Arch. Mech., 34, 4, pp. 437–454, Warszawa 1982

# Wave propagation in nonhomogeneous almost nonlinear thin elastic rods

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WAVE propagation in a thin nonhomogeneous almost nonlinear elastic semi-infinite rod subjected to a time-dependent velocity impact is studied. Similarity transformations are used to transform the equation of motion and the boundary condition at the impacted end and the similarity characteristic relation is applied for locating the wave front, to obtain a similarity representation in the form of a boundary value problem. A solution of the similarity representation has been obtained in the form of power series. Restrictions on the parameters of the problem and relations among them have been determined. The stress distribution in the rod is evaluated and graphically represented.

Rozważono propagację fal w cienkich niejednorodnych prawie nieliniowych prętach półnieskończonych poddanych działaniu zależnego od czasu impulsu prędkości. Dla przekształcenia równania ruchu i warunku brzegowego na uderzonym końcu pręta zastosowano transformację podobieństwa, a związek pomiędzy charakterystykami i zmienną transformacji wykorzystano do lokalizacji frontu fali, otrzymując reprezentację podobieństwa w postaci zagadnienia brzegowego. Rozwiązanie dla reprezentacji podobieństwa przedstawiono w postaci szeregu potęgowego. Określono ograniczenia dotyczące parametrów zagadnienia i związków między nimi. Wyznaczono i przedstawiono graficznie rozkład naprężeń w pręcie.

Рассмотрено распространение волн в тонких неоднородных почти нелинейных полубесконечных стержнях, подвергнутых действию зависящего от времени импульса скорости. Для преобразования уравнения движения и граничного условия на ударенным конце стержня применено преобразование подобия, а соотношение между характеристиками а подобием использовано для локализации фронта волны, получая представление подобия в виде краевой задачи. Решение для представления подобия представлено в виде степенного ряда. Определены ограничения, касающиеся параметров проблемы и связей между ними. Определено и представлено графически распределение напряжений в стержне.

## 1. Introduction

THE PROBLEMS of wave propagation in nonlinear [1, 2] and linear nonhomogeneous rods [3] have been dealt with by various techniques; however, no literature seems to be available on propagation of disturbances in rods which are simultaneously nonhomogenous and nonlinear. An attempt has been made in this study to deal with such a problem by the similarity analysis. Similarity transformations [4, 5] are determined from the equation of motion, the boundary and the initial conditions. The similarity characteristic relation [2] is applied to locate the wave front in the similarity coordinate. Using similarity transformations and the similarity characteristic relation a similarity representation is obtained of the original system of equations. This representation consists of a nonlinear, nonhomogeneous ordinary differential equation along with a well-defined boundary condition at the origin and another boundary condition at the point defined by the wave front.

The location of the wave front in terms of the similarity coordinate is dependent, in general, on the slope of the unknown similarity function at the front. Assuming the parameter of the material nonlinearity of the rod to be close to unity, i.e. considering the material to be homogeneous — almost nonlinear, the condition at the wave front is rendered independent of the slope of the unknown function and the nonlinear differential equation of the similarity representation becomes expressible as a linear ordinary differential equation with variable coefficients. The representation so obtained for the almost nonlinear rod is solved to obtain the solution in the form of power series. The results for displacement and stress distribution in the rod are at first expressed in terms of nondimensional quantities and then graphically presented for visual inspection.

## 2. Basic equations and similarity representation

The constitutive relation for an elastic nonlinear nonhomogeneous semi-infinite thin rod is assumed to be in the form

$$\sigma = E(x)e^{\frac{1}{q}}, \quad x \geqslant 0, \quad q > 0, \tag{2.1$_1$}$$

where

$$E(x) = E_0 x^n. \tag{2.1$_2$}$$

The equation of motion for the above case assumes the form [2]

$$\frac{E_0}{\varrho q} x^n \left(-\frac{\partial u}{\partial x}\right)^{\frac{1-q}{q}} \frac{\partial^2 u}{\partial x^2} - \frac{E_0}{\varrho} n x^{n-1} \left(-\frac{\partial u}{\partial x}\right)^{\frac{1}{q}} = \frac{\partial^2 u}{\partial t^2}, \quad x \geqslant 0, \quad t \geqslant 0, \quad q > 0. \tag{2.2}$$

In the above equations $x$ is the coordinate along the axis of the rod, $t$ is the time, $\sigma$ is the stress normal to the cross section of the rod, $e$ is the strain, $E(x)$ is the modulus of elasticity, $\varrho$ is the mass density. $E_0$, $n$ and $q$ are material parameters; $E_0$ and $n$ are associated with the nonhomogeneity and $q$ is associated with the nonlinearity of the rod. $\delta$ and $V_c$ are parameters of the velocity impact. In Eq. (2.2) and in the subsequent equations the compressive stress is assumed to be positive.

The boundary condition for a time-dependent velocity impact applied in the direction of the $x$ axis is assumed to be

$$\frac{\partial u}{\partial t}(x = 0, t) = V_c t^{\delta}, \quad t > 0. \tag{2.3}$$

The condition at and beyond the wave front is

$$u\big(x \geqslant x_w(t), t\big) = 0, \quad t > 0, \tag{2.4}$$

where $x = x_w(t)$ defines the wave front. The initial conditions are

$$u(x, t = 0) = 0, \quad x > 0, \qquad \frac{\partial u}{\partial t}(x, t = 0) = 0, \quad x > 0. \tag{2.5}$$

It may be noted that Eqs. (2.5) are redundant as Eq. (2.4) implies Eq. $(2.5)_1$ and Eq. $(2.5)_2$ follows consequently there from.

The similarity transformations for Eqs. (2.2)–(2.4) are obtained on the basis of [2, 6] as

(2.6)
$$\begin{aligned} u &= V_c t^{\delta+1} F(\eta), \quad 0 \leqslant x \leqslant x_w, \quad t > 0, \\ &= 0, \qquad\qquad x \geqslant x_w, \qquad t > 0, \end{aligned}$$

(2.7)
$$\begin{aligned} \eta &= \frac{K x^{\frac{1+q-nq}{1+q}}}{t^m}, \quad 0 \leqslant x \leqslant x_w, \quad t > 0, \\ &= 0, \quad \text{for } x = t = 0, \quad \text{by definition,} \end{aligned}$$

where

(2.8)
$$\begin{aligned} K &= \left(\frac{\varrho q}{E_0}\right)^{\frac{q}{q+1}} \left[\frac{1}{V_c}\right]^{\frac{1-q}{1+q}}, \\ m &= 1+\delta \frac{1-q}{1+q}. \end{aligned}$$

On the basis of Eqs. (2.6), (2.7), the equation of motion (2.2) can be written in terms of the similarity function $F(\eta)$ and its derivatives as

(2.9)
$$\left[\left(\frac{1+q-nq}{1+q}\right)^{\frac{1+q}{q}} (-F')^{\frac{1-q}{q}} - m^2\eta^2\right] F''(\eta) - \left(\frac{1+q-nq}{1+q}\right)^{\frac{1}{q}} \frac{nq^2}{1+q} \eta^{-1}$$
$$\times [-F'(\eta)]^{\frac{1}{q}} - m(m-2\delta-1)\eta F'(\eta) - \delta(\delta+1)F(\eta) = 0,$$
$$0 \leqslant \eta \leqslant \eta_w.$$

where $\eta_w$ is given by Eq. $(2.11)_2$.

