Arch. Mech., 34, 3, pp. 243–252, Warszawa 1982

On baroclinic instability in the case of vanishing viscosity

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IN THE TWO-LAYER model, describing baroclinic instability, viscosity plays a subtle role. The marginal stability curves for the non-viscous model and the one for the viscous model with viscosity tending to zero, do not coincide. An explanation in terms of polynomials with complex coefficients is given and the growth-rate for decreasing viscosity is calculated.

W modelu dwuwarstwowym opisującym niestateczność barokliniczną, lepkość odgrywa subtelną rolę. Krzywe stateczności dla modelu nielepkiego oraz modelu z zanikającą lepkością nie pokrywają się ze sobą. Przedstawiono uzasadnienie tego zjawiska posługując się wielomianami o współczynnikach zespolonych i obliczono prędkość wzrostu fali przy malejącej lepkości.

В двуслойной модели, описывающей бароклинную неустойчивость, вязкость играет деликатную роль. Кривая устойчивости невязкой модели и модели с исчезающей вязкостью не покрываются между собой. Представлено доказательство этого явления с помощью многочленов с комплексными коэффицентами, и расчитана скорость роста волны при уменьшающейся вязкости.

1. Introduction

THE PROBLEM of baroclinic instability in the earth's atmosphere has been studied extensively in the past, the theoretical work being often based on the well-known two-layer model introduced by PHILIPS [8]. It has been noticed that viscosity plays a subtle role. The marginal stability curve computed in a nonviscous model does not coincide with the limit (as viscosity tends to zero) of the marginal stability curve in a model that takes viscosity into account. Some attention has been paid to this phenomenon (NEWELL [5], ROMEA [9], HART [3]), giving thus a "physical" explanation.

The purpose of this paper is to present a mathematical explanation of the phenomenon. It turns out that the mathematical mechanism which produces the phenomenon is very simple. We start the paper by collecting the relevant information on the mathematical model describing baroclinic instability. In Sect. 2 the model is introduced and part of the linear stability analysis is given. In Sect. 3 we describe the singular behaviour due to viscosity, then, in Sect. 4, a mathematical explanation is given and the linear growth rate of the unstable wave is calculated.

2. The two-layer model

A model which has played an important role in linear and nonlinear stability analysis in geophysical fluid dynamics is the two-layer model introduced by PHILIPS [8]. It has been used by Pedlosky in a series of papers and later by many others. Experimental and theoretical work closely related to it was carried out by HART [1, 2].

The model is essentially a two-layer approximation of the atmosphere, in a closed rectangular ring around the earth, at middle lattitudes. The ring has north-south width L and depth D. The two-layers are both incompressible and have slightly different densities, with the heaviest fluid at the bottom. It is assumed that viscosity can be described by a single parameter ν for the two layers.

It is possible to derive by a rational procedure, i.e. through systematic scaling, expansion in small parameters and matching the interior flow with the boundary layers, the equations governing the fluid motion in the two-layer model, from the Navier-Stokes equations in spherical coordinates. In the process the following parameters appear:

(2.1)

ν	kinematic viscosity,
f_0, β	see Eq. (2.3),
U	typical horizontal velocity,
$\varepsilon = U/(f_0 L)$	Rossby-number,
$E = 2\nu/(f_0 D^2)$	Ekman-number,
$F = f_0^2 L^2/\{(\Delta\varrho/\varrho_2)g(D/2)\}$	stratification parameter.

The underlying assumptions are

$$E^{1/2} < \varepsilon \ll 1,\quad \Delta\varrho,\ \delta = D/L \ll 1,\quad F, \beta \sim 0(1). \tag{2.2}$$

Thus the effects of viscosity are limited to boundary layers at the top and bottom of the ring, the aspect ratio δ is small and the ratio ε of the inertial and the Coriolis force is small. We shall not reproduce the details of the analysis here. The result is identical to the set of equations PEDLOSKY [6] derived for the infinite channel, rotating with angular velocity Ω (Fig. 1).

$$\Omega = f_0 + \beta y, \tag{2.3}$$

i.e.

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial \psi_1}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi_1}{\partial x}\frac{\partial}{\partial y}\right)(\nabla^2\psi_1 + F(\psi_2 - \psi_1) + \beta y) &= -\frac{E^{1/2}}{\varepsilon}\nabla^2\psi_1,\\ \left(\frac{\partial}{\partial t} - \frac{\partial \psi_2}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi_2}{\partial x}\frac{\partial}{\partial y}\right)(\nabla^2\psi_2 - F(\psi_2 - \psi_1) + \beta y) &= -\frac{E^{1/2}}{\varepsilon}\nabla^2\psi_2.\end{aligned} \tag{2.4}$$

The variation of Ω with the north-south coordinate y, which is called the β-plane approximation, serves to bring into Eqs. (2.4) the term βy, which in our derivation is the only term resulting from working with spherical coordinates. Therefore β may be said to account for the earth's sphericity in the rectangular model.

