Constitutive equations for finite deformations of elastic-plastic metallic solids with induced anisotropy

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The object of this paper is to formulate the constitutive relations for elastic-plastic metallic solids at finite strain, making some plausible physical assumptions and employing Mandel's formalism. Thermodynamic terms are included. The equations are further simplified which should make them more useful for solving problems.

1. Introduction

ONE OF THE MOST important features of deformable metallic solids is the fact that their distortional elastic strains are small under arbitrary loading whereas they can undergo large elastic volumetric changes. The logarithmic measure of elastic strains and the formalism of MANDEL [6, 7, 8] were employed to establish the constitutive equations for isotropic metallic solids (cf. RANIECKI and NGUYEN [12]), assuming that the ratios of principal elastic stretches belong to the interval [5/6, 6/7]. Much attention is devoted to the formulation of strain-induced anisotropy for finite strains of rate-independent plastic materials in the recent years. These models apply to the deformation processes in which material rotation is of primary importance, e.g. large torsion of the test specimens. ·

In this paper, using the WILLIS approximation [17, 18] we derive the constitutive relations for elastic-plastic metallic bodies in the case of plastic strain induced anisotropy. The resulting equations are similar in form to those applying to problems of small deformation but are applicable for stresses up to those in the hydrodynamic regime of flow. The plastic strain induced anisotropy arises here from the residual stresses which is the result of the heterogeneous nature of plastic deformation in single crystals and polycrystalline material. It is modelled by the shift of the yield surface in stress-space and takes the form of combined isotropic-kinematic hardening.

Tensors will be denoted by boldface characters. With the summation over repeated indices implied, the following symbolic operations apply: $\mathbf{AB} \rightarrow A_{ij}B_j$, $\mathbf{A} \cdot \mathbf{B} \rightarrow A_{ij}B_{ij}$, $\mathbf{A} \otimes \mathbf{B} \rightarrow A_{ij}B_{kl}$ with proper extension to different orders tensor. The prefix tr indicates the trace, a superscript T the transpose and a superposed dot the material time derivative or rate. By $\overline{\mathbf{A}}$ we denote the deviatoric part of $\mathbf{A}$, by $\mathbf{1}$ the identity tensor and by a superscript -1 the inverse.

2. Background

Consider a homogeneous elastic-plastic material element $\mathcal{M}$ of unit mass. Let (k_0) be the initial configuration of this element under zero stress and at the temperature ϑ_0; (k) the actual configuration under the Cauchy's stress σ and at the temperature Assume that at the moment t the element is unloaded instantaneously and brought back to the temperature ϑ_0; this unloading is elastic. A relaxed instantaneous configuration (k_*) is

obtained, which is specified only to within a rotation. Denote by ρ_0, ρ_* and ρ the densities of $\mathcal{M}$ in (k_0), (k_*) and (k). The deformation gradient resolved according to [7] into elastic and plastic components gives

(2.1) $$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p.$$

Here $\mathbf{F}$, $\mathbf{F}^e$ and $\mathbf{F}^p$ maps $(k_0) \rightarrow (k)$, $(k_*) \rightarrow (k)$ and $(k_0) \rightarrow (k_*)$, respectively. The superscripts e and p refer to elastic and plastic deformation. Polar decomposition of $\mathbf{F}^e$ leads to

(2.2) $$\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e = \mathbf{S}^e\mathbf{R}^e,$$

where $\mathbf{R}^e$ is a proper rotation tensor, $\mathbf{U}^e$ and $\mathbf{S}^e$ are the right and left elastic stretches. $\mathbf{R}^e$ introduced in Eq. (2.2) rotates the principal directions $\mathbf{N}_i$ of $\mathbf{U}^e$ into the principal directions of the left elastic stretch tensor $\mathbf{S}^e$, i.e.

(2.3) $$\mathbf{n}_i = \mathbf{R}^e\mathbf{N}_i \quad \text{or} \quad \mathbf{R}^e = \mathbf{n}_i \otimes \mathbf{N}_i,$$

and $\mathbf{n}_i$ is the unit vector along the principal directions of $\mathbf{S}^e$.

