Arch. Mech., 44, 3, pp. 345–348, Warszawa 1992

BRIEF NOTES

Existence of a periodic solution for Kovalevskaya top

F. M. EL-SABAA (CAIRO)

THE EQUATIONS of motion of a heavy rigid body about a fixed point in Kovalevskaya's case are written in Deprit's canonical variables. The existence of periodic solution of equations of motion is demonstrated using the perturbation method due to Poincaré.

1. Introduction

THE PROBLEM of the motion of a heavy rigid body with a fixed point in Kovalevskaya case is one of three cases, where the problem is reduced to quadratures (the other two cases are Euler–Poinsot case wherein the fixed point is located at the centre of mass and rotation occurs freely without the influence of the torque; the Lagrange–Poisson case when the rigid body is a symmetric top with the centre of mass on the axis of symmetry). It is proved that, when the ellipsoid of inertia is not symmetric, a new algebraic integral cannot exist except in the mentioned three cases. A lot of special cases of the problem of the motion of a rigid body has been treated by many authors during the last 75 years. The special cases of the problem help us to know more about the general solution of the problem.

Returning to the case of Kovalevskaya problem of motion, the principal moments of inertia satisfy the relation $A = B = 2C$. The centre of mass of the body is x_c, y_c, z_c, where $y_c = z_c = 0$. The equations of motion was reduced to quadratures [1]. The general qualitative portrait of the problem in the Delone case ($k = 0$, where k is the Kovalevskaya constant) was ascertained in [2]. In case $k \neq 0$, a qualitative and numerical investigation of the motion was carried out in [3–6].

In 1967, DEPRIT [7] had investigated the motion of rigid body in Euler's case. He reduced the Hamiltonian function of the problem to a conservative function with only one degree of freedom by using new canonical variables L, G, H, l, g, h. This reduction helps a great deal in qualitative analysis in the phase plane.

Deprit's canonical variables were used in mechanics of a rigid body and in celestial mechanics to study the perturbation motion, for computing the higher order approximation, and for establishing the existence of periodic solutions. In this article, the Deprit's variables are used to study the perturbed motion of Kovalevskaya's case. Poincaré's method [8] of small parameter is used to show the existence of periodic solution.

2. Statement of the problem

We consider the motion of a heavy rigid body about a fixed point 0. Let $OXYZ$ be a fixed coordinate system with origin O fixed in the body, where the axis OZ is directed vertically upwards. The moving system is $Oxyz$ which is fixed relative to the body. The axes of the moving system are directed along the principal axes of inertia of the body. Let θ, φ, ψ be the Eulerian angles defining the position of the moving system $Oxyz$ relative to the fixed one.

The Hamiltonian function of the rigid body can be written in terms of $(\theta, \varphi, \psi, P_\theta, P_\varphi, P_\psi)$, where P_θ, P_φ and P_ψ are canonical momenta conjugated to Eulerian angles. Deprit introduced the canonical variables L, G, H, l, g, h; where G is the magnitude of the angular momentum of the body rotation; L and H are the projections of the angular momentum on z-axis and OZ-axis of the body, respectively; l, g and h are the canonical angular variables conjugate to L, G and H. The transformation from $(\theta, \varphi, \psi, P_\theta, P_\varphi, P_\psi)$ into (L, G, H, l, g, h) is canonical. The kinetic energy T and the force function U of the problem are given by

$$T = \frac{G^2 - L^2}{2AB}(A\cos^2 l + B\sin^2 l) + \frac{L^2}{2C}, \tag{2.1}$$

$$U = -P(x_c\gamma_1 + y_c\gamma_2 + z_c\gamma_3), \tag{2.2}$$

where A, B, C are the principal moments of inertia, P is the body's weight; x_c, y_c, z_c are the coordinates of the centre of mass of the body; $\gamma_1, \gamma_2, \gamma_3$ are the direction cosines of the radius vector in the moving system which can be expressed in terms of canonical variables as follows [9]:

$$\begin{aligned}
G^2\gamma_1 &= [H\sqrt{G^2 - L^2}\sin l + \sqrt{G^2 - H^2}(L\cos g \sin l + G\sin g\cos l)],\\
G^2\gamma_2 &= [H\sqrt{G^2 - L^2}\cos l + \sqrt{G^2 - H^2}(L\cos l\cos g - G\sin l\sin g)],\\
G^2\gamma_3 &= \frac{LH}{G^2}\sqrt{(G^2 - L^2)(G^2 - H^2)}\cos g.
\end{aligned} \tag{2.3}$$

