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# On deformation-induced plastic anisotropy of sheet metals

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PROBLEMS CONNECTED with the theoretical description of plastic flow of polycrystalline sheet metals are discussed and commented. Sheet metals display distinct anisotropic properties in the plastic range of deformation. It is observed that these properties may be different depending on the plastic deformation histories in the course of the manufacturing process. Much attention has been devoted to the experimental tests of plastic anisotropy of sheet metals and to the comparison of test results with various theoretical descriptions.

## 1. Introduction

POLYCRYSTALLINE SHEET metals display anisotropic plastic properties caused by complex deformation processes, like multistage rolling and stretching connected with heat treatment of the material during the manufacturing process. An important factor, difficult to be included into the theoretical description, is the non-homogeneity of anisotropy through the thickness of the sheet. This effect is usually averaged in theoretical analyses.

Depending on the history of deformations during the manufacturing process, the plastic anisotropy of sheet metals may be of different form. In sheet metals it is usually the orthotropy: two axes of orthotropy lie in sheet's plane, and the third axis is directed normally to this plane.

As a very important case of orthotropy, the so-called normal anisotropy, when the sheet is isotropic in its plane, while its plastic properties in the direction normal to the sheet's surface are different, should be mentioned. A more general case of anisotropy one can observe when the sheet under loadings acting in its plane displays quasi-orthotropic plastic properties, while the direction normal to sheet's surface does not constitute the orthotropy axis. Such a behaviour will be referred to in this paper as the quasi-orthotropy. Sheet metals may also display the so-called generalized Bauschinger effect.

The proper definition of the type of anisotropy of a sheet metal in question is especially important, when it is used as the material for manufacturing products with the use of various drawing techniques, since the anisotropy plays an important role in overall deformability of sheet metals. These problems are, for example, discussed in the book by MARCINIAK [1]. The practical significance of the anisotropy of sheet metals stimulated the development of various experimental techniques allowing to measure anisotropy parameters. The complexity of the phenomena connected with the plastic flow of anisotropic materials makes useful its theoretical description because, first of all, such a theory allows us to rationalize any experimental program and the proper interpretation of the obtained results.

In the present paper a certain attempt has been made to discuss more deeply the mechanics of plastic flow of sheet metals with the analysis of various experimental methods concerning such a flow. The possible practical applicability of the less studied variants of the theory for interpretation of experimental results has been also analysed.

## 2. Fundamentals of the classical mechanics of plastic flow of anisotropic bodies

The theory of plastic flow of anisotropic bodies originated in 1928 when the classical paper by MISES [2] appeared. It has been written in connection with plastic deformations of single crystals. In the following papers of other authors the concept proposed by Mises was used for theoretical description of plastic behaviour of polycrystalline metals with deformation-induced anisotropy.

MISES in his work [2] proposed the following general yield function for crystals:

$$
\begin{aligned}
(2.1)\qquad \psi = {} & h_{11}\sigma_x^2 + h_{22}\sigma_y^2 + h_{33}\sigma_y^2 + h_{44}\tau_{xy}^2 + h_{55}\tau_{yz}^2 + h_{66}\tau_{zx}^2 \\
& +2h_{12}\sigma_x\sigma_y + 2h_{13}\sigma_x\sigma_z + 2h_{14}\sigma_x\tau_{xy} + 2h_{15}\sigma_x\tau_{yz} + 2h_{16}\sigma_x\tau_{zx} \\
& +2h_{23}\sigma_y\sigma_z + 2h_{24}\sigma_y\tau_{xy} + 2h_{25}\sigma_y\tau_{yz} + 2h_{26}\sigma_y\tau_{zx} \\
& +2h_{34}\sigma_z\tau_{xy} + 2h_{35}\sigma_z\tau_{yz} + 2h_{36}\sigma_z\tau_{zx} \\
& +2h_{45}\tau_{xy}\tau_{yz} + 2h_{46}\tau_{xy}\tau_{zx} + 2h_{56}\tau_{yz}\tau_{zx}\,.
\end{aligned}
$$

It was assumed here that the yield function is quadratic with respect to stress components $\sigma_x, \sigma_y, \ldots, \tau_{zx}$. It contains 21 various coefficients (moduli) $h_{ij}(i,j = 1,2,\ldots,6)$ of anisotropy. Function (2.1) remains unchanged when the signs of all stress components are changed. Thus, when using it, we cannot take into account the Bauschinger effect. Since in the yield function (2.1) there are terms containing products of normal and shear stresses, it is necessary to establish a rule concerning the sign of the latter in the assumed Cartesian coordinate system $x$, $y$, $z$.

Yield condition for anisotropic bodies corresponding to yield function (2.1) can be written as

$$
(2.2)\qquad H_{ijkl}\sigma_{ij}\sigma_{kl} = \text{const.}
$$

For metals the hydrostatic component $p$ of the stress tensor has no influence on the plastic yielding. Thus, for obtaining the same form of the yield condition after introducing new stresses $\sigma_x - p$, $\sigma_y - p$, $\sigma_z - p$ instead of original stresses $\sigma_x, \sigma_y, \sigma_z$, it is necessary to reduce the number of anisotropy coefficients to 15. The yield condition takes then the form

$$
\begin{aligned}
(2.3)\qquad f(\sigma_{ij}) = {} & k_{12}(\sigma_x - \sigma_y)^2 + k_{23}(\sigma_y - \sigma_z)^2 + k_{31}(\sigma_z - \sigma_x)^2 \\
& +2\tau_{xy}[k_{16}(\sigma_z - \sigma_x) + k_{26}(\sigma_z - \sigma_y)] \\
& +2\tau_{yz}[k_{24}(\sigma_x - \sigma_y) + k_{34}(\sigma_x - \sigma_z)] \\
& +2\tau_{zx}[k_{35}(\sigma_y - \sigma_z) + k_{15}(\sigma_y - \sigma_x)] \\
& -2k_{45}\tau_{yz}\tau_{zx} - 2k_{56}\tau_{zx}\tau_{xy} - 2k_{64}\tau_{xy}\tau_{yz} \\
& +k_{44}\tau_{yz}^2 + k_{55}\tau_{zx}^2 + k_{66}\tau_{xy}^2 - 1 = 0\,.
\end{aligned}
$$

OLSZAK and URBANOWSKI [3, 4, 5] have generalized this yield condition for simultaneously anisotropic and non-homogeneous bodies. In such a case all coefficients $k_{ij}$ in Eq. (2.3) are known functions of the coordinates. The form of the yield condition remains the same as above.

Function (2.3) may be treated as the so-called plastic potential. Plastic strain rates are then connected with the stress components by the flow law

$$
(2.4)\qquad \dot{\varepsilon}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}}\,,
$$

where $\lambda$ is a proportionality factor. Relations (2.4) are referred to as the associated flow law.

Physical interpretation of anisotropy coefficients $k_{ij}$ in Eq. (2.3) may be found by considering various uniaxial stress states, each with one non-vanishing stress component only. By considering uniaxial tension (compression) we arrive at the relations (cf. [3])

$$\begin{aligned} k_{12} &= \frac{1}{2}\left(\frac{1}{Y_x^2} + \frac{1}{Y_y^2} - \frac{1}{Y_z^2}\right), \\ k_{23} &= \frac{1}{2}\left(-\frac{1}{Y_x^2} + \frac{1}{Y_y^2} + \frac{1}{Y_z^2}\right), \\ k_{31} &= \frac{1}{2}\left(\frac{1}{Y_x^2} - \frac{1}{Y_y^2} + \frac{1}{Y_z^2}\right), \end{aligned} \tag{2.5}$$

where $Y_x$, $Y_y$, $Y_z$ stand for the yield loci of the material under uniaxial tension in directions $x, y, z$, respectively. Exact realization of such tests is rather difficult for materials with the general type of anisotropy, since application of the standard testing techniques produces distortion of the tensioned specimens, which assume the form of the letter $S$. This problem has been analysed for example by Boehler et al. [6]. To minimize the arising curvature of the test piece they have designed special grips with knife-edged joints.

Even more difficult is realization of shear tests which are necessary, if yield loci in shear $Q_{xy}$, $Q_{yz}$, $Q_{zx}$ are to be directly measured. These yield loci are related to the respective anisotropy coefficients by the formulae

$$k_{44} = \frac{1}{Q_{yz}^2}, \qquad k_{55} = \frac{1}{Q_{zx}^2}, \qquad k_{66} = \frac{1}{Q_{xy}^2}. \tag{2.6}$$

The anisotropy coefficients are often measured indirectly, especially for sheet metals, by making use of the anisotropy of plastic strain increments that are connected with the yield condition (2.3) by the associated flow law (2.4). We obtain from that flow law the expressions for strain increments

$$\begin{aligned} d\varepsilon_x &= 2d\lambda[k_{12}(\sigma_x - \sigma_y) - k_{31}(\sigma_z - \sigma_x) - k_{16}\tau_{xy} + (k_{24} + k_{34})\tau_{yz} - k_{15}\tau_{zx}], \\ d\varepsilon_y &= 2d\lambda[-k_{12}(\sigma_x - \sigma_y) + k_{23}(\sigma_y - \sigma_z) - k_{26}\tau_{xy} - k_{24}\tau_{yz} + (k_{35} + k_{15})\tau_{zx}], \\ d\varepsilon_z &= 2d\lambda[-k_{23}(\sigma_y - \sigma_z) + k_{31}(\sigma_z - \sigma_x) + (k_{16} + k_{26})\tau_{xy} - k_{34}\tau_{yz} - k_{35}\tau_{zx}]. \end{aligned} \tag{2.7}$$

The corresponding expressions for shear strain increments are not given here, since they are difficult to be measured in the laboratory tests and therefore such expressions are not used in practice.

When using relations (2.7) for indirect measuring anisotropy coefficients we should be aware that often the values of these coefficients measured indirectly differ remarkably from those measured directly. Respective examples will be discussed later on. Nevertheless, it should be emphasized that in some cases indirect measuring methods based on relations (2.7) are the only methods possible for practical realization.

## 3. Fundamentals of the classical mechanics of plastic flow of orthotropic bodies

Orthotropy is an important particular case of general anisotropy. At each point of the body three orthogonal planes of symmetry of plastic properties can be distinguished. Intersections of these planes are called the axes of orthotropy. Directions of orthotropy axes may vary from point to point. Let us now discuss various types of orthotropy that are of practical significance.

### 3.1. Basic relations of plastic orthotropy

In the case of orthotropy the general yield condition (2.3) reduces to the form containing six plastic moduli only. Assuming that the axes of orthotropy coincide with the axes $x, y, z$ of the Cartesian coordinate system we can write (cf. [3])

$$(3.1)\quad k_{12}(\sigma_x - \sigma_y)^2 + k_{23}(\sigma_y - \sigma_z)^2 + k_{31}(\sigma_z - \sigma_x)^2 + k_{44}\tau_{yz}^2 + k_{55}\tau_{zx}^2 + k_{66}\tau_{xy}^2 = 1\,.$$

In HILL'S book [7] this yield condition is written in the equivalent form

$$(3.1\text{a})\quad F(\sigma_y - \sigma_z)^2 + G(\sigma_z - \sigma_x)^2 + H(\sigma_x - \sigma_y)^2 + 2L\tau_{yz}^2 + 2M\tau_{zx}^2 + 2N\tau_{xy}^2 = 1\,.$$

If directions of the principal stresses $\sigma_1$, $\sigma_2$, $\sigma_3$ coincide with the directions of orthotropy axes, condition (3.1) takes the form

$$(3.2)\qquad k_{12}(\sigma_1 - \sigma_2)^2 + k_{23}(\sigma_2 - \sigma_3)^2 + k_{31}(\sigma_3 - \sigma_1)^2 = 1\,.$$

By substituting relations (2.5) we obtain the equivalent form of yield condition (3.2)

$$(3.3)\quad \left(\frac{1}{Y_x^2} + \frac{1}{Y_y^2} - \frac{1}{Y_z^2}\right)(\sigma_1 - \sigma_2)^2 + \left(-\frac{1}{Y_x^2} + \frac{1}{Y_y^2} + \frac{1}{Y_z^2}\right)(\sigma_2 - \sigma_3)^2 + \left(\frac{1}{Y_x^2} - \frac{1}{Y_y^2} + \frac{1}{Y_z^2}\right)(\sigma_3 - \sigma_1)^2 = 2\,.$$

Here the orthotropy coefficients have been replaced by the yield loci in orthotropy directions.

Note that for orthotropic bodies, the measurements of yield loci under simple tension or compression in orthotropy direction can be performed with the use of standard experimental methods, because the test pieces cut out from the material in these directions will not suffer any distortion.