We have the following notations and kinematical relations [12]:

$$\mathbf{V} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{V}^e + \mathbf{V}^p,$$

$$\mathbf{V}^e = \dot{\mathbf{F}}^e\mathbf{F}^{e^{-1}}, \quad \mathbf{V}^p = \mathbf{F}^e\mathbf{V}^*\mathbf{F}^{e^{-1}}, \quad \mathbf{V}^* = \dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}},$$

$$2\mathbf{D} = \mathbf{V} + \mathbf{V}^T = 2(\mathbf{D}^e + \mathbf{D}^p), \quad 2\omega = \mathbf{V} - \mathbf{V}^T = 2(\omega^e + \omega^p),$$

$$2\mathbf{D}^e = \mathbf{V}^e + \mathbf{V}^{eT}, \quad 2\omega^e = \mathbf{V}^e - \mathbf{V}^{eT},$$

(2.4) $$2\mathbf{D}^p = \mathbf{V}^p + \mathbf{V}^{pT}, \quad 2\omega^p = \mathbf{V}^p - \mathbf{V}^{pT},$$

$$2\mathbf{D}^* = \mathbf{V}^* + \mathbf{V}^{*T}, \quad 2\omega^* = \mathbf{V}^* - \mathbf{V}^{*T},$$

$$2\mathbf{E}^e = \mathbf{F}^{eT}\mathbf{F}^e - 1, \quad \mathbf{D}^e = \mathbf{F}^{e^{-T}}\dot{\mathbf{E}}^e\mathbf{F}^{e^{-1}},$$

$$2\mathbf{D}^e = \mathbf{R}^e(\dot{\mathbf{U}}^e\mathbf{U}^{e^{-1}} + \mathbf{U}^{e^{-1}}\dot{\mathbf{U}}^e)\mathbf{R}^{eT},$$

$$2\omega^e = 2\dot{\mathbf{R}}^e\mathbf{R}^{eT} + \mathbf{R}^e(\dot{\mathbf{U}}^e\mathbf{U}^{e^{-1}} - \mathbf{U}^{e^{-1}}\dot{\mathbf{U}}^e)\mathbf{R}^{eT},$$

$$2\mathbf{D}^p = \mathbf{R}^e(\mathbf{U}^e\mathbf{V}^*\mathbf{U}^{e^{-1}} + \mathbf{U}^{e^{-1}}\mathbf{V}^{*T}\mathbf{U}^e)\mathbf{R}^{eT},$$

$$2\omega^p = \mathbf{R}^e(\mathbf{U}^e\mathbf{V}^*\mathbf{U}^{e^{-1}} - \mathbf{U}^{e^{-1}}\mathbf{V}^{*T}\mathbf{U}^e)\mathbf{R}^{eT},$$

$\mathbf{V}$ is the velocity gradient, $\mathbf{D}$ is the rate of total deformation — the symmetric part of $\mathbf{V}$ and ω— the material spin. $\mathbf{V}^*$, $\mathbf{D}^*$, ω^* are the corresponding terms for the purely plastic deformation which is envisaged to occur in the intermediate unstressed configuration. The name plastic rate of deformation and plastic spin can be associated either with $\mathbf{D}^*$ and ω^* at (k_*), or with $\mathbf{D}^p$ and ω^p at (k). $\mathbf{E}^e$ is the Green's deformation tensor.

An element $\mathcal{M}$ of an elastic-plastic material may be regarded as an elementary thermodynamic system being in constrained equilibrium at any instant of the deformation process. Under arbitrary loadings most metallic materials undergo small elastic distortional strains and possible large elastic dilatational chance in shape. For such materials, as shown in [12], it is more expedient to employ the logarithmic elastic strain measure $\mathbf{E}^* = \ln \mathbf{U}^e$ than the Green measure $\mathbf{E}^e$ as a state variable. This enables us to separate in a simple manner the dilatational elastic changes of $\mathcal{M}$ from the distortional ones and to

simplify the basic equations by assuming that elastic distortional strains are infinitesimal.

We have [12] :

$$\text{tr}\,\mathbf{E}^* = \ln\beta,$$

$$\beta = \frac{\rho_*}{\rho} = U_1 U_2 U_3 = \det\mathbf{F}^e,$$

here U_i^e, $(i = 1, 2, 3)$ are the principal elastic stretches. Let $\overline{\mathbf{E}}^*$ be the deviatoric part of $\mathbf{E}^*$, then

(2.5) $$\mathbf{U}^e = \beta^{\frac{1}{3}}\exp\overline{\mathbf{E}}^*.$$

Similarly, we have

(2.6) $$\mathbf{e} = \ln\mathbf{S}^e = \mathbf{R}^e\mathbf{E}^*\mathbf{R}^{eT}$$

with $\text{tr}\,\mathbf{e} = \ln\beta$ and the deviator $\overline{\mathbf{e}}$ describes the elastic distortional changes of M in actual configuration.