The Hamiltonian function of the system is

$$F = T - U. \tag{2.4}$$

We consider the perturbed motion in Kovalevskaya case ($A = B = 2C$, $y_c = z_c = 0$), and assume that the body is almost axisymmetric and that its fixed point lies near its centre of mass. In this case, the Hamiltonian (2.4) can be written in the form which allows for the application of the Poincaré method of small parameter

$$F = F_0 + \mu F_1, \tag{2.5}$$

$$F_0 = \frac{G^2 - L^2}{2AB} + \frac{L^2}{2C}, \tag{2.6}$$

$$\begin{aligned}
\mu F_1 = \frac{A - B}{2AB}(G^2 - L^2)\cos^2 l + \frac{Px_c}{G^2}[L\sqrt{G^2 - H^2}\sin l\cos g\\
+G\sqrt{G^2 - H^2}\cos l\sin g + H\sqrt{G^2 - H^2}\sin l],
\end{aligned} \tag{2.7}$$

where F_0 defines the generating solution and μF_1 is the perturbed Hamiltonian. The small parameter μ is taken as

$$\mu = \max\{A - B, x_c\}.$$

3. Periodic solutions

The equations of motion are

$$
\begin{aligned}
&\frac{dL}{dt} = \frac{\partial F}{\partial l}, \qquad \frac{dG}{dt} = \frac{\partial F}{\partial g}, \qquad \frac{dH}{dt} = \frac{\partial F}{\partial h},\\
&\frac{dl}{dt} = -\frac{\partial F}{\partial L}, \qquad \frac{dg}{dt} = -\frac{\partial F}{\partial G}, \qquad \frac{dh}{dt} = -\frac{\partial F}{\partial H}.
\end{aligned}
\tag{3.1}
$$

When $\mu = 0$, the equations have a set of periodic solutions of period τ:

$$
\begin{gathered}
L = L_0, \qquad G = G_0, \qquad H = H_0,\\
l = n_1 t + \beta_1, \qquad g = n_2 t + \beta_2, \qquad h = \beta_3,\\
n_1 = \frac{A-C}{AC} L_0, \qquad n_2 = \frac{G_0}{A},\\
k_1 n_1 = k_2 n_2, \qquad \tau = \frac{2\pi k_1}{n_2} = \frac{2\pi k_2}{n_2},
\end{gathered}
\tag{3.2}
$$

where k_1 and k_2 are integers (commensurability indicators), $L_0, G_0, H_0, \beta_1, \beta_2, \beta_3$ are arbitrary constants of integration.

In accordance with Poincaré's theorem [8], for small value of parameter μ, the system (2.5) admits a periodic solution of period τ if the generating solutions F_0 is nondegenerate,

$$
\frac{\partial(F_{0L_0,G_0})}{\partial(L_0, G_0)} = \frac{A-C}{A^2C} L_0 G_0 \neq 0,
\tag{3.3}
$$

and if the mean value $\overline{F}_1$ of the Hamiltonian F_1 over a period τ does not depend on the angular variables H_0 and β_i $(i = 1, 2)$

$$
\frac{\partial \overline{F}_1}{\partial H} = 0, \qquad \frac{\partial \overline{F}_1}{\partial \beta_i} = 0.
\tag{3.4}
$$

Put

$$
(n_1 - n_2)\tau + \beta_1 - \beta_2 = \lambda, \qquad (n_1 + n_2)\tau + \beta_1 + \beta_2 = \nu,
$$

then the conditions (3.4) for periodicity gives the following relations:

$$
\begin{aligned}
&\frac{P x_c L_0 H_0}{2\sqrt{G_0^2 - H_0^2}} \left[\frac{1}{n_1 + n_2} \{\cos\nu - \cos(\beta_1 - \beta_2)\} + \frac{1}{n_1 - n_2} \{\cos\lambda - \cos(\beta_1 - \beta_2)\} \right]\\
&+ \frac{P x_c H_0 G_0}{2\sqrt{G_0^2 - H_0^2}} \left[\frac{1}{n_1 + n_2} \{\cos\nu - \cos(\beta_1 - \beta_2\} - \frac{1}{n_1 - n_2} \{\cos\lambda - \cos(\beta_1 - \beta_2)\} \right]\\
&\qquad - \frac{P x_c}{n_1} \sqrt{G_0^2 - L_0^2} [\cos(n_1\tau + \beta_1) - \cos\beta_1] = 0,
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&\frac{A-B}{4ABn_1} (G_0^2 - L_0^2) [\cos 2(n_1\tau + \beta_1) - \cos 2\beta_1]\\
&+ \frac{P x_c L_0}{2G_0^2} \sqrt{G_0^2 - H_0^2} \left[\frac{1}{n_1 + n_2} \{\sin\nu - \sin(\beta_1 + \beta_2)\} + \frac{1}{n_1 - n_2} \{\sin\lambda - \sin(\beta_1 - \beta_2)\} \right]\\
&+ \frac{P x_c}{2G_0} \sqrt{G_0^2 - H_0^2} \left[\frac{1}{n_1 + n_2} \{\sin\nu - \sin(\beta_1 + \beta_2)\} + \frac{1}{n_1 - n_2} \{\sin\lambda - \sin(\beta_1 - \beta_2)\} \right]
\end{aligned}
\tag{3.6}
$$

$$\text{(3.6) [cont.]} \qquad + \frac{Px_c H_0}{n_1 G_0^2}\sqrt{G_0^2 - L_0^2}[\sin(n_1\tau + \beta_1) - \sin\beta_1] = 0,$$

$$\text{(3.7)} \quad \frac{Px_c L_0}{2G_0^2}\sqrt{G_0^2 - H_0^2}\left[\frac{1}{n_1+n_2}\{\sin\nu - \sin(\beta_1+\beta_2)\} - \frac{1}{n_1-n_2}\{\sin\lambda - \sin(\beta_1-\beta_2)\}\right]$$
$$+ \frac{Px_c}{2G_0^2}\sqrt{G_0^2 - H_0^2}\left[\frac{1}{n_1+n_2}\{\sin\nu - \sin(\beta_1+\beta_2)\} + \frac{1}{n_1-n_2}\{\sin\lambda - \sin(\beta_1-\beta_2)\}\right] = 0.$$

Under the additional assumption $|k_1| + |k_2| \geq 4$ concerning the nature of the commensurability of the frequencies, the equations (3.5), (3.6) and (3.7) give the following cases for k_1 and k_2:

$$|k_1| = |k_2|; \quad |k_1| = 1, \quad |k_2| = 2; \quad |k_1| = 2, \quad |k_2| = 1.$$

These results are compatible with the basic idea of Poincaré's method which consists in making a special choice for the arbitrary constants, such that the conditions (3.4) are satisfied for a certain commensurabilities of the unperturbed frequencies.

References

1. V. GOLUBEV, *Lecture on integration of the equation of motion of a rigid body about a fixed point*, Gostekhizdat, Moscow 1953.
2. N. DELONE, *Algebraic integrals of the motion of a heavy rigid body about a fixed point*, St. Petersburg 1892.
3. V. KOZLOV, *New periodic solutions for the problems of motion of a heavy solid around a fixed point*, PMM, **39**, 3, 1975.
4. A. DIKSHEVICH, *Two classes of motion of the Kovalevskaya top*, PMM, **45**, 4, 1982.
5. F. EL-SABAA, *Periodic solutions of the problem of the motion of the heavy rigid body around the fixed point in Kovalevskaya case and their stability*, Celestial Mech., **27**, 1982.
6. F. EL-SABAA, *A new class of periodic solutions in the Kovalevskaya case of a rigid body in rotation about a fixed point*, Celestial Mech., **37**, 1985.
7. A. DEPRIT, *Free rotation of a rigid body studied in the phase plane*, Amer. J. Phys., **35**, 1967.
8. H. POINCARÉ, *Les méthodes nouvelles de la mécanique céleste*, 3 Gauthier-Villars, Paris 1985.
9. V. KOZLOV, *Methods of qualitative analysis in dynamics of rigid body*, Izd. Mosk. Gos. Univ., Moscow 1980.

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION
AIN SHAMS UNIVERSITY, CAIRO, EGYPT.

Received October 21, 1991.