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Direct and variational methods in forming theories of plates

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IN THE PRESENT paper general kinematic assumptions used in theories of plates are adopted and, basing on them, rule for finding the energy-consistent governing equations is given by means of a direct method, hence circumventing the need of using the variational calculus.

1. Introduction

IN THE THEORIES of rods, plates and shells the original three-dimensional problems of deformable body mechanics are reduced to the one- or two-dimensional problems which are easier to handle. Therefore, all fields describing the response of slender or thin bodies should vary with respect to two or one direction in an *a priori* known manner. For instance, in the theory of plates all quantities should explicitly depend on the z coordinate characterizing the distance of a point from the reference plane. One can introduce this relation in various manners and that is why we face now a variety of plate theories developed in the past and being still refined. Various kinematic and static (stress) assumptions have been adopted. To arrive at the governing equations two approaches are at our disposal:

- a direct approach, called here the effective causes method;
- a variational approach (final causes method).

The following beautiful words written by Euler in 1744 refer to both the approaches, cf. [1, p. 31]:

"Since the fabric of the universe is most perfect, and is the work of a most wise Creator, nothing whatsoever takes place in the universe in which some relation of maximum and minimum does not appear. Wherefore there is absolutely no doubt that every effect in the universe can be explained as satisfactorily from final causes, by the aid of the method of maxima and minima, as it can from the effective causes themselves ... Therefore, two methods of studying effects in Nature lie open to us, one by means of effective causes, which is commonly called the direct method, the other by means of final causes ... One ought to make a special effort to see that both ways of approach to the solution of the problem be laid open; for this not only is one solution greatly strengthened by the other, but, more than that, from the agreement between the two solutions we secure the very highest satisfaction."

In the direct method the governing equations of the theory are obtained by using the local equations of the theory of elasticity. Some of these equations are satisfied pointwise and other are averaged. As usual we average the equilibrium equations by integrating over the thickness with the z^n weighting functions. The weighted functions can be suitably chosen for particular equations. Hence the problem of choice of weight functions arises. This choice would be optimal if for a given kinematic hypothesis both methods would lead to the identical governing equations.

The plan of the present paper is the following. We assume a certain general kinematic hypothesis suitable for plates. Then we put forward a method of construction of weighted functions and define a sequence of differential operations which make the governing equations obtained by the direct method coincide with those found by the variational method.

The domain occupied by the plate will be parametrized by a right-handed normal coordinate system $\{x^\alpha, x^3 = z\}$. This system is defined by one family of planes parallel to a reference plane and by two families of cylindrical surfaces orthogonal to these planes. The x^3 coordinate will be denoted by z. Partial differentiation with respect to x^α variables will be denoted briefly by a comma. In initial configuration the reference plane of the plate will by denoted by A.

Ω represent the domain occupied by the plate in this original configuration. It can be represented by a Cartesian product

$$\Omega = A \times (-h_2(x^\alpha), h_1(x^\alpha)),$$

where $z = h_1(x^\alpha)$ parametrizes the lower face of the plate and $z = -h_2(x^\alpha)$ determines its upper face. The boundary surface $\partial\Omega$ can be defined as follows

$$\partial\Omega = \overset{+}{A} \cup \overset{-}{A} \cup A_s, \qquad A_s = S \times (-h_2, h_1).$$

Here $\overset{+}{A}$, $\overset{-}{A}$ stand for the upper and lower faces of the plate, A_s being its lateral cylindrical surface. S denotes the boundary line of the reference plane. This boundary line is determined by the intersection of the A_s surface and the reference plane. The loads of densities $\overset{+}{\mathrm{p}}_\alpha$, $\overset{+}{\mathrm{p}}_3$, $\overset{-}{\mathrm{p}}_\alpha$, $\overset{-}{\mathrm{p}}_3$ per unit of the reference surfaces acting on the lower face (sign $+$) and on the upper face (sign $-$) are assumed to be conservative.

The summation convention over repeated Greek indices running over 1, 2 will be adopted.

2. Variational approach

The virtual work equation of the elasticity theory can be written in the form

$$\int_\Omega [t^{\beta\alpha}\delta u_{\alpha,\beta} + t^{3\alpha}\delta(u_{\alpha,3} + u_{3,\alpha}) + t^{33}\delta u_{3,3}]dV = \int_\Omega [X^\alpha \delta u_\alpha + X^3 \delta u_3]\, dV + \int_{\partial\Omega} [\mathrm{p}^\alpha \delta u_\alpha + \mathrm{p}^3 \delta u_3]\, dA. \tag{2.1}$$

Here $t^{\alpha\beta}, t^{3\alpha}, t^{33}$ are components of stresses; u_α, u_3 represent displacements; X_α, X_3 are components of the body force vector and $\mathrm{p}_\alpha, \mathrm{p}_3$ are components of the loads applied to the boundary surface of the body.

Let us represent the components of the displacement vector in the form

$$u_\alpha(x^\gamma, z) = -\sum_{k=1}^{M} \mathfrak{m}_k(z)\mathfrak{M}_k(w_k(x^\gamma)_{,\alpha}) - \sum_{k=1}^{P} \mathfrak{f}_k(z)\mathfrak{F}_k(\theta_{\alpha k}(x^\gamma)) + \sum_{k=1}^{N} \mathfrak{n}_k(z)\mathfrak{N}_k(\phi_{\alpha k}(x^\gamma)), \tag{2.2}$$

$$u_3(x^\gamma, z) = \sum_{k=1}^{M} \mathfrak{s}_k(z)\mathfrak{S}_k(w_k(x^\gamma)) + \sum_{k=1}^{P} \mathfrak{p}_k(z)\mathfrak{P}_k(\theta_k^\alpha(x^\gamma)_{,\alpha}) + \sum_{k=1}^{R} \mathfrak{r}_k(z)\mathfrak{R}_k(v_k(x^\gamma)), \tag{2.3}$$

where $\mathfrak{m}_k(z)$, $\mathfrak{n}_k(z)$, $\mathfrak{f}_k(z)$, $\mathfrak{p}_k(z)$, $\mathfrak{s}_k(z)$ and $\mathfrak{r}_k(z)$ are known functions of the z variable and the functions w_k, $\phi_{\alpha k}$, $\theta_{\alpha k}$, v_k are unknown functions defined on the plane of reference; M, N, P and R are arbitrary integers.