Yield condition (3.3) may be represented in the space of principal stresses $\sigma_1$, $\sigma_2$, $\sigma_3$ as an infinitely long cylinder with elliptical cross-section and the axis making the same angle with all three axes of principal stresses. In Fig. 1 two projections of that cylinder are presented. Cross-section of the cylinder by an octahedral plane

$$\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma^0$$

is also shown in the figure. In the lower part a triangular portion of the octahedral plane with this elliptical cross-section is presented. Dimensions and orientation of the ellipse depend on the yield loci in the directions of the orthotropy axes

$$Y_1 \equiv Y_x, \quad Y_2 \equiv Y_y, \quad Y_3 \equiv Y_z\,.$$

We shall find the equation of the ellipse on the octahedral plane with the axes $\sigma_1^0$, $\sigma_2^0$, $\sigma_3^0$ on that plane being projections of the original axes $\sigma_1$, $\sigma_2$, $\sigma_3$, respectively. Geometrical relations for the tetrahedron shown in Fig. 1 are used to find positions of points representing the respective yield loci on the axes $\sigma_1^0$, $\sigma_2^0$, $\sigma_3^0$ (Fig. 2)

$$Y_1^0 = Y_1\sqrt{\frac{2}{3}}, \quad Y_2^0 = Y_2\sqrt{\frac{2}{3}}, \quad Y_3^0 = Y_3\sqrt{\frac{2}{3}}\,.$$

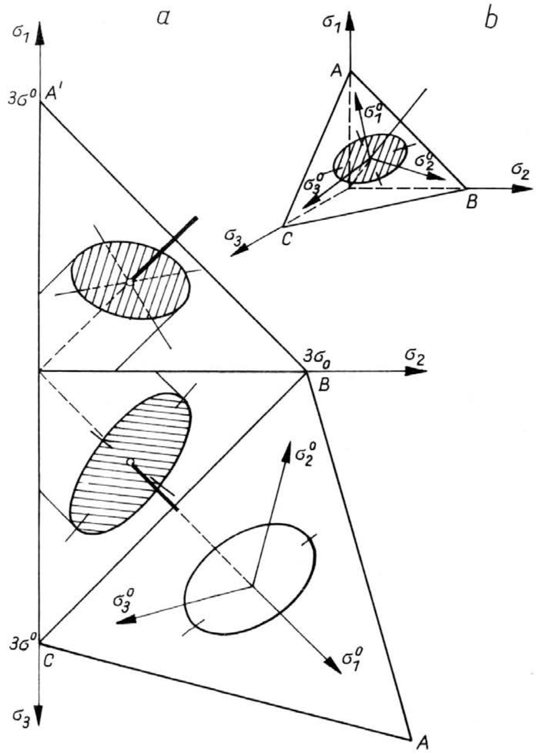


FIG. 1.

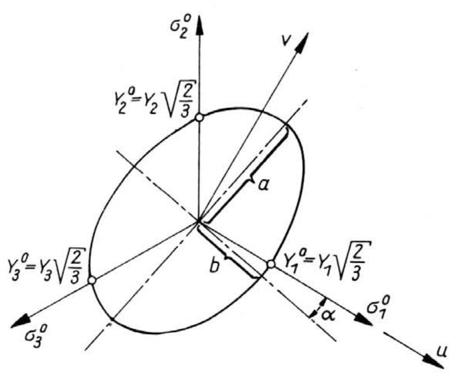


FIG. 2.

The ellipse with the centre at the origin of coordinates and passing through these three points has the equation

$$(3.4)\qquad \frac{3}{2Y_1^2}u^2 + \sqrt{3}\Big(\frac{1}{Y_2^2} - \frac{1}{Y_2^2})uv + (\frac{1}{Y_2^2} + \frac{1}{Y_3^2} - \frac{1}{2Y_1^2}\Big)v^2 = 1\,.$$

in the auxiliary coordinate system $u$, $v$ (Fig. 2).

Note that the yield surface presented in Fig. 1 does not determine uniquely the yield condition of an orthotropic body. It is valid for arbitrary values of anisotropy coefficients $k_{44}$, $k_{55}$, $k_{66}$ appearing in the generally written yield condition (3.1). Thus it corresponds only to particular stress states, when directions of principal stresses coincide with those of orthotropy.

Now we shall consider a certain particular case of orthotropy, that can occur in commercial metals, especially in sheet metals.

### 3.2. Transversal isotropy

Transversal isotropy is a particular case of orthotropy, when all directions perpendicular to one of orthotropy axes, say $z$-axis, are equivalent. Such a type of orthotropy characterises certain sedimentary soils and rocks, like e.g. diatomite (cf. [8, 9, 10]). However, for such materials the yield conditions discussed here do not apply, since plastic yielding of them depends on the first stress invariant, which is neglected in these conditions.

Transversal isotropy is often observed in sheet metals. Thus its practical significance is obvious. Yield condition for transversally isotropic materials results from Eq. (3.1), if the following relations and new notations (cf. [4])

$$k_{23} = k_{31} = k_1, \qquad k_{44} = k_{55} = k_2, \qquad k_{66} = 2(k_1 + 2k_{12})$$

are introduced. Then the yield condition for transversally isotropic materials takes the form

$$(3.5) \qquad k_1[\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + k_{12}(\sigma_x - \sigma_y)^2 + k_2(\tau_{zy}^2 + \tau_{zx}^2) + 2(k_1 + 2k_{12})\tau_{xy}^2 = 1 .$$

Since now we have $Y_x = Y_y = Y_0$, from Eq. (2.5) result the expressions for the anisotropy coefficients

$$k_1 = \frac{1}{2Y_z^2}, \qquad k_{12} = \frac{1}{Y_0^2} - \frac{1}{2Y_z^2} .$$

Now the yield condition (3.5) may be written in equivalent form

$$(3.5') \qquad \frac{1}{2Y_z^2}[(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + \left(\frac{1}{Y_0^2} - \frac{1}{2Y_z^2}\right)(\sigma_x - \sigma_y)^2 + k_2(\tau_{yz}^2 + \tau_{zx}^2) + \left(\frac{4}{Y_0^2} - \frac{1}{Y_z^2}\right)\tau_{xy}^2 = 1 .$$

Let us assume such a stress state when the direction of the principal stress $\sigma_3$ coincides with the orthotropy axis which has been chosen as the $z$-axis. The remaining two principal stresses can be oriented arbitrarily in the plane of transversal isotropy. For such stress states the yield condition (3.5′) can be written in the form

$$(3.6) \qquad \frac{1}{2Y_z^2}[(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] + \left(\frac{1}{Y_0^2} - \frac{1}{2Y_z^2}\right)(\sigma_1 - \sigma_2)^2 = 1 .$$

Yield condition (3.6) may be represented in the space of principal stresses by a certain cylinder with elliptical cross-section, in the similar manner as for the case of general orthotropy. Its elliptical cross-section is determined on the octahedral plane by the equation

$$(3.7) \qquad \frac{3}{2Y_0^2}u^2 + \sqrt{3}\left(\frac{1}{Y_z^2} - \frac{1}{Y_0^2}\right)uv + \left(\frac{1}{2Y_0^2} + \frac{1}{Y_z^2}\right)v^2 = 1 .$$

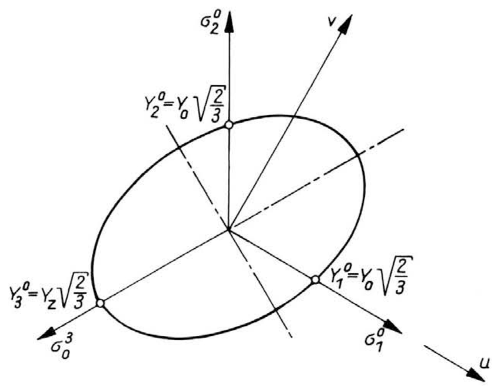


FIG. 3.

One of the principal axes of the ellipse coincides with the $\sigma_3^0$-axis on that plane (Fig. 3).

A certain peculiar type of transversal isotropy occurs when yield loci are identical for all three directions, that is when

$$Y_x = Y_y = Y_z = Y_0 . \tag{3.8}$$

In such a case yield condition (3.5a) reduces to the form

$$(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\tau_{xy}^2 + 2k_2 Y_0^2(\tau_{yz}^2 + \tau_{zx}^2) = 2Y_0^2 . \tag{3.9}$$

For particular stress states, when the direction of the principal stress $\sigma_3$ coincides with the orthotropy direction $z$, yield condition (3.9) reduces to the form

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y_0^2 . \tag{3.10}$$

This form of the yield condition of the material with a special type of orthotropy is identical with that of the Huber–Mises yield condition for isotropic materials. The cross-section of the cylinder in the space of principal stresses is circular. However, there is now a distinct difference with respect to isotropic materials, since yield condition (3.10) and the respective cylindrical yield surface in the space of principal stresses correspond to particular stress states only, when principal stress $\sigma_3$ is directed along the axis $z$ of orthotropy. For other stress states yield condition (3.10) is not valid.

### 3.3. Cubic orthotropy

Cubic orthotropy (cf. [4] is still another particular type of orthotropy characterized by identical values of yield stress under uniaxial tension (compression) in the directions of orthotropy — cf. Eq. (3.8). However, now all three orthotropy axes are fixed, while in the previous case the problem of transversal isotropy was dealt with. Let us assume that the axes $x$, $y$, $z$ of the coordinate system coincide with the axes of orthotropy.

In the yield condition

$$(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 2k_2 Y_0^2(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) = 2Y_0^2 \tag{3.11}$$

we have two material constants $Y_0$ and $k_2$.

When principal stresses are directed along the respective orthotropy axes, this yield condition reduces to the form Eq. (3.10), which is represented by the cylinder with circular cross-section in the space of principal stresses. Thus the Huber–Mises cylinder may correspond to various yield conditions including that for isotropic materials and those for materials with various particular cases of orthotropy.

## 4. Yield conditions for anisotropic bodies displaying Bauschinger effect

In sheet metals and in metal rods, a generalized form of the Bauschinger effect can be observed caused by plastic deformations induced during the manufacturing process. Generally speaking, this effect manifests itself as a difference between the absolute values of yield points of the material loaded by stress states of opposite signs. Bauschinger published his results in 1879 [11]. Interesting information concerning the history of investigations concerning this effect may be found in Bell's book [12]. Starting from the early fifties, scores of papers have appeared, in which a more generally understood Bauschinger effect induced in metals by previous plastic deformation was investigated. We shall mention only few earlier works, [13] to [17]. A comprehensive review of such papers was published by IKEGAMI [18, 19].

Interpretation of experimental results presented in the mentioned papers requires certain explanations if they are to be considered in terms of the theory of plastic anisotropy of commercial metals. Such metals underwent, during the manufacturing process, complex and usually not clearly known plastic deformations. In the experimental works referred to, the deformation history was strictly prescribed and registered. Thus it was possible to analyse the results of tests in terms of various strain-hardening hypothesis. Various definitions of yield point were assumed. In numerous papers the yield point was identified with the proportionality limit.

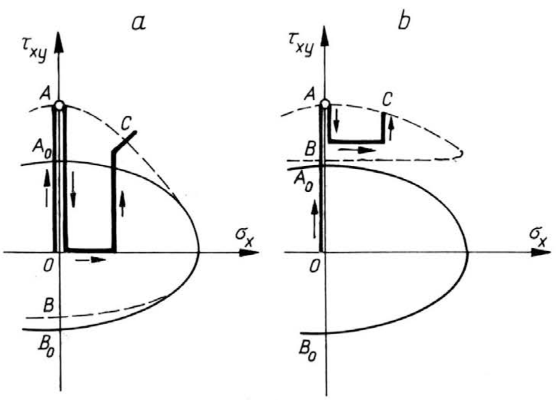


FIG. 4.

Two methods of performing experiments concerning the effect of plastic deformation on the yield condition may be distinguished. The scheme of the first of them is presented in Fig. 4a — which corresponds to the experimental procedure used in [13]. Thin-walled tubular specimens were twisted beyond the initial yield locus. Then, after total unloading, they were subsequently loaded by various combinations of axial force and twisting moment. One of such subsequent loading paths is shown in the figure. During each subsequent loading a point $C$ corresponding to the new conventional yield limit was found. The experimentally determined new yield surface of prestressed material, shown in the figure by the dashed line, differs remarkably from the initial yield surface shown by the continuous line. The new yield surface may be treated as that of the material with deformation-induced anisotropy.

Such a statement cannot be formulated if the new yield surface is determined according to the other experimental procedure presented schematically in Fig. 4b. It corresponds to the procedure being used e.g. in [15]. It was used also in numerous investigations of other authors. As in the first method, thin-walled tubular specimens were prestressed by twisting up to the point $A$, beyond the initial yield point $A_0$. Then they were only partly unloaded to a certain value of the twisting moment and subsequently reloaded along various loading paths — cf. Fig. 4b. In such a manner consecutive points $C$ corresponding to the new conventional yield surface have been experimentally determined. If the predeformation is sufficiently large, the origin of the reference system lies outside the new yield surface identified with the surface of the proportionality limit.

In both papers [13] and [15] subsequent loading was interrupted when the first nonlinear deformations appeared. Thus the information on the plastic behaviour of prestressed material was limited. More information may be found in the works of other authors. Interesting results were obtained by DEAK [20], who measured (with great accuracy) the deformations in twisted thin-walled tubes during loading, and also during unloading. It was found that, even after a comparatively small plastic deformations, the unloading diagrams are curvilinear almost from the beginning of the unloading process. It means that the position of point $B$ in Fig. 4b depends on the accuracy of strain measurements.

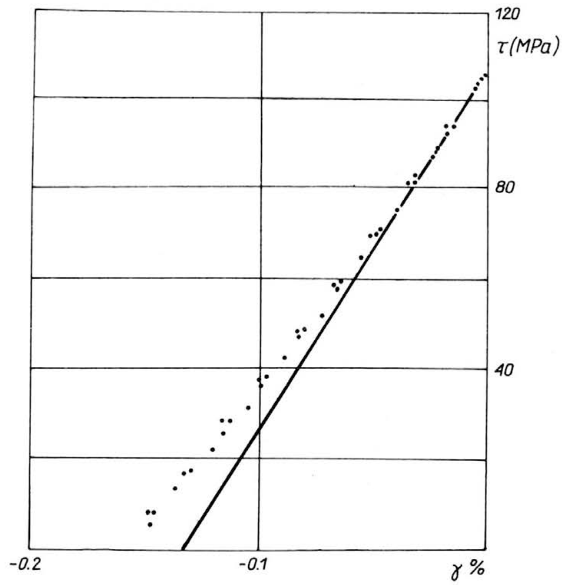


FIG. 5.