Assume that the elastic distortional strains are so small that, instead of the closed form (2.5), the following approximation is acceptable:

(2.7) $$\mathbf{U}^e = \beta^{\frac{1}{3}}(\mathbf{1} + \overline{\mathbf{E}}^*) + O(|\overline{\mathbf{E}}^*|^2).$$

In that case, to the order of $|\overline{\mathbf{E}}^*|^2$, from the relations (2.3) we have:

(2.8)
$$\mathbf{D}^e = \frac{1}{3}(\text{tr}\,\mathbf{D})\mathbf{1} + \mathbf{R}^e\dot{\overline{\mathbf{E}}}^*\mathbf{R}^{eT}, \qquad \omega^e = \dot{\mathbf{R}}^e\mathbf{R}^{eT},$$

$$\mathbf{D}^p = \mathbf{R}^e\mathbf{D}^*\mathbf{R}^{eT} - \mathbf{R}^e(\omega^*\overline{\mathbf{E}}^* - \overline{\mathbf{E}}^*\omega^*)\mathbf{R}^{eT},$$

$$\omega^p = \mathbf{R}^e\omega^*\mathbf{R}^{eT} - \mathbf{R}^e(\mathbf{D}^*\overline{\mathbf{E}}^* - \overline{\mathbf{E}}^*\mathbf{D}^*)\mathbf{R}^{eT}$$

or equivalently:

(2.9)
$$\mathbf{D}^e = \breve{\mathbf{e}} + O(|\overline{\mathbf{e}}|^2),$$

$$\mathbf{R}^e\mathbf{D}^*\mathbf{R}^{eT} = \mathbf{D}^p + (\omega^p\overline{\mathbf{e}} - \overline{\mathbf{e}}\omega^p) + O(|\overline{\mathbf{e}}|^2),$$

$$\mathbf{R}^e\omega^*\mathbf{R}^{eT} = \omega^p + (\mathbf{D}^p\overline{\mathbf{e}} - \overline{\mathbf{e}}\mathbf{D}^p) + O(|\overline{\mathbf{e}}|^2);$$

by $\breve{\mathbf{a}} = \dot{\mathbf{a}} - \omega^e\mathbf{a} + \mathbf{a}\omega^e$ we denote here the Zaremba–Jaumann derivative of tensor $\mathbf{a}$.

From the additive decomposition of the rate of the total work $\dot{W}$ into elastic $\dot{W}^e$ and plastic $\dot{W}^p$ parts

(2.10)
$$\dot{W} = \frac{1}{\rho}\sigma \cdot \mathbf{D} = \dot{W}^e + \dot{W}^p,$$

$$\dot{W}^e = \frac{1}{\rho}\sigma \cdot \mathbf{D}^e = \mathbf{T}^e \cdot \dot{\mathbf{E}}^e,$$

$$\dot{W}^p = \frac{1}{\rho}\sigma \cdot \mathbf{D}^p = \text{tr}(\mathbf{P}^*\mathbf{V}^*),$$

we have the following relations between the different stress measures:

(2.11) $$\mathbf{T}^e = \frac{1}{\rho}\mathbf{F}^{e-1}\sigma\mathbf{F}^{e-T}, \qquad \mathbf{P}^* = \frac{1}{\rho}\mathbf{F}^{e-1}\sigma\mathbf{F}^e.$$

Here $\mathbf{T}^e$ denotes the stress measure conjugate to Green's elastic strain $\mathbf{E}^e$ [12].

For materials that are isotropic with respect to the elastic properties in (k_*), the stress $\mathbf{T}^*$ energy-conjugate to the logarithmic $\mathbf{E}^*$ elastic strain measure is [12]

$$(2.12) \qquad\qquad \mathbf{T}^* = \frac{1}{\rho}\mathbf{R}^{eT}\sigma\mathbf{R}^e.$$

The decomposition (2.1) is unique assuming the instantaneous unloaded configuration to be isoclinic so that its orientation is fixed. It can be used as a basic reference configuration for the formulation of the constitutive equations of an elastic-plastic solid. Its use simplifies the description of material properties since the usual material derivatives of the constitutive quantities have simple meanings. The physical laws describing the properties of the elastic-plastic bodies can be then divided in two groups. The first group describes the elastic properties and the second group the plastic ones. Within the framework of MANDEL'S formulation [7], the constitutive equations established first in terms of the starred quantities, can be subsequently transformed to the Euler's configuration (k) or the Lagrange's configuration (k_0) using the relations (2.4), (2.10)–(2.12).