The differential operators $\mathfrak{M}_k$, $\mathfrak{N}_k$, $\mathfrak{S}_k$, $\mathfrak{P}_k$, $\mathfrak{F}_k$ and $\mathfrak{R}_k$ are defined by

$$\begin{aligned} \mathfrak{M}_k &= \sum_{i=0}^{M_k} a_i\nabla^{2i}, & \mathfrak{N}_k &= \sum_{i=0}^{N_k} b_i\nabla^{2i}, & \mathfrak{S}_k &= \sum_{i=0}^{M_k} c_i\nabla^{2i}, \\ \mathfrak{R}_k &= \sum_{i=0}^{R_k} d_i\nabla^{2i}, & \mathfrak{P}_k &= \sum_{i=0}^{P_k} e_i\nabla^{2i}, & \mathfrak{F}_k &= \sum_{i=0}^{P_k} f_i\nabla^{2i}, \end{aligned} \tag{2.4}$$

where M_k, N_k, R_k and P_k are given integers and a_i, b_i, c_i, d_i, e_i and f_i are given coefficients. These operators will be called accompanying operators with proper functions of the z variable (e.g. operator $\mathfrak{M}_k(..)_{,\alpha}$ accompanies function $\mathfrak{m}_k(z)$ and so on).

Let us define the averaged quantities

$$\begin{aligned} \underset{k}{M}_{\beta\alpha}(x^\gamma) &= \int_{h_2}^{h_1} t_{\beta\alpha}\mathfrak{m}_k(z)\,dz, & \underset{k}{N}_{\beta\alpha}(x^\gamma) &= \int_{-h_2}^{h_1} t_{\beta\alpha}\mathfrak{n}_k(z)\,dz, \\ \underset{k}{H}_{\beta\alpha}(x^\gamma) &= \int_{-h_2}^{h_1} t_{\beta\alpha}\mathfrak{f}_k(z)\,dz, & \underset{k}{Q}_{\alpha}(x^\gamma) &= \int_{-h_2}^{h_1} t_{3\alpha}\frac{d\mathfrak{m}_k(z)}{dz}\,dz, \\ \underset{k}{P}_{\alpha}(x^\gamma) &= \int_{-h_2}^{h_1} t_{3\alpha}\frac{d\mathfrak{n}_k(z)}{dz}\,dz, & \underset{k}{F}_{\alpha}(x^\gamma) &= \int_{-h_2}^{h_1} t_{3\alpha}\frac{d\mathfrak{f}_k(z)}{dz}\,dz \\ \underset{k}{V}_{\alpha}(x^\gamma) &= \int_{-h_2}^{h_1} \mathfrak{r}_k(z)t_{\alpha 3}\,dz, & \underset{k}{T}_{\alpha}(x^\gamma) &= \int_{-h_2}^{h_1} \mathfrak{s}_k(z)t_{\alpha 3}\,dz, \\ \underset{k}{G}_{\alpha}(x^\gamma) &= \int_{-h_2}^{h_1} \mathfrak{p}_k(z)t_{\alpha 3}\,dz, & \underset{k}{S}(x^\gamma) &= \int_{-h_2}^{h_1} t_{33}\frac{d\mathfrak{s}_k(z)}{dz}\,dz, \\ \underset{k}{R}(x^\gamma) &= \int_{-h_2}^{h_1} t_{33}\frac{d\mathfrak{r}_k(z)}{dz}\,dz, & \underset{k}{K}(x^\gamma) &= \int_{-h_2}^{h_1} t_{33}\frac{d\mathfrak{p}_k(z)}{dz}\,dz. \end{aligned} \tag{2.5}$$

On substituting (2.2), (2.3) into the virtual work equation (2.1), performing z-integration, making some rearrangements and using the notations (2.5), one arrives at

$$\sum_{k=1}^{M} \int_A [\mathfrak{M}_k(\underset{k}{M}^{\beta\alpha}{}_{,\beta\alpha} - \underset{k}{Q}^{\alpha}{}_{,\alpha} + \underset{k}{m}^{\alpha}{}_{,\alpha}) + \mathfrak{S}_k(\underset{k}{T}^{\alpha}{}_{,\alpha} + q_k)]\delta w_k\,dA \tag{2.6}$$

$$\text{(2.6) [cont.]}\qquad +\sum_{k=1}^{N}\int_A \mathfrak{N}_k(\underset{k}{N}^{\beta}{}_{\alpha,\beta} - \underset{k}{P}_{\alpha} + \underset{k}{\mathfrak{p}}_{\alpha})\delta\phi_k^{\alpha}\,dA + \sum_{k=1}^{R}\int_A \mathfrak{R}_k(\underset{k}{V}^{\alpha}{}_{,\alpha} + \mathfrak{g}_k)\delta v_k\,dA$$
$$-\sum_{k=1}^{P}\int_A [\mathfrak{F}_k(\underset{k}{H}^{\beta}{}_{\alpha,\beta} - \underset{k}{F}_{\alpha} + \underset{k}{\mathfrak{m}}_{\alpha}) + \mathfrak{P}_k(\underset{k}{G}^{\beta}{}_{,\beta\alpha} + \mathfrak{q}_{k,\alpha})]\delta\theta_k^{\alpha}\,dA +$$
$$\int_S [\ldots]\,dS + [[\ldots]] = 0.$$

The following new quantities have been introduced

$$\begin{aligned}
\underset{k}{m}_{\alpha} &= \mathfrak{m}_k(h_1)\,\overset{+}{p}_{\alpha} + \mathfrak{m}_k(-h_1)\,\overline{p}_{\alpha} + \int_{-h_2}^{h_1}\mathfrak{m}_k(z)X_{\alpha}\,dz,\\
\underset{k}{\mathfrak{m}}_{\alpha} &= \mathfrak{f}_k(h_1)\,\overset{+}{p}_{\alpha} + \mathfrak{f}_k(-h_2)\,\overline{p}_{\alpha} + \int_{-h_2}^{h_1}\mathfrak{f}_k(z)X_{\alpha}\,dz,\\
q_k &= \mathfrak{s}_k(h_1)\,\overset{+}{p}_3 + \mathfrak{s}_k(-h_2)\,\overline{p}_3 + \int_{-h_2}^{h_1}\mathfrak{s}_k(z)X_3\,dz - \underset{k}{S},\\
\mathfrak{q}_k &= \mathfrak{p}_k(h_1)\,\overset{+}{p}_3 + \mathfrak{p}_k(-h_2)\,\overline{p}_3 + \int_{-h_2}^{h_1}\mathfrak{p}_k(z)X_3\,dz - \underset{k}{K},\\
\mathfrak{p}_k^{\alpha} &= \mathfrak{n}_k(h_1)\,\overset{+}{p}_{\alpha} + \mathfrak{n}_k(-h_2)\,\overline{p}_{\alpha} + \int_{-h_2}^{h_1}\mathfrak{n}_k(z)X_{\alpha}\,dz,\\
\underset{k}{\mathfrak{g}} &= \mathfrak{r}_k(h_1)\,\overset{+}{p}_3 + \mathfrak{r}_k(-h_2)\,\overline{p}_3 + \int_{-h_2}^{h_1}\mathfrak{r}_k(z)X_3\,dz - \underset{k}{R},
\end{aligned}\tag{2.7}$$