The boundary condition assumes the form

(2.10)
$$F(\eta = 0) = \frac{1}{1+\delta}.$$

The similarity characteristic relationship is employed to express the condition at the wave front (2.4) in the form [2]

$(2.11)_1$
$$F(\eta = \eta_w) = 0,$$

where

$(2.11)_2$
$$\eta_w = \left(\frac{1+q-nq}{1+q}\right)^{\frac{1+q}{2q}} \frac{1}{m} [-F'(\eta_w)]^{\frac{1-q}{2q}},$$

and

$$q > 0, \quad m > 0, \quad n < \frac{1+q}{q}.$$

### 3. Similarity solution of a non-homogeneous and almost nonlinear rod

For an almost nonlinear rod we assume that the parameter $q$ assumes values close to unity such that

$$(-F')^{\frac{1-q}{q}} \simeq 1. \tag{3.1}$$

It is understood in Eq. (3.1) that the slope of the similarity function $F(\eta)$ is not zero and does not tend to infinity at any point $0 \leqslant \eta \leqslant \eta_w$. With the above approximation, the similarity representation given by Eqs. (2.9)–(2.11) assumes the form

$$(3.2)_1 \quad [(1+q-nq)^{\frac{1+q}{q}}\eta-(1+q)^{\frac{1+q}{q}}m^2\eta^3]F''(\eta)+[nq^2(1+q-nq)^{\frac{1}{q}} -(1+q)^{\frac{1+q}{q}}m(m-2\delta-1)\eta^2]F'(\eta)-(1+q)^{\frac{1+q}{q}}\delta(1+\delta)\eta F(\eta) = 0,$$
$$0 \leqslant \eta \leqslant \eta_w,$$

$$(3.2)_2 \quad F(\eta = 0) = \frac{1}{1+\delta},$$

$$(3.2)_3 \quad F(\eta = \eta_w) = 0,$$

where

$$(3.2)_4 \quad \eta_w = \frac{1}{m}\left(\frac{1+q-nq}{1+q}\right)^{\frac{1+q}{2q}}.$$

and

$$q > 0, \quad m > 0, \quad n < \frac{1+q}{q}. \tag{3.3}$$

Equation $(3.2)_1$ is a linear ordinary differential equation of the second order with variable coefficients with the points

$$(3.4)_1 \quad \eta = 0,$$

and

$$(3.4)_2 \quad \eta = \pm\frac{1}{m}\left(\frac{1+q-nq}{1+q}\right)^{\frac{1+q}{2q}},$$

as its regular singular points.

Equation $(3.2)_1$ can be expressed in an operational form as

$$(3.5)_1 \quad LF = 0,$$

where the differential operator $L$ is

$$(3.5)_2 \quad L = [(1+q-nq)^{\frac{1+q}{q}}\eta-(1+q)^{\frac{1+q}{q}}m^2\eta^3]D^2+[nq^2(1+q-nq)^{\frac{1}{q}} -(1+q)^{\frac{1+q}{q}}m(m-2\delta-1)^2]D-(1+q)^{\frac{1+q}{q}}\delta(1+\delta)$$

and $D$ is an ordinary differential operator,

$$D = \frac{d}{d\eta}. \tag{3.5$_3$}$$

It may be noted that the equation represented by Eq. $(3.5)_1$ is a linear ordinary differential equation of the second order with variable coefficients. It has regular singular points at $\eta = 0$ and $\eta = \pm \frac{1}{m}\left(\frac{1+q-nq}{1+q}\right)^{\frac{1+q}{2q}}$. Its solution can be obtained, in principle, by expressing $F(\eta)$ in terms of power series. However, on the basis of the nature of the differential operator $L$ and the presence of a regular singular point at the origin, we can make use of a relevant theorem to solve the problem directly. For convenience, the theorem is stated below in terms of the notation used in this paper. For an original statement of the theorem and its proof the reader is referred to KAPLAN, [7] page 369, theorem 5.

THEOREM. *Consider the differential operator*

$$L = a_0(\eta)D^2 + a_1(\eta)D + a_2(\eta), \tag{3.6$_1$}$$

*where $a_0(\eta)$, $a_1(\eta)$, $a_2(\eta)$ are polynomials without a common factor. Let the differential equation $LF = 0$ have a regular singular point at $\eta = 0$.*

*Let*

$$L(\eta^l) = f(l)\eta^{l+h} + g(l)\eta^{l+h+k} \quad (k > 0), \tag{3.6$_2$}$$

*where $f(l) \not\equiv 0$. Then $h = 0$ or $-1$, $f(l)$ is a polynomial in $l$ of the second degree, and $g(l)$ is a polynomial in $l$ of a degree at the most 2. The substitution*

$$F(\eta) = \eta^l \sum_{s=0}^{\infty} c_s \eta^s, \tag{3.6$_3$}$$

*in the differential equation leads to the indicial equation $f(l) = 0$ and to a two-term recursion formula for the $c_s$. If $l_1$, $l_2$ are the roots of the indicial equation and $(l_1 - l_2)/k$ is not an integer or zero, then two linearly independent solutions $F_1(\eta)$, $F_2(\eta)$ are given by*

$$F_1(\eta) = \phi(\eta, l_1), \quad F_2(\eta) = \phi(\eta, l_2), \tag{3.6$_{4,5}$}$$

where

$$\phi(\eta, l) = \eta^l\left\{1 + \sum_{s=1}^{\infty} (-1)^s \eta^{ks} \frac{g(l)g(l+k)\ldots g[l+(s-1)k]}{f(l+k)f(l+2k)\ldots f(l+sk)}\right\}. \tag{3.6$_6$}$$

*If $g(\eta)$ is of the degree 0 or 1, the solutions are valid for all $\eta$ except perhaps $\eta = 0$. If $g(l)$ is of the degree 2, then the solutions are valid for $0 < |\eta| < a$, where*

$$a = \left|\frac{f_0}{g_0}\right|^{1/k}, \quad f(l) = f_0 l^2 + \ldots, \quad g(l) = g_0 l^2 + \ldots. \tag{3.6$_{7,8,9}$}$$

A comparison of Eqs. (3.5) and $(3.6)_1$ gives the coefficients

$$\begin{aligned} a_0(\eta) &= [(1+q-nq)^{\frac{1+q}{q}}\eta - (1+q)^{\frac{1+q}{q}} m^2\eta^3], \\ a_1(\eta) &= [nq^2(1+q-nq)^{\frac{1}{q}} - (1+q)^{\frac{1+q}{q}} m(m-2\delta-1)\eta^2], \\ a_2(\eta) &= -(1+q)^{\frac{1+q}{q}}\delta(1+\delta)\eta, \end{aligned} \tag{3.7}$$

which do not have a common factor. Moreover, as already mentioned, Eq. (3.5) has a regular singular point at the origin, $\eta = 0$. Thus the above theorem is applicable to the solution of the equation. Now, making use of Eq. $(3.5)_1$ we construct

$$L(\eta^l) = (1+q-nq)^{\frac{1}{q}} l[(1+q-nq)l+(1+q)(nq-1)] \times \eta^{l-1} - (1+q)^{\frac{1+q}{q}} [m^2l^2 - m(2\delta+1)l + \delta(\delta+1)]. \tag{3.8}$$

