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On the dynamics of films of viscous and elastoviscous liquids

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THE ISOTHERMAL dynamics of thin liquid film is discussed. General dynamical equations are derived. The effects of complicated rheology of the liquid are accounted for. Constitutive equations of elastoviscous liquid of Maxwellian type are discussed. Stationary motions as well as small perturbations of film about its stationary position are considered. Some examples are given including the equations of capillary waves on the plane film of viscous liquid and the growth of thin spherical film of the elastic liquid due to internal pressure.

Rozważono problem dynamiki izotermicznej cienkich błonek cieczy. Wyprowadzono ogólne równania dynamiki uwzględniające złożone efekty reologiczne. Omówiono równania konstytutywne cieczy lepkosprężystych typu Maxwella. Przedyskutowano ruch ustalony błonek, jak również małe odchylenia od takiego ruchu. Podano kilka przykładów uwzględniających równania fal kapilarnych na płaskiej błonce cieczy lepkiej oraz problem wzrostu cienkiej błonki sferycznej pod wpływem ciśnienia wewnętrznego.

Рассматривается изотермическая динамика тонких пленок жидкости. Выведены общие динамические уравнения, учитывающие эффекты усложненной реологии жидкости. Обсуждаются определяющие соотношения для упруговязких жидкостей максвелловского типа. Рассмотрены стационарные движения и малые колебания относительно стационарного положения. Приведены некоторые примеры, в том числе уравнения капиллярных волн на плоской пленке вязкой жидкости и расширение тонкой сферической пленки упругой жидкости под действием внутреннего давления.

UNLIKE liquid films moving along solid surface which have been treated rather thoroughly, free dynamical liquid films have not been sufficiently investigated. Except for the quasi-static problems usual in the theory of polymer processing [1, 2], only a few problems of dynamical equilibrium of thin films of ideal or viscous fluid have been studied [3–11]. Most important are the ideas of G. TAYLOR [3–6] who considered the experiments with dynamical liquid films or sheets ("water-bells") not as a curiosity but as a means of investigating liquid behaviour and its disintegration. The ideas appear to be very fruitful if applied to rheologically complex fluids, but in order to proceed to the next step we need an adequate theoretical framework. The aim of this paper is to develop such a framework. Our study includes the derivation of general dynamical equations of a liquid film with complex rheology under arbitrary type of motion. Equations of small disturbances of the film about its stationary position and some simple examples are treated as well.

1.

Consider a thin liquid film, assuming its thickness h to be small in comparison with the characteristic length along the middle surface M of the film. The surface M is described by the equation

$$\mathbf{r} = \mathbf{r}(\theta^1, \theta^2, t),$$

θ^α being the curvilinear coordinates, t — time. We suppose that the surface of the film are free of tangential tractions, so there is no shear across the thickness of the film. Hence the local dynamical state of sufficiently thin film with traction-free surfaces can be described by two-dimensional (2D) fields of liquid density ϱ, velocity $\mathbf{V}$ and film thickness h

$$\varrho = \varrho(\theta^1, \theta^2, t), \quad h = h(\theta^1, \theta^2, t), \quad \mathbf{V} = \mathbf{V}(\theta^1, \theta^2, t).$$

The parametrization θ^α of the surface M may be chosen at will. The corresponding metrics is characterized with the covariant base vectors

$$\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} \equiv \partial \mathbf{r}/\partial\theta^\alpha, \quad \alpha = 1, 2 \tag{1.1}$$

contravariant base vectors $\mathbf{a}^\alpha$ and co- and contravariant components of the metric tensor $a_{\alpha\beta}$, $a^{\alpha\beta}$, det $a_{\alpha\beta} = a$. (For the required geometrical formulae see, for example, [11]).

Consider an element of liquid moving along the surface M. Its motion can be described in two ways: by an explicit formula for the radius-vector of the element $\mathbf{R}$ or parametrically through the dependence of the coordinates of the element on time

$$\mathbf{R} = \mathbf{R}(t) \quad \text{or} \quad \mathbf{R} = \mathbf{r}(\Theta^\alpha(t)), \quad \theta^\alpha = \Theta^\alpha(t), \tag{1.2}$$

$$\mathbf{V} = \dot{\mathbf{R}} = \mathbf{r}_{,\alpha}\dot{\Theta}^\alpha + \mathbf{r}_{,t} = \mathbf{a}_\alpha\dot{\Theta}^\alpha + \mathbf{r}_{,t}. \tag{1.3}$$

On the other hand, a point of M with the fixed coordinates θ^α possess velocity $\mathbf{U} = \mathbf{r}_{,t}$ so that

$$\mathbf{V} = \mathbf{U} + \mathbf{a}_\alpha\dot{\Theta}^\alpha, \quad \mathbf{W} = \mathbf{V} - \mathbf{U} = W^\alpha\mathbf{a}_\alpha, \quad W^\alpha = \dot{\Theta}^\alpha. \tag{1.4}$$

The relative velocity $\mathbf{W}$ of a point on the surface M is a "surface vector". Now, the fluid motion along the surface M induces some fictitious motion on the $\theta^1\theta^2$ plane with velocity $\dot{\theta}^1, \dot{\theta}^2$. It is seen readily that the liquid mass within the element $d\theta^1 d\theta^2$ of the film equals

$$\varrho^* d\theta^1 d\theta^2 = \varrho h a^{1/2} d\theta^1 d\theta^2 \tag{1.5}$$

so that the fictitious flow in the $\theta^1\theta^2$ plane has mass density ϱ^* and mass velocity $\varrho^*\,\mathbf{W}$ and the continuity equation has the form

$$(\varrho h a^{1/2})_{,t} + (\varrho h a^{1/2} W^\alpha)_{,\dot{\alpha}} = 0. \tag{1.6}$$

For the momentum balance refer to Fig. 1 in which the $d\theta^1 d\theta^2$ element of the film is shown, the forces and momentum flux being indicated. The overall momentum within

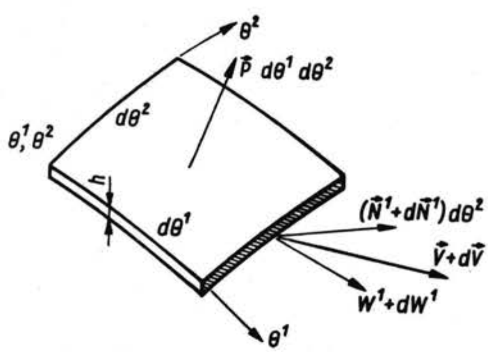

FIG. 1.

the element equals $\varrho h a^{1/2}\mathbf{V}\, d\theta^1 d\theta^2$, the outflow of momentum across the shaded face is $[\varrho h a^{1/2}\mathbf{V}W^1]_{\theta^1+d\theta^1}d\theta^2$, all other fluxes being evaluated in the same way. The resulting momentum balance has the form

$$(1.7)\qquad (\varrho h a^{1/2}\mathbf{V})_{,t}+(\varrho h a^{1/2}W^\alpha\mathbf{V})_{,\alpha} = \mathbf{P}+\mathbf{N}^\alpha_{,\alpha}.$$