Introduce now the free energy function $\phi = u - \vartheta s$, where u is the internal energy, s is the entropy per unit of mass in the configuration (k_*). Assume that the independent variables of free energy are $\{\vartheta, \mathbf{E}^*, \alpha^*_{(j)}, \alpha_{(i)}\}$ denoted collectively by $Y^{\vartheta E}$, where $\alpha^*_{(j)}$ are internal tensorial variables and $\alpha_{(i)}$ are internal scalar ones. The fundamental equation is adopted in the form

$$(2.13) \qquad du = dW^0 + \vartheta ds, \quad dW^0 = \mathbf{T}^* \cdot d\mathbf{E}^* + \pi_{(i)}d\alpha_{(i)} + \pi^*_{(j)} \cdot d\alpha^*_{(j)},$$

$\pi_{(i)}$ and $\pi^*_{(j)}$ are scalar and tensorial internal thermodynamic forces, respectively. We employ here for convenience the internal thermodynamic forces and the work-conjugate stress per unit of mass, by multiplying by ρ_0 we obtain the usual ones per unit of volume. The term dW^0 represents the part of external work that is accumulated in the form of energy. The thermostatic properties of $\mathcal{M}$ in (k_*) are described by equations

$$(2.14) \ \mathbf{T}^* = \frac{\partial\phi(Y^{\vartheta E})}{\partial\mathbf{E}^*}, \quad -s = \frac{\partial\phi(Y^{\vartheta E})}{\partial\vartheta}, \quad \pi^*_{(j)} = \frac{\partial\phi(Y^{\vartheta E})}{\partial\alpha^*_{(j)}}, \quad \pi_{(i)} = \frac{\partial\phi(Y^{\vartheta E})}{\partial\alpha_{(i)}}.$$

When ϕ has the form

$$(2.15) \qquad \phi(\vartheta, \mathbf{E}^*, \alpha^*_{(j)}, \alpha_{(i)}) = \phi_1(\vartheta, \mathbf{E}^*) + \phi_2(\vartheta, \alpha^*_{(j)}, \alpha_{(i)}),$$

where ϕ_1, ϕ_2 are isotropic functions of E^* and $\alpha^*_{(j)}$ respectively; the equations (2.14) can easily be transformed to the actual configuration (k) [12]:

$$(2.16) \ \tau = \frac{\partial\phi(Y^{\vartheta E})}{\partial\mathbf{e}}, \quad -s = \frac{\partial\phi(Y^{\vartheta E})}{\partial\vartheta}, \quad \pi^*_{(j)} = \frac{\partial\phi(Y^{\vartheta E})}{\partial\alpha_{(j)}}, \quad \pi_{(i)} = \frac{\partial\phi(Y^{\vartheta E})}{\partial\alpha_{(i)}};$$

here we denote by $Y^{\vartheta e}$ the set of variables $\{\vartheta, \mathbf{e}, \alpha_{(j)}, \alpha_{(i)}\}$ and

$$(2.17) \qquad \tau = \frac{\sigma}{\rho}, \quad \alpha_{(j)} = \mathbf{R}^e\alpha^*_{(j)}\mathbf{R}^{eT}, \quad \pi_{(j)} = \mathbf{R}^e\pi^*_{(j)}\mathbf{R}^{eT}.$$

The rate equations in Eulerian description can be derived similarly as in [12] by calculating the Zaremba–Jaumann derivatives of the relations (2.16) with the spin ω^e. For the plastic part, we know that the rate of dissipation $\mathcal{F}$ of mechanical work is [12]

(2.18) $$\mathcal{F} = \dot{W} - \dot{W}^{0} = \text{tr}(\mathbf{P}^*\mathbf{V}^*) - \pi_{(i)}\alpha_{(i)} - \boldsymbol{\pi}^*_{(j)} \cdot \dot{\boldsymbol{\alpha}}^*_{(j)}.$$

The plastic flow rules in (k_*) relate the thermodynamical forces $(\mathbf{P}^{*T}, \pi_{(i)}, \boldsymbol{\pi}^*_{(j)})$ to the rate $(\mathbf{V}^*, \alpha_{(i)}, \boldsymbol{\alpha}^*_{(j)})$ occurring in Eq. (2.18). For materials that are isotropic with respect to elastic properties in (k_*) $\mathbf{P}^*$ is symmetric [12] and

(2.19) $$\mathbf{P}^* = \mathbf{T}^*,$$

hence

(2.20) $$\mathcal{F} = \text{tr}(\mathbf{T}^*\mathbf{V}^*) - \pi_{(i)}\alpha_{(i)} - \boldsymbol{\pi}^*_{(j)} \cdot \dot{\boldsymbol{\alpha}}^*_{(j)}.$$

Our task is now to specialize and simplify the equations of this section to obtain a set of equations which will be useful for solving practical problems of dynamical plasticity.