Setting

$$L(\eta^l) = f(l)\eta^{l+h} + g(l)\eta^{l+h+k}, \tag{3.9}$$

and on the basis of comparison of Eqs. (3.8) and (3.9), we obtain

$$\begin{aligned} f(l) &= (1+q-nq)^{\frac{1}{q}} l[(1+q-nq)l+(1+q)(nq-1)],\\ g(l) &= -(1+q)^{\frac{1+q}{q}} [m^2l^2 - m(2\delta+1)l + \delta(\delta+1)],\\ h &= -1,\\ k &= 2. \end{aligned} \tag{3.10}$$

The roots of the polynomial $f(l)$, Eq. $(3.10)_1$, on the basis

$$f(l) = 0, \tag{$3.11_1$}$$

are obtained in the form

$$l_1 = 0, \tag{$3.11_2$}$$

and

$$l_2 = \frac{1+q-nq-nq^2}{1+q-nq}. \tag{$3.11_3$}$$

From Eqs. $(3.11)_{2,3}$ it may be noted that

$$l_1 \neq l_2 \quad \text{for} \quad n \neq \frac{1}{q}. \tag{$3.11_4$}$$

Thus we can apply the above theorem directly to obtain two linearly independent solutions of Eq. $(3.2)_1$. Accordingly, the first solution $F_1(\eta)$ can be written as

$$F_1(\eta) = 1 + \sum_{s=1}^{\infty} (-1)^s \eta^{2s} \left[\frac{g(0)}{f(2)}\right]\left[\frac{g(2)}{f(4)}\right] \dots \left[\frac{g(2(s-1))}{f(2s)}\right]. \tag{3.12}$$

With the help of Eqs. $(3.10)_{1,2}$ we can write

$$f(2s) = 2(1+q-nq)^{\frac{1}{q}} s[(1+q-nq)(2s-1)+nq^2], \tag{$3.13_1$}$$

and

$$g(2(s-1)) = -(1+q)^{\frac{1+q}{q}} [\delta - 2m(s-1)+1][\delta - 2m(s-1)]. \tag{$3.13_2$}$$

From Eqs. $(3.13)_{1,2}$ we can write

$$(3.14)\qquad \frac{g\big(2(s-1)\big)}{f(2s)} = -\frac{(1+q)^{\frac{1+q}{q}}[\delta-2m(s-1)+1][\delta-2m(s-1)]}{2s(1+q-nq)^{1/q}[(1+q-nq)(2s-1)+nq^2]}.$$

Making use of Eq. (3.14) in Eq. (3.12), the first solution is obtained as

$$(3.15)\qquad F_1(\eta) = 1+\sum_{s=1}^{\infty}\eta^{2s}\frac{(1+q)^{\frac{1+q}{q}s}}{2^s s!(1+q-nq)^{s/q}}\frac{\delta(\delta+1)}{(1+q-nq+nq^2)}$$
$$\frac{(\delta-2m+1)(\delta-2m)}{[3(1+q-nq)+nq^2]}\frac{(\delta-4m+1)(\delta-4m)}{[5(1+q-nq)+nq^2]}\cdots\frac{\big(\delta-2m(s-1)+1\big)\big(\delta-2m(s-1)\big)}{[(2s-1)(1+q-nq)+nq^2]}.$$

The second solution $F_2(\eta)$ assumes the form

$$(3.16)\qquad F_2(\eta) = \eta^{\frac{1+q-nq-nq^2}{1+q-nq}}\left\{1+\sum_{s=1}^{\infty}(-1)^s\eta^{2s}\frac{g\left(\dfrac{1+q-nq-nq^2}{1+q-nq}\right)}{f\left(\dfrac{1+q-nq-nq^2}{1+q-nq}+2\right)}\right.$$
$$\left.\times\frac{g\left(\dfrac{1+q-nq-nq^2}{1+q-nq}+2\right)}{f\left(\dfrac{1+q-nq-nq^2}{1+q-nq}+4\right)}\frac{g\left(\dfrac{1+q-nq-nq^2}{1+q-nq}+2(s-1)\right)}{f\left(\dfrac{1+q-nq-nq^2}{1+q-nq}+2s\right)}\right\},$$

where, as before,

$$(3.17)\qquad \frac{g\left(\dfrac{1+q-nq-nq^2}{1+q-nq}+2(s-1)\right)}{f\left(\dfrac{1+q-nq-nq^2}{1+q-nq}+2s\right)} = -\frac{(1+q)^{\frac{1+q}{q}}}{2s(1+q-nq)^{1/q}}$$
$$\times\frac{\left(\delta-2m(s-1)+\dfrac{mnq^2}{1+q-nq}+1-m\right)\left(\delta-m(2s-1)+\dfrac{mnq^2}{1+q-nq}\right)}{[(2s+1)(1+q-nq)-nq^2]}.$$

Substituting Eq. (3.17) into Eq. (3.16) and simplifying, the second solution can be written as

$$(3.18)\qquad F_2(\eta) = \eta^{\frac{1+q-nq-nq^2}{1+q-nq}}\left\{1+\sum_{s=1}^{\infty}\eta^{2s}\frac{(1+q)^{\frac{1+q}{q}s}}{2^s s!(1+q-nq)^{s/q}}\right.$$
$$\times\frac{\left(\delta+\dfrac{mnq^2}{1+q-nq}+1-m\right)\left(\delta-m+\dfrac{mnq^2}{1+q-nq}\right)}{[3(1+q-nq)-nq^2]}$$
$$\times\frac{\left(\delta-3m+\dfrac{mnq^2}{1+q-nq}+1\right)\left(\delta-3m+\dfrac{mnq^2}{1+q-nq}\right)}{[5(1+q-nq)-nq^2]}$$
$$\cdots\frac{\left(\delta-2m(s-1)+\dfrac{mnq^2}{1+q-nq}+1-m\right)\left(\delta-m(2s-1)+\dfrac{mnq^2}{1+q-nq}\right)}{[(2s+1)(1+q-nq)-nq^2]}.$$

It follows from the stated theorem that the solutions $F_1(\eta)$ and $F_2(\eta)$ are convergent for

$$(3.19)_1 \qquad 0 < |\eta| < a,$$

where

$$(3.19)_2 \qquad a = \left|\frac{f_0}{g_0}\right|^{1/k} = \left|\frac{(1+q-nq)^{\frac{1+q}{q}}}{-(1+q)^{\frac{1+q}{q}} m^2}\right|^{1/2} = \left(\frac{1+q-nq}{1+q}\right)^{\frac{1+q}{2q}} \frac{1}{m}.$$

As in this problem $\eta \geqslant 0$, the series are convergent for

$$(3.20) \qquad 0 < \eta < \left(\frac{1+q-nq}{1+q}\right)^{\frac{1+q}{2q}} \frac{1}{m}.$$