Here $\mathbf{P}$ is the external force (body force and surface traction on the "free" surfaces of the film) per unit area in the $\theta^1\theta^2$ plane, $\mathbf{N}^\alpha$ is the "internal surface force", the force acting across the line θ^α = const per unit of θ^β. It can be shown readily [11] that

$$(1.8)\qquad \mathbf{N}^\alpha = N^{\alpha\varrho}\mathbf{a}_\varrho,$$

$N^{\alpha\varrho}$ being the symmetrical contravariant surface stress tensor. In the component form Eq. (I.7) becomes

$$(1.9)\qquad \begin{aligned}&(\varrho^*V^\beta)_{,t}+(\varrho^*W^\alpha V^\beta)_{,\alpha}+\varrho^*W^\alpha V^\varrho\Gamma^\beta_{\alpha\varrho}\\ &\quad+\varrho^*V^\alpha(U^\beta_{,\alpha}+U^\varrho\Gamma^\beta_{\alpha\varrho}-U^3b_{\alpha\varrho}a^{\varrho\beta})-\varrho^*V^3(U^3_{,\alpha}+U^\gamma b_{\alpha\gamma})a^{\alpha\beta} = P^\beta+N^{\alpha\beta}_{,\alpha}+N^{\alpha\varrho}\Gamma^\beta_{\alpha\varrho},\\ &(\varrho^*V^3)_{,t}+(\varrho^*W^\alpha V^3)_{,\alpha}+\varrho^*W^\alpha V^\beta b_{\alpha\beta}+\varrho^*V^\alpha(U^3_{,\alpha}+U^\beta b_{\alpha\beta}) = P+N^{\alpha\beta}b_{\alpha\beta};\end{aligned}$$

$$(1.10)\qquad \varrho^* = \varrho h a^{1/2},\quad \mathbf{V} = V^\alpha\mathbf{a}_\alpha+V^3\mathbf{a}_3,\quad \mathbf{P} = P^\alpha\mathbf{a}_\alpha+P\mathbf{a}_3.$$

Here $\mathbf{a}_3$ is the unit normal to M,

$$(1.11)\qquad \begin{aligned}&\mathbf{a}_{\alpha,\beta} = \Gamma^\varrho_{\alpha\beta}\mathbf{a}_\varrho+b_{\alpha\beta}\mathbf{a}_3,\quad \mathbf{a}_{3,\alpha} = -b_{\alpha\beta}\mathbf{a}^\beta,\\ &\Gamma^\varrho_{\alpha\beta} = \mathbf{a}_{\alpha,\beta}\mathbf{a}^\varrho,\quad b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta}\mathbf{a}_3.\end{aligned}$$

Equations (1.6)–(1.10) can also be derived from the general theory of Cosserat thin shells developed by GREEN and NAGHDI [12]. The author thanks a referee for mentioning the reference.

For steady motion the surface M is stationary and its parametrization can be chosen time-independent

$$(1.12)\qquad \mathbf{r} = \mathbf{r}(\theta^1,\theta^2),\quad \mathbf{W} = \mathbf{V},\quad V^3 = 0$$

so that for the steady case the equations of continuity and momentum reduce to

$$(1.13)\qquad (\varrho h a^{1/2}V^\alpha)_{,\alpha} = 0,$$

$$(1.14)\qquad \varrho^*V^\alpha V^\beta_{,\alpha}+\varrho^*V^\alpha V^\varrho\Gamma^\beta_{\alpha\varrho} = P^\beta+N^{\alpha\beta}_{,\alpha}+N^{\alpha\varrho}\Gamma^\beta_{\alpha\varrho},$$

$$(1.15)\qquad \varrho^*V^\alpha V^\beta b_{\alpha\beta} = P+N^{\alpha\beta}b_{\alpha\beta}$$

and can be further simplified by introducing the stream function Ψ and choosing the streamlines for the coordinate curves ($\Psi = \Psi(\theta^2)$).

2.

Generally we are interested in those cases when the form of the film (i.e. the surface M) is unknown and must be found simultaneously with the solution of the dynamical problem. First of all we must correlate the evolution of parametrization of M with its motion. Sufficiently general parametrization of M can be introduced in the following way. Let $\theta^1,\theta^2,\theta^3$ be an arbitrary time-dependent coordinate system in space:

$$(2.1)\qquad \mathbf{r} = \mathbf{R}(\theta^1,\theta^2,\theta^3,t)$$

and the coordinate curves of the third family (θ^1 = const, θ^2 = const) have not more than one point of intersection with M. Under these assumptions we can choose θ^1 and θ^2 as surface coordinates on M and prescribe the surface by equations

$$\theta^3 = \Theta(\theta^1, \theta^2, t), \quad \mathbf{r} = \mathbf{R}(\theta^1, \theta^2, \Theta, t). \tag{2.2}$$

Now we have for a material element moving along the surface

$$\mathbf{V} = d\mathbf{r}/dt = \mathbf{A}_\alpha \dot{\theta}^\alpha + \mathbf{A}_3(\Theta_{,\alpha}\dot{\theta}^\alpha + \Theta_{,t}) + \mathbf{u}, \tag{2.3}$$

$$\mathbf{u} = \mathbf{R}_{,t}, \quad \mathbf{A}_i = \partial \mathbf{R}/\partial \theta^i. \tag{2.4}$$

It follows from Eqs. (2.2)–(2.4) that

$$\begin{gathered} \dot{\theta}^\alpha = W^\alpha = (\mathbf{V}-\mathbf{u})\mathbf{A}^\alpha, \quad \Theta_{,t} = (\mathbf{V}-\mathbf{u})\mathbf{A}^3 - \Theta_{,\alpha}W^\alpha, \\ \mathbf{a}_\alpha = \mathbf{A}_\alpha + \mathbf{A}_3\Theta_{,\alpha}. \end{gathered} \tag{2.5}$$

The second equation in the set (2.5) serves to describe the evolution of the surface M, as soon as the instantaneous velocity field $\mathbf{V}$ is known.

Now we have to correlate the surface stress in the film with its deformation. First of all we need some "surface" ($2D$) form of kinematics of continua which may be formulated by analogy with the $3D$ case [13].