One of the unloading diagrams obtained in [20] for a tubular steel specimen prestressed by twisting until 3.6 percent of the permanent deformation is shown in Fig. 5. The diagram is curvi-linear almost from the beginning. Thus, assuming the point at which the first

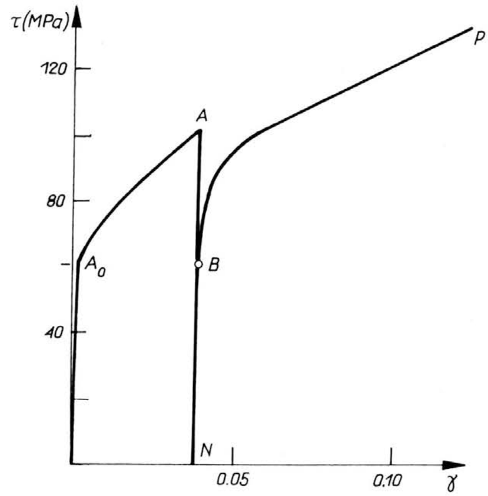


FIG. 6.

symptoms of nonlinearity of the unloading diagram are observed as the yield point, one obtains a subsequent yield surface such as that shown in Fig. 4b.

If experimental results are to be used for the description of the deformation-induced anisotropy, then the subsequent yield locus should rather be identified with the stresses at which plastic yielding during subsequent loading begins from the stress-free state. The diagram (Fig. 6) for a steel tube twisted in opposite directions, taken from [20], demonstrates how points $A$ and $B$ have been obtained in the testing procedure shown in Fig. 4a. The specimen was initially twisted in one direction (sector $A_0 - A$) and then, after total unloading, it was twisted in opposite direction (sector $N - B - P$) — absolute values of stresses and strains are shown in the figure. Points $A_0$, $A$ and $B$ correspond to the respective points in Fig. 4a. In [13] the subsequent loading was interrupted immediately after point $B$ was reached.

Plastic anisotropy induced in the material by deformations connected with the manufacturing process will be treated as an existing property of the material, without connecting it with the previous deformation history.

#### 4.1. On a general form of yield condition for anisotropic bodies with Bauschinger effect

If the Bauschinger effect in metals with deformation-induced anisotropy is to be accounted for in the yield condition, linear terms with respect to stress components should be introduced. There were no such terms in the yield conditions discussed in previous Sections. However, they exist in the yield condition

$$(4.1)\quad F(\sigma_y - \sigma_z)^2 + G(\sigma_z - \sigma_x)^2 + H(\sigma_x - -\sigma_y)^2 \\ +2L\tau_{yz}^2 + 2M\tau_{zx}^2 + 2N\tau_{xy}^2 - C\sigma_x - D\sigma_y - E\sigma_z = 1$$

proposed by OTA *et al.* [21] — cf. also [22, 23].

Constants $C$, $D$, $E$ must satisfy the equality

$$C + D + E = 0,$$

if the plastic yielding is to be independent of the hydrostatic component of the stress tensor.

We shall analyse a more general form of the yield condition for materials displaying the Bauschinger effect. By analogy to Eq. (2.3), we shall write

$$(4.2)\qquad \begin{aligned} f(\sigma_{ij}) = {} & k_{12}(\sigma_x-\sigma_y)^2+k_{23}(\sigma_y-\sigma_z)^2+k_{31}(\sigma_z-\sigma_x)^2 \\ & +2\tau_{xy}[k_{16}(\sigma_z-\sigma_x)+k_{26}(\sigma_z-\sigma_y)] \\ & +2\tau_{yz}[k_{24}(\sigma_x-\sigma_y)+k_{34}(\sigma_x-\sigma_z)] \\ & +2\tau_{zx}[k_{35}(\sigma_y-\sigma_z)+k_{15}(\sigma_y-\sigma_x)] \\ & -2k_{45}\tau_{yz}\tau_{zx}-2k_{56}\tau_{zx}\tau_{xy}-2k_{64}\tau_{xy}\tau_{yz} \\ & +k_{44}\tau_{yz}^2+k_{55}\tau_{zx}^2+k_{66}\tau_{xy}^2 \\ & -b_{12}(\sigma_x-\sigma_y)-b_{23}(\sigma_y-\sigma_z)-b_{31}(\sigma_z-\sigma_x) \\ & +b_{44}\tau_{yz}+b_{55}\tau_{zx}+b_{66}\tau_{xy}=1\,. \end{aligned}$$

Physical interpretation of some of the anisotropy coefficients $k_{ij}$ and $b_{ij}$ may be determined by analysing uniaxial stress states, each with only one non-vanishing stress component. By considering uniaxial tension (compression) we obtain the relations (cf. [22])

$$(4.3)\qquad \begin{aligned} k_{12} &= \frac{1}{2}\left(\frac{1}{Y_xZ_x}+\frac{1}{Y_yZ_y}-\frac{1}{Y_zZ_z}\right), \\ k_{23} &= \frac{1}{2}\left(-\frac{1}{Y_xZ_x}+\frac{1}{Y_yZ_y}+\frac{1}{Y_zZ_z}\right), \\ k_{31} &= \frac{1}{2}\left(\frac{1}{Y_xZ_x}-\frac{1}{Y_yZ_y}+\frac{1}{Y_zZ_z}\right), \end{aligned}$$

where $Y_x$, $Y_y$, $Y_z$ stand for the yield limits of the material uniaxially tensioned in the directions $x$, $y$, $z$, respectively, and $Z_y$, $Z_y$, $Z_z$ are the yield limits under uniaxial compressive loading.

Considering uniaxial tensile (compressive) loadings, we obtain also the following relations:

$$(4.4)\qquad \begin{aligned} b_{31}-b_{12} &= \frac{1}{Y_x}-\frac{1}{Z_x}, \\ b_{12}-b_{23} &= \frac{1}{Y_y}-\frac{1}{Z_y}, \\ b_{23}-b_{31} &= \frac{1}{Y_z}-\frac{1}{Z_z}, \end{aligned}$$

for the moduli $b_{12}$, $b_{23}$, $b_{31}$. This system of equations has no unique solution since its characteristic determinant is equal to zero. By assuming that one of the moduli, say $b_{31}$, is equal to zero we obtain

$$(4.5)\qquad b_{31}=0,\quad b_{12}=-\frac{1}{Y_x}+\frac{1}{Z_x},\quad b_{13}=\frac{1}{Y_z}-\frac{1}{Z_z}\,.$$

Another variants are

$$(4.5')\qquad b_{12}=0,\quad b_{23}=-\frac{1}{Y_y}+\frac{1}{Z_y},\quad b_{31}=\frac{1}{Y_x}-\frac{1}{Z_x},$$

or

$$b_{23} = 0, \quad b_{12} = \frac{1}{Y_y} - \frac{1}{Z_y}, \quad b_{31} = -\frac{1}{Y_z} + \frac{1}{Z_z}. \tag{4.5''}$$

From Eqs. (4.4) the relation follows

$$\frac{1}{Y_x} + \frac{1}{Y_y} + \frac{1}{Y_z} = \frac{1}{Z_x} + \frac{1}{Z_y} + \frac{1}{Z_z}. \tag{4.6}$$

Let $R_{xy}$ be the yield limit under simple shearing by the shear stress $\tau_{xy}$ acting in positive direction, and let $S_{xy}$ be the yield limit under shear stress $\tau_{xy}$ acting in negative direction. Similarly $R_{yz}$, $S_{yz}$ and $R_{zx}$, $S_{zx}$ stand for the respective yield limits under loading by shear stresses $\tau_{yz}$ and $\tau_{zx}$, respectively. Then the anisotropy coefficients $k_{44}$, $k_{55}$, $k_{66}$ and $b_{44}$, $b_{55}$, $b_{66}$ can be related to these yield limits

$$\begin{aligned} k_{44} &= \frac{1}{R_{yz}S_{yz}}, & b_{44} &= \frac{1}{R_{yz}} - \frac{1}{S_{yz}}, \\ k_{55} &= \frac{1}{R_{zx}S_{zx}}, & b_{55} &= \frac{1}{R_{zx}} - \frac{1}{S_{zx}}, \\ k_{66} &= \frac{1}{R_{xy}S_{xy}}, & b_{66} &= \frac{1}{R_{xy}} - \frac{1}{S_{xy}}. \end{aligned} \tag{4.7}$$

#### 4.2. Associated plastic strain increments

The anisotropy of plastic strain increments may be used, as it was mentioned above, for undirect measuring of the values of the anisotropy coefficients appearing in yield conditions. For materials with Bauschinger effect the expressions for plastic strain increments associated with yield condition (4.2) are — cf. expressions (2.7) — as follows:

$$\begin{aligned} d\varepsilon_x &= d\lambda\{2[k_{12}(\sigma_x - \sigma_y) - k_{31}(\sigma_z - \sigma_x) - k_{16}\tau_{xy} \\ &\qquad + (k_{24} + k_{34})\tau_{yx} - k_{15}\tau_{zx}] - b_{12} + b_{31}\}, \\ d\varepsilon_y &= d\lambda\{2[-k_{12}(\sigma_x - \sigma_y) + k_{23}(\sigma_y - \sigma_z) - k_{26}\tau_{xy} \\ &\qquad - k_{24}\tau_{yz} + (k_{35} + k_{15})\tau_{zx}] + b_{12} - b_{23}\}, \\ d\varepsilon_z &= d\lambda\{2[-k_{23}(\sigma_y - \sigma_z) + k_{31}(\sigma_z - \sigma_x) \\ &\qquad + (k_{16} + k_{26})\tau_{xy} - k_{34}\tau_{yz} - k_{35}\tau_{zx}] + b_{23} - b_{31}\}. \end{aligned} \tag{4.8}$$

These expressions will be used in the analysis of stress states in sheet metals.

#### 4.3. Particular cases of anisotropic yield conditions for materials displaying Bauschinger effect

The following particular form of the yield condition (4.2)

$$\begin{aligned} k_{12}(\sigma_x - \sigma_y)^2 + k_{23}(\sigma_y - \sigma_z)^2 + k_{31}(\sigma_z - \sigma_x)^2 + k_{44}\tau_{yz}^2 + k_{55}\tau_{zx}^2 + k_{66}\tau_{xy}^2 \\ - b_{12}(\sigma_x - \sigma_y) - b_{23}(\sigma_y - \sigma_z) - b_{31}(\sigma_z - \sigma_x) \\ + b_{44}\tau_{yz} + b_{55}\tau_{zx} + b_{66}\tau_{xy} = 1 \end{aligned} \tag{4.9}$$

is of practical significance. By substituting Eqs. (4.3), (4.5) and (4.7) we can obtain a more convenient, equivalent form of this condition which, for the sake of brevity, is not given here. Another equivalent forms may be obtained if relations (4.5′) or (4.5″) are used.

If directions of principal stresses $\sigma_1$, $\sigma_2$, $\sigma_3$ coincide with the axes of the reference system $x$, $y$, $z$, yield condition (4.9) can be written as

$$(4.10)\quad k_{12}(\sigma_1-\sigma_2)^2+k_{23}(\sigma_2-\sigma_3)^2+k_{31}(\sigma_3-\sigma_1)^2 -b_{12}(\sigma_1-\sigma_2)-b_{23}(\sigma_2-\sigma_3)-b_{31}(\sigma_3-\sigma_1)=1,$$

or in the equivalent form

$$(4.10')\quad \left(\frac{1}{Y_xZ_x}+\frac{1}{Y_yZ_y}-\frac{1}{Y_zZ_z}\right)(\sigma_1-\sigma_2)^2 +\left(-\frac{1}{Y_xZ_x}+\frac{1}{Y_yZ_y}+\frac{1}{Y_zZ_z}\right)(\sigma_2-\sigma_3)^2 +\left(\frac{1}{Y_xZ_x}-\frac{1}{Y_yZ_y}+\frac{1}{Y_zZ_z}\right)(\sigma_3-\sigma_1)^2 -2\left(\frac{1}{Y_x}-\frac{1}{Z_x}\right)(\sigma_1-\sigma_2)-2\left(\frac{1}{Y_z}-\frac{1}{Z_z}\right)(\sigma_2-\sigma_3)=2.$$

This yield condition may be represented in the space of principal stresses as an infinitely long cylinder with elliptical cross-section. The axis of the cylinder is inclined at the same angle to each of the axes of principal stresses. However, now it does not pass through the origin of the reference system.

In the particular case when

$$(4.11)\quad \frac{1}{Y_xZ_x}=\frac{1}{Y_yZ_y}=\frac{1}{Y_zZ_z},$$

yield condition (4.10′) takes the following form:

$$(4.12)\quad (\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 -2Y_xZ_x\left[\left(\frac{1}{Y_x}-\frac{1}{Z_x}\right)(\sigma_1-\sigma_2)+\left(\frac{1}{Y_z}-\frac{1}{Z_z}\right)(\sigma_2-\sigma_3)\right]=2Y_xZ_x.$$

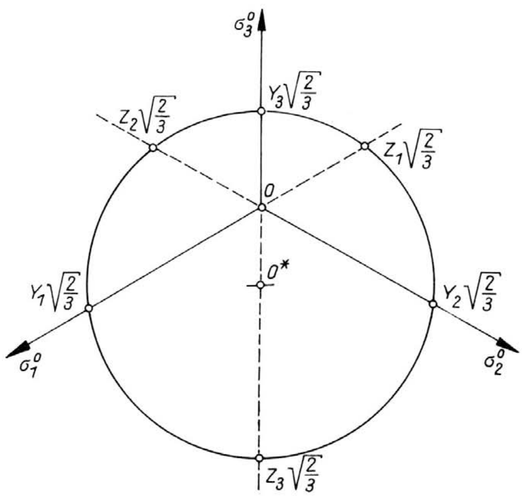


FIG. 7.

This particular form of the yield condition is represented in the space of principal stresses by the cylinder with circular cross-section. Such a cross-section is shown on the octahedral plane in Fig. 7.

#### 4.4. Comparison of theoretical yield conditions with experimental data

Numerous experimental data show that plastic anisotropy induced in metals by previous complex plastic deformations is very complicated. Thus it is difficult to expect that it can be described by an universal theory. All the forms of yield conditions discussed above are based on certain simplifications and, therefore, they should be treated as approximate conditions only. Let us use the following examples to illustrate how far this approximation can be from the actual plastic properties of metals which underwent previous plastic deformation.

