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Cylindrical wave solutions to the Korteweg–de Vries equation

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CYLINDRICAL wave solutions for the Korteweg–de Vries equation are obtained within a reasonable approximation. They are shown to be representable as infinite sums of cylindrical solitons.

1. Introduction

The Korteweg–de Vries equation (referred to as KdV equation henceforth) is a nonlinear partial differential equation which arises in the study of many physical problems, such as water waves, plasma waves, lattice waves, waves in elastic rods, etc. For a survey we cite the article by MIURA [1].

Whitham demonstrated the representation of periodic waves as infinite sums of solitons for the one-dimensional KdV equation and modified KdV equation [2], CHEN and WEN showed the similar results for the two-dimensional KdV equation and modified KdV equation using entirely different methods [3]. In this paper we apply Chen and Wen's method to the cylindrical KdV equation. A cnoidal wave solution is obtained, and we prove that the cnoidal wave solution can be expressed as a sum of infinite number of solitons by using Fourier series expansions and Poisson's summation formula. We have also established a criterion for the existence of single soliton solution, it is $C > 0$, where C is a constant, or $X < \dfrac{\xi}{\delta\sqrt{\mu}}$ (see Sec. 3).

2. KDV equation

We start from the cylindrical KdV equation of the form [4, 5]

$$2u_\tau + \frac{u}{\tau} + 3uu_\xi + \frac{h_0^2}{3}u_{\xi\xi\xi} = 0, \tag{2.1}$$

where $u = u(\xi, \tau)$ is a function of ξ and τ. This equation was first derived from the study of acoustic wave propagating in a collisionless plasma by MAXON and VIECELLI [5], and it was also derived from the shallow water wave equations by the authors [4]. We shall establish the solitary wave and cnoidal wave solutions to the cylindrical KdV equation. Motivated by the one-dimensional results by MAXON and VIECELLI [5], we introduce the following transformation.

Let

$$\tau' = \frac{\tau}{h_0}, \quad \xi' = \frac{2\xi}{3h_0} \quad \text{and} \quad u(\xi,\tau) = u(\xi',\tau'),$$

then

$$2u_\tau = 2u_{\tau'}(1/h_0), \quad u/\tau = u/(\tau' h_0),$$
$$3uu_\xi = 2uu_{\xi'}/h_0 \quad \text{and} \quad (h_0^2/3)u_{\xi\xi\xi} = (8/81)u_{\xi'\xi'\xi'}/h_0.$$

Therefore we can write Eq. (2.1) as

$$u_{\tau'} + \frac{u}{2\tau'} + uu_{\xi'} + \frac{4}{81}u_{\xi'\xi'\xi'} = 0.$$

For convenience, we drop the primes in the above equation and obtain

$$u_\tau \tau + \frac{u}{2} + uu_\xi \tau + \frac{4}{81}u_{\xi\xi\xi}\tau = 0. \tag{2.2}$$

We now define $\mu = 2\sqrt{\tau}$ and $U(\xi,\mu) = \sqrt{\tau}u$. Then

$$U_\mu = \frac{\mu}{2}u_\tau \frac{d\tau}{d\mu} + \frac{u}{2} = u_\tau \tau + \frac{u}{2}, \qquad UU_\xi = uu_\xi \tau,$$

$$\frac{1}{2}\mu U_{\xi\xi\xi} = \sqrt{\tau}\sqrt{\tau}u_{\xi\xi\xi} = u_{\xi\xi\xi}\tau.$$

Substituting these quantities into Eq. (2.2) we obtain

$$U_\mu + UU_\xi + \frac{2}{81}\mu U_{\xi\xi\xi} = 0. \tag{2.3}$$

We look for real-valued wave solutions of the form $G(X) = \dfrac{81\delta^2}{2}U(\xi,\mu)$ with $X = \dfrac{\xi - (4/81)\delta^{-2}C\mu}{\delta\sqrt{\mu}}$, where C is a constant number, $\delta \ll 1$ is a small positive parameter introduced by CUMBERBATCH [6], and G is a C^3 function of its argument. Since

$$U_\mu = \frac{2}{81}\cdot\frac{G'(X)}{\delta^2}\cdot\frac{\partial X}{\partial \mu} = \frac{2}{81}\cdot\frac{G'(X)}{\delta^2}\left[-\xi\delta\frac{1}{2\sqrt{\mu}} - \frac{2}{81}\delta^{-1}C\sqrt{\mu}\right]\frac{1}{\delta^2\mu}$$