Let us choose the film configuration at $t = t_0$ as reference configuration. Corresponding coordinates of material elements can be used as Lagrangian coordinates

$$\Theta^\alpha = \theta^\alpha(t_0).$$

As usual, deformation is the mapping of the reference configuration onto the actual one:

$$\Theta^\alpha \to \theta^\alpha(\Theta^\alpha, t).$$

Locally the deformation is characterized by the deformation gradient $\mathbf{F}$ which can be defined as a linear operator transforming any infinitesimal vector $d\mathbf{X}$ of reference configuration into the infinitesimal vector $d\mathbf{x}$ of the actual configuration consisting of the same material points

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \tag{2.6}$$

All the consequent kinematics of $2D$ deformation of continua can now be formulated by direct analogy with the $3D$ case. The only significant difference consists in the fact that there is no common $2D$ Cartesian coordinate system for the reference and actual configurations. Thus it becomes necessary to distinguish explicitly between tensors (operators) which act in the reference configuration ($\mathbf{C} = \mathbf{F}^T\mathbf{F}$), in the actual configuration ($\mathbf{B} = \mathbf{F}\mathbf{F}^T$) and from one of the configurations into another ($\mathbf{F}$ and $\mathbf{F}^T$). The tensors $\mathbf{B}$ and $\mathbf{C}$ are the familiar measures of deformation, the left and the right Cauchy–Green tensors.

Now we proceed along the well-known path. Let $\mathbf{F}_t(\tau)$ be the relative deformation gradient, i.e. the gradient of deformation at time τ with the actual configuration as the reference one and let

$$\mathbf{G} = \dot{\mathbf{F}}_t(t) = \frac{d}{d\tau}\mathbf{F}_t(\tau)\bigg|_{\tau=t} = \frac{d\mathbf{F}(t)}{dt}\mathbf{F}^{-1}(t), \tag{2.7}$$

G is the deformation rate and its symmetric and skew-symmetric parts are the strain rate tensor **D** and rotation rate tensor **Ω** respectively

(2.8) $$\mathbf{D} = \frac{1}{2}(\mathbf{G}+\mathbf{G}^T), \quad \mathbf{\Omega} = \frac{1}{2}(\mathbf{G}-\mathbf{G}^T).$$

For the time derivatives of the measures of deformation we get [13]

(2.9) $$d\mathbf{C}/dt \equiv \dot{\mathbf{C}} = 2\mathbf{F}^T\mathbf{D}\mathbf{F},$$
$$d\mathbf{B}/dt \equiv \dot{\mathbf{B}} = \mathbf{DB}+\mathbf{BD}+\mathbf{\Omega B}-\mathbf{B\Omega}.$$

Some staightforward but tedious calculations lead to the following explicit formulae:

$$G^{\alpha\cdot}_{\cdot\beta} = \mathbf{a}^\alpha\mathbf{G}\mathbf{a}_\beta = V^\alpha_{,\beta}+V^\mu\Gamma^\alpha_{\mu\beta}-V^3b^\alpha{}_\beta = \nabla_\beta V^\alpha - V^3 b_{\gamma\beta}a^{\gamma\alpha},$$

(2.10) $$D^{\alpha\cdot}_{\cdot\beta} = \frac{1}{2}(G^{\alpha\cdot}_{\cdot\beta}+a^{\alpha\mu}G^{\nu\cdot}_{\cdot\mu}a_{\nu\beta}),$$

$$\Omega^{\alpha\cdot}_{\cdot\beta} = \frac{1}{2}(G^{\alpha\cdot}_{\cdot\beta}-a^{\alpha\mu}G^{\nu\cdot}_{\cdot\mu}a_{\nu\beta});$$

(2.11) $$B^{\alpha\beta}_{,t} = D^{\alpha\cdot}_{\cdot\nu}B^{\nu\beta}+D^{\beta\cdot}_{\cdot\nu}B^{\nu\alpha}+\Omega^{\alpha\cdot}_{\cdot\nu}B^{\nu\beta}-\Omega^{\beta\cdot}_{\cdot\nu}B^{\alpha\nu}$$
$$-W^\mu\nabla_\mu B^{\alpha\beta}-B^{\nu\beta}\nabla_\nu U^\alpha - B^{\alpha\nu}\nabla_\nu U^\beta + U^3(B^{\nu\beta}b^{\alpha\cdot}_{\cdot\nu}+B^{\nu\alpha}b^\beta{}_\nu).$$

For the stationary film these equations become

(2.12) $$G^{\alpha\cdot}_{\cdot\beta} = \nabla_\beta W^\alpha,$$

(2.13) $$W^\mu\nabla_\mu B^{\alpha\beta} = D^\alpha{}_\nu B^{\nu\beta}+D^\beta{}_\nu B^{\nu\alpha}+\Omega^{\alpha\cdot}_{\cdot\nu}B^{\nu\beta}-\Omega^{\beta\cdot}_{\cdot\nu}B^{\alpha\nu},$$

∇_β being the covariant derivative.

3.

Now we have all the prerequisites necessary to obtain the 2*D* constitutive equations for the liquid film. Consider, along with the 2*D* tensors, their three-dimensional counterparts ($\mathbf{D}^*$, $\mathbf{F}^*$ etc). Due to the thinness of the film, the deformation of the material in the direction normal to the film surface reduces to pure elongation. So we have

(3.1) $$\mathbf{F}^* = \begin{pmatrix} F^{\alpha\cdot}_{\cdot\beta} & 0 \\ 0 & \lambda_3 \end{pmatrix}, \quad \mathbf{B}^* = \begin{pmatrix} B^{\alpha\beta} & 0 \\ 0 & \lambda_3^2 \end{pmatrix},$$
$$\mathbf{C}^* = \begin{pmatrix} C_{\alpha\beta} & 0 \\ 0 & \lambda_3^2 \end{pmatrix}, \quad \det\mathbf{B}^* = \det\mathbf{C}^* = 1,$$
$$\mathbf{D}^* = \begin{pmatrix} D^\alpha{}_\beta & 0 \\ 0 & \gamma_3 \end{pmatrix}, \quad \boldsymbol{\sigma}^* = \begin{pmatrix} \sigma^{\alpha\beta} & 0 \\ 0 & \sigma_3 \end{pmatrix}.$$

Here $\boldsymbol{\sigma}^*$ is the 3*D* stress tensor, $\boldsymbol{\sigma}$ is its 2*D* counterpart, the fluid is considered to be incompressible. It is easy to show that

(3.2) $$N^{\alpha\varrho} = ha^{1/2}\sigma^{\alpha\varrho},$$

(3.3) $$D^*_{33} = \gamma_3 = \partial v^3/\partial\theta^3 = h^{-1}(h_{,t}+h_{,\alpha}W^\alpha).$$

The tensors $\boldsymbol{\sigma}^*$, $\mathbf{D}^*$ and $\mathbf{B}^*$ are connected by means of the constitutive law of the liquid considered. For the extremes of viscous fluid and incompressible elastic material we get, respectively,

$$\boldsymbol{\sigma}^* = -p\boldsymbol{\delta}+2\eta\mathbf{D}^*, \tag{3.4}$$
$$\boldsymbol{\sigma}^* = -p\boldsymbol{\delta}+2L_{,I_1}\mathbf{B}-2L_{,I_2}\mathbf{B}^{-1},$$

$$I_1 = B^{\alpha\beta}a_{\alpha\beta}, \quad I_2 = B^{\alpha\gamma}B^{\mu\beta}a_{\gamma\beta}a_{\mu\alpha}. \tag{3.5}$$

The intermediate case of the elastoviscous liquid is much more complicated and is considered separately. The $3D$ constitutive laws are transformed into the respective $2D$ constitutive laws taking into account the fact that σ_3^* is constant across the film due to its thinness and, consequently, is prescribed by the boundary conditions. For the free film $\sigma_3^* = 0$.