Let us analyse the experimental yield surface of prestressed material shown in Fig. 4a by dashed line. It has been determined in [13] by a simultaneous twisting and tension of thin-walled tubular specimen. The stress state reduces to two stress components $\sigma_x$ and $\tau_{xy}$ if the $x$-axis is directed along the generatrix on tube's surface and the $y$-axis has circumferential direction.

For such a particular loading mode, the yield condition (4.9) reduces to the form

$$\frac{1}{Y_x^2}\sigma_x^2 + \frac{1}{R_{xy}S_{xy}}\tau_{xy}^2 + \left(\frac{1}{R_{xy}} - \frac{1}{S_{xy}}\right)\tau_{xy} = 1\,, \tag{4.13}$$

if the assumption that $Y_x = Z_x$, corresponding to the conditions of this experimental test, is introduced.

The experimental values $Y_x$, $R_{xy}$, $S_{xy}$ can be estimated from the experimental curve as shown in Fig. 8. Thus the ellipse (4.13) is uniquely determined. A remarkable difference between this theoretical ellipse and the experimental curve is clearly seen. Similar differences can be observed when other experimental results concerning the effect of plastic deformation on the yield condition are compared with various anisotropic theoretical yield conditions.

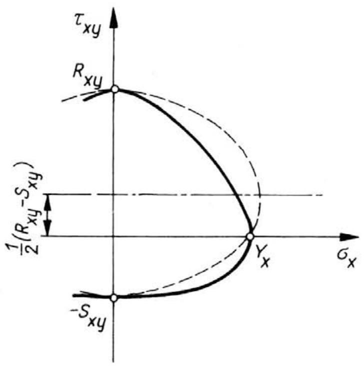


FIG. 8.

In numerous experimental investigations of the effect of plastic deformation on the yield condition, the yield limit is identified with the stresses at which the first deviation from the proportionality between strains and stresses appears. Such an experimental procedure has been used in the mentioned paper [13] from which the experimental curve shown in Fig. 8 was taken. Another experimental procedure in which several conventional yield surfaces have been determined was used by JAGN and SHISHMARIEV [14] and by SZCZEPIŃSKI [16]. Usually such conventional yield loci are determined as the stress levels at which permanent deformations are reaching a certain prescribed conventional small magnitude. The shape and dimensions of a conventional yield surface strongly depend on that prescribed magnitude.

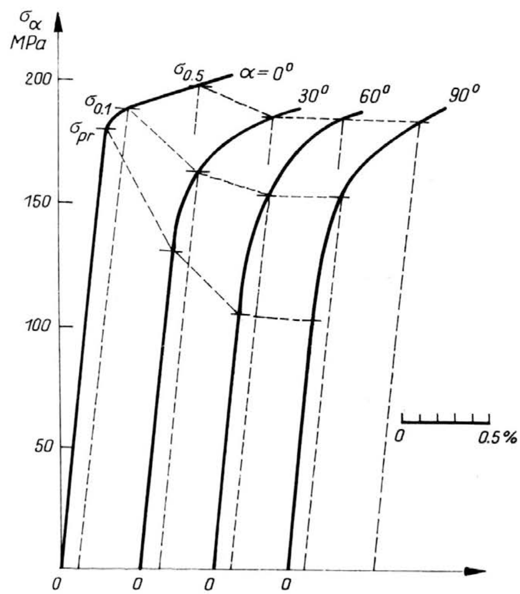


FIG. 9.

As an example, we shall discuss in brief the experimental results obtained in [16]. From a sheet of an Al-2%Mg aluminium alloy, prestressed by uniaxial tension in $x$-direction until 1.92 percent of permanent deformation was reached, small specimens were cut out in different directions making various angles $\alpha$ with the $x$-axis. Then all these small specimens were loaded by uniaxial tension. Initial portions of some stress-strain diagrams are shown in Fig. 9. It is also shown in the figure how the conventional yield loci were determined starting from the proportionality limit $\sigma_{\mathrm{pr}}$ to the conventional yield limit $\sigma_{0.5}$, when the permanent deformation is 0.5 percent. Having found such experimentally determined conventional yield limits $\sigma_\alpha$ for various angles $\alpha$, we can calculate stress components $\sigma_x$, $\sigma_y$, $\tau_{xy}$ from the formulae

$$\begin{aligned} &\sigma_x = \sigma_\alpha \cos^2 \alpha\,, \\ &\sigma_y = \sigma_\alpha \sin^2 \sigma\,, \qquad \tau_{xy} = \sigma_\alpha \sin\alpha \cos\alpha\,. \end{aligned} \tag{4.14}$$

The limit stress states calculated in this manner can be represented in the space of stresses $\sigma_x$, $\sigma_y$, $\tau_{xy}$. Then the corresponding curves passing through the respective points can be drawn (Fig. 10).

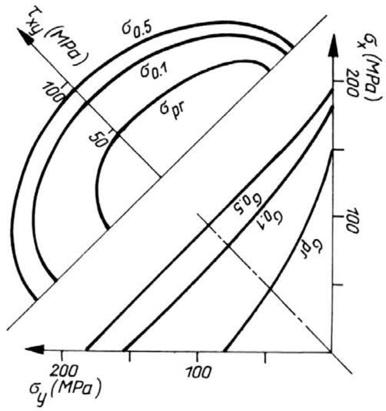


FIG. 10.

In the discussed example, when there exist three stress components only, yield condition (4.9) can be represented by a certain ellipsoid. However, the number of experimental data is not sufficient for determining all the parameters of such an ellipsoid. Nevertheless, the experimental curves shown in Fig. 10 clearly indicate that for each conventional yield limit the ellipsoid will have different dimensions and position. This simple example demonstrates how important is the proper choice of the convention concerning the definition of the yield limit, when the anisotropy coefficients are to be determined.

## 5. Plane stress states in anisotropic sheet metals

Plane stress state exists in a thin metal sheet when all external forces are acting in its median plane. If the coordinate axes are directed as shown in Fig. 11, we have

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0\,. \tag{5.1}$$

The distribution of stress components $\sigma_x$, $\sigma_y$, $\tau_{xy}$ is in general non-uniform across the thickness of the sheet, but their variations are usually insignificant. Hence they are assumed to be uniformly distributed.

Various yield conditions for anisotropic bodies discussed above reduce, for plane stress conditions, to particular forms, that are represented in the space of existing stress components $\sigma_x$, $\sigma_y$, $\tau_{xy}$ by a certain ellipsoid. Such a representation of the yield condition under plane stress states was (for isotropic bodies) used in [16]. For such bodies the yield condition concerning plane stress states takes the well-known form

$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = Y_0^2\,. \tag{5.2}$$

This condition is represented in the stress space $\sigma_x$, $\sigma_y$, $\tau_{xy}$ by an ellipsoid (Fig. 12). One of the axes of the ellipsoid coincides with the $\tau_{xy}$-axis, whereas the two others lie in the $\sigma_x$, $\sigma_y$-plane; they are bisectrices of the right angles between the coordinate axes. Various loading modes used in experimental investigations are represented by certain ellipses lying on the surface of the ellipsoid. For example, ellipse $AB$ corresponds to a

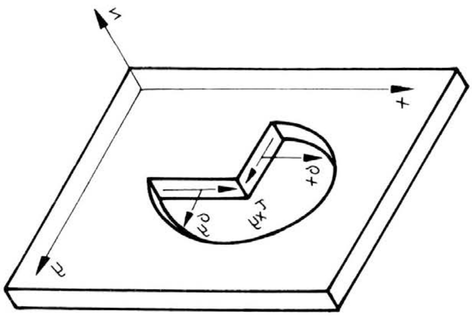

FIG. 11.

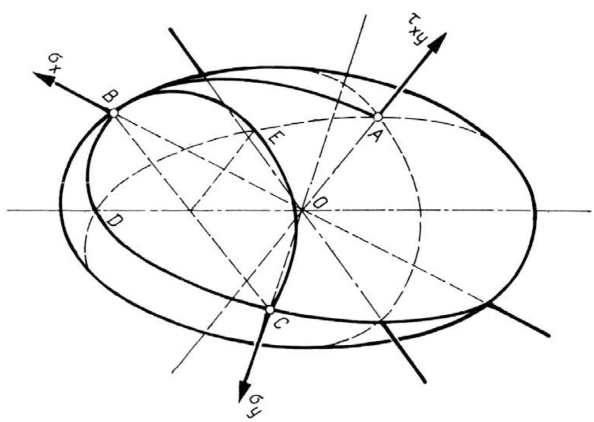


FIG. 12.

combined torsion and tension of tubular specimens. Such a loading mode was used in [13] and [15] — cf. Fig. 4. Ellipse $BDC$ represents the states of biaxial tensile loadings.

Ellipse $BEC$ formed by intersection of the ellipsoid with the plane $\sigma_x + \sigma_y = Y_0$ perpendicular to the $\sigma_x$, $\sigma_y$-plane, corresponds to the states of uniaxial tension in various directions with respect to the $x$-axis. For example, point $E$ represents uniaxial tension in the direction making an angle of 45° with the $x$-axis.

We shall use such a geometrical representation of yield conditions for anisotropic materials under the conditions of plane stress states. Geometrical representation of this kind is useful when experiments for determining the anisotropy coefficients of sheet metals are programmed.

## 6. Yield conditions for sheet metals that do not display Bauschinger effect

We shall discuss now various cases of plastic anisotropy of sheet metals in which no Bauschinger effect is observed. When analysing such yield conditions, it is necessary to make certain assumptions concerning the uniformity of anisotropy in the direction perpendicular to sheet's surface. The plastic deformation induced in the sheet during the rolling process is remarkably non-homogeneous across the thickness of the sheet. Such a non-homogeneity depends, among others, on the frictional conditions that exist at the interfaces between the rolls and the work-piece. Cold rolling is normally carried out with lubricated, polished rolls on material that possesses a fairly high yield stress in shear. Under these conditions, relative movement is assumed to occur between the rolls and the strip at the interfaces; this is termed the slipping friction — cf. CRANE and ALEXANDER [25].

In hot rolling, which is usually performed using rough unlubricated rolls on a very plastic hot material, the frictional drag is assumed to be large enough to attain the yield stress in shear of the rolled strip. It is then assumed that there is no relative motion at the interfaces, and the condition is termed the sticking friction.

Such a classification of frictional conditions during rolling has been repeated here after CRANE and ALEXANDER [25], who performed also the fundamental experiments concerning the distribution of deformations in rolled strips. Such experiments were performed also by MILSON and ALEXANDER [24]. Most important results of these works are discussed below.

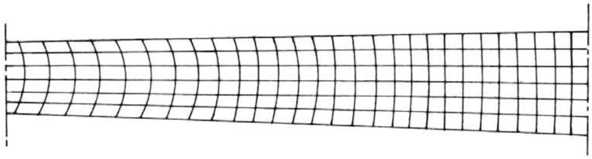

FIG. 13.

A typical distortion of an initially rectangular grid after part-rolling is shown in Fig. 13 — cf. [24]. The non-homogeneous distortion of the originally vertical lines can be clearly seen. When formulating the yield conditions for sheet metals with such induced non-homogeneous plastic strain distribution, their properties in the direction perpendicular to sheet's surface are usually averaged if the sheet is treated as a homogeneous body. Since the distortion in sheet metals is, in most cases, symmetrical with respect to the median plane, it is justified to assume that the direction normal to sheet's surface is the principal direction of anisotropy. However, not always such a symmetrical distribution of distortion occurs in rolled sheets.

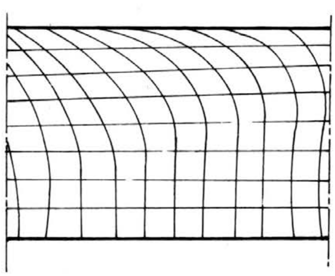

FIG. 14.

If friction at one roll surface is very different from that at the other, very asymmetric grids may be produced (Fig. 14 — cf. [25]). In such cases the averaged axis of anisotropy

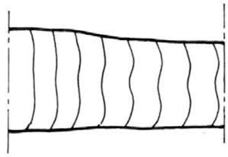

FIG. 15.

will not be normal to sheet's surface. Non-homogeneity of distortion through the thickness of the rolled strips may be connected also with multi-stage rolling. In Fig. 15 is presented the deformation of an initially rectangular grid after three stages of rolling. This figure was prepared on the basis of a photograph obtained by MCGREGOR and COFFIN [26] — cf. also [27].

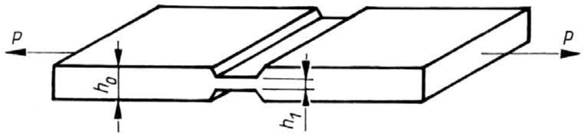


FIG. 16.

Experimental evidence of the non-uniform distribution of distortional deformation through the thickness of rolled sheets indicates that, when special test-pieces (such as that shown in Fig. 16) are prepared, we should be aware that layers of the material deformed differently from that in the vicinity of the median plane have been removed. Specimens of this kind are used for testing the sheet metals under plane strain conditions. However, the properties of the non-removed central part of sheet's material may be different from the overall properties of the whole sheet. Nevertheless, such specimens are used in experimental testing of sheet metals — cf. e.g. [42].