$$\overset{+}{p}_{\alpha} = \overset{+}{\mathrm{p}}_{\alpha}\overset{+}{\omega},\quad \overset{+}{p}_3 = \overset{+}{\mathrm{p}}_3\overset{+}{\omega},\quad \overline{p}_{\alpha} = \overline{\mathrm{p}}_{\alpha}\overline{\omega},\quad \overline{p}_3 = \overline{\mathrm{p}}_3\overline{\omega},$$
$$\overset{+}{\omega} = \sqrt{1 + h_{1,\alpha}h_{1,}{}^{\alpha}},\quad \overline{\omega} = \sqrt{1 + h_{2,\alpha}h_{2,}{}^{\alpha}}.$$

Since we shall not deal with boundary conditions, we shall not specify the integrands of the boundary integral and the possible jumps [[...]] at points of the boundary.

The equation (2.6) should be satisfied for arbitrary variations of displacements within the plate domain and at its boundary. Due to this arbitrariness all the integrals can be (separately) equated to zero. In general the variations δw_k, $\delta\phi_{\alpha k}$, $\delta\theta_{\alpha k}$ and δv_k can be assumed as independent and arbitraty within the A domain. Hence we obtain the following local equations of equilibrium

$$\mathfrak{M}_k(\underset{k}{M}^{\beta\alpha}{}_{,\beta\alpha} - \underset{k}{Q}^{\alpha}{}_{,\alpha} + \underset{k}{m}^{\alpha}{}_{,\alpha}) + \mathfrak{S}_k(\underset{k}{T}^{\alpha}{}_{,\alpha} + q_k) = 0,\quad k = 1,2,\ldots,M,\tag{2.8}$$

$$\mathfrak{F}_k(\underset{k}{H}^{\beta}{}_{\alpha,\beta} - \underset{k}{F}_{\alpha} + \underset{k}{\mathfrak{m}}_{\alpha}) + \mathfrak{P}_k(\underset{k}{G}^{\beta}{}_{,\beta\alpha} + \mathfrak{q}_{k,\alpha}) = 0,\quad k = 1,2,\ldots,P,\tag{2.9}$$

$$\mathfrak{N}_k(\underset{k}{N}^{\beta}{}_{\alpha,\beta} - \underset{k}{P}_{\alpha} + \underset{k}{\mathfrak{p}}_{\alpha}) = 0,\quad k = 1,2,\ldots,N,\tag{2.10}$$

$$\mathfrak{R}_k(\underset{k}{V}^{\alpha}{}_{,\alpha} + \mathrm{g}_k) = 0, \quad k = 1, 2, \ldots, R. \tag{2.11}$$

If we assume for instance $w_1 = w_2 = w_3 = w$, $\phi_{\alpha 1} = \phi_{\alpha 2} = \phi_\alpha$, $v_1 = v_2 = v$ then Eq. (2.6) implies the local equations of the form

$$\begin{aligned}
&\mathfrak{M}_1(\underset{1}{M}^{\beta\alpha}{}_{,\beta\alpha} - \underset{1}{Q}^{\alpha}{}_{,\alpha} + \underset{1}{m}^{\alpha}{}_{,\alpha}) + \mathfrak{S}_1(\underset{1}{T}^{\alpha}{}_{,\alpha} + q_1)\\
&\qquad + \mathfrak{M}_2(\underset{2}{M}^{\beta\alpha}{}_{,\beta\alpha} - \underset{2}{Q}^{\alpha}{}_{,\alpha} + \underset{2}{m}^{\alpha}{}_{,\alpha}) + \mathfrak{S}_2(\underset{2}{T}^{\alpha}{}_{,\alpha} + q_2)\\
&\qquad + \mathfrak{M}_3(\underset{3}{M}^{\beta\alpha}{}_{,\beta\alpha} - \underset{3}{Q}^{\alpha}{}_{,\alpha} + \underset{3}{m}^{\alpha}{}_{,\alpha}) + \mathfrak{S}_3(\underset{3}{T}^{\alpha}{}_{,\alpha} + q_3) = 0,\\
&\mathfrak{M}_k(\underset{k}{M}^{\beta k}{}_{,\beta\alpha} - \underset{k}{Q}^{\alpha}{}_{,\alpha} + \underset{k}{m}^{\alpha}{}_{,\alpha}) + \mathfrak{S}_k(\underset{k}{T}^{\alpha}{}_{,\alpha} + q_k) = 0, \quad k = 4, 5, \ldots, M,\\
&\mathfrak{N}_1(\underset{1}{N}^{\beta\alpha}{}_{,\beta} - \underset{1}{P}^{\alpha} + \underset{1}{\mathrm{p}}^{\alpha}) + \mathfrak{N}_2(\underset{2}{N}^{\beta\alpha}{}_{,\beta} - \underset{2}{P}^{\alpha} + \underset{2}{\mathrm{p}}^{\alpha}) = 0,\\
&\mathfrak{N}_k(\underset{k}{N}^{\beta\alpha}{}_{,\beta} - \underset{k}{P}^{\alpha} + \underset{k}{\mathrm{p}}^{\alpha}) = 0, \quad k = 3, 4, \ldots, N,\\
&\mathfrak{R}_1(\underset{1}{V}^{\alpha}{}_{,\alpha} + \mathrm{g}_1) + \mathfrak{R}_2(\underset{2}{V}^{\alpha}{}_{,\alpha} + \mathrm{g}_2) = 0,\\
&\mathfrak{R}_k(\underset{k}{V}^{\alpha}{}_{,\alpha} + \mathrm{g}_k) = 0, \quad k = 3, 4, \ldots, R,\\
&\mathfrak{F}_k(\underset{k}{H}^{\beta}{}_{\alpha,\beta} - \underset{k}{F}_{\alpha} + \underset{k}{\mathrm{m}}_{\alpha}) + \mathfrak{P}_k(\underset{k}{G}^{\beta}{}_{,\beta\alpha} + \mathrm{q}_{k,\alpha}) = 0, \quad k = 1, 2, \ldots, P.
\end{aligned} \tag{2.12}$$

If we put

$$\begin{aligned}
\mathfrak{M}_k &= \mathfrak{S}_k, \quad \mathfrak{s}_k = \frac{d\mathrm{m}_k(z)}{dz},\\
\mathfrak{F}_k &= \mathfrak{P}_k, \quad \mathfrak{p}_k = \frac{d\mathfrak{f}_k(z)}{dz},
\end{aligned} \tag{2.13}$$

then one can reduce Eqs. (2.8), (2.9) to the form

$$\mathfrak{M}_k(\underset{k}{M}^{\beta\alpha}{}_{,\beta\alpha} + \underset{k}{m}^{\alpha}{}_{,\alpha} + q_k) = 0, \quad k = 1, 2, \ldots, M, \tag{2.14}$$

$$\mathfrak{F}_k(\underset{k}{H}^{\beta}{}_{\alpha,\beta} - \underset{k}{F}_{\alpha} + \underset{k}{F}^{\beta}{}_{,\beta\alpha} + \underset{k}{\mathrm{m}}_{\alpha} + \mathrm{q}_{k,\alpha}) = 0, \quad k = 1, 2, \ldots, P. \tag{2.15}$$

All equations obtained above are energy-consistent.