Now we investigate the convergence of the similarity functions $F_1(\eta)$ and $F_2(\eta)$ at the boundary points. It is readily seen that $F_1(\eta)$ is convergent at the point $\eta = 0$. However, at the other point

$$(3.21) \qquad \eta = \eta_w = \frac{1}{m}\left(\frac{1+q-nq}{1+q}\right)^{\frac{1+q}{2q}},$$

its convergence needs to be investigated and also that of $F_2(\eta)$ at both the end points. The well-known d'Alembert's criteria of convergence cannot be used here as the limit of the ratio of two consecutive terms of the series tends to unity. Cauchy's criteria is also inapplicable as, in this case, the root of the ratio of two consecutive terms equals unity in the limit. Applying Rabbe's criteria of convergence [8], we obtain at the point $\eta_w$

$$(3.22) \qquad \overset{(1)}{\lim_{s\to\infty}}\left(s\left(\frac{a_{s+1}}{a_s}-1\right)+2\right) = \frac{(1+q-nq)(m-1-2\delta)-mnq^2}{2m(1+q-nq)},$$

for the series of $F_1(\eta)$, Eq. (3.15), and

$$(3.23)_1 \qquad \overset{(2)}{\lim_{s\to\infty}}\left(s\left(\frac{a_{s+1}}{a_s}-1\right)+2\right) = -\frac{(2\gamma+1+5m)(1+q-nq)-mnq^2}{2m(1+q-nq)+2}+2,$$

for $F_2(\eta)$, Eq. (3.18). In the above equations $a_s$ and $a_{s+1}$ are two consecutive terms of the series under consideration and

$$(3.23)_2 \qquad \gamma \equiv \delta + \frac{mnq^2}{1+q-nq} - m.$$

For convergence, setting both the limits less than unity, the following inequalities are obtained at the point $\eta_w$:

$$(3.24) \qquad (1+q-nq)(1+2\delta+m)+mnq^2 > 0,$$

and

$$(3.25) \qquad 1+q-nq > 0, \quad m > 0, \quad q > 0, \quad n \neq \frac{1}{q}.$$

The series for $F_2(\eta)$ converges at the origin $\eta = 0$ if

$$\frac{1+q-nq-nq^2}{1+q-nq} > 0. \tag{3.26}$$

Making use of Eq. $(2.8)_2$ in Eq. (3.24) and combining Eqs. (3.25) and (3.26), we obtain the following inequalities:

$$\begin{gathered}(1+q-nq)(1+2\delta)+\left(1+\delta\frac{1-q}{1+q}\right)(1+q-nq+nq^2) > 0,\\ n < \frac{1}{q}, \quad 1+\delta\frac{1-q}{1+q} > 0, \quad q > 0.\end{gathered} \tag{3.27}$$

The general solution of Eq. $(3.2)_1$ in terms of two linearly independent functions $F_1(\eta)$ and $F_2(\eta)$ can be written as

$$F(\eta) = C_1F_1(\eta)+C_2F_2(\eta), \tag{3.28}$$

where the functions $F_1(\eta)$ and $F_2(\eta)$ are given by Eqs. (3.16) and (3.18), respectively. $C_1$ and $C_2$ are constants to be determined from the boundary conditions. Making use of the boundary condition $(3.2)_2$, the value of $C_1$ is obtained as

$$C_1 = \frac{1}{1+\delta}. \tag{3.29}$$

On the basis of the boundary condition $(3.2)_3$, we obtain

$$C_2 = -\frac{F_1(\eta_w)}{(1+\delta)F_2(\eta_w)}, \tag{3.30}$$

where $\eta_w$ is qiven by Eq. $(3.2)_4$. Thus, on the basis of Eqs. (3.29) and (3.30), the solution (3.23) can be written as

$$F(\eta) = \frac{1}{1+\delta}\left[F_1(\eta)-\frac{F_1(\eta_w)}{F_2(\eta_w)}F_2(\eta)\right], \tag{3.31}$$

for

$$0 \leqslant \eta \leqslant \frac{1}{m}\left(\frac{1+q-nq}{1+q}\right)^{\frac{1+q}{2q}}, \tag{3.32}$$

and under the condition that the parameters $n$, $q$ and $\delta$ must satisfy the inequalities (3.27). Furthermore, it may be remembered that the solution holds for the values of $q$ close to unity.

### 3.1. Special cases

*Case 1.* Nonhomogeneous linear elastic rod

$$q = 1, \quad n < 1.$$

On the basis of the expressions (3.15) and (3.18), the values of $F_1(\eta)$ and $F_2(\eta)$ assume the following form in this case [9]:

$$(3.33)_1 \quad F_1(\eta) = 1+\sum_{s=1}^{\infty}\eta^{2s}\frac{1}{(2-n)^s s!} \times \frac{(\delta+1)\,\delta(\delta-1)(\delta-2)\,\ldots\,[\delta-(2s-3)][\delta-2(s-1)]}{(3-n)(5-2n)\,\ldots\,[(2-n)s+n-1]},$$

and

$$(3.33)_2 \quad F_2(\eta) = \eta^{\frac{2-2n}{2-n}}\left\{1+\sum_{s=1}^{\infty}\eta^{2s}\frac{1}{(2-n)^s s!}\frac{\left(\delta+\frac{n}{2-n}\right)\left(\delta-1+\frac{n}{2-n}\right)}{(3-2n)} \ldots \frac{\left[\delta-2(s-1)+\frac{n}{2-n}\right]\left[\delta-(2s-1)+\frac{n}{2-n}\right]}{[(2-n)s+1-n]}\right\}.$$

Equation (3.31) remains valid for $0 \leqslant \eta \leqslant \frac{2-n}{2}$. The above series $(3.33)_1$ and $(3.33)_2$ are convergent when the parameters satisfy the inequalities

$$(3.34)_1 \qquad (n-4)-2(2-n)\delta < 0$$

and

$$(3.34)_2 \qquad n < 1.$$

*Case 2.* Homogeneous linear rod subjected to time-dependent velocity impact

$$q = 1, \quad n = 0, \quad \delta \neq 0.$$

Functions $F_1(\eta)$ and $F_2(\eta)$ assume the simplified forms as [9]

$$(3.35) \qquad \begin{aligned} F_1(\eta)|_{n=0} &= \frac{(1+\eta)^{1+\delta}+(1-\eta)^{1+\delta}}{2}, \\ F_2(\eta)|_{n=0} &= \frac{(1+\eta)^{1+\delta}-(1-\eta)^{1+\delta}}{2(1+\delta)}, \end{aligned}$$

giving

$$(3.36) \qquad F(\eta) = \frac{1}{1+\delta}(1-\eta)^{1+\delta},$$

for $0 \leqslant \eta \leqslant 1$, $\delta > -1$.

## 4. Displacement and stress distribution in the rod

The displacement $u(x, t)$ at a point is given by Eq. (2.6). The stress at a point in the rod is given on the basis of Eqs. (2.1) as

$$(4.1) \qquad \sigma(x, t) = E(x)e^{1/q} = E_0 x^n\left(-\frac{\partial u}{\partial x}\right)^{1/q}.$$

The displacement gradient is obtained from Eqs. (2.6) and (2.7) and substituted in Eq. (4.1) to obtain the stress distribution as

$$\sigma(x,t) = \left(\frac{1+q-nq}{1+q}\right)^{1/q} V_c^{1/q} K^{\frac{1+q}{(1+q-nq)q}} t^{(\delta+1)\frac{1}{q}-\frac{m(1+\delta)}{(1+q-nq)q}} \times \eta^{-\frac{n}{1+q-nq}} E_0 x^n(-F'(\eta))^{1/q}. \tag{4.2}$$