### 6.1. General form of yield condition for anisotropic sheet metals

For plane stress states the general anisotropic yield condition (2.3) may be written as

$$(6.1)\quad (k_{12}+k_{31})\sigma_x^2 - 2k_{12}\sigma_x\sigma_y + (k_{12}+k_{23})\sigma_y^2 + k_{66}\tau_{xy}^2 + 2k_{16}\sigma_x\tau_{xy} + 2k_{26}\sigma_y\tau_{xy} = 1\,.$$

By substituting Eqs. (2.5) and (2.6) we obtain an equivalent form of this condition, more convenient in practical interpretation of experimental results

$$(6.1')\quad \frac{1}{Y_x^2}\sigma_x^2 - \left(\frac{1}{Y_x^2}+\frac{1}{Y_y^2}-\frac{1}{Y_z^2}\right)\sigma_x\sigma_y + \frac{1}{Y_y^2}\sigma_y^2 + \frac{1}{Q^2}\tau_{xy}^2 + 2\tau_{xy}(k_{16}\sigma_x + k_{26}\sigma_y) = 1\,.$$

In the space of existing stress components $\sigma_x, \sigma_y, \tau_{xy}$, condition (6.1') is represented by an ellipsoid with the central point at the origin of the coordinate system and with the symmetry axes inclined, in general, to all three coordinate axes. The dimensions and orientation of the ellipsoid are determined by the values of all six anisotropy coefficients (moduli) appearing in the condition (6.1). Experimental measurements of some of the anisotropy coefficients is rather difficult and requires an advanced equipment. However, some of the coefficients can be measured in a relatively simple manner.

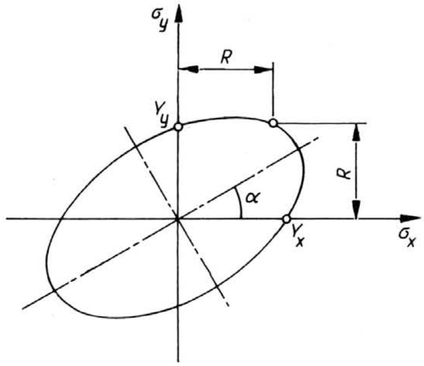


FIG. 17.

Consider at first the ellipse on the $\sigma_x$, $\sigma_y$-plane (Fig. 17) formed by intersection of the ellipsoid with the $\tau_{xy} = 0$ plane. Points on this ellipse represent various limit states of biaxial tension (compression) in the $x$ and $y$ directions. The equation of the ellipse can be written as

$$\text{(6.2)} \qquad \frac{1}{Y_x^2}\sigma_x^2 - \left(\frac{1}{Y_x^2} + \frac{1}{Y_y^2} - \frac{1}{Y_z^2}\right)\sigma_x\sigma_y + \frac{1}{Y_y^2}\sigma_y^2 = 1\,.$$

Yield loci $Y_x$ and $Y_y$ can be easily measured by simple tension tests of specimens cut out in $x$ and $y$ directions, respectively. Certain problems arising during realization of such tests have been mentioned in Sec. 2.

The yield locus $Y_z$ can be measured by a compression through-thickness test, with several test-pieces made to adhere to each other by using some adhesive — cf. e.g. [42, 43], or by a plane strain compression of a strip — cf. [43]. Another method of determining the value of $Y_z$ is the equal-biaxial ($\sigma_x = \sigma_y$) test with the use of cruciform specimens. During such a test the yield stress $\sigma^p = R$ can be measured. By substituting $\sigma_x = \sigma_y = R$ to the yield condition (6.1′) we obtain

$$\text{(6.3)} \qquad Y_z = R\,.$$

Tests in which the cruciform specimens are used are rather difficult to perform. A review of special testing devices was given by MIASTKOWSKI [28] and also in [29]. The main problem is to obtain the uniform stress and strain distribution in the central part of the specimen.

Still another method of measuring the yield stress $Y_z$ is the realization of equi-biaxial tension by the hydraulic circular bulging test — cf. [33, 43], which may be applied with good approximation also for anisotropic sheet metals.

For cruciform specimens loaded by equi-biaxial tension ($\sigma_x = \sigma_y = \sigma$) general relations (2.7) for plastic strain increments reduce, after substituting Eq. (2.5), to the following simple relations:

$$
\begin{aligned}
d\varepsilon_x &= d\lambda\left(\frac{1}{Y_x^2} - \frac{1}{Y_y^2} + \frac{1}{Y_z^2}\right)\sigma\,,\\
d\varepsilon_y &= d\lambda\left(-\frac{1}{Y_x^2} + \frac{1}{Y_y^2} + \frac{1}{Y_z^2}\right)\sigma\,,\\
d\varepsilon_z &= 2d\lambda\frac{1}{Y_z^2}\sigma\,.
\end{aligned}
\tag{6.4}
$$

For example, by measuring experimentally the strain increment ratio $d\varepsilon_x/d\varepsilon_y$ we obtain a relation between the yield limit $Y_z$ and the two remaining yield limits $Y_x$ and $Y_y$, which have been previously measured during uniaxial tension tests. Such a undirect experimental procedure can provide complementary experimental information concerning anisotropic plastic properties of the sheet, and may also be used for cross-checking the values of $Y_z$ measured by other experimental techniques.

The value of the directional yield limit $Y_z$ measured non-directly with the use of formulae (6.4), or of the respective relations for uniaxial tension, say in $x$-direction

$$
\begin{aligned}
d\varepsilon_x &= 2d\lambda\frac{1}{Y_x^2}\sigma_x\,,\\
d\varepsilon_y &= -d\lambda\left(\frac{1}{Y_x^2} + \frac{1}{Y_y^2} - \frac{1}{Y_z^2}\right)\sigma_x\,,\\
d\varepsilon_z &= -d\lambda\left(\frac{1}{Y_x^2} - \frac{1}{Y_y^2} + \frac{1}{Y_z^2}\right)\sigma_x\,,
\end{aligned}
\tag{6.5}
$$

may be remarkably different from its value measured by other methods. Such a phenomenon is termed in some works (e.g. [31]) the "anomaly" of plastic behaviour of sheet metals. We shall discuss this problem later on.

The ingenious method of determining the anisotropy coefficients of sheet metals by measuring plastic strain increments during uniaxial tension tests was proposed in 1949 by KRUPKOWSKI and KAWIŃSKI [30]. Such a possibility has been mentioned also by HILL [7], who refers to the original work by Klinger and Sachs.

Experimental measurements of the remaining three anisotoropy coefficients (moduli) $Q$, $k_{16}$, $k_{26}$, appearing in yield condition (6.1′), would require special tests under shearing load and other tests with shearing accompanied by tension or compression. Such tests are rather difficult to perform.

### 6.2. Particular cases of anisotropy of non-orthotropic types

It is often observed in sheet metals that yield loci $Y_x$ and $Y_y$ are of the same magnitude, while yield loci measured during uniaxial tension tests on specimens cut out in directions making a certain angle with the rolling direction are different from $Y_x$. Such experimental observations are cited e.g. in HILL'S book [7]. This effect was also observed in numerous works, e.g. [16, 42]. This phenomenon is usually discussed in terms of the theory of plastic orthotropy. Below we shall analyse it in more general terms.

Consider a particular case of anisotropy when

$$Y_x = Y_y = Y_0 \quad \text{and} \quad k_{16} = k_{26} = k_0\,.$$

Then yield condition (6.1′) reduces to the particular form

$$(6.6)\qquad \sigma_x^2 - \left[2 - \left(\frac{Y_0}{Y_z}\right)^2\right]\sigma_x\sigma_y + \sigma_y^2 + \left(\frac{Y_0}{Q}\right)^2\tau_{xy}^2 + 2k_0Y_0^2\tau_{xy}(\sigma_x + \sigma_y) = Y_0^2 .$$

In the space of stress components $\sigma_x$, $\sigma_y$, $\tau_{xy}$ yield condition (6.6) is represented by an ellipsoid shown in Fig. 18. The longer axis of the ellipsoid makes an angle $\beta$ with the $\sigma_x$, $\sigma_y$-plane. The value of that angle is

$$\beta = \frac{1}{2}\operatorname{Arc\,cot}\left[\frac{1}{4k_0}\left(\frac{1}{Y_z^2} - \frac{1}{Q^2}\right)\right].$$

Projection of that axis on the $\sigma_x$, $\sigma_y$-plane coincides with the bisectrix of the right angle between the $\sigma_x$ and $\sigma_y$-axes.

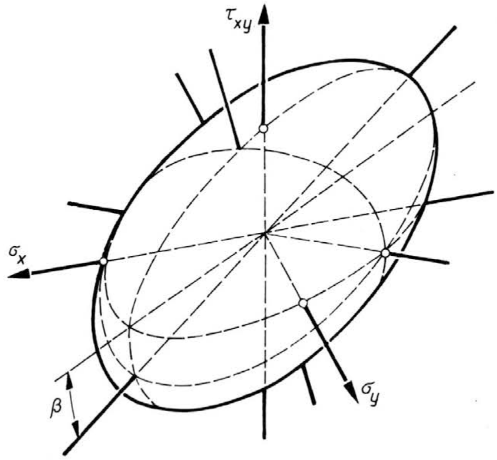


FIG. 18.

Tensile specimens cut out at the angles of +45° and −45° to the $x$-direction from a sheet obeying the yield condition (6.6) will have different yield stresses. For a specimen cut out at the angle of +45° we shall have $\sigma_x = \sigma_y = \tau_{xy} = \frac{1}{2}\sigma_0$, where $\sigma_0$ is the directional yield stress. From (6.6) we obtain

$$(6.7)\qquad \sigma_0^+ = \frac{1}{\sqrt{\frac{1}{4}\left(\frac{1}{Y_z^2} + \frac{1}{Q^2}\right) + k_0}}.$$

For the specimen cut out at the angle of −45° we have $\sigma_x = \sigma_y = -\tau_{xy} = \frac{1}{2}\sigma_0$. Then we can write

$$(6.7')\qquad \sigma_0^- = \frac{1}{\sqrt{\frac{1}{4}\left(\frac{1}{Y_z^2} + \frac{1}{Q^2}\right) - k_0}}.$$

No experimental studies devoted to the investigation of this hypothetical effect are as yet known to the author. However, plastic deformations induced in sheet metals during the manufacturing processes are so complex that the possibility of existence of such an effect should not be rejected without experimental confirmation. It is not excluded that a non-zero value of the coefficient $k_0$ may be observed in deep drawn non-circular cups, whose material was exposed to considerable shearing during the drawing operation.

### 6.3. Sheet metals with quasi-orthotropy

Consider the important case when $k_{16} = k_{26} = 0$. Then the yield condition (6.1′) reduces to the particular form

$$\frac{1}{Y_x^2}\sigma_x^2 - \left(\frac{1}{Y_x^2} + \frac{1}{Y_y^2} - \frac{1}{Y_z^2}\right)\sigma_x\sigma_y + \frac{1}{Y_y^2}\sigma_y^2 + \frac{1}{Q^2}\tau_{xy}^2 = 1 \tag{6.8}$$

identical with the yield condition (6.12) for sheet metals with true plastic orthotropy. Sheet metals may be considered as fully orthotropic provided certain additional conditions concerning plastic strain increments are satisfied. These additional conditions require that distortional plastic strain increments $d\varepsilon_{xz}$ and $d\varepsilon_{yz}$ should not appear when the sheet is loaded in its plane. Assuming the associated flow law (2.4) and plastic potential in the general form (2.3), these distortional strain increments are

$$\begin{aligned} d\varepsilon_{xz} &= 2d\lambda[-k_{15}\sigma_x + (k_{15} + k_{35})\sigma_y - k_{56}\tau_{xy}]\,, \\ d\varepsilon_{yz} &= 2d\lambda[(k_{24} + k_{34})\sigma_x - k_{24}\sigma_y - k_{64}\tau_{xy}]\,. \end{aligned} \tag{6.9}$$

Thus a sheet metal is fully orthotropic if all six anisotropy coefficients in Eqs. (6.9) are equal to zero. It has been pointed out in Sec. 6 that, owing to complex deformation histories during the manufacturing processes, these coefficients do not always vanish. If some of them have non-zero values, the specimens cut out from the sheet may deform during the tension test in the manner similar to that shown in Fig. 19. Thus a through-thickness distortion may appear. Such a deformation of quasi-orthotropic sheets is rather difficult to measure and usually is neglected in laboratory tests. No experimental evidence of this effect is available as yet.

A particular type of quasi-orthotropy corresponds to the case when

$$Y_x = Y_y = Y_0\,.$$

Then the yield condition (6.8) reduces to the well-known form

$$\sigma_x^2 - \left[2 - \left(\frac{Y_0}{Y_z}\right)^2\right]\sigma_x\sigma_y + \sigma_y^2 + \left(\frac{Y_0}{Q}\right)^2\tau_{xy}^2 = Y_0^2\,, \tag{6.10}$$

identical with the yield condition (6.14) for exactly orthotropic sheet metals. However, now the sheet may suffer a through-thickness distortion even in the case when it is loaded by stresses acting in its plane.

Still another modification of yield condition (6.8) leads to the following condition:

$$\sigma_x^2 - \left[2 - \left(\frac{Y_0}{Y_z}\right)^2\right]\sigma_x\sigma_y + \sigma_y^2 + \left[4 - \left(\frac{Y_0}{Y_z}\right)^2\right]\tau_{xy}^2 = Y_0^2\,, \tag{6.11}$$

which may be termed quasi-normal anisotropy. Tensile specimens cut out at arbitrary angles with respect to the $x$-axis will have the same yield stress equal to $Y_0$. Yield condition (6.11) has the same form as the yield condition (6.15) for sheet metals with normal anisotropy. However, those with quasi-normal anisotropy display through-thickness distortion when loaded in their plane.

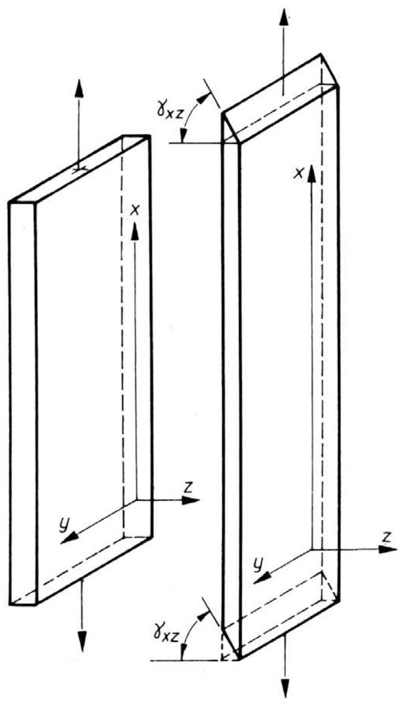


FIG. 19.