3. Direct method

The differential equilibrium equations of the body can be written in the form

$$t^{\beta\alpha}{}_{,\beta} + t^{3\alpha}{}_{,3} + X^{\alpha} = 0, \tag{3.1}$$

$$t^{\beta 3}{}_{,\beta} + t^{33}{}_{,3} + X^{3} = 0. \tag{3.2}$$

Let us multiply Eq. (3.1) by m_k and integrate with respect to z in the limits $(-h_2, h_1)$. We obtain

$$\int_{-h_2}^{h_1} \mathrm{m}(z) t^{\beta\alpha}{}_{,\beta}\, dz + \int_{-h_2}^{h_1} \mathrm{m}_k(z) t^{3\alpha}{}_{,3}\, dz + \int_{-h_2}^{h_1} \mathrm{m}_k(z) X^{\alpha}\, dz = 0. \tag{3.3}$$

The following identities hold

$$\int\limits_{-h_2}^{h_1} \mathfrak{m}_k(z) t^{\beta\alpha}{}_{,\beta}\, dz = \Big\{ \int\limits_{-h_2}^{h_1} \mathfrak{m}_k(z) t^{\beta\alpha}\, dz \Big\}_{,\beta}$$

$$-\mathfrak{m}_k(h_1) h_{1,\beta} t^{\beta\alpha}(x^\gamma, h_1) - \mathfrak{m}_k(-h_2) h_{2,\beta} t^{\beta\alpha}(x^\gamma, -h_2)$$

$$= \underset{k}{M}{}^{\beta\alpha}{}_{,\beta} - h_{1,\beta}\mathfrak{m}_k(h_1) t^{\beta\alpha}(x^\gamma, h_1) - h_{2,\beta}\mathfrak{m}_k(-h_2) t^{\beta\alpha}(x^\gamma, -h_2),$$

$$\int\limits_{-h_2}^{h_1} \mathfrak{m}_k(z) t^{3\alpha}{}_{,3} dz = - \int\limits_{-h_2}^{h_1} t^{3\alpha} \frac{d\mathfrak{m}_k(z)}{dz}\, dz + \mathfrak{m}_k(h_1) t^{3\alpha}(x^\gamma, h_1) \tag{3.4}$$

$$-\mathfrak{m}_k(-h_2) t^{3\alpha}(x^\gamma, -h_2)$$

$$= -\underset{k}{Q}{}^{\alpha} + \mathfrak{m}_k(h_1) t^{3\alpha}(x^\gamma, h_1) - \mathfrak{m}_k(-h_2) t^{3\alpha}(x^\gamma, -h_2).$$

The following boundary conditions should be satisfied on the faces $z = h_1(x^\alpha)$, $z = -h_2(x^\alpha)$ of the plate

$$\begin{aligned}
\overset{+}{\mathrm{p}}_\alpha &= t^\beta{}_\alpha(x^\gamma, h_1)\, \overset{+}{n}_\beta + t_{3\alpha}(x^\gamma, h_1)\, \overset{+}{n}_3,\\
\overset{-}{\mathrm{p}}_\alpha &= t^\beta{}_\alpha(x^\gamma, -h_2)\, \overset{-}{n}_\beta + t_{3\alpha}(x^\gamma, -h_2)\, \overset{-}{n}_3,\\
\overset{+}{\mathrm{p}}_3 &= t^\beta{}_3(x^\gamma, h_1)\, \overset{+}{n}_\beta + t_{33}(x^\gamma, h_1)\, \overset{+}{n}_3,\\
\overset{-}{\mathrm{p}}_3 &= t^\beta_3(x^\gamma, -h_2)\, \overset{-}{n}_\beta + t_{33}(x^\gamma, -h_2)\, \overset{-}{n}_3,
\end{aligned} \tag{3.5}$$

where $\overset{+}{n}_\alpha, \overset{+}{n}_3, \overset{-}{n}_\alpha, \overset{-}{n}_3$ represent components of vectors outwardly normal to the lower and upper plate faces. These components are given by the formulae

$$\begin{aligned}
\overset{+}{n}_\alpha &= -\frac{h_{1,\alpha}}{\overset{+}{\omega}}, & \overset{+}{n}_3 &= \frac{1}{\overset{+}{\omega}},\\
\overset{-}{n}_\alpha &= -\frac{h_{2,\alpha}}{\overset{-}{\omega}}, & \overset{-}{n}_3 &= -\frac{1}{\overset{-}{\omega}}.
\end{aligned} \tag{3.6}$$

Using notation (2.5), (2.7) and considering (3.4)–(3.6) one can rearrange Eq. (3.3) to the form

$$\underset{k}{M}{}^{\beta}{}_{\alpha,\beta} - \underset{k}{Q}{}_\alpha + \underset{k}{m}{}_\alpha = 0, \qquad k = 1, 2, \ldots, M. \tag{3.7}$$

Performing now the differential operation $\mathfrak{M}_k(\ldots)_{,\alpha}$ for the subsequent values of k we obtain (do not sum over k)

$$\mathfrak{M}_k(\underset{k}{M}{}^{\beta\alpha}{}_{,\beta\alpha} - \underset{k}{Q}{}^{\alpha}{}_{,\alpha} + \underset{k}{m}{}^{\alpha}{}_{,\alpha}) = 0, \qquad k = 1, 2, \ldots, M, \tag{3.8}$$

Now, let us multiply both sides of Eq. (3.2) by $\mathfrak{s}_k$, perform integration through the thickness and then perform the differential operation $\mathfrak{S}_k$. After making rearrangements similar to those that had lead us to Eq. (3.3), we find the equation

$$\mathfrak{S}_k(\underset{k}{T}{}^{\alpha}{}_{,\alpha} + q_k) = 0, \qquad k = 1, 2, \ldots, M. \tag{3.9}$$

Adding Eqs. (3.8) and (3.9) we obtain Eq. (2.8).