The ψ_i $(i = 1, 2)$ are the (pressure-) stream functions in each layer. To zeroth order in ε, the motion in each layer is geostrophic and hydrostatic $\left(\text{meaning: } u_i = -\frac{\partial \psi_i}{\partial y},\right.$ $\left. v_i = \frac{\partial \psi_i}{\partial x}, \frac{\partial \psi_i}{\partial z} = 0,\ u_i \text{ velocity in the } x\text{-direction, } v_i \text{ in the } y\text{-direction}\right)$.

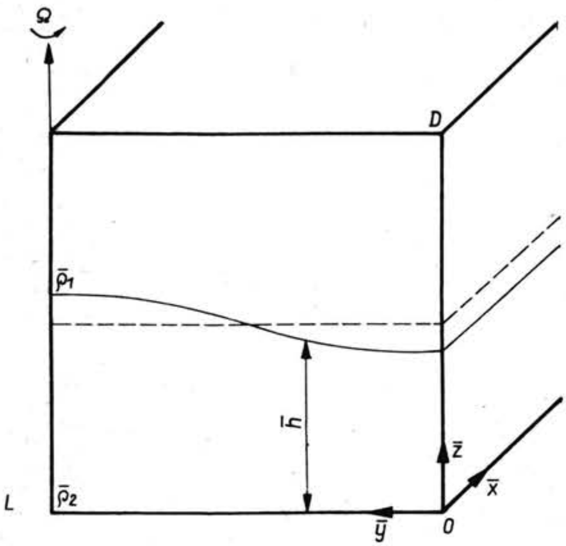

FIG. 1. The two-layer model, with dimensional coordinates $-\infty < \bar{x} < \infty$, $0 \leqslant \bar{y} \leqslant L$, $0 \leqslant \bar{z} \leqslant \bar{D}$ and interface $\bar{h}$. $\bar{\varrho}_1 < \bar{\varrho}_2$.

The boundary conditions used are

(2.5)
$$\frac{\partial \psi_i}{\partial x} = 0,$$
$$\lim_{x\to\infty} \frac{1}{x} \int_{-x}^{x} \frac{\partial^2 \psi_i}{\partial y\, \partial t}\, dx = 0, \quad y = 0, 1, \quad i = 1, 2,$$

i.e. to zeroth and first order in ε, there is no transport of fluid across the north and south boundaries of the channel. SMITH [10, 11] has noted that these boundary conditions introduce an artificial energy source in the model but that this does not influence the results in PEDLOSKY [7], qualitatively.

The main advantages in using the two-layer model are that it leaves the study of the nonlinear developement of instabilities tractable, while on the other hand there is a good correspondence with experimental work. The model is closely related to, for example, the cylindrical configurations used by HART [1], and his results have influenced theoretical developments.

We conclude this section by giving the first part of the linear stability analysis. We are interested in the behaviour of small perturbations ϕ_i on a basic zonal flow $\psi_{B,i}$, where

(2.6)
$$\psi_{B,i} = -U_i y, \quad i = 1, 2.$$

So the total stream function is

(2.7)
$$\psi_i = \psi_{B,i} + \phi_i, \quad i = 1, 2.$$

Initially the two layers move with different velocities U_1 and U_2 along the channel. $V = U_1 - U_2$ is called the shear.

Substituting Eq. (2.7) in Eqs. $(2.4)_1$ and $(2.4)_2$ and subsequently linearizing the result gives

$$\tag{2.8} \begin{aligned} &\left[\frac{\partial}{\partial t}(\nabla^2-F)+(\beta-FV)\frac{\partial}{\partial x}+r\nabla^2\right]\phi_1+F\frac{\partial}{\partial t}\phi_2=0,\\ &\left[\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)(\nabla^2-F)+(\beta+FV)\frac{\partial}{\partial x}+r\nabla^2\right]\phi_2+F\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)\phi_1=0, \end{aligned}$$

where $r = E^{1/2}/\varepsilon$.

If we choose ϕ_i of the form

$$\tag{2.9} \begin{aligned} \phi_1 &= Ae^{i(k_x x-\omega t)}\sin m\pi y,\\ \phi_2 &= \gamma e^{i(k_x x-\omega t)}\sin m\pi y, \qquad m\in\mathcal{N} \end{aligned}$$

and substitute this in Eq. (2.8), we obtain the following dispersion relation:

$$\tag{2.10} \begin{aligned} -k^2(k^2+2F)\frac{\omega^2}{k_x^2}+\left\{k^2(k^2+2F)V-(k^2+F)2\beta-2\frac{rk^2}{k_x}(k^2+F)i\right\}\frac{\omega}{k_x}\\ +(k^2+F)V\beta-FV^2k^2-\beta^2+\frac{r^2k^4}{k_x^2}+i\frac{rk^2}{k_x}\{(k^2+F)V-2\beta\}=0 \end{aligned}$$

with

$$k^2 = k_x^2+m^2.$$

Equation (2.10) is a second-degree polynomial in ω/k_x, with complex coefficients, so its roots will be complex, but in general not complex conjugate. The roots are