$$= -\frac{1}{81}\delta^{-3}G'(X)\xi\mu^{-3/2} - \left(\frac{2}{81}\right)^2 C\delta^{-5}G'(X)\mu^{-1/2},$$

$$U_\xi = \frac{2}{81}\delta^{-3}G'(X)\mu^{-1/2}, \quad \text{and} \quad U_{\xi\xi\xi} = \frac{2}{81}\delta^{-5}G'''(X)\mu^{-3/2},$$

then substitution of the above results into Eq. (2.3) yields

$$-\frac{1}{81}\delta^{-3}G'(X)\xi\mu^{-3/2} - \left(\frac{2}{81}\right)^2 C\delta^{-5}G'(X)\mu^{-1/2}$$
$$+\left(\frac{2}{81}\right)^2\delta^{-5}\mu^{-1/2}[G(X)G'(X) + G'''(X)] = 0,$$

i.e.,

$$-\frac{1}{81}\delta^2 G'(X)\frac{\xi}{\mu} + \left(\frac{2}{81}\right)^2[G'''(X) + G(X)G'(X) - CG'(X)] = 0. \tag{2.4}$$

The first term in Eq. (2.4) is of order δ^2 if $G'(X)$ and (ξ/μ) are bounded. One can argue that since in the original derivation $\xi = \varepsilon^{1/2}(r-t)$, $\tau = \varepsilon^{3/2}t$, and $\mu = 2\sqrt{\tau}$, where ε is a small parameter, r the radial distance and t the time, it seems to be reasonable to assume $|\xi/\mu|$ to be bounded. This is the case in particular, when both r and t are large and of the same order, or in the domain where $|\xi/\mu| \ll \delta^{-\alpha}$ with $\alpha < 2$. Therefore, a good approximation to Eq. (2.4) is:

$$G'''(X) + G(X)G'(X) - CG'(X) = 0. \tag{2.5}$$

Integrating both sides of Eq. (2.5) and using the fact $G''(X) = \frac{1}{2}\frac{d[G'(X)]^2}{dG(X)}$, we have

$$[G'(X)]^2 = \frac{1}{3}[-G^3(X) + 3CG^2(X) + AG(X) + B] = \frac{1}{3}F(G), \tag{2.6}$$

where A and B are two integration constants and F is the cubic function $-G^3 + 3CG^2 + AG + B$.

3. Solitary wave solution

For a solitary wave solution we implose the boundary conditions $G, G', G'', G''' \to 0$ when $X \to \pm\infty$. Therefore, $A = B = 0$ in Eq. (2.6), and we obtain from Eq. (2.6)

$$[G'(X)]^2 = \frac{1}{3}G^2(X)[3C - G(X)]. \tag{3.1}$$

If $C < 0$, i.e., $X > \frac{\xi}{\delta\sqrt{\mu}}$, we shall have the solution

$$G(X) = 3C\{1 + \tan^2[\sqrt{-C}(X - X_0)]\},$$

where X_0 is an integration constant. Clearly, $G(X)$ is unbounded, and hence, it is not of much physical interest.

If $C \geq 0$, i.e., $X \leq \frac{\xi}{\delta\sqrt{\mu}}$, then solution to Eq. (3.1) becomes

$$G(X) = 3C \cdot \text{sech}^2[\sqrt{C}(X - X_0)], \tag{3.2}$$

where X_0 is an integration constant. We note that $C > 0$, i.e., $X < \frac{\xi}{\delta\sqrt{\mu}}$, gives a condition under which a nontrivial solitary wave solution exists. In particular, if we choose $C = 1$ and $X_0 = 0$, then from Eq. (3.2) we have

$$G(X) = 3\,\text{sech}^2\left[\frac{\xi - (4/81)\delta^{-2}\mu}{\delta\sqrt{\mu}}\right]. \tag{3.3}$$