Making use of the similarity transformation, relation (2.7), we obtain

$$x^n = \left(\eta\frac{t^m}{K}\right)^{\frac{(1+q)n}{1+q-nq}}. \tag{4.3}$$

Substituting (4.3) in (4.2), we obtain the stress in terms of $\eta$ and $t$ as

$$\sigma(x,t) = E_0 V_c^{1/q} K^{\frac{1+q-nq-nq^2}{(1+q-nq)q}} \left(\frac{1+q-nq}{1+q}\right)^{1/q} \times t^{(\delta+1)\frac{1}{q}+\frac{m(1+q)(nq-1)}{(1+q-nq)q}} \eta^{\frac{nq}{1+q-nq}}(-F'(\eta))^{1/q}, \tag{4.4}$$

where

$$F'(\eta) = \frac{1}{1+\delta}\left[F_1'(\eta) - \frac{F_1(\eta_w)}{F_2(\eta_w)}F_2'(\eta)\right]. \tag{4.5}$$

The derivatives in Eq. (4.5) are given by

$$F_1'(\eta) = \sum_{s=1}^{\infty} 2s\eta^{2s-1} \frac{(1+q)^{\frac{1+q}{q}s}}{2^s s!(1+q-nq)^{s/q}} \frac{\delta(\delta+1)}{[(1+q-nq)+nq^2]} \times \frac{(\delta-2m+1)(\delta-2m)}{[3(1+q-nq)+nq^2]} \frac{(\delta-4m+1)(\delta-4m)}{[5(1+q-nq)+nq^2]} \cdots \frac{(\delta-2m(s-1)+1)(\delta-2m(s-1))}{[(2s-1)(1+q-nq)+nq^2]}, \tag{4.6}$$

and

$$F_1'(\eta) = \eta^{-\frac{nq^2}{1+q-nq}}\left\{\frac{1+q-nq-nq^2}{1+q-nq} + \sum_{s=1}^{\infty}\left(2s+\frac{1+q-nq-nq^2}{1+q-nq}\right) \times \eta^{2s}\frac{(1+q)^{\frac{1+q}{q}s}}{2^s s!(1+q-nq)^{s/q}} \frac{\left(\delta+\frac{mnq^2}{1+q-nq}+1-m\right)\left(\delta-m+\frac{mnq^2}{1+q-nq}\right)}{[3(1+q-nq)-nq^2]} \times \frac{\left(\delta-2m(s-1)+\frac{mnq^2}{1+q-nq}+1-m\right)\left(\delta-m(2s-1)+\frac{mnq^2}{1+q-nq}\right)}{[(2s+1)(1+q-nq)-nq^2]}\right\}. \tag{4.7}$$

It may also be noted that in Eq. (4.4)

$$\lim_{\eta\to 0^+}\{\eta^{\frac{nq}{1+q-nq}}(-F'(\eta))^{1/q}\} = \frac{1}{1+\delta}\frac{(1+q-nq-nq^2)}{(1+q-nq}\frac{F_1(\eta_w)}{F_2(\eta_w)}. \tag{4.8}$$

We can express the above relations in nondimensional form for convenience in the evaluation of numerical results. For this purpose we set

$$\bar{x} = \frac{x}{x_0} \quad \text{and} \quad \bar{t} = \frac{t}{t_0}, \tag{4.9}$$

where $\bar{x}$ and $\bar{t}$ are dimensionless, $x_0$ and $t_0$ have the same dimension as $x$ and $t$ respectively, otherwise they have nonzero but arbitrary magnitudes. On this basis we obtain the following nondimensional expressions:

$$\bar{u} = \frac{u(\tilde{x}, \bar{t})}{V_c t_0^{\delta+1}} = \bar{t}^{\delta+1} F(\eta), \tag{4.10}$$

$$\bar{\sigma} = -\frac{\sigma(\tilde{x}, \bar{t})}{E_0 V_0^{1/q} K^{\frac{(1+q)(1-nq)}{1+q-nq}} t_0^{\left[\frac{(\delta+1)}{q} + \frac{m(1+q)(nq-1)}{(1+q-nq)}\right]}}, \tag{4.11}$$

where

$$\tilde{x} = \bar{x} \frac{x_0}{\left(\frac{t_0^m}{K}\right)^{\frac{1+q}{1+q-nq}}} = (\eta \bar{t}^m)^{\frac{1+q}{1+q-nq}}. \tag{4.12}$$

On the basis of Eq. (4.4), (4.5), (4.8) and (4.12) the nondimensional stress distribution assumes the form

$$\bar{\sigma} = \left[\frac{1}{1+\delta} \bar{t}^{\delta+1-m+\frac{mnq^2}{1+q-nq}} \left(\frac{1+q-nq-nq^2}{1+q}\right) \frac{F_1(\eta_w)}{F_2(\eta_w)}\right]^{1/q}, \quad \eta = 0, \tag{4.13}$$

$$= \left[-\frac{(1+q-nq)}{(1+q)} \bar{t}^{\left(\delta+1-m\frac{mnq^2}{1+q-nq}\right)} \eta^{\frac{nq^2}{1+q-nq}} F'(\eta)\right]^{1/q}, \quad \eta \neq 0.$$

On the basis of Eqs. (4.13) and (4.5) the results can be numerically evaluated.

In the case of a nonhomogeneous linear rod, the stress distribution assumes the form

$$\bar{\sigma} = \begin{cases} \dfrac{1}{1+\delta} \bar{t}^{\delta+\frac{n}{2-n}} (1-n) \dfrac{F_1(\eta_w)}{F_2(\eta_w)}, & \eta = 0, \\ -\bar{t}^{\delta+\frac{n}{2-n}} \dfrac{2-n}{2} \eta^{\frac{n}{2-n}} F'(\eta), & \eta \neq 0. \end{cases} \tag{4.14}$$

When, in the above case, Eq. (4.14), the velocity impact is constant, $\delta = 0$, the stress distribution becomes

$$\bar{\sigma} = \bar{t}^{\frac{n}{2-n}} (1-n) \frac{F_1(\eta_w)}{F_2(\eta_w)} \left[1 - \left(\frac{2}{2-n}\eta\right)^2\right]^{\frac{n}{2(2-n)}}. \tag{4.15}$$

In the relation (4.13) there is no discontinuity for $\bar{\sigma}$ at the point $\eta_w = \frac{2-n}{2}$. However, in the neighbourhood of the point $\eta_w = \frac{2-n}{2}$ the derivative of $\bar{\sigma}(\bar{t}_{\text{fix}}, \eta)$ for $\eta$ tending to $\eta_w$ from the left side is

$$(4.16)_1 \qquad \bar{\sigma}'(\bar{t}_{\rm fix}, \eta) = -(\bar{t}_{\rm fix})^{\frac{n}{2-n}} \frac{4n(1-n)\eta}{(2-n)^3 F_2(\eta_w)} \left[1-\left(\frac{2}{2-n}\eta\right)^2\right]^{\frac{3n-4}{2(2-n)}},$$

and

$$(4.16)_2 \qquad \lim_{\eta \to \left(\frac{2-n}{2}\right)^-} \bar{\sigma}'(\bar{t}_{\rm fix}, \eta) = -\infty, \quad \text{for} \quad 0 < n < 1.$$

For $n = 0$ the limit equals zero. Equation $(4.16)_1$ and the property $(4.16)_2$ imply that for $0 < n < 1$ the curves in Fig. 9 sharply tend to zero in the neighbourhood of the point $\eta_w = \dfrac{2-n}{2}$ and they are perpendicular to the $x$-axis at point $\tilde{x}_w = (\eta_w \bar{t}_{\rm fix})^{\frac{2}{2-n}}$.

## 5. Numerical results

On the basis of the results obtained in the previous section, numerical evaluation of the similarity function $F(\eta)$ and the nondimensional stress $\sigma$ has been carried out.