In the case of viscous liquid we have, for example,

$$\sigma_3^* = -p+2\eta D_{33}^* = 0, \quad p = 2\eta h^{-1}(h_{,t}+h_{,\alpha}W^\alpha), \tag{3.6}$$
$$\sigma^{\alpha\varrho} = -pa^{\alpha\varrho}+2\eta D^{\alpha\varrho} = 2\eta\,[D^{\alpha\varrho}-h^{-1}(h_{,t}+W^\alpha h_{,\alpha})a^{\alpha\varrho}].$$

For very thin films it is necessary to take into account the surface tension; thus instead of Eqs. (3.2) and (3.6), we get

$$N^{\alpha\varrho} = \sigma^{\alpha\varrho}ha^{1/2}+2a^{1/2}\chi a^{\alpha\varrho}, \tag{3.7}$$

$$p = \chi h_{,\alpha\beta}a^{\alpha\beta}+2\eta h^{-1}(h_{,t}+h_{,\alpha}W^\alpha), \tag{3.8}$$

where χ is the surface tension.

4.

Now we shall try to formulate a simple model of elastoviscous Maxwellian liquid which is similar to the model of [14, 15] but differs from it in some aspects.

By "Maxwellian liquid" we understand a liquid in which instantaneous elimination of stress is followed by instantaneous elastic recoil. So for an arbitrary state of strain and stress of an element of the liquid it is possible to define the gradient of elastic deformation $\boldsymbol{\Phi}$ corresponding to the deformation from the unloaded to the actual state. It is then possible to write for the deformation gradient and deformation tensors

$$\mathbf{F} = \boldsymbol{\Phi}\mathbf{N}, \tag{4.1}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \boldsymbol{\Phi}\mathbf{A}\boldsymbol{\Phi}^T, \quad \mathbf{A} = \mathbf{N}\mathbf{N}^T, \tag{4.2}$$

where $\mathbf{A}$ is the local inelastic deformation tensor.

Let

$$\mathbf{G}^e = \dot{\boldsymbol{\Phi}}\boldsymbol{\Phi}^{-1}, \quad \mathbf{D}^e = \frac{1}{2}(\mathbf{G}^e+\mathbf{G}^{eT}), \quad \boldsymbol{\Omega}^e = \frac{1}{2}(\mathbf{G}^e-\mathbf{G}^{eT}), \tag{4.3}$$
$$\dot{\mathbf{N}}\mathbf{N}^{-1} = \mathbf{G}^p, \quad \mathbf{D}^p = \frac{1}{2}(\mathbf{G}^p+\mathbf{G}^{pT}), \quad \boldsymbol{\Omega}^p = \frac{1}{2}(\mathbf{G}^p-\mathbf{G}^{pT}), \quad \mathbf{B}^e = \boldsymbol{\Phi}\boldsymbol{\Phi}^T.$$

It is seen readily that

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p, \quad \mathbf{\Omega} = \mathbf{\Omega}^e + \mathbf{\Omega}^p, \tag{4.4}$$

$$\dot{\mathbf{B}}^e = \mathbf{D}^e\mathbf{B}^e + \mathbf{B}^e\mathbf{D}^e + \mathbf{\Omega}^e\mathbf{B}^e - \mathbf{B}^e\mathbf{\Omega}^e. \tag{4.5}$$

The main problem of closure is that we must introduce some constitutive hypotheses relating the stress $\boldsymbol{\sigma}$ to the elastic and inelastic deformation. We assume the following:

a. The elastic as well as the total deformation is isochoric

$$I_{3\mathbf{B}} = I_{3\mathbf{B}^e} = 1. \tag{4.6}$$

b. The elastic deformation tensor $\mathbf{B}^e$ is connected with the stress tensor $\boldsymbol{\sigma}$ in the same way as in an elastic material. This means that Eqs. (3.5) are valid with $\mathbf{B}^e$ substituted for $\mathbf{B}$.

c. The inelastic deformation rate $\mathbf{D}^p$ is determined uniquely by the elastic deformation and stress. Keeping in mind our assumption b, we see that $\mathbf{D}^p$ is uniquely determined by $\mathbf{B}^e$ (or $\boldsymbol{\sigma}$). In an isotropic material the tensors $\mathbf{B}^e$, $\mathbf{D}^p$ and $\boldsymbol{\sigma}$ have common principal axes. So we postulate that

$$\mathbf{D}^p = \frac{1}{2}\left(\mathbf{f}(I_1^e, I_2^e)\boldsymbol{\sigma} - I_{\sigma f}\boldsymbol{\delta}\right). \tag{4.7}$$

In the case of isotropic viscosity

$$\mathbf{f} = \eta^{-1}\boldsymbol{\delta}. \tag{4.8}$$

Viscosity η generally speaking depends on the invariants of $\mathbf{B}^e$.

These constitutive assumptions are almost identical with those of [14, 15]. However, they are not sufficient for the closure of the model as the elastic rotation rate $\mathbf{\Omega}^e$ remains indefinite. In [14, 15] it is assumed that

$$\dot{\mathbf{B}}^e = \mathbf{D}^e\mathbf{B}^e + \mathbf{B}^e\mathbf{D}^e + \mathbf{\Omega}\mathbf{B}^e - \mathbf{B}^e\mathbf{\Omega}. \tag{4.9}$$

Comparing Eqs (4.9) and (4.3), we see that $\mathbf{\Omega}$ and $\mathbf{\Omega}^e$ are implicitly identified. The reasons for such identification are not clear. Taking into account the intrinsic isotropy of the material, it seems more reasonable to assume that

d. Unloading of an element of the Maxwellian liquid proceeds without rotation.

This means that the tensors $\mathbf{C}^e$, $\mathbf{\Phi}$ and $\mathbf{B}^e$ are coaxial and the following additional identity holds:

$$\dot{\mathbf{B}}^e\mathbf{B}^{e-1} - \mathbf{B}^{e-1}\dot{\mathbf{B}}^e = 2\mathbf{\Omega}^e. \tag{4.10}$$

Using Eq. (4.5) we can rewrite the equation as

$$\mathbf{B}^e\mathbf{\Omega}^e\mathbf{B}^{e-1} + \mathbf{B}^{e-1}\mathbf{\Omega}^e\mathbf{B}^e = \mathbf{B}^e\mathbf{D}^e\mathbf{B}^{e-1} - \mathbf{B}^{e-1}\mathbf{D}^e\mathbf{B}^e. \tag{4.11}$$

Equation (4.11) serves to determine $\mathbf{\Omega}^e$ through $\mathbf{D}^e$ and the attained state of elastic deformation $\mathbf{B}^e$.