3. Assumptions

With certain loss of generality, for the sake of simplicity, the structural variables will be restricted to a second-order symmetric tensor α^* and a scalar α, which will be specified later. The following assumptions seem plausible for metallic solids:

1. The material is plastically incompressible:

(3.1) $$\rho_* = \rho_0 = \text{const}, \qquad \beta = \frac{\rho_0}{\rho}.$$

2. The plastic hardening does not influence the thermoelastic properties.

3. Elastic distortional strain is linear and infinitesimal so that the approximation (2.7) is permissible. Material is isotropic with respect to the elastic properties in (k_*).

4. The specific heat at constant volume c_v is constant (the classical Debye theory of metals).

5. The free energy is an isotropic and quadratic function of the internal parameter α^*.

Under those assumptions the free energy function has the form [12, 13]

(3.2) $$\phi = \varphi_1(\beta, \vartheta) + \frac{\beta\mu}{\rho_0}\overline{\mathbf{E}}^* \cdot \overline{\mathbf{E}}^* + u^*(\alpha, \alpha^*) - \vartheta s^*(\alpha^*),$$

where μ is the usual isothermal shear modulus assumed constant (although it could be assumed to depend upon β, ϑ [12]). u^*, s^* are stored entropy and stored internal energy at the natural state where $\sigma = 0$ and $\vartheta = 0$

(3.3)
$$u^*(\alpha, \alpha^*) = \frac{1}{\rho_0}\varphi_2(\alpha) + \frac{1}{2\rho_0}(c_0 + c_T\vartheta_0)\alpha^* \cdot \alpha^*,$$
$$s^*(\alpha^*) = \frac{1}{2\rho_0}c_T\alpha^* \cdot \alpha^*,$$

where c_0 and c_T are constants. For isotropic solids the specific form (3.2) was first derived by WILLIS [17] under the assumption that the yield stress in simple shear was much less than the elastic modulus. The various forms of φ_1 are discussed in [12, 14, 17].

4. Rate constitutive equations

The Eqs. (2.16), (2.17), (3.2), (3.3) yield

(4.1)
$$\check{T} = \beta(LD^e - B\dot{\vartheta}) + O(|\bar{e}|^2),$$

$$\rho\vartheta\dot{s} = \vartheta B \cdot D^e + \rho c_v \dot{\vartheta} + \frac{\rho}{\rho_0}\vartheta c_T \alpha \cdot \check{\alpha} + O(|\bar{e}|^2),$$

assuming, as in the linear theory of elasticity, that $\dot{e}$ is of the same order as e. Here $T = \rho_0 \tau = \beta\sigma$ is the Kirchhoff stress tensor. L and B are the generalized tensor of isothermal elastic stiffness and elastic thermal stress in (k), respectively,

(4.2)
$$L_{ijkl} = \delta_{ij}\delta_{kl}(K_T - p) + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}),$$

$$B_{ij} = \alpha_v K_T \delta_{ij}.$$

Here we have the following relations [12]:

(4.3)
$$K_T = \rho_0 \beta \frac{\partial^2 \varphi_1}{\partial \beta^2} \qquad \text{isothermal bulk modulus,}$$

$$\alpha_v = -\frac{\partial^2 \varphi_1}{\partial \beta \partial \vartheta}\left(\beta \frac{\partial^2 \varphi_1}{\partial \beta^2}\right)^{-1} \qquad \text{volumetric thermal expansion,}$$

$$c_v = -\vartheta \frac{\partial^2 \varphi_1}{\partial \vartheta^2} + O(|\bar{e}|^2) \qquad \text{specific heat at constant volume,}$$

$$-p = \frac{\sigma_{ii}}{3} = \rho_0 \frac{\partial \varphi_1}{\partial \beta} + O(|\bar{e}|^2) \qquad \text{mean pressure,}$$

$$\bar{\sigma} = 2\mu\bar{e} + O(|\bar{e}|^2) \qquad \text{deviator of the Cauchy stress.}$$