Let us average Eq. (3.1) with a weighting function $\mathfrak{f}_k(z)$ and then perform the differential operation $\mathfrak{F}_k$. We arrive at

$$\mathfrak{F}(\underset{k}{H}{}^{\beta}{}_{\alpha,\beta} - \underset{k}{F}{}_{\alpha} + \underset{k}{\mathfrak{m}}{}_{\alpha}) = 0, \quad k = 1, 2, \ldots, P. \tag{3.10}$$

Let us average now Eq. (3.2) with a weighting function $\mathfrak{p}_k(z)$ and then perform the differential operation $\mathfrak{P}_k(\ldots)_{,\alpha}$. We find

$$\mathfrak{P}_k(\underset{k}{G}{}^{\beta}{}_{,\beta\alpha} + \mathfrak{q}_{k,\alpha}) = 0, \quad k = 1, 2, \ldots, P. \tag{3.11}$$

Upon adding equations (3.10), (3.11) we obtain Eq. (2.9).

Similarly, let us multiply Eq. (3.1) by $\mathfrak{n}_k(z)$ and perform the integration in $\int_{-h_2}^{h_1}(\ldots)\,dz$, and then perform the differential operation $\mathfrak{N}_k$. After appropriate rearrangements the equation (2.10) is found.

Let us multiply Eq. (3.2) by $\mathfrak{r}_k(z)$, integrate over the thickness and perform the operation $\mathfrak{R}_k$. We arrive then at (2.11).

Therefore one can readily see that correct, energy-consistent, two-dimensional equilibrium equations can be obtained by the direct method if we perform a series of operations as follows:

1. The differential equilibrium equations (3.1) are multiplied by the known functions $(\mathfrak{m}_k(z), \mathfrak{f}_k(z), \mathfrak{n}_k(z))$ of z variable and integrated across the plate thickness, respectively. As a result, for given α ($\alpha = 1, 2$) the three groups of differential equations of the two variables x^α are obtained.

2. The differential equilibrium equation (3.2) is multiplied by the known functions $(\mathfrak{p}_k(z), \mathfrak{s}_k(z), \mathfrak{r}_k(z))$ of z variable and integrated along the plate thickness, respectively. As a result, the next three groups of differential equations of the two variables x^α are obtained.

3. The differential equations obtained in the step 1 are subjected to the differentiation operations by means of the accompanying operators $\mathfrak{M}_k(..)_{,\alpha}, \mathfrak{F}_k, \mathfrak{N}_k$, respectively.

4. The differential equations obtained in the step 2 are subjected to the differentiation operations by means of the accompanying operators $\mathfrak{S}_k, \mathfrak{P}_k(..)_{,\alpha}, \mathfrak{R}_k$, respectively.

5. Since the unknown function $w_k(x^\alpha)$ simultaneously described the displacements u_α and u_3, hence we sum up the equations obtained as a result of the operations $\mathfrak{M}_k(..)_{,\alpha}$ and $\mathfrak{S}_k$, respectively.

6. Since the unknown function $\theta_{\alpha k}(x^\beta)$ simultaneously described the displacements u_α and u_3, hence we sum up the equations obtained as a result of the operations $\mathfrak{F}_k$ and $\mathfrak{P}_k(..)_{,\alpha}$, respectively.

7. If the unknown functions of the x^α variables (e.g. w_1, w_2) are the same (i.e. $w_1 = w_2 = w$) but the differential operators or the known functions of z variable (accompanying them) are different, then we sum up the appropriate differential equations obtained in the way given above.

Thus we conclude that the method described above, which augments the direct method, makes it possible to derive the correct differential equations coinciding with those derived via the variational calculus.

4. Examples

The method of deriving energy-consistent equations of equilibrium will be illustrated by some examples.

In many energy-inconsistent plate theories the equations of equilibrium are usually derived by averaging the Eqs. (3.1) with the weights $(1, z)$ and averaging Eq. (3.2) with the weight (1). Then we obtain the following five differential equations

$$N^{\beta\alpha}{}_{,\beta} + p^{\alpha} = 0, \quad M^{\beta\alpha}{}_{,\beta} - Q^{\alpha} + m^{\alpha} = 0, \quad V^{\alpha}_{,\alpha} + q = 0, \tag{4.1}$$

where

$$N_{\beta\alpha}(x^{\gamma}) = \int_{-h_2}^{h_1} t_{\beta\alpha}\, dz, \quad M_{\beta\alpha}(x^{\gamma}) = \int_{-h_2}^{h_1} z t_{\beta\alpha}\, dz, \quad Q_{\alpha}(x^{\gamma}) = \int_{-h_2}^{h_1} t_{3\alpha}\, dz,$$

$$V_{\alpha}(x^{\gamma}) = \int_{-h_2}^{h_1} t_{\alpha 3}\, dz = Q_{\alpha}(x^{\gamma}), \quad p_{\alpha} = \overset{+}{p}_{\alpha} + \overset{-}{p}_{\alpha} + \int_{-h_2}^{h_1} X_{\alpha}\, dz, \tag{4.2}$$

$$m_{\alpha} = h_1 \overset{+}{p}_{\alpha} - h_2 \overset{-}{p}_{\alpha} + \int_{-h_2}^{h_1} z X_{\alpha}\, dz, \quad q = \overset{+}{p}_{3} + \overset{-}{p}_{3} + \int_{-h_2}^{h_1} X_3\, dz.$$

These equations are compatible with the Hencky–Bolle hypothesis [2–4]

$$\begin{aligned} u_{\alpha}(x^{\gamma}, z) &= \underset{0}{u}_{\alpha}(x^{\gamma}) - z\phi_{\alpha}, \\ u_3(x^{\gamma}, z) &= w(x^{\gamma}), \end{aligned} \tag{4.3}$$

which can be found from (2.2) by assuming

$$\begin{gathered} \mathfrak{n}_1 = 1, \quad \mathfrak{n}_2 = -z, \quad \mathfrak{N}_1 = \mathfrak{N}_2 = 1, \quad \mathfrak{R}_1 = 1, \quad \mathfrak{r}_1 = 1, \\ \phi_{\alpha 1} = \underset{0}{u}_{\alpha}(x^{\beta}), \quad \phi_{\alpha 2} = \phi_{\alpha}, \quad v_1 = w, \\ \underset{1}{R}(x^{\gamma}) = 0, \quad \underset{1}{N}_{\beta\alpha}(x^{\gamma}) = N_{\beta\alpha}(x^{\gamma}), \quad \underset{2}{N}_{\beta\alpha}(x^{\gamma}) = -M_{\beta\alpha}, \quad \mathfrak{g}_1 = q, \\ \underset{1}{P}_{\alpha}(x^{\gamma}) = 0, \quad \underset{1}{V}_{\alpha} = V_{\alpha} = -\underset{2}{P}_{\alpha}(x^{\gamma}) = Q_{\alpha}, \quad \underset{1}{\mathfrak{p}}_{\alpha} = p_{\alpha}, \quad \underset{2}{\mathfrak{p}}_{\alpha} = -m_{\alpha}. \end{gathered} \tag{4.4}$$