Below we shall use the following constitutive equations for the isotropic elastic Maxwellian liquid:

$$\dot{\mathbf{B}}^e = \mathbf{D}^e\mathbf{B}^e + \mathbf{B}^e\mathbf{D}^e + \mathbf{\Omega}^e\mathbf{B}^e - \mathbf{B}^e\mathbf{\Omega}^e, \tag{4.12}$$

$$\boldsymbol{\sigma} + p\boldsymbol{\delta} = 2(W_{,I_1^e}\mathbf{B}^e - W_{,I_2^e}\mathbf{B}^{e-1}), \tag{4.13}$$

$$\mathbf{D}^e = \mathbf{D} - \mathbf{D}^p, \tag{4.14}$$

$$\mathbf{D}^p = (2\eta)^{-1}\left(\boldsymbol{\sigma} - \frac{1}{3} I_\sigma \boldsymbol{\delta}\right), \tag{4.15}$$

$$\mathbf{B}^e\boldsymbol{\Omega}^e\mathbf{B}^{e-1} + \mathbf{B}^{e-1}\boldsymbol{\Omega}^e\mathbf{B}^e = \mathbf{B}^e\mathbf{D}^e\mathbf{B}^{e-1} - \mathbf{B}^{e-1}\mathbf{D}^e\mathbf{B}^e. \tag{4.16}$$

Assuming the Mooney–Rivlin form of the elastic potential [16]

$$W = G[(I_1^e - 3) + \alpha(I_2^e - 3)], \tag{4.17}$$

we get from Eq. (4.13)

$$\boldsymbol{\sigma} = -p\boldsymbol{\delta} + 2G(\mathbf{B}^e - \alpha\mathbf{B}^{e-1}). \tag{4.18}$$

5.

The main goal of this section is to obtain linearized equations for small perturbations of the steady motion of the liquid film. On the surface M_0 of the unperturbed film we introduce the stationary coordinate system θ^1, θ^2, so that the dynamical equations take the form (1.13)–(1.15). The perturbed film surface M will be referred to the same parameters θ^α. Let $\mathbf{a}_0^3$ be the unit normal to M_0 at the point (θ^1, θ^2). The point of M which lies on the same normal will, by definition, have the same coordinates θ^1, θ^2. So we get

$$\mathbf{r}(\theta^1, \theta^2, t) = \mathbf{r}_0(\theta^1, \theta^2) + \boldsymbol{\delta}(\theta^1, \theta^2, t), \tag{5.1}$$

$$\boldsymbol{\delta}(\theta^1, \theta^2, t) = \delta(\theta^1, \theta^2, t)\mathbf{a}_0^3(\theta^1, \theta^2). \tag{5.2}$$

(All the unperturbed quantities have the sub- or superscript "0"). A straightforward calculation gives for the geometry of M

$$\begin{gathered} \mathbf{a}_\alpha = \mathbf{a}_\alpha^0 + \delta_{,\alpha} a_3^0 - \delta b_{\alpha\beta}^0 \mathbf{a}_0^\beta, \qquad \mathbf{a}_3 = \mathbf{a}_3^0 - \delta_{,\alpha}\mathbf{a}_0^\alpha, \\ a_{\alpha\beta} = a_{\alpha\beta}^0 - 2\delta b_{\alpha\beta}^0, \qquad a = a_0(1 - 2\delta\varkappa^0), \qquad \varkappa = a^{\alpha\beta} b_{\alpha\beta}, \\ \Gamma_{\alpha\beta}^\varrho = \Gamma_{\alpha\beta}^{0\varrho} - \delta_{,\alpha} b_{\beta\gamma}^0 a_0^{\gamma\varrho} - \delta_{,\beta} b_{\alpha\gamma}^0 a_0^{\gamma\varrho} + \delta b_{\alpha\gamma}^0 \Gamma_{\beta\nu}^{0\gamma} a_0^{\nu\varrho} - \delta b_{\alpha\gamma,\beta}^0 a_0^{\gamma\varrho} + \delta \Gamma_{\alpha\beta}^{0\gamma} b_{\gamma\nu}^0 a_0^{\nu\varrho}; \end{gathered} \tag{5.3}$$

$$b_{\alpha\beta} = b_{\alpha\beta}^0 + \delta_{,\alpha\beta} - \delta_{,\nu}\Gamma_{\alpha\beta}^{0\nu} - \delta b_{\alpha\varrho}^0 b_{\beta\gamma}^0 a_0^{\varrho\gamma}. \tag{5.4}$$

Further, let

$$\begin{gathered} h(\theta^1, \theta^2, t) = h^0(\theta^1, \theta^2) + \lambda(\theta^1, \theta^2, t), \\ \mathbf{W} = \mathbf{W}_0 + \mathbf{w}, \qquad \mathbf{N} = \mathbf{N}_0 + \mathbf{n} \qquad \text{etc.} \end{gathered} \tag{5.5}$$

Perturbations of other quantities are also designated by respective small letters. Perturbations of the components of the vectors and tensors include contributions of both these "proper" perturbations and of the geometry perturbations. Let $\mathbf{L} = \mathbf{L}_0 + \mathbf{l}$ be an arbitrary vector on M. Using Eqs. (5.3) and (5.4), we get

$$\begin{gathered} L^\alpha = \mathbf{L}\mathbf{a}^\alpha = L_0^\alpha + l^\alpha + L_0^3 \delta_{,\varrho} a_0^{\alpha\varrho} + \delta b_{\varrho 0}^\alpha L_0^\varrho; \\ L^3 = L_0^3 + l^3 - L_0^\alpha \delta_{,\alpha}, \qquad l^\alpha = \mathbf{l}\mathbf{a}^\alpha \simeq \mathbf{l}\mathbf{a}_0^\alpha. \end{gathered} \tag{5.6}$$

We also have

$$\begin{gathered} W^\alpha = W_0^\alpha + w^\alpha + \delta a_0^{\alpha\beta} b_{\beta\varrho}^0 W^\varrho, \qquad u^\alpha = 0, \qquad u^3 = \delta_{,t}, \\ V^\alpha = W^\alpha + u^\alpha = W^\alpha, \qquad V^3 = u^3 = \delta_{,t}. \end{gathered} \tag{5.7}$$

Now let $\mathbf{S} = \mathbf{S}_0 + \mathbf{s}$ be a $2D$ tensor on M. It is convenient to consider $\mathbf{S}$ as a $3D$ tensor having the normal to M as one of the principal axes with the corresponding eigenvalue equal to unity. Using Eqs. (5.3) and (5.4), we get

$$S^{\alpha\beta} = S_0^{\alpha\beta} + \delta(b^{\beta}_{0\nu} S^{\alpha\nu} + b^{\alpha}_{0\nu} S^{\nu\beta}) + s_0^{\alpha\beta}. \tag{5.8}$$