Equations (3.26) to (3.28) give the general solution of the similarity representation under the restrictions on the parameters imposed by Eqs. $(3.3)_{1,2}$, (3.24) and (3.25). In the evaluation of numerical results the values of $n$, $\delta$ and $q$ are assumed to be non--negative, for convenience.

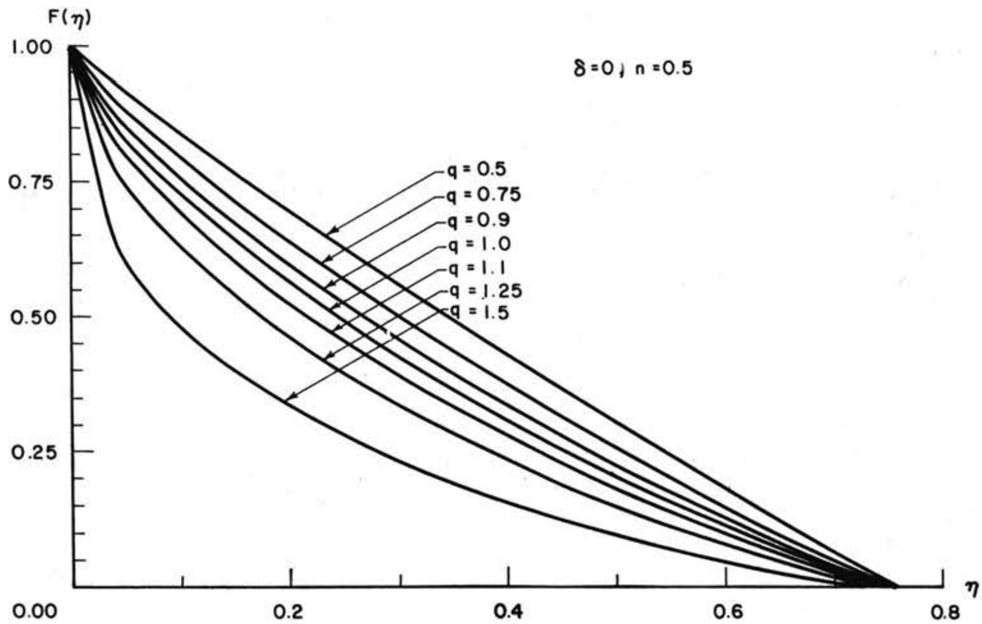


FIG. 1

In Fig. 1 the similarity function $F(\eta)$ is plotted against $\eta$ for varying $q$ and fixed $\delta = 0$, and $n = 0.5$. For $q = 1$, the results hold for a linear nonhomogeneous rod and agree with those previously obtained for this case [9]. At $\eta = 0$, $F(\eta)$ is independent of $n$ and $q$ and depends only on $\delta$. Thus, for $\delta = 0$ the value $F(\eta = 0)$ remains fixed at unity. The value of $F(\eta = \eta_w) = 0$ as required by the continuity condition at the wave front, however, $\eta_w$ itself being dependent on the parameters $n$, $\delta$ and $q$, varies with $q$ even though $n$ and

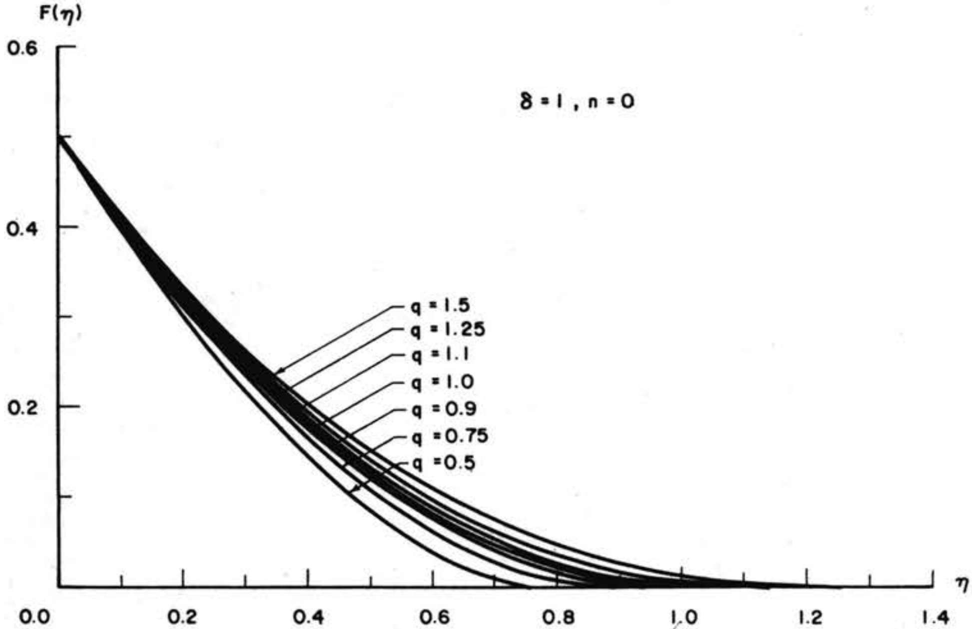


FIG. 2.

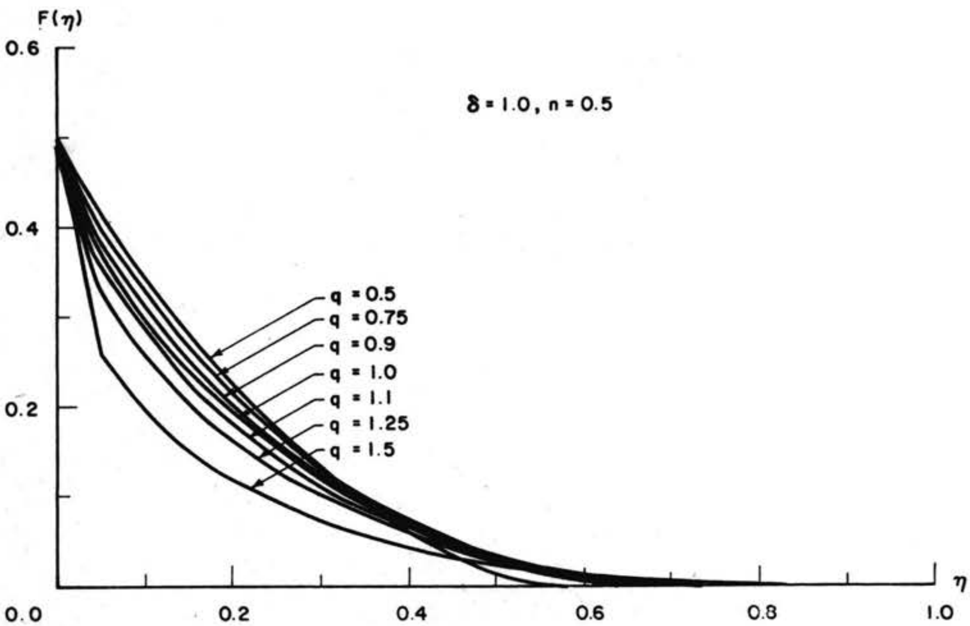


FIG. 3.

$\delta$ are fixed. There is gradual change in the curves $F(\eta)$ with the gradual change in the values of $q$ around $q = 1$ as can be seen in Fig. 1.