In the case of Navier–Kirchhoff hypothesis

$$\begin{aligned} u_{\alpha}(x^{\gamma}, z) &= \underset{0}{u}_{\alpha}(x^{\gamma}) - z w(x^{\gamma})_{,\alpha}, \\ u_3(x^{\gamma}, z) &= w(x^{\gamma}), \end{aligned} \tag{4.5}$$

one should put

$$\begin{gathered} \mathfrak{m}_1 = z, \quad \mathfrak{M}_1 = 1, \quad \mathfrak{n}_1 = 1, \quad \mathfrak{N}_1 = 1, \quad \mathfrak{s}_1 = 1, \\ \mathfrak{S}_1 = 1, \quad v_k = 0, \quad \phi_{\alpha 1} = \underset{0}{u}_{\alpha}(x^{\beta}), \quad w_1 = w, \quad M_{\beta\alpha}(x^{\gamma}) = \underset{1}{M}_{\beta\alpha}(x^{\gamma}), \\ \underset{1}{N}_{\beta\alpha}(x^{\gamma}) = N_{\beta\alpha}(x^{\gamma}), \quad Q_{\alpha}(x^{\gamma}) = \underset{1}{T}_{\alpha}(x^{\gamma}), \\ \underset{k}{P}_{\alpha}(x^{\gamma}) = 0, \quad \underset{k}{S}(x^{\gamma}) = 0. \end{gathered} \tag{4.6}$$

In this case, according to the rules indicated in Sec. 3, one should perform integration with respect to x^{α} on Eq. $(4.1)_2$, and then add this new equation to the Eq. $(4.1)_3$. In this way we arrive at the equilibrium equation of the Kirchhoff theory, cf. (2.14)

$$M^{\beta\alpha}{}_{,\beta\alpha} + q + m^{\alpha}{}_{,\alpha} = 0. \tag{4.7}$$

The theory of VLASOV [5, 6] is based upon the following kinematical assumption

$$\begin{aligned} u_\alpha(x^\gamma, z) &= \underset{0}{u}_\alpha(x^\gamma) - \frac{4z^3}{3h^2}w_{,\alpha} - z\left(1 - \frac{4z^2}{3h^2}\right)\phi_\alpha, \\ u_3(x^\gamma, z) &= w(x^\gamma), \end{aligned} \tag{4.8}$$

where $h_1 = h_2 = h/2 = \text{const}$.

By comparing (4.8) with (2.2) one finds

$$\begin{gathered} \mathfrak{m}_1 = \frac{4z^2}{3h^2}, \quad \mathfrak{n}_1 = 1, \quad \mathfrak{n}_2 = -z\left(1 - \frac{4z^2}{3h^2}\right), \\ \mathfrak{M}_1 = \mathfrak{N}_1 = \mathfrak{N}_2 = \mathfrak{S}_1 = 1, \quad w = w_1, \quad \phi_\alpha = \phi_{\alpha 1}, \quad \mathfrak{s}_1 = 1. \end{gathered} \tag{4.9}$$

On using (4.9) we can express the stress resultants (2.4) by the formulae

$$\begin{aligned} &\underset{1}{M}_{\beta\alpha}(x^\gamma) = \frac{4}{3h^2}\int_{-h_2}^{h_1} z^3 t_{\beta\alpha}\,dz, \quad \underset{1}{N}_{\beta\alpha}(x^\gamma) = N_{\beta\alpha}(x^\gamma), \\ &\underset{2}{N}_{\beta\alpha}(x^\gamma) = -\int_{-h_2}^{h_1} z\left(1 - \frac{4z^2}{3h^2}\right)t_{\beta\alpha}\,dz, \quad \underset{1}{P}_\alpha(x^\gamma) = 0, \\ &\underset{1}{Q}_\alpha(x^\gamma) = \frac{4}{h^2}\int_{-h_2}^{h_1} z^2 t_{3\alpha}\,dz, \quad \underset{2}{P}_\alpha(x^\gamma) = -\int_{-h_2}^{h_1}\left(1 - 4\frac{z^2}{h^2}\right)t_{3\alpha}\,dz, \\ &\underset{1}{T}_\alpha(x^\gamma) = \int_{-h_2}^{h_1} t_{\alpha 3}\,dz, \quad \underset{1}{S}(x^\gamma) = 0. \end{aligned} \tag{4.10}$$

The energy-consistent equilibrium equations of the theory of Vlasov can be obtained via the variational method by using (2.7), (2.8), (4.9) and (4.10), or via the direct method following the rule of Sec. 3. Here the equations are(1)

$$\begin{aligned} &N^{\beta\alpha}{}_{,\beta} + p^\alpha = 0, \\ &\underset{1}{M}^{\beta\alpha}{}_{,\beta\alpha} - \underset{1}{Q}^{\alpha}{}_{,\alpha} + \underset{1}{m}^{\alpha}{}_{,\alpha} + \underset{1}{T}^{\alpha}{}_{,\alpha} + q_1 = 0, \\ &\underset{1}{N}^{\beta\alpha}{}_{,\beta} - \underset{1}{P}^{\alpha} + \underset{1}{\mathrm{p}}^{\alpha} = 0, \end{aligned} \tag{4.11}$$

where

$$\underset{1}{m}_\alpha = \frac{h}{6}\left(\overset{+}{p}_\alpha - \overset{-}{p}_\alpha\right) + \frac{4}{3}\int_{-h/2}^{h/2}\frac{z^3}{h^2}X_\alpha\,dz, \quad q_1 = \overset{+}{p}_3 + \overset{-}{p}_3 + \int_{-h/2}^{h/2} X_3\,dz,$$

(1) In the paper of Vlasov the equations of equilibrium were given in the form (4.1). The displacement equations of the layered plate theory based on the Vlasov kinematical assumptions at $\phi_\alpha = w_{,\alpha} - \psi_{,\alpha}$ were derived by the BOLOTIN and NOVICHKOV [7] variational method and the BHIMARADDI and STEVENS [8] one. Certain case of Eqs. (4.11) was found by REDDY [9] by the variational method.