Substitution into the dynamical equations (1.6)–(1.7) gives

$$a_0^{1/2}\lambda_{,t} - h_0 a_0^{1/2}\varkappa_0\delta_{,t} + h_0 a_0^{1/2} W_0^{\alpha}(\lambda h_0^{-1} - \delta\varkappa_0)_{,\alpha} + [h_0 a_0^{1/2}(w^{\alpha} + \delta a_0^{\alpha\beta} b^0_{\beta\gamma} W_0^{\gamma})]_{,\alpha} = 0, \tag{5.9}$$

$$\begin{aligned}
&[\varrho\lambda a_0^{1/2} W_0^{\beta} - \varrho h_0 \delta\varkappa_0 W_0^{\beta} + \varrho h_0 a_0^{1/2} w^{\beta} + \varrho h_0 \sqrt{a_0}\,\delta \\
&\quad \times a_0^{\beta\alpha} b^0_{\alpha\gamma} W_0^{\gamma})_{,t} + (\varrho\lambda a_0^{1/2} W_0^{\alpha} W_0^{\beta} - \varrho h_0 \delta\varkappa_0 W_0^{\alpha} W_0^{\beta} + \varrho h_0 a_0^{1/2} W_0^{\alpha} w^{\beta} + \varrho h_0 a_0^{1/2} W_0^{\beta} w^{\alpha} \\
&\quad + \varrho h_0 a_0^{1/2}\delta a_0^{\beta\gamma} b^0_{\gamma\nu} W_0^{\alpha} W_0^{\nu} + \varrho h_0 a_0^{1/2}\delta a_0^{\alpha\nu} b^0_{\nu\gamma} W_0^{\gamma} W_0^{\beta})_{,\alpha} \\
&\quad + \varrho h_0 a_0^{1/2} W_0^{\alpha} W_0^{\varrho}\gamma^{\beta}_{\alpha\varrho} + \varrho\lambda a_0^{1/2} W_0^{\alpha} W_0^{\varrho}\mathring{\Gamma}^{\beta}_{\alpha\varrho} - \varrho h_0\delta\varkappa_0 W_0^{\alpha} W_0^{\varrho}\mathring{\Gamma}^{\beta}_{\alpha\varrho} + \varrho h_0 a_0^{1/2} W_0^{\alpha} w^{\varrho}\mathring{\Gamma}^{\beta}_{\alpha\varrho} \\
&\quad + \varrho h_0 a_0^{1/2}\delta a_0^{\varrho\delta} b^0_{\delta\gamma} W_0^{\alpha} W_0^{\gamma}\mathring{\Gamma}^{\beta}_{\alpha\varrho} + \varrho h_0 a_0^{1/2} W_0^{\varrho} w^{\alpha}\mathring{\Gamma}^{\beta}_{\alpha\varrho} + \varrho h_0 a_0^{1/2}\delta a_0^{\alpha\delta} b^0_{\delta\gamma} W_0^{\gamma} W_0^{\varrho}\mathring{\Gamma}^{\beta}_{\alpha\varrho} \\
&\quad - \varrho h_0 a_0^{1/2}\delta_{,t} W_0^{\alpha} b^0_{\alpha\varrho} a_0^{\varrho\beta} = p^{\beta} + n^{\alpha\beta}_{\;\;\cdot\,,\alpha} + (\delta N_0^{\alpha\varrho} a_0^{\beta\gamma} b^0_{\gamma\varrho})_{,\alpha} + n^{\alpha\varrho}\mathring{\Gamma}^{\beta}_{\alpha\varrho} \\
&\quad + \delta N_0^{\alpha\gamma} a_0^{\varrho\nu} b^0_{\nu\gamma}\mathring{\Gamma}^{\beta}_{\alpha\varrho} + N_0^{\alpha\varrho}\gamma^{\beta}_{\alpha\varrho} + (\delta N_0^{\varrho\beta} a_0^{\gamma\alpha} b^0_{\gamma\varrho})_{,\alpha} + \delta N_0^{\gamma\varrho} a_0^{\alpha\nu} b^0_{\nu\gamma}\mathring{\Gamma}^{\beta}_{\alpha\varrho} \\
&\quad + P_0\delta_{,\varrho} a_0^{\beta\varrho} + \delta P_0^{\varrho} b_0{}^{\beta}{}_{\varrho};
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
&\varrho h_0 N a_0 \delta_{,tt} + 2\varrho h_0 a_0^{1/2} W_0^{\alpha}\delta_{,t\alpha} + \varrho\lambda a_0^{1/2} W_0^{\alpha} W_0^{\beta} b^0_{\alpha\beta} \\
&\quad - \varrho h_0 \delta\varkappa_0 W_0^{\alpha} W_0^{\beta} b^0_{\alpha\beta} + 2\varrho h_0 a_0^{1/2} w^{\alpha} W_0^{\beta} b^0_{\alpha\beta} + 2\varrho h_0 a_0^{1/2}\delta a_0^{\beta\nu} b^0_{\nu\gamma} b^0_{\alpha\beta} W_0^{\alpha} W_0^{\gamma} \\
&\quad + \varrho h_0 a_0^{1/2} W_0^{\alpha} W_0^{\beta}\beta_{\alpha\beta} = p + n^{\alpha\beta} b^0_{\alpha\beta} + 2\delta N_0^{\alpha\varrho} a_0^{\beta\gamma} b^0_{\gamma\varrho} b^0_{\alpha\beta} + N_0^{\alpha\beta}\beta_{\alpha\beta} - P_0^{\alpha}\delta_{,\alpha};
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
\gamma^{\varrho}_{\alpha\beta} &= -\delta_{,\alpha} b^0_{\gamma\beta} a_0^{\gamma\varrho} - \delta_{,\beta} b^0_{\alpha\gamma} a_0^{\gamma\varrho} - \delta b^0_{\alpha\gamma,\beta} a_0^{\gamma\varrho} + \delta b^0_{\alpha\gamma}\mathring{\Gamma}^{\gamma}_{\nu\beta} a_0^{\nu\varrho} + \delta\mathring{\Gamma}^{\gamma}_{\alpha\beta} b^0_{\gamma\nu} a_0^{\nu\varrho}, \\
\beta_{\alpha\beta} &= \delta_{,\alpha\beta} - \delta b^0_{\alpha\gamma} b^0_{\nu\beta} a_0^{\gamma\nu} - \delta_{,\varrho}\mathring{\Gamma}^{\varrho}_{\alpha\beta}.
\end{aligned} \tag{5.12}$$