One can also distinguish quasi-isotropic sheet metals obeying yield condition (5.2), but displaying through-thickness distortion if even only one of anisotropy coefficients in expressions (6.9) has non-zero value.

### 6.4. Orthotropic sheet metals

Yield condition for orthotropic sheet metals is written in the following form:

$$\frac{1}{Y_x^2}\sigma_x^2 - \left(\frac{1}{Y_x^2} + \frac{1}{Y_y^2} - \frac{1}{Y_z^2}\right)\sigma_x\sigma_y + \frac{1}{Y_y^2}\sigma_y^2 + \frac{1}{Q^2}\tau_{xy}^2 = 1 \tag{6.12}$$

identical with the condition (6.8) for quasi-orthotropic sheets. However, now there is no possibility of through-thickness distortion during tension tests, because for orthotropic sheet metals all the anisotropy coefficients which are present in relations (6.9) must be equal to zero.

In the space of existing stress components $\sigma_x$, $\sigma_y$, $\tau_{xy}$ equation (6.12) is represented by an ellipsoid with the center at the origin of the reference system and with two symmetry axes lying in the $\sigma_x$, $\sigma_y$-plane (Fig. 20).

This type of anisotropy induced in a M-63 brass by previous plastic deformation has been observed, for example, by MIASTKOWSKI and SZCZEPIŃSKI [17]. In Fig. 21 are presented the yield curves for two definitions of yield stress, namely $\sigma_{pr}$ and $\sigma_{0.01}$. For comparison, the initial isotropic yield curve $\sigma_0$ is also shown in the figure.

The shearing yield point of orthotropic sheet metals can be directly measured by using for example specimens of the type shown in Fig. 22. When such a specimen is loaded by

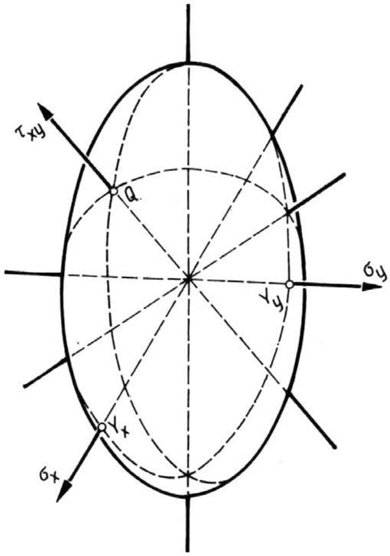


FIG. 20.

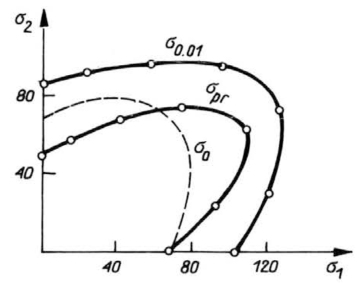


FIG. 21.

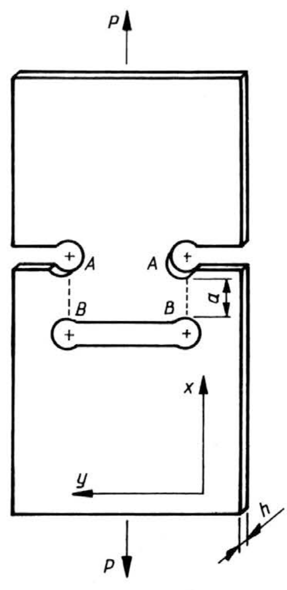


FIG. 22.

a tensile force, a state of deformation close to pure shearing develops in sections $A - B$. However, it should be noticed that interpretation of shear tests is always connected with certain difficulties. As an example, in Fig. 23 is shown the stress-elongation diagram for an ordinary tensile specimen cut out in the rolling direction from a sheet 2 mm thick of an Al-2%Mg aluminium alloy. The yield stress $Y_x$ can easily be determined. The corresponding diagram (Fig. 24) for the specimen of the type shown in Fig. 22 cut out from the same sheet indicates that the yield stress under shear can be estimated only conventionally. These tests have been performed in [45], where also optimal shape of specimens for shearing tests has been analysed by means of methods similar to those used by ALBERTINI *et al.* [46].

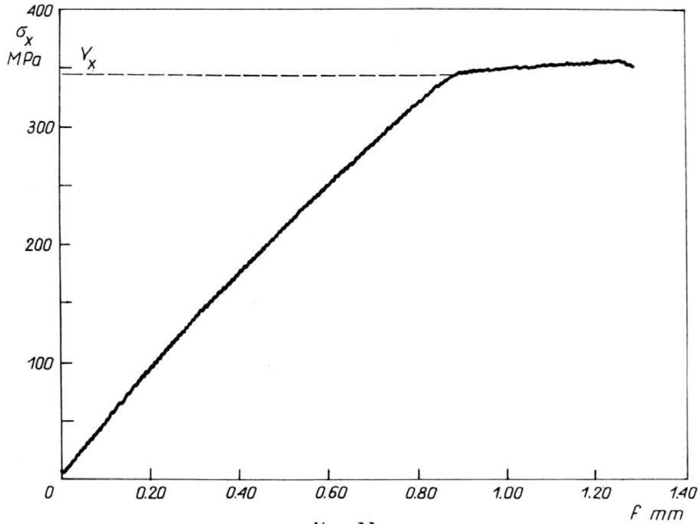


FIG. 23.

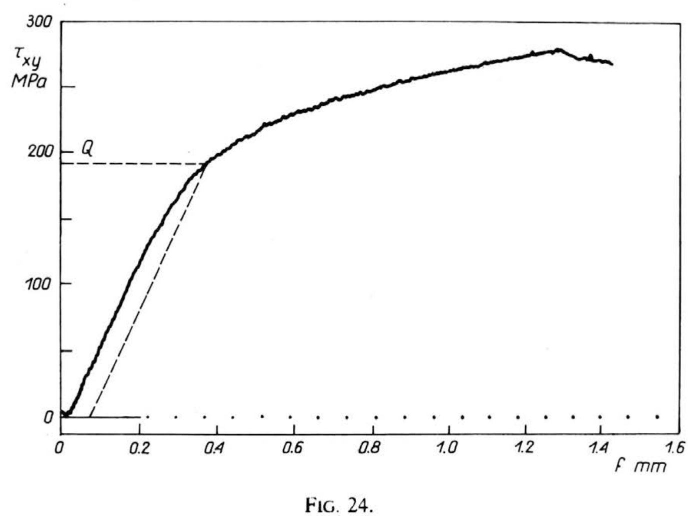


FIG. 24.

Let us notice that in both shearing zones of the specimen shown in Fig. 22 shear stresses $\tau_{xy}$ have opposite signs. It is of no practical significance in the case of the yield condition (6.12) with quadratic terms only. Note that such specimens may also be used for sheets obeying yield consitions (6.1) or (6.6), because terms depending linearly on $\tau_{xy}$ vanish for shear test.

The value of the yield stress $Q$ may easily be estimated in a simple indirect manner, namely by the tensile test on a specimen cut out from the sheet at the angle of 45° with respect to the $x$-direction. If the yield stress of such specimen equals $\sigma_0$, then $Q$ can be calculated from the formula

$$Q = \frac{\sigma_y Y_z}{\sqrt{4Y_x^2 - \sigma_0^2}}. \tag{6.13}$$

Theoretically all four moduli of orthotropy in Eq. (6.12) could be experimentally determined by simple tension tests on specimens, each of them being cut out from the sheet at a different angle $\alpha$ with respect to the $x$-axis (cf. HILL [7]). For each specimen its yield stress should be experimentally determined. Then, by substituting relations (4.14) to (6.12), we obtain four equations with four unknowns $Y_x$, $Y_y$, $Y_z$, $Q$.

Using such indirect methods of measuring the moduli $Y_z$ and $Q$, we should be aware that yield condition (6.12) describes only approximately the real deformation-induced anisotropy of sheet metals. Thus their values measured in this manner may remarkably differ from the exact values measured directly.

#### 6.5. Particular cases of plastic orthotropy

Consider the important particular case when

$$Y_x = Y_y = Y_0 .$$

Yield stresses of specimens cut out at certain angles with respect to the $x$-direction are assumed to be not equal to $Y_0$. Then the yield condition (6.12) takes the particular form

$$\sigma_x^2 - \left[2 - \left(\frac{Y_0}{Y_z}\right)^2\right]\sigma_x\sigma_y + \sigma_y^2 + \left(\frac{Y_0}{Q}\right)^2\tau_{xy}^2 = Y_0^2 \tag{6.14}$$

identical with the yield condition (6.10) for quasi-orthotropic sheets. However, now, as in the case of the general orthotropy, all the anisotropy coefficients in Eq. (6.9) must be equal to zero.

Yield stresses $Y_0$ and $Y_z$ may be estimated by tensioning uniaxially a specimen cut out from the sheet, for example in $x$-direction. The value of $Y_0$ results directly from the stress-strain diagram. The value of $Y_z$ can be estimated non-directly by measuring plastic strain increments in the transversal direction $d\varepsilon_y$ and in the through-thickness direction $d\varepsilon_z$. Then $Y_z$ can be calculated from the relation

$$\varkappa = \frac{d\varepsilon_y}{d\varepsilon_z} = 2\left(\frac{Y_z}{Y_0}\right)^2 - 1 . \tag{6.15}$$

The yield shear stress $Q$ may be estimated indirectly in the manner described in Sec. 6.4 by applying uniaxial tension to a specimen cut out at the angle of 45° with respect to the $x$-axis. Formula (6.13) holds valid also in the present case.

The value of $Y_z$ estimated from the relation (6.15) may differ significantly from that measured directly, or by using other indirect methods. Such differences found by WOODTHROPE and PIERCE [33] for an aluminium sheet were termed the "anomaly" of plastic properties. To eliminate this inconsistency, various non-quadratic yield criteria were proposed. They will be discussed in Sec. 6.7.

If in the yield criterion (6.14) we shall additionally assume that also $Y_z = Y_0$, then this criterion takes the simple form

$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + \left(\frac{Y_0}{Q}\right)^2 \tau_{xy}^2 = Y_0^2 \tag{6.16}$$

corresponding to the cubic orthotropy. This particular form of the yield condition may also be directly obtained from the general yield condition (3.11) for cubic orthotropy by substituting there $\sigma_z = \tau_{yz} = \tau_{zx} = 0$.

#### 6.6. Normal anisotropy

Specimens cut out from sheet metals with normal anisotropy at any arbitrary angle have the same value of yield stress under uniaxial tension. The yield condition for sheets with such plastic properties has the following form

$$\sigma_x^2 - \left[2 - \left(\frac{Y_0}{Y_z}\right)^2\right]\sigma_x\sigma_y + \sigma_y^2 + \left[4 - \left(\frac{Y_0}{Y_z}\right)^2\right]\tau_{xy}^2 = Y_0^2, \tag{6.17}$$

identical with the yield condition (6.11) for sheet metals with the quasi-normal anisotropy. However, now the direction orthogonal to sheet's plane constitutes the principal direction of orthotropy. Yield condition (6.17) results also directly from the general yield condition (3.5′) for the so-called transversal isotropy by substituting there $\sigma_z = \tau_{yz} = \tau_{zx} = 0$.

In the space of stress components $\sigma_x, \sigma_y, \tau_{xy}$, the yield condition (6.17) is represented by an ellipsoid (Fig. 25). For $Y_z > Y_0$ the ellipsoid is more elongated in the 0-D direction than that for fully isotropic sheets (cf. Fig. 12), when $Y_z = Y_0$.

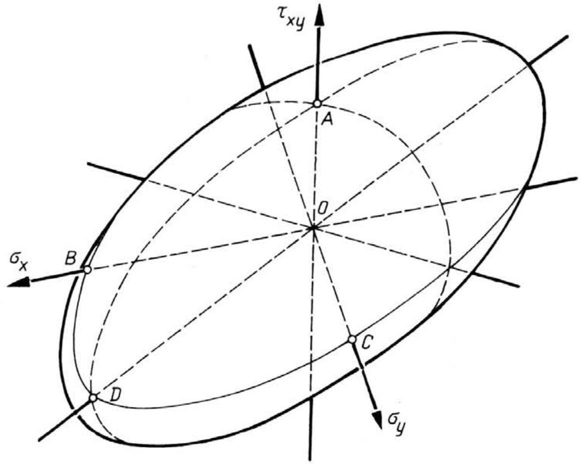


FIG. 25.

If the directions of principal stresses $\sigma_1$, $\sigma_2$ coincide with the reference axes $x$ and $y$, the yield condition (6.17) is often written in the following form:

$$\sigma_y^2 - \frac{2r}{1+r}\sigma_y\sigma_2 + \sigma_2 = Y_0^2, \tag{6.18}$$

where

$$r = 2\left(\frac{Y_z}{Y_0}\right)^2 - 1 \tag{6.19}$$

is the so-called coefficient of normal anisotropy.