Consider now the rate equations for the plastic part. Assume the simplest form of the internal tensorial parameter

(4.4)
$$\dot{\alpha}^* = D^*, \quad \text{tr}\,\alpha^* = 0.$$

From (2.14), (3.2), (3.3) we can calculate the thermodynamical force π^*

(4.5)
$$\pi = \frac{1}{\rho_0}\frac{d\varphi_2}{d\alpha}, \quad \pi^* = \frac{1}{\rho_0}[c_0 - c_T(\vartheta - \vartheta_0)]\alpha^* \overset{\text{def}}{=} \frac{1}{\rho_0}c\alpha^*.$$

We shall identify this force with the back stress that determines the position of the symmetry centre of the yield surface. Then the following evolution equation for the internal force in the unloaded configuration (k_*) is obtained:

(4.6)
$$\dot{\pi}^* = \frac{1}{\rho_0}cD^* - \frac{c_T}{c}\dot{\vartheta}\pi^*,$$

and the dissipation rate (2.18) for the considered model is

(4.7)
$$\mathcal{F} = (T^* - \pi^*) \cdot D^* - \pi\dot{\alpha}.$$

Adopt the generalized Huber–Mises yield criterion

(4.8)
$$f = \frac{3}{2}(\overline{T}^* - \pi^*) \cdot (\overline{T}^* - \pi^*) - Y^2(\vartheta, \pi) = 0.$$

Then the plastic flow law associated with (4.13) yields:

(4.9)
$$\mathbf{D}^* = j\Lambda_* \frac{\partial f}{\partial \overline{\mathbf{T}}^*},$$
$$-\dot{\boldsymbol{\alpha}}^* = j\Lambda_* \frac{\partial f}{\partial \boldsymbol{\pi}^*},$$
$$-\dot{\alpha} = j\Lambda_* \frac{\partial f}{\partial \pi},$$

where

(4.10)
$$j = \begin{cases} 1 & \text{if} \quad f = 0 \quad \text{and} \quad \Lambda_* \geq 0, \\ 0 & \text{if} \quad f = 0 \quad \text{and} \quad \Lambda_* < 0 \quad \text{or} \quad f < 0, \end{cases}$$

and the parameter Λ_* at the yield point $f = 0$ is calculated from the consistency relation

(4.11)
$$\Lambda_* = \rho_0 \frac{3(\overline{\mathbf{T}}^* - \boldsymbol{\pi}^*) \cdot \left(\dot{\overline{\mathbf{T}}}^* + \frac{c_T}{c} \dot{\vartheta} \boldsymbol{\pi}^* \right) - 2Y \frac{\partial Y}{\partial \vartheta} \dot{\vartheta}}{2Y^2 \left[3c + 2 \left(\frac{\partial Y}{\partial \pi} \right)^2 \frac{d^2 \varphi_2}{d\alpha^2} \right]}.$$

The set of constitutive equations for the plastic part in the unloaded configuration (k_*) consists of Eqs. (4.5)–(4.9). They can be transformed to the actual configuration on account of Eqs. (2.4), (2.8) and (2.17). It is evident from Eq. (2.9) that for solids anisotropic with respect to plastic properties, the normality in (k_*) does not imply the normality in (k). When the terms $(\omega^p \overline{\mathbf{e}} - \overline{\mathbf{e}} \omega^p)$ and $(\mathbf{D}^p \overline{\mathbf{e}} - \overline{\mathbf{e}} \mathbf{D}^p)$ may be neglected, $\mathbf{D}^p$, ω^p and τ defined (k) differ from $\mathbf{D}^*$, ω^* and T^* merely by the rotation resulting from the elastic deformation. In this case, normality in (k) implies normality in (k) and vice-versa.