Figure 2 shows the variation of the similarity function $F(\eta)$ for a homogeneous, $n = 0$, nonlinear rod subjected to time dependent velocity impact, $\delta = 1$. Here the variation of $F(\eta)$ for varying $q$ around $q = 1$ is not only gradual but also small as can be compared

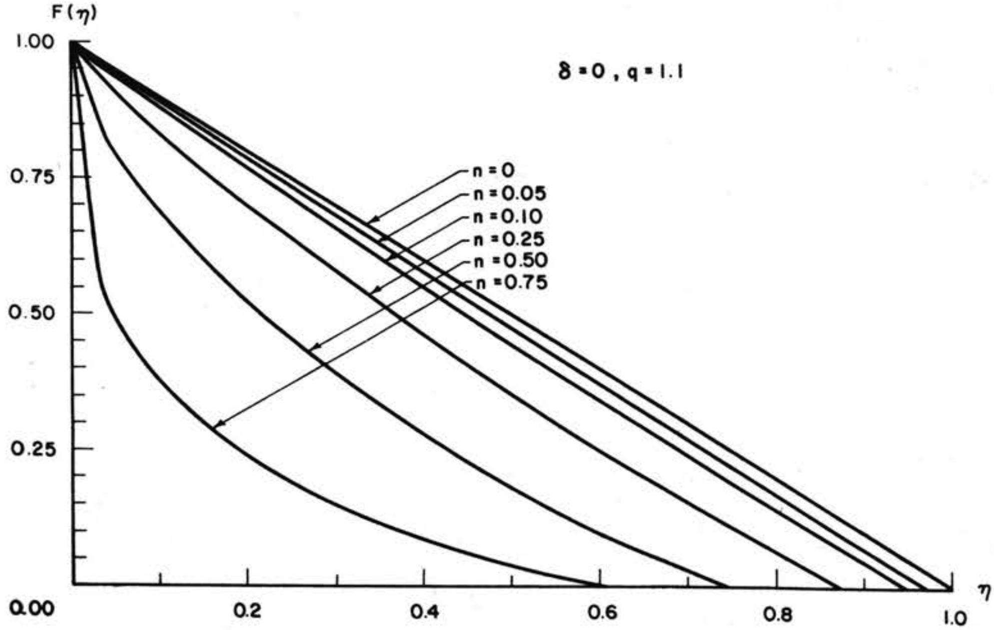


FIG. 4.

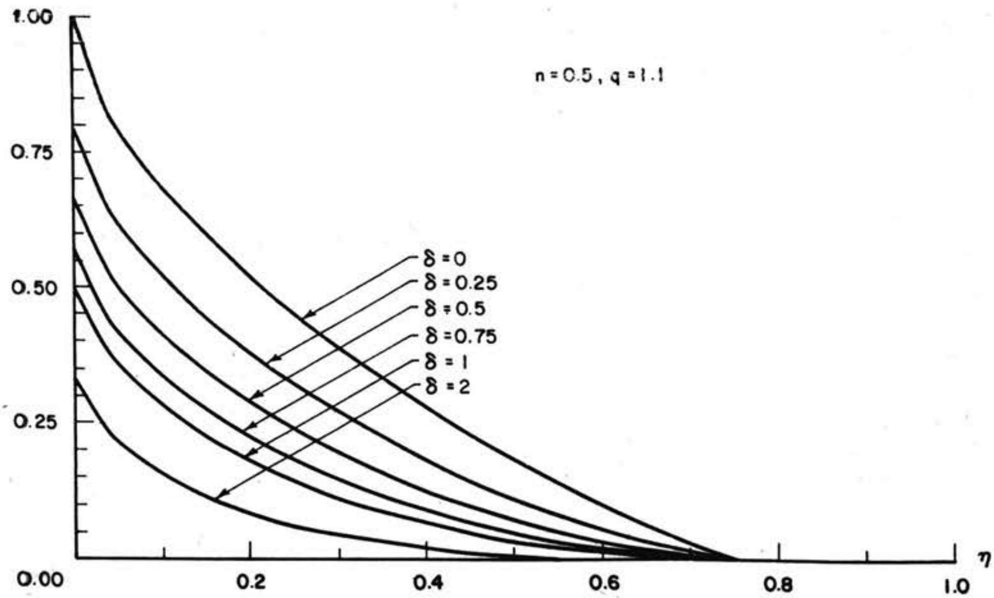


FIG. 5.

for the three curves $q = 0.9$, 1.0 and 1.1, respectively. Also, it may be observed in comparing Figs. 1 and 2 that whereas in Fig. 1, for a fixed $\eta$, $F(\eta)$ decreases with increasing $q$, in Fig. 2 it is just the reverse, i.e. $F(\eta)$ increases with increasing $q$.

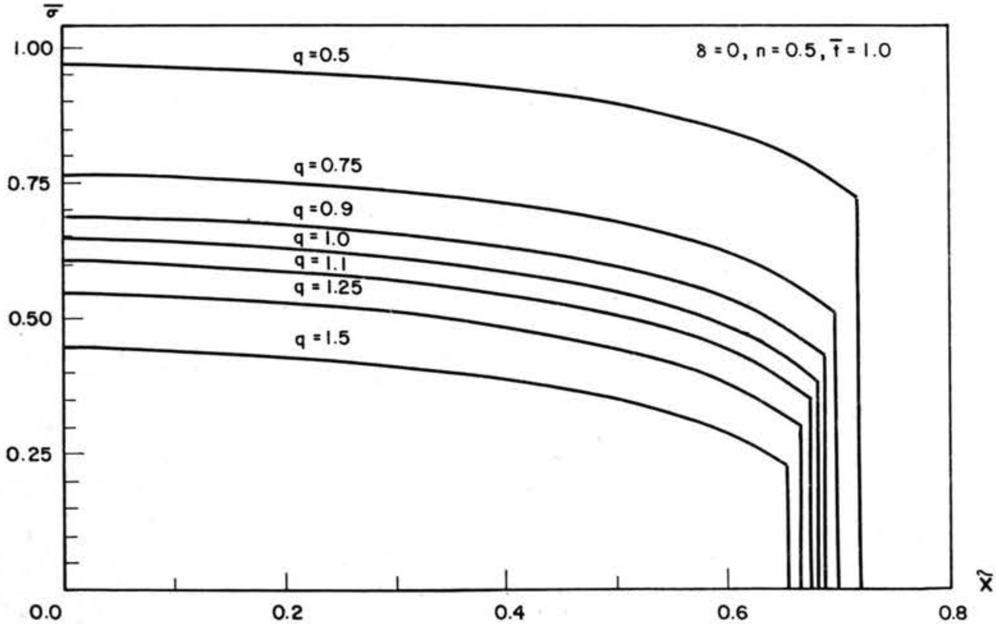


FIG. 6.