$$\text{(4.12)}\qquad \underset{1}{\mathrm{p}}_\alpha = p_\alpha = \overset{+}{p}_\alpha + \overset{-}{p}_\alpha + \int_{-h_2}^{h_1} X_\alpha\, dz,$$

$$\underset{2}{\mathrm{p}}_\alpha = -\frac{1}{3}h\left(\overset{+}{p}_\alpha - \overset{-}{p}_\alpha\right) - \int_{-h_2}^{h_1} z\left(1-\frac{4z^2}{3h^2}\right)X_\alpha\, dz.$$

Mushtari assumed the following displacement field, [10]

$$\text{(4.13)}\qquad \begin{aligned} u_\alpha &= -zv_{,\alpha} - \frac{\nu z^3}{6(1-\nu)}\nabla^2 v_{,\alpha} + z\left(1-\frac{4z^2}{3h^2}\right)\phi_\alpha,\\ u_3 &= v + \frac{\nu z^2}{2(1-\nu)}\nabla^2 v. \end{aligned}$$

This field follows from (2.2), (2.3) by putting

$$\text{(4.14)}\qquad \begin{gathered} \mathfrak{m}_1(z) = z, \quad \mathfrak{m}_2(z) = \frac{\nu z^2}{6(1-\nu)}, \quad \mathfrak{n}_1(z) = z\left(1-\frac{4z^2}{3h^2}\right),\\ \mathfrak{s}_1(z) = 1, \quad \mathfrak{s}_2(z) = \frac{\nu z^2}{2(1-\nu)}, \quad h_1 = h_2 = \frac{h}{2},\\ \mathfrak{M}_1 = 1, \quad \mathfrak{M}_2 = \nabla^2, \quad \mathfrak{N}_1 = 1, \quad \mathfrak{S}_1 = 1, \quad \mathfrak{S}_2 = \nabla^2. \end{gathered}$$

To find by the direct method the energy-consistent equilibrium equations one should perform the following operations:

1. Average Eq. (3.1) with the weighted function $\mathfrak{m}_1(z)$ and then perform the differential operation $\mathfrak{M}_1(\ldots)_{,\alpha}$;

2. Average Eq. (3.1) with the weighted function $\mathfrak{m}_2(z)$ and then perform the differential operation $\mathfrak{M}_2(\ldots)_{,\alpha}$;

3. Average Eq. (3.2) with the weighted function $\mathfrak{s}_1(z)$ and then perform the differential operation $\mathfrak{S}_1(\ldots)$;

4. Average Eq. (3.2) with the weighted function $\mathfrak{s}_2(z)$ and then perform the differential operation $\mathfrak{S}_2(\ldots)$;

5. Average Eq. (3.1) with the weighted function $\mathfrak{n}_1(z)$ and then perform the differential operation $\mathfrak{N}_1(\ldots)$.

At the first four weighted functions $(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{s}_1, \mathfrak{s}_2)$ the same function $v(x^\alpha)$ appears. Hence all the four equations obtained according to the rules given above should be added. Then we find

$$\text{(4.15)}\qquad \underset{1}{M}{}^{\beta\alpha}{}_{,\beta\alpha} + \nabla^2 \underset{2}{M}{}^{\beta\alpha}{}_{,\beta\alpha} + \underset{1}{m}{}^{\alpha}{}_{,\alpha} + q_1 + \nabla^2(\underset{2}{m}{}^{\alpha}{}_{,\alpha} + q_2) = 0.$$

The algorithm given in clause (point) 5 leads to the system of equations

$$\text{(4.16)}\qquad (\underset{1}{N}{}^{\beta}{}_{\alpha,\beta} - \underset{1}{P}_\alpha + \underset{1}{\mathrm{p}}_\alpha) = 0.$$

In Eqs. (4.15), (4.16) the following averaged quantities have been used

$$\underset{1}{M}_{\beta\alpha}(x^\gamma) = \int\limits_{-h/2}^{h/2} z t_{\beta\alpha}\, dz, \quad \underset{2}{M}_{\beta\alpha}(x^\gamma) = \frac{\nu}{6(1-\nu)} \int\limits_{-h/2}^{h/2} z^3 t_{\beta\alpha}\, dz,$$

$$\underset{1}{N}_{\beta\alpha}(x^\gamma) = \int\limits_{-h/2}^{h/2} z\left(1 - \frac{4z^2}{3h^2}\right) t_{\beta\alpha} dz, \quad \underset{1}{Q}_{\alpha}(x^\gamma) = \underset{1}{T}_{\alpha}(x^\gamma) = \int\limits_{-h/2}^{h/2} t_{\alpha 3}\, dz,$$

$$\underset{2}{Q}_{\alpha}(x^\gamma) = \underset{2}{T}_{\alpha}(x^\gamma) = \frac{\nu}{2(1-\nu)} \int\limits_{-h/2}^{h/2} z^2 t_{\alpha 3}\, dz, \quad \underset{1}{S}(x^\gamma) = 0,$$

$$\underset{1}{P}_{\alpha}(x^\gamma) = \int\limits_{-h/2}^{h/2} \left(1 - 4\frac{z^2}{h^2}\right) t_{3\alpha}\, dz, \quad \underset{2}{S}(x^\gamma) = \frac{\nu}{1-\nu} \int\limits_{-h/2}^{h/2} z t_{33}\, dz,$$

$$(4.17) \qquad \underset{1}{m}_{\alpha} = \frac{h}{2}(\overset{+}{p}_{\alpha} - \overset{-}{p}_{\alpha}) + \int\limits_{-h/2}^{h/2} z X_{\alpha}\, dz, \quad q_1 = \overset{+}{p}_3 + \overset{-}{p}_3 + \int\limits_{-h/2}^{h/2} X_3\, dz,$$

$$\underset{2}{m}_{\alpha} = \frac{\nu}{48(1-\nu)} h^3(\overset{+}{p}_{\alpha} - \overset{-}{p}_{\alpha}) + \frac{\nu}{6(1-\nu)} \int\limits_{-h/2}^{h/2} z^3 X_{\alpha}\, dz$$

$$q_2 = \frac{\nu}{8(1-\nu)} h^2(\overset{+}{p}_3 + \overset{-}{p}_3) + \frac{\nu}{2(1-\nu)} \int\limits_{-h/2}^{h/2} z^2 X_3\, dz - \underset{2}{S},$$

$$\underset{1}{\mathfrak{p}}_{\alpha} = \frac{h}{3}(\overset{+}{p}_{\alpha} - \overset{-}{p}_{\alpha}) + \int\limits_{-h_2}^{h_1} z\left(1 - \frac{4z^2}{3h^2}\right) X_{\alpha}\, dz,$$

$$\overset{+}{p}_{\alpha} = \overset{+}{\mathfrak{p}}_{\alpha}, \quad \overset{+}{p}_3 = \overset{+}{\mathfrak{p}}_3, \quad \overset{-}{p}_{\alpha} = \overset{-}{\mathfrak{p}}_{\alpha}, \quad \overset{-}{p}_3 = \overset{-}{\mathfrak{p}}_3, \quad \overset{+}{\omega} = \overset{-}{\omega} = 1.$$