Comparison of Eq. (6.19) with Eq. (6.15) indicates that $r$ can be measured as the ratio of strain increments $d\varepsilon_y/d\varepsilon_z$ in a specimen cut out from the sheet and loaded by uniaxial tension. As it was mentioned in Sec. 6.1, the idea of such measuring method was proposed by KRUPKOWSKI and KAWIŃSKI [30]. It was also proposed independently by LANKFORD *et al.* [35]. In the latter work, the following formula was recommended:

$$r = \frac{\ln(b/b_0)}{\ln(h/h_0)}, \tag{6.20}$$

where $h_0$ and $b_0$ are the initial thickness and width of the tensile specimen, respectively, and $h$ and $b$ are the corresponding dimensions after deformation. In [30] the coefficient of normal anisotropy was formulated as

$$K = \frac{(b_0/b)^2 - 1}{(h_0/h)^2 - 1}. \tag{6.21}$$

The main difficulty arising when such non-direct methods of measuring the anisotropy coefficients of sheet metals are used, consists in changing the anisotropy itself during the progressing deformation of the specimen tested. As an example let us analyse Fig. 10. An initially isotropic sheet metal pulled uniaxially in tension by stresses $\sigma_x$ to produce a relatively small permanent deformation (about 2 percent) displays a significant anisotropy. There is a remarkable decrease of $Y_y$ yield stress and an increase of $Y_x$ with respect to the initial values before deformation.

Thus it can be expected that the value of the anisotropy coefficient $r$ will be dependent on the amount of plastic deformation induced in the specimen cut out from the sheet in question. This has been experimentally demonstrated by TRUSZKOWSKI in several works, e.g. [36, 37]. To estimate the initial anisotropy of a tested sheet metal, TRUSZKOWSKI (cf. e.g. [38]) extrapolates the experimentally determined curve $r(\varepsilon)$ for the changing coefficient $r$ to the starting point, where $\varepsilon = 0$.

For practical purposes the anisotropy coefficient $r$ is sometimes used in the cases when the sheet material does not accurately obey the yield condition (6.18) for normal anisotropy. In some works (cf. e.g. [42, 43]) the average value of $r$ is estimated for sheet materials for which rather the yield condition (6.12) for general orthotropy is more adequate.

#### 6.7. Non-quadratic yield conditions for sheet metals

It was mentioned in Sec. 6.5 that experiments on certain sheet metals show that they do not obey the flow law associated with standard yield conditions. Such a phenomenon was termed the "anomaly" of plastic properties of the sheet. "Anomalies" of this kind observed in [33] and later in [42] are characterized by the inequalities

$$\varkappa = \frac{d\varepsilon_y}{d\varepsilon_z} < 1, \qquad \frac{Y_z}{Y_0} > 1.$$

Such relations are in contradiction with the relation (6.15) resulting from the flow law associated with yield conditions discussed above.

To overcome such contradictions sometimes observed HILL [31] proposed some new, the so-called non-quadratic yield conditions; in the cases when directions of principal stresses coincide with those of principal axes of orthotropy, the conditions were written as

$$
\begin{aligned}
(6.22)\qquad & F|\sigma_2-\sigma_3|^m + G|\sigma_3-\sigma_1|^m + H|\sigma_1-\sigma_2|^m \\
& +A|2\sigma_1-\sigma_2-\sigma_3|^m + B|2\sigma_2-\sigma_3-\sigma_1|^m + C|2\sigma_3-\sigma_1-\sigma_1|^m = \sigma^m ,
\end{aligned}
$$

where $m > 1$, and $\sigma$ is a certain constant. Coefficients $F$, $G$, $H$ are positive. Yield condition (6.22) contains seven material parameters, not counting the constant $\sigma$. For practical purposes one can assume, for a sheet metal in question, various particular forms of yield condition (6.22), by assuming that certain parameters vanish.

In [31] such particular cases were analysed for the case of transversal isotropy. E.g., by assuming, that

$$F = G = 0 \quad \text{and} \quad A = B = 0 ,$$

the following form of non-quadratic yield condition is obtained

$$(6.23)\qquad H|\sigma_1-\sigma_2|^m + C|2\sigma_3-\sigma_1-\sigma_2|^m = \sigma^m .$$

From the flow law associated with (6.23) we obtain for uniaxial tension by stresses $\sigma_1$ the following expression for the strain increments ratio

$$(6.24)\qquad \varkappa = \frac{d\varepsilon_2}{d\varepsilon_3} = \frac{H-C}{2C} .$$

Assuming that the direction of principal stress $\sigma_3$ coincides with the $z$-axis of the Cartesian coordinate system we denote, as previously, by $Y_z$ the yield stress under uniaxial tension in $z$-direction, and by $Y_0$ — the yield stress for uniaxial tension in the $x, y$-plane. Then the following relations can be written:

$$(6.25)\qquad C = \left(\frac{\sigma}{2Y_z}\right)^m , \quad H = \left(\frac{\sigma}{Y_0}\right)^m - \left(\frac{\sigma}{2Y_z}\right)^m .$$

Now, instead of Eq. (6.24), we can write

$$(6.26)\qquad \varkappa = \frac{d\varepsilon_2}{d\varepsilon_3} = 2^{m-1}\left(\frac{Y_z}{Y_0}\right)^m - 1 .$$

For $m = 2$ this formula takes the form (6.15).

For plane stress state, when $\sigma_3 = 0$, yield condition (6.23) reduces to the form

$$(6.27)\qquad |\sigma_1+\sigma_2|^m - \left[1-\left(\frac{2Y_z}{Y_0}\right)^m\right]|\sigma_1-\sigma_2|^m = 2Y_z^m .$$

For $m = 2$ this yield condition becomes identical with Eq. (6.17), if the latter were presented in principal stresses. Now we have an additional parameter $m$. If, for example, $Y_z$ were experimentally determined, say by equi-biaxial tension test, then measuring the value $Y_0$ by means of the uniaxial tension and also the strain increments ratio $\varkappa$ we could calculate from Eq. (6.26) the value of $m$. Thus for the non-quadratic yield condition (6.27) the problem of "anomaly" does not exist.

Since the yield condition (6.27) is formulated in principal stresses, it does not ensure the specimens cut out at various angles to have the same yield stress when they are uniaxially pulled in tension. To secure this requirement of the transversal isotropy, a more general formulation would be necessary.

Nevertheless, the condition (6.27) has been applied in a number of investigations. The value of exponent $m$ was estimated to vary between $m = 1.7$ and $m = 2$, depending on the material (cf. [39]). Condition (6.27) has also been used with certain modifications in some numerical solutions to various practical problems (cf. e.g. [39, 40, 42]).

A particular form of yield condition (6.22) when $A = B = C = 0$ was independently formulated by Hosford in 1979. In his next work [47] HOSFORD modified this yield condition by assuming that principal directions of stress and strain increment tensors coincide independently of whether they are parallel to the orthotropy axes or not. Bearing in mind that such an assumption may introduce certain errors, the author argues on the basis of experimental data that in real materials the mutual inclination of the two directions is rather small. Thus the errors connected with introduction of the hypothesis of transversal isotropy will be larger than those resulting from his assumption. The modified Hosford's yield criterion is

$$R_2|\sigma_1|^m + R_1|\sigma_2|^m + R_1R_2|\sigma_1 - \sigma_2|^m = R_2(R_1 + 1)Y_1^m , \tag{6.28}$$

where $R_1$ and $R_2$ are the ratios of strain increments, corresponding to $\varkappa$ determined by Eq. (6.15), but measured in the directions 1 and 2, respectively. Directional yield stress in the direction 1 is denoted by $Y_1$.

ZHOU WEIXIAN [48] formulated the following, more general form of the non-quadratic yield condition

$$\begin{aligned} f = F(\sigma_y^2 + 3\tau_{xy}^2)^{m/2} + G(\sigma_x^2 + 3\tau_{xy}^2)^{m/2} \\ + H[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]^{m/2} + 2N(\tau_{xy}^2)^{m/2} \\ - \frac{2}{3}(F + G + H)\sigma_i^m = 0 , \end{aligned} \tag{6.29}$$

accounting for the influence of shear stress $\tau_{xy}$. Here $\sigma_i$ is the equivalent stress. The exponent $m$ was estimated on the basis of experimental tests to be approximately equal to $m = 8$.

In connection with the problem of formation of the so-called "ears" in the sheet metals during the deep drawing operations, HILL [7] proposed still another form of the yield condition for plane stress states. It contains the polynomial of degree $n$

$$\sum A_{ijk}\sigma_x^i\sigma_y^j\tau_{xy}^k , \tag{6.30}$$

where the powers $i$, $j$, $k$ are positive integers or zero ($i + j + k \leq n$). Note that for $n = 2$ one obtains the yield condition (6.1) resulting from the von Mises general yield function (2.1).

A particular form of this yield condition for $n = 4$ was analysed by GOTOH [43, 44]. It was found for some metals that yield curves on the $\sigma_1$, $\sigma_2$-plane differ only slightly from those resulting from the quadratic yield condition.

Owing to non-quadratic yield conditions it was possible to obtain a more accurate description of the plastic behaviour of sheet metals. However, even these conditions are not able to encompass fully all the "anomalies" (cf. also [48]) resulting from the very complex deformation pattern induced in sheet metals during the manufacturing operations.

## 7. Yield conditions for sheet metals with Bauschinger effect

Similarly as in the case of yield conditions discussed in Sec. 6 also now, when the Bauschinger effect is accounted for, the through-thickness averaging of plastic properties of the sheet is necessary. Experimental investigations indicate that in sheet metals delivered by the manufacturers the Bauschinger effect may be present in a clearly visible form. A distinct difference between the yield stress under tension and compression was observed for example by LITEWKA [23] in a sheet 1.5mm thick, made of an aluminium alloy PA2-M, and also in a sheet 5mm thick, made of another aluminium alloy PA4-T1. For the two materials the absolute value of the yield stress under compression was larger than that under tension in the plane of the sheet.

### 7.1. The case of general anisotropy

The general anisotropic yield condition (4.2) reduces for plane stress states to the following form

$$(7.1)\qquad (k_{12}+k_{31})\sigma_x^2 - 2k_{12}\sigma_x\sigma_y + (k_{12}+k_{23})\sigma_y^2 + k_{66}\tau_{xy}^2 + 2k_{16}\sigma_x\tau_{xy} + 2k_{26}\sigma_y\tau_{xy} \\ -b_{12}(\sigma_x-\sigma_y) - b_{23}\sigma_y + b_{66}\tau_{xy} = 1,$$

if relations (4.5) for the coefficients $b_{ij}$ are assumed.

By substituting Eqs. (4.3) and (4.5), this yield condition may be written in the form more suitable for applications,

$$(7.1')\qquad \frac{1}{Y_xZ_x}\sigma_x^2 - \left(\frac{1}{Y_xZ_x}+\frac{1}{Y_yZ_y}-\frac{1}{Y_zZ_z}\right)\sigma_x\sigma_y + \frac{1}{Y_yZ_y}\sigma_y^2 \\ +k_{66}\tau_{xy}^2 - 2\tau_{xy}(k_{16}\sigma_x + k_{26}\sigma_y) + b_{66}\tau_{xy} \\ +\left(\frac{1}{Y_x}-\frac{1}{Z_x}\right)\sigma_x + \left(\frac{1}{Y_y}-\frac{1}{Z_y}\right)\sigma_y = 1.$$

For $Y_i = Z_i (i = x, y, z)$ and, moreover, for $b_{66} = 0$ this yield condition reduces to the condition (6.1′) for sheet metals free of the Bauschinger effect.

In the $\sigma_x$, $\sigma_y$, $\tau_{xy}$-space equation (7.1′) is represented by a certain ellipsoid with the central point shifted with respect to the origin of the reference system. The principal axes of the ellipsoid are, in a general case, inclined to the reference axes. In the yield condition (7.1′) the number of material moduli is ten, but relation (4.6) reduces that number to nine. To determine all the moduli experimentally it is necessary to make (among other tests) also the uniaxial compression tests on specimens cut out in the $x$ and $y$ directions. For such compression tests special devices are used — cf. e.g. DIETRICH and TURSKI [41].

The material constants (moduli) in (7.1′) can also be measured non-directly with the use of expressions for the associated plastic strain increments. For plane stress state we obtain from the general relations (4.8) the following formulae:

$$(7.2)\qquad \begin{aligned} d\varepsilon_x &= d\lambda[2k_{12}(\sigma_x-\sigma_y) + 2k_{31}\sigma_x - 2k_{16}\tau_{xy} - b_{12} + b_{31}],\\ d\varepsilon_y &= d\lambda[2k_{12}(\sigma_y-\sigma_x) + 2k_{23}\sigma_y - 2k_{26}\tau_{xy} + b_{12} - b_{23}],\\ d\varepsilon_z &= d\lambda[-2k_{23}\sigma_y - 2k_{31}\sigma_x + 2(k_{16}+k_{26})\tau_{xy} + b_{23} - b_{31}]. \end{aligned}$$

For example for uniaxial tension in the $x$-direction we obtain, accounting for (4.3) and (4.5), the formulae for practical use

$$
\begin{aligned}
d\varepsilon_x &= d\lambda\left(\frac{2}{Y_x Z_x}\sigma_x + \frac{1}{Y_x} - \frac{1}{Z_x}\right),\\
d\varepsilon_y &= -d\lambda\left[\left(\frac{1}{Y_x Z_x} + \frac{1}{Y_y Z_y} - \frac{1}{Y_z Z_z}\right)\sigma_x - \frac{1}{Y_y} + \frac{1}{Z_y}\right],\\
d\varepsilon_z &= -d\lambda\left[\left(\frac{1}{Y_x Z_x} - \frac{1}{Y_y Z_y} + \frac{1}{Y_z Z_z}\right)\sigma_x - \frac{1}{Y_z} + \frac{1}{Z_z}\right].
\end{aligned}
\tag{7.3}
$$

Similar relations can be written for uniaxial tension by stresses $\sigma_y$.

If the moduli $Y_x$, $Z_x$ and $Y_y$, $Z_y$ are measured directly by tension and compression tests, then the values of $Y_z$ and $Z_z$ can be evaluated by measuring the respective plastic strain increments. However, such non-direct measuring methods are of questionable accuracy.