Denote by $\Pi = \rho_0 \pi$ and $\sigma_Y = \rho_0 Y$ the true internal and the yield stress, $\Pi = \rho_0 \pi$, the flow laws in Eulerian description are

(4.12)
$$\mathbf{D}^p = 3j\Lambda(\overline{\mathbf{T}} - \Pi) - (\omega^p \overline{\mathbf{e}} - \overline{\mathbf{e}} \omega^p),$$
$$\dot{\alpha} = \frac{1}{\sigma_Y} \frac{\partial \sigma_Y}{\partial \Pi} [\mathbf{D}^p + (\omega^p \overline{\mathbf{e}} - \overline{\mathbf{e}} \omega^p)] \cdot (\overline{\mathbf{T}} - \Pi),$$
$$\check{\Pi} = c[\mathbf{D}^p + (\omega^p \overline{\mathbf{e}} - \overline{\mathbf{e}} \omega^p)] - \frac{c_T}{c} \dot{\vartheta} \Pi.$$

The yield criterion in terms in (k)-description preserves it form

(4.13)
$$f = \frac{3}{2} (\overline{\mathbf{T}} - \Pi) \cdot (\overline{\mathbf{T}} - \Pi) - \sigma_Y^2(\vartheta, \Pi) = 0$$

and we have the following expression for Λ:

(4.14)
$$\Lambda = \frac{3(\overline{\mathbf{T}} - \Pi) \cdot \left(\check{\overline{\mathbf{T}}} + \frac{c_T}{c} \dot{\vartheta} \Pi \right) - 2\sigma_Y \frac{\partial \sigma_Y}{\partial \vartheta} \dot{\vartheta}}{2\sigma_Y^2 \left[3c + 2 \left(\frac{\partial \sigma_Y}{\partial \Pi} \right)^2 \frac{d^2 \varphi_2}{d\alpha^2} \right]}.$$

The combined plastic hardening modulus H

$$(4.15) \qquad H = 3c + 2(\frac{\partial \sigma_Y}{\partial \Pi})^2 \frac{d^2\varphi_2}{d\alpha^2}$$

is composed of the kinematic hardening part $h_\alpha = 3c$ and the isotropic hardening term h_i.

The changes in the temperature are caused by the dissipation of mechanical work, the heat exchange with environment, the heat of elastic deformation and the heat of internal rearrangement [11]. Let us present here without derivation the equation for the rate of temperature.

$$(4.16) \qquad \rho_0 c_v \dot{\vartheta} = \left(1 - \frac{\Pi}{\sigma_Y}\frac{\partial \sigma_Y}{\partial \Pi}\right)(\overline{\mathbf{T}} - \mathbf{\Pi}) \cdot [\mathbf{D}^p + (\omega^p \overline{\mathbf{e}} - \overline{\mathbf{e}}\omega^p)] - \beta \operatorname{div} \mathbf{q}$$

$$-\vartheta \beta \alpha_v K_T \operatorname{tr} \mathbf{D} - \vartheta \frac{c_T}{c}\mathbf{\Pi} \cdot [\mathbf{D}^p + (\omega^p \overline{\mathbf{e}} - \overline{\mathbf{e}}\omega^p)].$$

Here $\mathbf{q}$ is the heat flux. From this set of equations we can get, in the case of isotropic materials, the equations of [12] and the equations presented by Raniecki and Samanta in [13] for the rigid-plastic solids. In particular, following the procedure presented in [16], we can derive the system of equations describing the isothermal and adiabatic processes.

5. The plastic spin

To complete the constitutive equations of the previous section, we have to specify three equations for ω^* or ω^p (cf. Kleiber and Raniecki [3]). Decision upon what form of plastic spin to admit was left open here. As we know, the plastic spin ω^* describes the rate of material element rotation with respect to the isoclinic unloaded configuration and thus, roughly speaking, it represents the mean relative spin of all material fibers measured with respect to a certain chosen triad. In the case of materials fully isotropic with respect to plastic properties in (k_*), this relative angular velocity may be neglected [12] so that $\omega^* = 0$ and then $\omega^p = 0$. Anisotropy results from the fact that different material fibers have different angular velocities with respect to the director triad chosen. As the plastic spin is not explicitly measurable, the representation theorems for isotropic functions have been used in conjunction with the concept of tensorial structure variables to provide explicit forms for it (Loret [4], Dafalias [1, 2] Paulun and Pęcherski [9]). Another possibility was proposed in Raniecki and Mróz [10] for a model of rigid-plastic solids. Supposing that the texture orientation can be specified by a rigidly rotating triad and that the texture orientation at consecutive steps of plastic deformation can be measured, the plastic spin is then the difference of the material and texture spins. Such an approach closely follows the ideas of Mandel [5, 7]. We present in brief the extension of this idea to elasto-plastic bodies.