In the present author's paper [11] it has been assumed that in isotropic, homogeneous plates of constant thickness the following state of displacement holds.([2])

$$(4.18) \qquad \begin{aligned} u_{\alpha} &= -z v_{,\alpha} + z\left(1 - \frac{4z^2}{3h^2}\right)\theta_{\alpha}, \\ u_3 &= v + \frac{\nu}{1-\nu}\frac{h^2}{48}\left(1 - 4\frac{z^2}{h^2}\right)\left[\left(5 - 4\frac{z^2}{h^2}\right)\theta^{\alpha}{}_{,\alpha} - 6\nabla^2 v\right], \end{aligned}$$

where

$$v = u_3(x^{\alpha}, \pm h/2).$$

This hypothesis follows from (2.2), (2.3) by putting

$$h_1 = h_2 = \frac{h}{2}, \quad \mathfrak{m}_1(z) = z, \quad \mathfrak{s}_1(z) = 1, \quad \mathfrak{s}_2 = -\frac{\nu}{1-\nu}\frac{h^2}{8}\left(1 - 4\frac{z^2}{h^2}\right),$$

$$(4.19) \qquad \mathfrak{f}_1(z) = -z\left(1 - \frac{4z^2}{3h^2}\right), \quad \mathfrak{p}_1(z) = \frac{\nu}{1-\nu}\frac{h^2}{48}\left(1 - 4\frac{z^2}{h^2}\right)\left(5 - 4\frac{z^2}{h^2}\right),$$

$$\mathfrak{M}_1 = 1, \quad \mathfrak{F}_1 = 1, \quad \mathfrak{S}_1 = 1, \quad \mathfrak{S}_2 = \nabla_2, \quad \mathfrak{P}_1 = 1.$$

([2]) In this paper only the case of bending was considered. We recall here a simplified version of the hypothesis used in [11].

Upon considering (4.19) in (2.5) we find the stress resultants

$$\underset{1}{M}_{\beta\alpha}(x^\gamma) = \int_{-h/2}^{h/2} z t_{\beta\alpha}\,dz, \quad \underset{1}{H}_{\beta\alpha}(x^\gamma) = -\int_{-h/2}^{h/2} z\left(1-\frac{4z^2}{3h^2}\right)t_{\beta\alpha}\,dz,$$

$$\underset{1}{Q}_{\alpha}(x^\gamma) = \underset{1}{T}_{\alpha}(x^\gamma) = \int_{-h/2}^{h/2} t_{3\alpha}\,dz, \quad \underset{1}{F}_{\alpha}(x^\gamma) = -\int_{-h/2}^{h/2} \left(1-4\frac{z^2}{h^2}\right)t_{3\alpha}\,dz,$$

$$\underset{2}{T}_{\alpha}(x^\gamma) = -\frac{\nu}{1-\nu}\frac{h^2}{8}\int_{-h/2}^{h/2}\left(1-4\frac{z^2}{h^2}\right)t_{\alpha 3}\,dz, \tag{4.20}$$

$$\underset{1}{G}_{\alpha}(x^\gamma) = \frac{\nu}{1-\nu}\frac{h^2}{48}\int_{-h/2}^{h/2}\left(1-4\frac{z^2}{h^2}\right)\left(5-4\frac{z^2}{h^2}\right)t_{\alpha 3}\,dz, \quad S_1(x^\gamma) = 0,$$

$$\underset{2}{S}(x^\gamma) = \frac{\nu}{1-\nu}\int_{-h/2}^{h/2} z t_{33}\,dz, \quad \underset{1}{K}(x^\gamma) = \frac{\nu}{1-\nu}\int_{-h/2}^{h/2} z\left(1-\frac{4z^2}{3h^2}\right)t_{33}\,dz.$$

After performing the operations described in Sec. 3 on Eqs. (3.1), (3.2) we find, by a direct method, the energy-consistent equations of equilibrium relevant to the (4.18) hypothesis

$$\begin{aligned}(\underset{1}{M}^{\beta\alpha}{}_{,\beta\alpha} - \underset{1}{Q}^{\alpha}{}_{,\alpha} + \underset{1}{m}^{\alpha}{}_{,\alpha}) + (\underset{1}{T}^{\alpha}{}_{,\alpha} + q_1) + \nabla^2(\underset{2}{T}^{\alpha}{}_{,\alpha} + q_2) &= 0,\\ (\underset{1}{H}^{\beta}{}_{\alpha,\beta} - \underset{1}{F}_{\alpha} + \underset{1}{\mathrm{m}}_{\alpha}) + (\underset{1}{G}^{\beta}{}_{,\beta\alpha} + \mathrm{q}_{1,\alpha}) &= 0,\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}\underset{1}{m}_{\alpha} &= \frac{h}{2}(\overset{+}{p}_{\alpha} - \bar{p}_{\alpha}) + \int_{-h_2}^{h_2} zX_\alpha\,dz, \quad q_1 = \overset{+}{p}_3 + \bar{p}_3 + \int_{-h_2}^{h_1} X_3\,dz,\\ q_2 &= -\frac{\nu}{1-\nu}\frac{h^2}{8}\int_{-h_2}^{h_1}\left(1-4\frac{z^2}{h^2}\right)X_3\,dz - \underset{2}{S},\\ \mathrm{q}_1 &= \frac{\nu}{1-\nu}\frac{h^2}{48}\int_{-h_2}^{h_1}\left(1-4\frac{z^2}{h^2}\right)\left(5-4\frac{z^2}{h^2}\right)X_3\,dz - \underset{1}{K},\end{aligned} \tag{4.22}$$

$$\overset{+}{\omega} = \bar{\omega} = 1, \quad \overset{+}{p}_{\alpha} = \overset{+}{\mathrm{p}}_{\alpha}, \quad \overset{+}{p}_3 = \overset{+}{\mathrm{p}}_3, \quad \bar{p}_{\alpha} = \bar{\mathrm{p}}_{\alpha}, \quad \bar{p}_3 = \bar{\mathrm{p}}_3.$$

These equations were found by LEWIŃSKI [12] by the variational method.

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