### 7.2. Particular cases of anisotropy of sheet metals displaying Bauschinger effect

From among a large variety of possible particular yield conditions, which may be deduced from the general formulation (7.1′), we shall discuss only a few most important, for which there exists a certain experimental evidence. Unfortunately, experimental data concerning the Bauschinger effect in sheet metals are scarce. However, the few available data demonstrate that the yield conditions with terms linearly dependent on the stress components may be of practical significance.

Consider a particular form of the yield condition (7.1′) when

$$Y_x = Y_y = Y_0, \quad Z_x = Z_y = Z_0, \quad k_{16} = k_{26} = 0.$$

Under these assumptions the yield condition takes the following form

$$
\begin{aligned}
\sigma_x^2 - \left(2 - \frac{Y_0 Z_0}{Y_z Z_z}\right)\sigma_x\sigma_y + \sigma_y^2 + k_{66} Y_0 Z_0 \tau_{xy}^2 \\
+ b_{66} Y_0 Z_0 \tau_{xy} + (Z_0 - Y_0)(\sigma_x + \sigma_y) = Y_0 Z_0 .
\end{aligned}
\tag{7.4}
$$

Let us denote the yield stress due to loading of a sheet by positively directed shear stress by $Q^+$, and by $Q^-$ — the absolute value of the yield stress for the case when shear stresses are acting in the negative direction. Then we can write

$$k_{66} = \frac{1}{Q^+ Q^-}, \quad b_{66} = \frac{1}{Q^+} - \frac{1}{Q^-}. \tag{7.5}$$

In the space of stress components $\sigma_x$, $\sigma_y$, $\tau_{xy}$ the yield condition (7.4) is represented by an ellipsoid shown in Fig. 26. Its central point 0* is shifted with respect to the origin 0 of the reference system.

The experimental confirmation of such a type of anisotropy in sheet metals is as yet lacking. However, anisotropy of this kind may be induced in sheet metals during deep drawing operations, when a flat blank, before entering the drawing ring of the die, may suffer considerable distortion. Thus in the drawn object the anisotropy of such a type may be present.

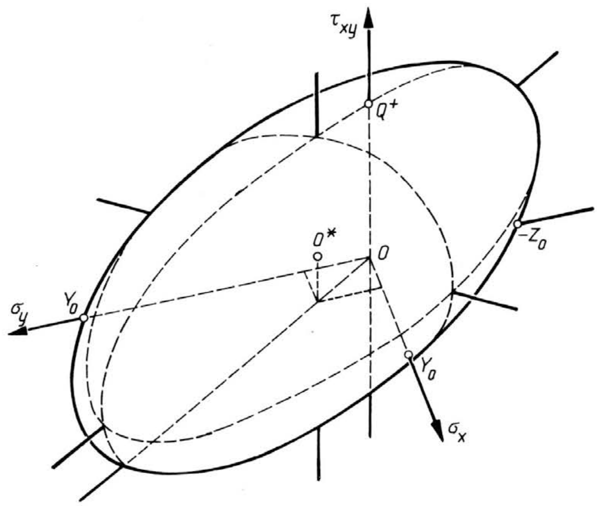


FIG. 26.

For most sheet metals it may be expected that $b_{66} = 0$. This leads to a still simpler particular form of the yield condition (7.4)

$$(7.6) \qquad \sigma_x^2 - \left(2 - \frac{Y_0 Z_0}{Y_z Z_z}\right)\sigma_x\sigma_y + \sigma_y^2 + \frac{Y_0 Z_0}{Q^2}\tau_{xy}^2 + (Z_0 - Y_0)(\sigma_x + \sigma_y) = Y_0 Z_0 .$$

Now the central point of the ellipsoid is shifted in the $\sigma_x$, $\sigma_y$-plane along the bisectrix of the right angle between the axes $\sigma_x$ and $\sigma_x$.

If we shall assume the yield condition in the form

$$(7.7) \qquad \sigma_x^2 - \left(2 - \frac{Y_0 Z_0}{Y_z Z_z}\right)\sigma_x\sigma_y + \sigma_y^2 + \left(4 - \frac{Y_0 Z_0}{Y_z Z_z}\right)\tau_{xy}^2 + (Z_0 - Y_0)(\sigma_x + \sigma_y) = Y_0 Z_0,$$

then the yield stresses under uniaxial tension of specimens cut out from the sheet at an arbitrary angle will be of the same magnitude $Y_0$. The same concerns uniaxial compression, when all specimens will have the yield stress equal to $Z_0$.

Yield condition (7.7) corresponds to the plastic properties of sheets of an aluminium alloy PA2 and also PA4 observed by LITEWKA [22]. Yield stresses of specimens cut out in different directions under uniaxial loadings were of the same magnitude. However, Litewka noticed also another "anomaly" in these materials, because the parameter $\varkappa$ [cf. (6.15)] was different for specimens cut out in various directions. This is in contradiction with the flow law associated with the yield condition (7.7).

All the "anomalies" mentioned in this paper indicate that indirect methods of measuring the anisotropy coefficients should be used cautiously and with a certain criticism.

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## References


1. Z. MARCINIAK, *Limit strains in the processes of sheet metals forming* [in Polish], WNT, Warszawa 1971.
2. R.V. MISES, *Mechanik der plastischen Formänderung von Kristallen*, Zeitsch. Angew. Math. Mech., **8**, 3, 161–185, 1928.
3. W. OLSZAK, W. URBANOWSKI, *The orthotropy and the non-homogeneity in the theory of plasticity* [in Polish], Arch. Mech. Stos., **8**, 1, 85–110, 1956.
4. W. OLSZAK, W. URBANOWSKI, *The plastic potential and the generalized distortion energy in the theory of non-homogeneous anisotropic elastic-plastic body*, Arch. Mech. Stos., **8**, 4, 671–694, 1956.
5. W. OLSZAK, W. URBANOWSKI, *The flow function and the yield condition for non-homogeneous orthotropic bodies*, Bull. Acad. Polon. Sci., **5**, 4, 191–203, 1957.
6. J.P. BOEHLER, L. EL AOUFI and J. RACLIN, *On experimental testing methods for anisotropic materials*, Res Mechanica, **21**, 73–95, 1987.
7. R. HILL, *The mathematical theory of plasticity*, Oxford at the Clarendon Press, 1956.
8. J.P. BOEHLER, A. SAWCZUK, *Equilibre limite des sol anisotropes*, J. de Mecanique, **9**, 1, 5–33, 1970.
9. J.P. BOEHLER and A. SAWCZUK, *On yielding of oriented solids*, Acta Mechanica, **27**, 185–206, 1977.
10. D. ALLIROT, J.P. BOEHLER and A. SAWCZUK, *Yielding and failure of transversely isotropic solids. Part I. Experiment*, Res Mechanica, **4**, 97–113, 1982.
11. J. BAUSCHINGER, *Über die Quercontraction und Dilatation bei der Längenausdehnung und Zusammendrücken prismatischer Körper*, Zivilingenieur, Leipzig, **25**, 81–124, 1879.
12. J.F. BELL, *The experimental foundations of solid mechanics*, Handbuch der Physik, Band VIa/1, Springer Verlag, 1973.
13. P.M. NAGHDI, F. ESSENBURG, W. KOFF, *An experimental study of initial and subsequent yield surfaces in plasticity*, J. Appl. Mech., **25**, 201–209, 1958.
14. J.I. JAGN, O.A. SHISHMARIEV, *On certain results of the investigation of elastic state limits of plastically deformed nickel specimens* [in Russian], Dokl. Akad. Nauk USSR, **119**, 46–48, 1958.
15. H.J. IVEY, *Plastic stress-strain relations and yield surfaces for aluminium alloys*, J. Mech. Engng. Sci., **3**, 15–31, 1961.
16. W. SZCZEPINSKI, *On the effect of plastic deformation on yield condition*, Arch. Mech. Stos., **15**, 275–296, 1963.
17. J. MIASTKOWSKI, W. SZCZEPIŃSKI, *An experimental study of yield surfaces of prestrained brass*, Int. J. Solids and Structures, **1**, 189–194, 1965.
18. K. IKEGAMI, *An historical perspective of the experimental study of subsequent yield surfaces for metals* [in Japanese], Japan Soc. Mat. Sci., **24**, Part 1, 491–505, Part 2, 709–719, 1975, English translation BISI 14420, Sept. 1976.
19. K. IKEGAMI, *Experimental plasticity on the anisotropy of metals*, Coll. Intern. CNRS, No. 295, 201–242, 1982.
20. G.J. DEAK, *A study on the causes of the Bauschinger effect*, Ph. D. Thesis, Massachusetts Institute of Technology, 1962.
21. T. OTA, A. SHINDO, H. FUKUOKA, *A consideration on anisotropic yield criterion*, Proc. 9th Jap. Nat. Congr. Appl. Mech., 117–120, 1959.
22. A. LITEWKA, *Non-associated flow law for plastically anisotropic aluminium alloys* [in Polish], Mech. Teor. Stos., **15**, 491–499, 1977.
23. A. LITEWKA, *Plastic flow of anisotropic aluminium alloy sheet metals*, Bull. Acad. Polon. Sci., **25**, 475–484, 1977.
24. B.E. MILSOM and J.M. ALEXANDER, *An experimental determination of the detailed distortion in hot rolling*, J. Mech. Phys. Solids, **9**, 105–113, 1961.
25. F.A.A. CRANE and J.M. ALEXANDER, *Friction in hot rolling*, J. Inst. of Metals, **91**, 188–189, 1962–63.
26. C.W. GREGOR, L.F. COFFIN, *The distribution of strains in the rolling process*, J. Appl. Mech., **10**, 1943.
27. O. PIISPANEN, R. PIISPANEN, *Die plastischen Vorgänge beim Walzen*, Bänder, Bleche, Röhre, **7**, 7, 189–193, 1966.
28. J. MIASTKOWSKI, *Experimental methods of static investigations of plastic flow of metals* [in Polish], Mech. Teor. Stos., **13**, 225–252, 1975.
29. W. SZCZEPIŃSKI, L. DIETRICH, J. MIASTKOWSKI, *Plastic properties of metals, Part I.* [in:] Experimental Methods in Mechanics of Solids, W. SZCZEPIŃSKI [Ed.], Elsevier, PWN, 1990.

30. A. KRUPKOWSKI and S. KAWIŃSKI, *The phenomenon of anisotropy in annealed polycrystalline metals*, J. Inst. of Metals, **75**, Part II, 869–880, 1949.
31. R. HILL, *Theoretical plasticity of textured aggregates*, Math. Proc. Cambridge Phil. Soc., **85**, 179–191, 1979.
32. K. MIYAUCHI, *On a simple shear deformation*, English text reprinted from Scientific Papers of the Institute of Physical and Chemical Research Rikagaku Kenkyusho, Wako-shi, Saitama, Japan, **81**, 57–67, 1987.
33. J. WOODTHROPE and R. PEARCE, *The anomalous behaviour of aluminium sheet under balanced biaxial tension*, Intern. J. Mech. Sci., **12**, 341–347, 1970.
34. Z. MARCINIAK, *Influence of the sign change of the load on the strain hardening curve of a copper test piece subject to torsion*, Arch. Mech. Stos., **6**, 743–752, 1961.
35. W.T. LANKFORD, S.G. SNYDER, J.A. BAUSCHER, *New criteria for predicting the press performance of deep drawing sheets*, Trans. ASM, **42**, 1950.
36. W. TRUSZKOWSKI, J. KLOCH, *Application of the maximal error method for the calculation of the $r(\varepsilon)$ function*, Bull. Acad. Polon. Sci., **34**, 11–12, 691–701, 1986.
37. W. TRUSZKOWSKI, *Sur le sens physique du rapport des allongements obtenu par la methode d'extrapolation*, Memoria presentata al XIII Convegno Nazionale AIM — Milano, 1968, 17–20.
38. W. TRUSZKOWSKI, *Stress-strain relation for polycrystalline non-homogeneous anisotropic metals* [in Polish], Arch. Hutn., **28**, 4, 429–440, 1983.
39. A. PAMAR and P.B. MELLOR, *Plastic expansion of a circular hole in sheet metal subjected to biaxial tensile stress*, Intern. J. Mech. Sci., **20**, 707–720, 1978.
40. M.J. SARAN, D.J. ZHOU and R.H. WAGONER, *Numerical simulation of arbitrary sheet stampings using CFS formulation with anisotropic material models*, Proc. Third Intern. Conf. on Computational Plasticity, Barcelona 1992, Pineridge Press, 1992, 1213–1225.
41. L. DIETRICH, K. TURSKI, *A new method of compression tests of sheet metals* [in Polish], Rozpr. Inż., **26**, 1, 91–99, 1978.
42. F. MONTHEILLET, J.J. JONAS and M. BENFERRAH, *Development of anisotropy during the cold rolling of aluminium sheet*, Intern. J. Mech. Sci., **33**, 197–209, 1991.
43. M. GOTOH, *A theory of plastic anisotropy based on a yield function of fourth order (plane stress state) — I*, Intern. J. Mech. Sci., **19**, 505–512, 1977.
44. M. GOTOH, *A theory of plastic anisotropy based on a yield function of fourth order (plane stress state) — II*, Intern. J. Mech. Sci., **19**, 513–520, 1977.
45. G. SOCHA, W. SZCZEPIŃSKI, *Optimal design of a specimen for pure shear tests of sheet metals*, (in preparation).
46. C. ALBERTINI, M. MONTAGNANI, M. ZYCZKOWSKI and S. LACZEK, *Optimal design of a specimen for pure double shear tests*, Intern. J. Mech. Sci., **32**, 729–741, 1990.
47. W.F. HOSFORD, *Comments on anisotropic yield criteria*, Intern. J. Mech. Sci., **27**, 423–427, 1985.
48. ZHOU WEIXIAN, *A new non-quadratic orthotropic yield criterion*, Intern. J. Mech. Sci., **32**, 513–520, 1990.

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