Consider the plastic transformation $\mathbf{F}^p$. Let $\mathbf{t}_i^0$, $i = 1, 2, 3$ be the three orthonormal vectors representing the initial texture orientation in (k_0). Assuming that at each subsequent instant t the orientation of texture is specified by the triad $\mathbf{t}_i^*$, the instantaneous texture spin ω^t is

$$(5.1) \qquad \dot{\mathbf{t}}_i^* = \omega^t \mathbf{t}_i^*.$$

The texture reference frame $\mathbf{m}_i^*$ is selected in the following way: vector $\mathbf{m}_1^*$ represents the material fiber lying in a chosen material plane with normal $\mathbf{m}_2^*$ and $\mathbf{m}_3^* = \mathbf{m}_1^* \wedge \mathbf{m}_2^*$

(the vector product). It is shown in [3] that the spin $\hat{\omega}$ of the triad $\mathbf{m}_i^*$ is a very simple function of the rate of permanent strain tensor D^* and the chosen triad:

$$(5.2) \quad 2\hat{\omega}(m,t) = \hat{\omega}_{ij}(\mathbf{m}_i^* \otimes \mathbf{m}_j^* - \mathbf{m}_j^* \otimes \mathbf{m}_i^*),$$

$$\hat{\omega}_{12} = \mathbf{m}_1^* \cdot D^* \mathbf{m}_2^*, \quad \hat{\omega}_{13} = \mathbf{m}_1^* \cdot D^* \mathbf{m}_3^*, \quad \hat{\omega}_{32} = \mathbf{m}_2^* \cdot D^* \mathbf{m}_3^*.$$

If the transformation from $\mathbf{m}_i^*$ to $\mathbf{t}_i^*$ is specified by an orthogonal tensor Q, then the relative texture spin ω^r is defined as follows:

$$(5.3) \quad \omega^r(m,t) = \dot{Q}Q^T.$$

The following relation holds true [10]:

$$(5.4) \quad \omega^* - \omega^t = \hat{\omega}(m,t) - \omega^r(m,t).$$

In Eqs. (5.2)–(5.4) the symbol m emphasizes the fact that $\hat{\omega}(m,t)$ and $\omega^r(m,t)$ depend on the selection of the texture reference frame. If **a generic** instant t the texture reference triad $\mathbf{m}_i^*$ is assumed to coincide with $\mathbf{t}_i^*$, then the corresponding tensors are denoted by $\hat{\omega}$ and ω^r. The relation (5.4) remains valid, so that

$$(5.5) \quad \omega^* - \omega^t = \hat{\omega} - \omega^r,$$

When the motion of the texture frame is isoclinic $\omega^t = 0$, Eq. (5.5) takes the form

$$(5.6) \quad \omega^* = \hat{\omega} - \omega^r.$$

For the sake of simplicity we drop the term $\omega^p \bar{\mathbf{e}} - \bar{\mathbf{e}}\omega^p$ and $(D^p \bar{\mathbf{e}} - \bar{\mathbf{e}}D^p)$ in (2.9). Passing to the Eulerian description, Eq. (5.6) yields

$$(5.7) \quad \begin{aligned} \omega^p &= \omega_1^p - \omega_2^p, \\ \omega_1^p &= R^e \hat{\omega} R^{eT} = \mathbf{g}_1, \\ \omega_2^p &= R^e \omega^r R^{eT} = \mathbf{g}_2. \end{aligned}$$

Here $\mathbf{t}_i$ is the texture triad in actual configuration. The plastic spin is thus divided into two parts. From Eq. (5.2), $\mathbf{g}_1$ is a known function of $\mathbf{t}_i$ and D^p. To complete the description, a constitutive equation for $\mathbf{g}_2$ should be formulated. This could be verified by metallographic measurements of texture orientation changes during deformation. In general this equation has the form

$$(5.8) \quad \mathbf{g}_2 = \mathbf{g}_2(D^p, T, \Pi, \alpha, \mathbf{t}_i),$$

and $\mathbf{g}_2$ is an isotropic function of all tensor arguments. For rate-independent materials it is a homogeneous function of degree one with respect to D^p. Due to the lack of appropriate experimental data, some theoretical assumptions may be proposed for the form of $\mathbf{g}_2$ [15].

6. Conclusions

The simple model of metallic solids with combined isotropic linear kinematic hardening modified by thermal effects is considered here. The complexity of the equations is minimized by assuming that the material satisfies the Huber–Mises yield condition and the associated flow rule, rate dependence of plastic properties being excluded. The choice of flow rule other than (4.7) could be made if some particular mechanism of flow required such a choice. Mandel's formalism combined with the concepts of thermodynamics may be used to generalize the more complex models of infinitesimal plasticity theory.

Acknowledgements

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