The set of four equations of the set (5.9)–(5.12) serves to find the four unknown variables δ, λ, w^{α} provided the external force perturbation vector $\mathbf{p}$ is known and the perturbation of the surface stress tensor $\mathbf{n}$ is expressed through the kinematical variables. To get the last expression δ, we must make use of the liquid rheology (see below). The simplest possible case is that of the initially plane film. For the case in question

$$\begin{aligned}
&\mathring{\Gamma}^{\varrho}_{\alpha\beta} = b^0_{\alpha\beta} = \varkappa_0 = 0, \quad a^0_{\alpha\beta} = \delta_{\alpha}{}^{\beta}, \quad a_0 = 1, \\
&a_{\alpha\beta} = \delta_{\alpha}{}^{\beta}, \quad b_{\alpha\beta} = \delta_{,\alpha\beta}, \quad \Gamma^{\varrho}_{\alpha\beta} = 0, \quad a = 1
\end{aligned} \tag{5.13}$$

and Eqs. (5.9)–(5.12) give

$$\lambda_{,t} + h_0 W_0^{\alpha}(\lambda h_0^{-1})_{,\alpha} + (h_0 w^{\alpha})_{,\alpha} = 0, \tag{5.14}$$

$$\lambda_{,t} W_0^{\beta} + h_0 w^{\beta}_{,t} + (\lambda W_0^{\alpha} W_0^{\beta} + h_0 W_0^{\alpha} w^{\beta} + h_0 W_0^{\beta} w^{\alpha})_{,\alpha} = (p^{\beta} + n^{\alpha\beta}_{\;\;\cdot\,,\alpha} + P_0\delta_{,\beta})\varrho^{-1}, \tag{5.15}$$

$$h_0\delta_{,tt} + 2h_0 W_0^{\alpha}\delta_{,\alpha t} = \varrho^{-1}(N_0^{\alpha\beta}\delta_{,\alpha\beta} + p + \delta_{,\alpha} P_0^{\alpha}). \tag{5.16}$$

For the film of constant thickness ($h_0 = \mathrm{const}$) moving with constant velocity ($W_0^1 = W, W_0^2 = 0$) in the absence of external force ($\mathbf{P}_0 = \mathbf{p} = 0$) we get

$$
(5.17)\quad
\begin{aligned}
&h_0^{-1}\lambda_{,t}+W\lambda_{,1}+w^{\alpha}_{.\alpha}=0,\\
&\lambda_{,t}W+h_0w^1_{.t}+(\lambda W^2+2h_0Ww^1)_{,1}+h_0Ww^2_{.2}=\varrho^{-1}n^{\alpha 1}_{\cdot\,\alpha},\\
&h_0w^2_{.t}+h_0Ww^2_{.1}=\varrho^{-1}n^{\alpha 2}_{\cdot\,\alpha};\\
&\delta_{,tt}+2W\delta_{,1t}+W^2\delta_{,11}=(\varrho h_0)^{-1}N_0^{\alpha\beta}\delta_{,\alpha\beta}.
\end{aligned}
$$

Consider this last equation in more detail assuming that the principal axes of N_0 coincide with the coordinate curves. We have

$$
(5.18)\quad
\begin{aligned}
\delta_{,tt}+2W\delta_{,1t}+W^2\delta_{,11}&=c_1^2\delta_{,11}+c_2^2\delta_{,22},\\
c_1^2=N_0^{11}/\varrho h_0,\quad c_2^2&=N_0^{22}/\varrho h_0.
\end{aligned}
$$

For the wave solutions of the form

$$
(5.19)\quad \delta=\Delta\exp[i(\omega t+k\theta^1+l\theta^2)]
$$

the dispersion relation is

$$
(5.20)\quad \omega=-kW\pm(c_1^2k^2+c_2^2l^2)^{1/2}.
$$

If $0<W<c_1$, then for the arbitrary wave vector (k,l) there are two waves — one propagating downflow and the other propagating in the opposite direction. Equation (5.18) in this case is a hyperbolic equation analogous to the equation of the vibrations of a membrane. It is therefore not unreasonable to consider two-sided boundary value problems, for example, the problem of free oscillations with prescribed values at the entrance and exit. Thus we may consider unidimensional oscillations of a finite length of the film, $0<\theta_1<L$, under the conditions $\delta_{,2}=0$, $\delta(0,t)=\delta(l,t)=0$.

If $W>c_1$, then all the waves run in the downflow direction $(\omega/k<0)$ and the correctly stated problem for Eq. (5.18) is the problem of propagation of disturbances in the direction of the flow. For example, we can prescribe $\delta(\theta^2,t)$ and $\delta_{,1}(\theta^2,t)$ at $\theta_1=0$. In that case the problem of free oscillations of the film fixed along a closed contour is incorrectly posed.

It is worth noting that the equation of transverse oscillations of liquid film (5.17) does not depend on the perturbed stress tensor $\mathbf{n}$. This means that the transverse membrane oscillations of the film do not depend on the liquid rheology and are determined exclusively by the initial (stationary) surface stress in the film.

Consider now the perturbations of the thickness of the film. In the Cartesian coordinate system $x=\theta^1-Wt$, $y=\theta^2$, $w^1=u$, $w^2=v$ and Eqs. (5.17) give

$$
(5.21)\quad
\begin{aligned}
\frac{1}{h_0}\frac{\partial\lambda}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}&=0,\\
\varrho h_0\frac{\partial u}{\partial t}&=\frac{\partial n^{11}}{\partial x}+\frac{\partial n^{21}}{\partial y},\\
\varrho h_0\frac{\partial v}{\partial t}&=\frac{\partial n^{12}}{\partial x}+\frac{\partial n^{22}}{\partial y}.
\end{aligned}
$$

It is obvious that stress perturbations are of primary significance if we are interested in the change of thickness. Consequently the material rheology must be taken into account. For the viscous liquid we then get

$$
(5.22)\quad n^{\alpha\varrho}=h_0a_0^{1/2}[\eta(w^{\alpha}_{.\gamma}+w^{\gamma}_{.\alpha})-2\eta h_0^{-1}(\lambda_{,t}+\lambda_{,x}W)\delta^{\alpha}{}_{\gamma}+\lambda_{,\nu\beta}\chi a_0^{\nu\beta}\delta^{\alpha}{}_{\gamma}]a_0^{\gamma\varrho}.
$$

Substitution into Eq. (5.21) results in

$$\text{(5.23)}\qquad \begin{aligned} \frac{1}{h_0}\frac{\partial\lambda}{\partial t} &= -\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y},\\ \varrho\frac{\partial u}{\partial t} &= 4\eta\frac{\partial^2 u}{\partial x^2}+\eta\frac{\partial^2 v}{\partial x\partial y}-\eta\frac{\partial^2 u}{\partial y^2}+\chi\Delta\frac{\partial\lambda}{\partial x},\\ \varrho\frac{\partial v}{\partial t} &= 4\eta\frac{\partial^2 v}{\partial y^2}+\eta\frac{\partial^2 u}{\partial x\partial y}-\eta\frac{\partial^2 v}{\partial x^2}+\chi\Delta\frac{\partial\lambda}{\partial y}. \end{aligned}$$