4. Cnoidal wave solution

The cubic function $F(G)$ in the right-hand side of Eq. (2.6) plays an important role. Applying a similar argument as that given in Ref. [3], we can show that a cnoidal wave solution exists only if $F(G)$ has three distinct real simple zeros G_1, G_2 and G_3 such that $G_1 > G_2 > G_3$ and $G_2 \leq G(X) \leq G_1$ [3]. If this is the case, we have

$$\sqrt{\frac{1}{3}}(X_1 - X) = \int_G^{G_1} \frac{dG}{\sqrt{F(G)}} = \int_G^{G_1} \frac{dG}{\sqrt{(G_1 - G)(G - G_2)(G - G_3)}}, \tag{4.1}$$

where X_1 is a value such that $G(X_1) = G_1$, and the period $2T$ in X is

$$(4.2) \qquad 2T = 2\sqrt{3}\int\limits_{G_1}^{G_2} \frac{dG}{\sqrt{(G_1 - G)(G - G_2)(G - G_3)}}.$$

By Ref. [7], we can write Eq. (4.1) as

$$(4.3) \qquad \sqrt{\frac{1}{3}}(X_1 - X) = \frac{2}{\sqrt{G_1 - G_3}} sn^{-1}(\sin\phi, k) = \frac{2}{\sqrt{G_1 - G_3}} F(\phi, k),$$

where

$$\phi = \sin^{-1}\sqrt{\frac{G_1 - G}{G_1 - G_2}}, \qquad k^2 = \frac{G_1 - G_2}{G_1 - G_3},$$

and $F(\phi, k) = sn^{-1}(\sin\phi, k)$ is the normal elliptic integral of the first kind with modulus k. If we define $\nu = F(\phi, k)$, then

$$\nu = \frac{1}{2\sqrt{3}}\sqrt{G_1 - G_3}(X_1 - X),$$

and the cnoidal wave solution is obtained

$$(4.4) \qquad G(X) = G_1 - (G_1 - G_2)sn^2(\nu, k) = G_2 + (G_1 - G_2)cn^2(\nu, k)$$
$$= G_3 + (G_1 - G_3)dn^2(\nu, k) = G_3 + (G_1 - G_3)\, dn^2\left(\frac{1}{2\sqrt{3}}\sqrt{G_1 - G_3}(X - X_1), k\right),$$

where

$$sn(\nu, k) = \sin\phi, cn(\nu, k) = \cos\phi \quad \text{and} \quad dn(\nu, k) = \sqrt{1 - k^2\sin^2\phi}.$$

It should be noted that here the C can be positive, zero or negative as long as $C = \frac{1}{3}(G_1 + G_2 + G_3)$. In particular, if $X_1 = 0$, then

$$G(X) = G_2 + (G_1 - G_2)cn^2\left(\frac{1}{2\sqrt{3}}\sqrt{G_1 - G_3}\frac{\xi - (4/81)\delta^{-2}C\mu}{\delta\sqrt{\mu}}, k\right).$$

Using the Fourier series expansion of $dn^2(\nu, k)$ [8] and the Poisson summation formula [9], we obtain [3]

$$(4.5) \qquad dn^2(\nu, k) = \frac{E}{K} - \frac{\pi}{2KK'} + \frac{\nu^2}{4K'^2}\sum_{m=-\infty}^{\infty} \operatorname{sech}^2\left[\frac{\pi}{2K'}(\nu - 2mK)\right],$$

where $K = \int\limits_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$ is the complete elliptic integral of the first kind with modulus k; $K' = \int\limits_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2\sin^2\theta}}$ is the complete elliptic integral of the first kind with modulus $k' = \sqrt{1 - k^2}$; $E = \int\limits_0^{\pi/2} \sqrt{1 - k^2\sin^2\theta}\,d\theta$ is the complete elliptic integral of the second kind with modulus k. Therefore, the cnoidal wave solution $G(X)$ in Eq. (4.4) can be written as

$$G(X) = P + Q \sum_{m=-\infty}^{\infty} \operatorname{sech}^2 R(X - X_1 + 2mT), \tag{4.6}$$

where

$$P = G_3 + (G_1 - G_3)\left[\frac{E}{K} - \frac{\pi}{2KK'}\right],$$

$$Q = (G_1 - G_3)\frac{\pi^2}{4K'},$$

$$2T = \frac{4\sqrt{3}}{\sqrt{G_1 - G_3}} F\left(\frac{\pi}{2}, k\right) = \frac{4\sqrt{3}K}{\sqrt{G_1 - G_3}},$$

$$R = \frac{\pi K}{2K'T},$$

where K, K' and E are defined following Eq. (4.5). In Eq. (4.6), G is clearly a periodic function of X with period $2T$. Each term in the infinite series is a soliton. This gives a representation of a periodic function by an infinite number of solitons. It should be mentioned that the representation is valid within the order of δ^2.

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