3^{k+1} \\ &+ \sum_{k=0}^{n+2} \frac{1}{k!} \sum_{s=1}^{4} a_s^{(k)} \otimes w_\alpha^{(s)} x_3^k + \sum_{k=0}^{n+1} \frac{1}{k!} v_\alpha^{(k)} x_3^k, \\ u_3 = &\sum_{k=0}^{n+2} \frac{1}{(k+1)!} [a_\rho^{(k)}(t) x_\rho + a_3^{(k)}(t)] x_3^{k+1} \\ &+ \sum_{k=0}^{n+2} \frac{1}{k!} \sum_{s=1}^{4} a_s^{(k)} \otimes w_3^{(s)} x_3^k + \sum_{k=0}^{n+1} \frac{1}{k!} v_3^{(k)} x_3^k. \end{aligned} \tag{5.3}$$

We now proceed to describe the procedure for determination of the unknown functions $a_s^{(k)}(t)$, $(s = 1, 2, 3, 4;\ k = 0, 1, \ldots, n+2)$ and the unknown vector fields $\mathbf{v}^{(k)}$ $(k = 0, 1, \ldots, n+1)$ in order the displacement defined by relation (5.3) to be a solution of the relaxed Almansi's problem.

We first note that the relations (4.15), (4.16) and (4.18) and (5.1) give

$$\mathcal{R}_i^0 = -R_i, \quad \mathcal{M}_i^0 = -M_i. \tag{5.4}$$

By means of the relations (4.44)–(4.47), the conditions (4.15) and (4.16) are satisfied if

$$\sum_{s=1}^{4} D_{ms} \otimes a_s^{(k)} = H_m^{(k)} \quad (m = 1, 2, 3, 4;\ k = 0, 1, \ldots, n+2), \tag{5.5}$$

where

$$\begin{aligned} H_\alpha^{(0)} &= \varepsilon_{3\alpha\beta} M_\beta - \int_D x_\alpha [T_{33}(\mathbf{v}^{(0)}) + K_{33}^{(0)} + L_{33}^{(0)}]\, da, \\ H_3^{(0)} &= -R_3 - \int_D [T_{33}(\mathbf{v}^{(0)}) + K_{33}^{(0)} + L_{33}^{(0)}]\, da, \\ H_4^{(0)} &= -M_3 - \int_D \varepsilon_{3\alpha\beta} x_\alpha [T_{3\beta}(\mathbf{v}^{(0)}) + K_{3\beta}^{(0)} + L_{3\beta}^{(0)}]\, da; \end{aligned} \tag{5.6}$$

$$\begin{aligned} H_\alpha^{(1)} &= -R_\alpha + \varepsilon_{3\alpha\beta} M_{\beta 0}(\mathbf{f}, \mathbf{p}) - \int_D x_\alpha [T_{33}(\mathbf{v}^{(1)}) + K_{33}^{(1)} + L_{33}^{(1)}]\, da, \\ H_3^{(1)} &= -R_{30}(\mathbf{f}, \mathbf{p}) - \int_D [T_{33}(\mathbf{v}^{(1)}) + K_{33}^{(1)} + L_{33}^{(1)}]\, da, \end{aligned} \tag{5.7}$$

(5.7) [cont.]
$$H_4^{(1)} = -M_{30}(\mathbf{f},\mathbf{p}) - \int_D \varepsilon_{3\alpha\beta}x_\alpha[T_{3\beta}(\mathbf{v}^{(1)}) + K_{3\beta}^{(1)} + L_{3\beta}^{(1)}]\,da\,;$$

$$H_\alpha^{(k+1)} = -R_{\alpha(k-1)}(\mathbf{f},\mathbf{p}) + \varepsilon_{3\alpha\beta}M_{\beta k}(\mathbf{f},\mathbf{p}) - \int_D x_\alpha[T_{33}(\mathbf{v}^{(k+1)}) + K_{33}^{(k+1)} + L_{33}^{(k+1)}]\,da\,,$$

(5.8)
$$H_3^{(k+1)} = -R_{3k}(\mathbf{f},\mathbf{p}) - \int_D [T_{33}(\mathbf{v}^{(k+1)}) + K_{33}^{(k+1)} + L_{33}^{(k+1)}]\,da\,,$$

$$H_4^{(k+1)} = -M_{3k}(\mathbf{f},\mathbf{p}) - \int_D \varepsilon_{3\alpha\beta}x_\alpha[T_{3\beta}(\mathbf{v}^{(k+1)}) + K_{3\beta}^{(k+1)} + L_{3\beta}^{(k+1)}]\,da,$$

$$k = 1,2,\ldots,n-1;$$

$$H_\alpha^{(n+1)} = -R_{\alpha(n-1)}(\mathbf{f},\mathbf{p}) + \varepsilon_{3\alpha\beta}M_{\beta n}(\mathbf{f},\mathbf{p}) - \int_D x_\alpha[T_{33}(\mathbf{v}^{(n+1)}) + K_{33}^{(n+1)}]\,da\,,$$

(5.9)
$$H_3^{(n+1)} = -R_{3n}(\mathbf{f},\mathbf{p}) - \int_D [T_{33}(\mathbf{v}^{(n+1)}) + K_{33}^{(n+1)}]\,da\,,$$

$$H_4^{(n+1)} = -M_{3n}(\mathbf{f},\mathbf{p}) - \int_D \varepsilon_{3\alpha\beta}x_\alpha[T_{3\beta}(\mathbf{v}^{(n+1)}) + K_{3\beta}^{(n+1)}]\,da\,;$$