In the inviscid case ($\eta = 0$) Eq. (5.23) reduces to the well-known equation [17]

$$\text{(5.24)}\qquad \frac{\varrho}{h_0}\frac{\partial^2\lambda}{\partial t^2} = -\chi\Delta^2\lambda.$$

For the unidimensional motion we get from Eq. (5.23) ($v = 0$, $\partial/\partial y = 0$)

$$\text{(5.25)}\qquad \frac{\partial^2\lambda}{\partial t^2} = -\frac{\chi h_0}{\varrho}\frac{\partial^4\lambda}{\partial x^4}+4\nu\frac{\partial^3\lambda}{\partial x^2\partial t},\qquad \nu = \eta/\varrho.$$

The spatially-periodic solutions of Eq. (5.25)

$$\text{(5.26)}\qquad \lambda = \Lambda e^{\mu t+ikx},\qquad \mu = k^2[-2\nu\pm(4\nu^2-\chi h_0/\varrho)^{1/2}]$$

describe running ($\chi h_0/\varrho > 2\nu$) or stationary ($\chi h_0/\varrho < 2\nu$) harmonic waves of decreasing amplitude. Consider further the unidimensional wave pattern generated by a stationary generator. The waves are described by the solution $\lambda = \exp(i\omega t)X(\xi)$

$$\text{(5.27)}\qquad -(\chi h_0/\varrho)X^{\mathrm{IV}}+4\nu WX^{\mathrm{III}}+(4\nu i\omega-W^2)X^{\mathrm{II}}-2i\omega WX'+\omega^2X = 0,$$

ξ being the distance from the generator. It can be shown that Eq. (5.27) has two linearly independent solutions which vanish at infinity ($\xi\to\infty$). Hence the physically reasonable problem with prescribed values of film thickness and velocity at $\xi = 0$ is stated correctly and unidimensional perturbations do not lead to film disintegration.

6.

As another elementary example we shall consider the dynamics of the spherical film of the elastoviscous liquid under internal pressure. Let R be the radius of the film, $h \ll R$ its thickness, $V = -dR/dt$, q the internal pressure. Introducing θ^1 and θ^2 as geographical coordinates on the sphere, $\theta^1 = \theta, \theta^2 = \varphi$, we get

$$\text{(6.1)}\qquad \begin{gathered} a_{11} = R^2,\quad a_{12} = a_{21} = a^{21} = a^{12} = 0,\quad a^{11} = R^{-2},\\ a^{22} = R^{-2}(\sin\theta)^{-2},\quad a = R^4\sin^2\theta,\quad b_{\alpha\beta} = R^{-1}a_{\alpha\beta},\\ V^\alpha = U^\alpha_\cdot = W^\alpha = 0,\quad V^3 = V. \end{gathered}$$

The equations of continuity and dynamics (1.6) and (1.10) give

$$\text{(6.2)}\qquad \begin{gathered} (ha^{1/2})_{,t} = 0\\ \varrho(ha^{1/2}V)_{,t} = P+N^{\alpha\beta}b_{\alpha\beta}. \end{gathered}$$

According to Eq. (2.10),

(6.3) $$G^{\alpha\cdot}_{\cdot\beta} = D^{\alpha}{}_{\beta} = -VR^{-1}a^{\alpha}{}_{\beta}$$

so that from Eq. (2.11)

(6.4) $$B^{\alpha\beta}{}_{,t} = 0.$$

This means that for the isotropic and homogeneous initial state

(6.5) $$B^{\alpha\beta} = Ba^{\alpha\beta},$$

where B is a scalar. From Eq. (4.11) it also follows for the elastic deformation that

(6.6) $$K^{\alpha\beta}_{,t} - (2V/R)K^{\alpha\beta} = 2K^{\alpha\gamma}L^{\beta}_{\cdot\gamma}, \quad \mathbf{K} \equiv \mathbf{B}^{e}, \quad \mathbf{L} \equiv \mathbf{D}^{e}.$$

For the isotropic initial state

(6.7) $$\mathbf{K} = K\boldsymbol{\delta}, \quad \mathbf{L} = L\boldsymbol{\delta}.$$

Using Eqs (4.17) and (4.16) with $\alpha = 0$, we get

(6.8) $$\boldsymbol{\sigma} = \Sigma\boldsymbol{\delta} = 2G(K-K^{-2})\boldsymbol{\delta},$$

Substitution into Eqs. (4.14) and (4.13) results in

(6.9) $$\mathbf{D}^{p} = \left(\frac{1}{3\eta}\right)G(K-K^{-2})\boldsymbol{\delta}, \qquad L = -V/R-(G/3\eta)(K-K^{-2}).$$

Using Eqs. (6.9) and (6.1), Eq. (6.6) reduces to

(6.10) $$\dot{K}-(2\dot{R}/R)K = -(G/3\eta)(K-K^{-2}).$$

The last equation is in a sense a "net constitutive equation" of the spherical elastoviscous film.

Using Eqs. (6.8) and (3.7), we get

(6.11) $$N^{\alpha\varrho} = 2ha^{1/2}[G(K-K^{-2})+\chi R^{2}/h_0 R_0^{2}]a^{\alpha\varrho},$$

(6.12) $$h_0 = h(0), \quad R_0 = R(0), \quad P = -qa^{1/2}.$$

Substitution into Eq. (6.2) results in

(6.13) $$\varrho\frac{d^{2}R}{dt^{2}} = \frac{q}{h_0}\frac{R^{2}}{R_0^{2}} - \frac{4}{R}\left[G(K-K^{-2})+\chi\frac{R^{2}}{h_0 R_0^{2}}\right].$$

Equations (6.13) and (6.10) are the resulting set of equations of the problem under consideration. For a sufficiently thin film it is possible to neglect the inertial forces putting $\varrho = 0$. Assuming in addition that $K \gg 1$, we get from Eq. (6.13)

(6.14) $$K = \frac{1}{Gh_0}\left(q\frac{R^{3}}{4R_0^{2}} - \chi\frac{R^{2}}{R_0^{2}}\right)$$

and from Eq. (6.10)

(6.15) $$\frac{K}{R^{2}} = C\exp\left(-\frac{t}{3\Theta}\right), \quad \Theta = \frac{\eta}{G}.$$

Assuming that the volume of gas within the sphere is constant and that the elastic strain is initially equal to zero ($K_0 = 1$), we get

$$q = q_0(R_0/R)^3, \quad C = R_0^{-2},$$

$$R = \frac{1}{2} R_0 \left(\frac{qR_0}{\chi + Gh_0 \exp(-t/3\Theta)} \right)^{1/2}. \tag{6.16}$$

This equation may be used to estimate the properties of the liquid.

In conclusion we mention that a more comprehensive treatment of the problem under consideration can be found in the preprint by the author [18].

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