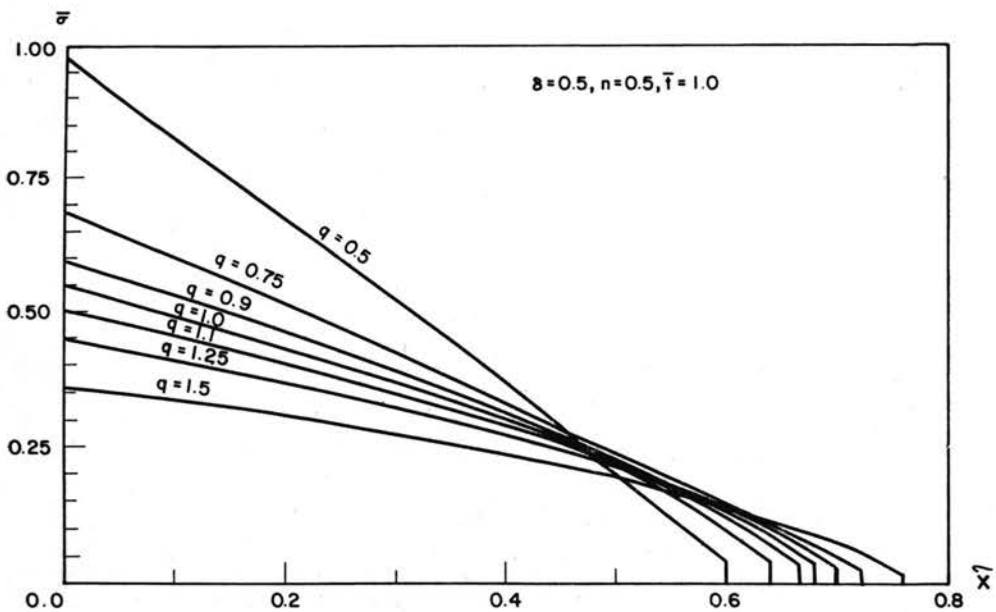


FIG. 7.

Figure 3 presents the variation of $F(\eta)$ against $\eta$ for a nonhomogeneous, $n = 0.5$, time-dependent velocity impact, $\delta = 1.0$ and varying $q$. For values of $\eta$ less than 0.4, approximately, the effect of nonhomogeneity dominates in the sense that variation of

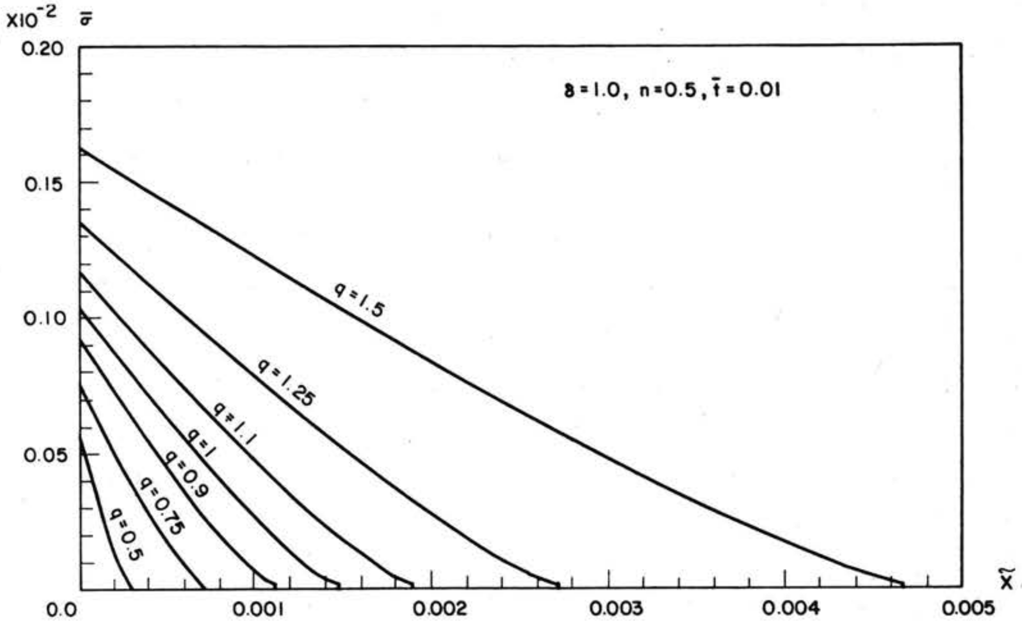


FIG. 8.

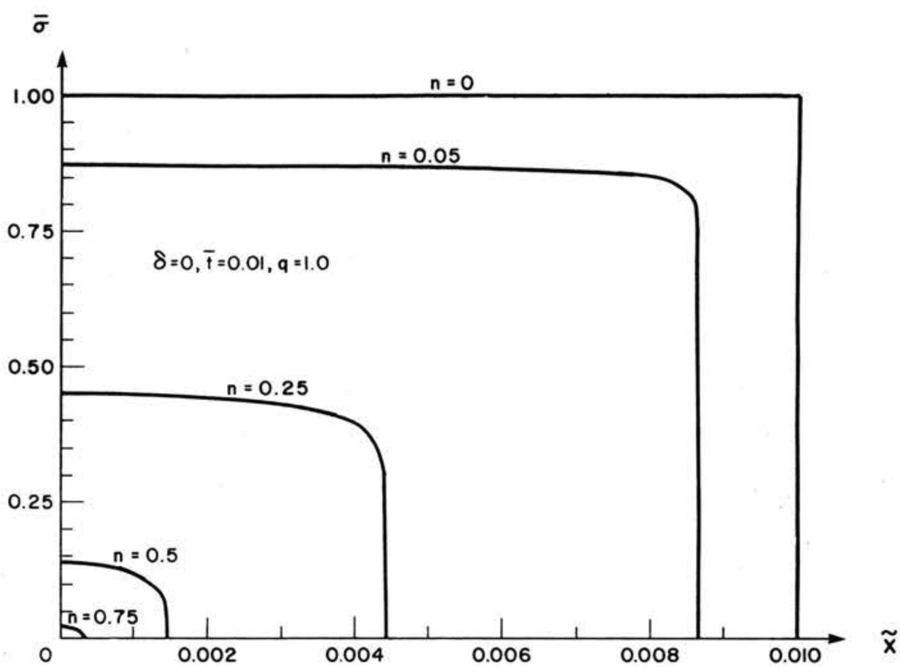


FIG. 9.

$F(\eta)$ is in accordance with Fig. 1, whereas for values of $\eta$ greater than that of about 0.4 its variation is like that in Fig. 2, that is, the effect of $\delta$ dominates. Figure 4 shows the variation of the similarity function against the similarity variable for $\delta = 0$, constant $q = 1.1$ and varying $n$. The results are in harmony with those for a linear rod having the same values of $\delta$ and $n$ but $q = 1$, [9].

Finally, in Fig. 5 the variation of $F(\eta)$ is shown for fixed $n = 0.5$ and $q = 1.1$. For varying $\delta$, in this case again, the behaviour is in accordance with that expected when compared with the results for $n = 0.5$ and $q = 1$ [9]. The last three figures, 6, 7, and 8 show the variation of nondimensional stress $\bar{\sigma}$ against the nondimensional coordinate $\tilde{x}$. Figure 9 presents the stress distribution for a nonhomogeneous linear rod subjected to time-independent velocity impact.

In conclusion, similarity transforms and similarity-characteristic relations have been applied to obtain a nonlinear similarity representation for a nonhomogeneous nonlinaer semi-infinite rod. Assuming the parameter of nonlinearity of the problem $q$ close to unity, the similarity representation is rendered linear and its solution is obtained in the form of power series. For the values of $q = 1$, the series gives a solution for the nonhomogeneous linear elastic rod. When the nonlinear representation is solved numerically, it agrees with the series solution for values of $q$ close to unity. Restrictions on the physical and impact parameters have been obtained in the form of an inequality on the basis of the criterion of convergence of the power series of the solution.

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*Received September* 8, 1981.