(5.10)
$$H_\alpha^{(n+2)} = -R_{\alpha n}(\mathbf{f},\mathbf{p}), \qquad H_3^{(n+2)} = H_4^{(n+2)} = 0\,.$$

In view of the notation (4.27), the relation (5.5) can be written in the following matrix form:

(5.11)
$$\mathcal{D}(0)a^{(k)}(t) + \int_0^t \dot{\mathcal{D}}(t-z)a^{(k)}(z)\,dz = H^{(k)}(t), \qquad k = 0,1,\ldots,n+2\,,$$

where

$$a^{(k)}(t) = (\widehat{a}^{(k)}(t))^T \quad \text{and} \quad H^{(k)}(t) = (H_1^{(k)}(t), H_2^{(k)}(t), H_3^{(k)}(t), H_4^{(k)}(t))^T\,.$$

Since $\mathcal{D}(0)$ is invertible, from relation (5.11) we deduce

(5.12)
$$a^{(k)}(t) + \int_0^t [\mathcal{D}(0)]^{-1}\dot{\mathcal{D}}(t-z)a^{(k)}(z)\,dz = [\mathcal{D}(0)]^{-1}H^{(k)}(t), \qquad k = 0,1,\ldots,n+2\,.$$

For $H^{(k)}(t)$ continuous on $[0,T)$, the Volterra integral equation (5.12) has one and only one solution $a^{(k)}(t)$ continuous in $[0,T)$, which can be obtained by the method of successive approximation [7].

Now, by means of the relations (4.41) and (4.42), the quasi-static equilibrium equations (2.9) and lateral boundary conditions (2.10) are satisfied if

(5.13)
$$T_{\alpha i,\alpha}(\mathbf{v}^{(k)}) + (K_{\alpha i}^{(k)} + L_{\alpha i}^{(k)})_{,\alpha} + \sum_{s=1}^{4} a_s^{(k+1)} \otimes S_{3i}(\mathbf{u}^{(s)}) + K_{3i}^{(k+1)} + L_{3i}^{(k+1)} + T_{3i}(\mathbf{v}^{(k+1)}) + f_{ik} = 0 \quad \text{in } D\,,$$

$$T_{\alpha i}(\mathbf{v}^{(k)})n_\alpha = -(K^{\cdot(k)}_{\alpha i} + L^{(k)}_{\alpha i})n_\alpha + p_{ik} \quad \text{on } \partial D, \quad k = 0,1,\ldots,n-1; \tag{5.13 [cont.]}$$

$$\begin{gathered} T_{\alpha i,\alpha}(\mathbf{v}^{(n)}) + (K^{\cdot(n)}_{\alpha i} + L^{(n)}_{\alpha i})_{,\alpha} + \sum_{s=1}^{4} a_s^{(n+1)} \otimes S_{3i}(\mathbf{u}^{(s)}) \\ + K^{\cdot(n+1)}_{3i} + T_{3i}(\mathbf{v}^{(n+1)}) + f_{in} = 0 \quad \text{in } D, \\ T_{\alpha i}(\mathbf{v}^{(n)})n_\alpha = -(K^{\cdot(n)}_{\alpha i} + L^{(n)}_{\alpha i})n_\alpha + p_{in} \quad \text{on } \partial D; \end{gathered} \tag{5.14}$$

$$\begin{gathered} T_{\alpha i,\alpha}(\mathbf{v}^{(n+1)}) + K^{\cdot(n+1)}_{\alpha i,\alpha} + \sum_{s=1}^{4} a_s^{(n+2)} \otimes S_{3i}(\mathbf{u}^{(s)}) = 0 \quad \text{in } D, \\ T_{\alpha i}(\mathbf{v}^{(n+1)})n_\alpha = -K^{\cdot(n+1)}_{\alpha i} n_\alpha \quad \text{on } \partial D. \end{gathered} \tag{5.15}$$

It is easy to see that the necessary and sufficient conditions for the existence of a solution for each plane boundary value problem defined by relations (5.13), (5.14) or (5.15) are satisfied on the basis of the relations (5.5).

Finally, we remark that the calculation must be made in the following order. First, we determine $a_s^{(n+2)}$ ($s = 1, 2, 3, 4$) as a solution of the Volterra integral equation (5.12) for $k = n+2$. Then we substitute these values into (5.15) and determine $v_i^{(n+1)}$ as a solution of the plane boundary-value problem defined by (5.15). Further, we determine $a_s^{(n+1)}$ ($s = 1, 2, 3, 4$) from relation (5.12) for $k = n+1$ and then we get $v_i^{(n)}$ as a solution of the plane boundary-value problem defined by (5.14). We determine $a_s^{(n)}$ ($s = 1, 2, 3, 4$) from relation (5.12) for $k = n$ and then we get $v_i^{(n-1)}$ from relation (5.13) for $k = n-1$, and so on.

Therefore, a solution of the relaxed Almansi's problem has the form (5.3) in which $a_s^{(k)}(t)$ ($s = 1, 2, 3, 4$; $k = 0, 1, \ldots, n+2$) are determined from relation (5.12), and the unknown functions $v_i^{(k)}$ ($k = 0, 1, \ldots, n+1$) are determined from the plane boundary value problems defined by relations (5.13)–(5.15).

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