$$\tag{2.11} \frac{\omega_{1,2}}{k_x}=\frac{V}{2}-\frac{k^2+F}{k^2+2F}\left(\frac{\beta}{k^2}+\frac{ir}{k_x}\right)\pm\frac{\left[V^2k^4(k^4-4F^2)+4F^2\left(\beta+\frac{irk^2}{k_x^2}\right)^2\right]^{\frac{1}{2}}}{2k^2(k^2+2F)}$$

with imaginary parts:

$$\tag{2.12} \operatorname{Im}\frac{\omega_{1,2}}{k_x}=-\frac{k^2+F}{k^2+2F}\frac{r}{k_x}\pm\frac{\operatorname{Im}\left[V^2k^4(k^4-4F^2)+4F^2\left(\beta+\frac{irk^2}{k_x^2}\right)^2\right]^{\frac{1}{2}}}{2k^2(k^2+2F)}.$$

If one of the imaginary parts is greater than zero, the corresponding perturbation will grow exponentially with a growth-rate determined by Eq. (2.12). In the linear model there is no way to stop this growth. To achieve this, nonlinear effects must be taken into account. The growth-rate determines the time-scale on which nonlinear effects will begin to play their part. We will not go into the details of the nonlinear analysis but only indicate the results as far as they are influenced by the parameters β and r.

For details we refer to PEDLOSKY [6] and ROMEA [9].

3. Viscosity in the two-layer model

We remark that the parameter β is the only effect of sphericity left in Eq. (2.4) and that the same goes for $r = E^{1/2}/\varepsilon$ with respect to viscosity. The values of β and r influence the growth-rate and therefore the results of the nonlinear analysis (qualitatively and/or quantitatively). We distinguish at first two cases

$$\beta \neq 0, \quad r \neq 0, \tag{3.1}$$

$$\beta \neq 0, \quad r = 0. \tag{3.2}$$

The nonspherical model with $\beta = 0$ will be treated in Sect. 5.

We want to pay attention to the following phenomenon: The transition from the case (3.1) to the case (3.2) via $r \downarrow 0$ is not smooth. In other words, the marginal stability curve, on which $\mathrm{Im}\,\frac{\omega}{k_x} = 0$, resulting from $r \downarrow 0$ in the case (3.1) does not coincide with the curve in the nonviscous model (3.2). This singularity in r was first mentioned by HOLOPAINEN [4], in a two-layer model for the atmosphere, in the form of an unexpected destabilization of long waves upon introducing a small viscosity term. In the present context it was first discussed by NEWELL [5].

We give the marginal stability curves for the cases (3.1) and (3.2) and the corresponding growth rates for the growing and the decaying wave. From Eq. (2.12) it follows that the general form of the marginal stability curve is

$$V^2(\beta, r, k) = \frac{4r^2k^2}{k_x^2(2F-k^2)} + \frac{4F^2\beta^2}{k^2(k^2+F)^2(2F-k^2)}. \tag{3.3}$$

If the shear V is varied by a small amount Δ

$$V = V(\beta, r, k) + \Delta, \quad \Delta \ll V(\beta, r, k), \tag{3.4}$$

we find

$$\mathrm{Im}\,\frac{\omega_{1,2}}{k_x} = -\frac{k^2+F}{k^2+2F}\,\frac{r}{k_x} \pm \left[\frac{k^2+F}{k^2+2F}\,\frac{r}{k_x} + g_1 r\Delta + g_2\Delta^2 + \ldots\right]. \tag{3.5}$$

We find that the unstable wave has a growth-rate $O(\Delta)$ ($r = O(1)$), while the decaying wave decays like $O(r)$.

Now let $r \downarrow 0$ in Eq. (3.3), then

$$V(\beta, 0, k) = 2F\beta \Big/ \left(k(k^2+F)\,(2F-k^2)^{\frac{1}{2}}\right). \tag{3.6}$$

If, on the other hand, we put $r = 0$ in Eq. (2.10), reducing it to a polynomial with real coefficients, we find

$$V_0(k) = 2F\beta \Big/ \left(k^2(2F-k^2)^{\frac{1}{2}}\,(2F+k^2)^{\frac{1}{2}}\right), \tag{3.7}$$

V_0 has its minimum at $k^2 = \sqrt{2}F$ and then $V = \beta/F$, whereas $V(\beta, 0, k)$ has its minimum at $k^2 = \left((1+\sqrt{3})/2\right)F$ and then $V \approx 0.91\ \beta/F$ (see Fig. 2). So $V(\beta, 0, k)$ and V_0 are at a distance $O(1)$ when they have their minima and the wavelength of the most unstable wave is longer for $V(\beta, 0, k)$ than for V_0. Finally we remark that the growth-rate in the nonviscous model (3.2) is not linear in Δ:

$$V = V_0 + \Delta, \quad \Delta \ll V_0,$$

$$\mathrm{Im}\,\frac{w}{k_x} = \pm\frac{\sqrt{2}\,\beta F}{k^2(k^2+2F)}\left(\frac{\Delta}{V_0}\right)^{\frac{1}{2}}. \tag{3.8}$$

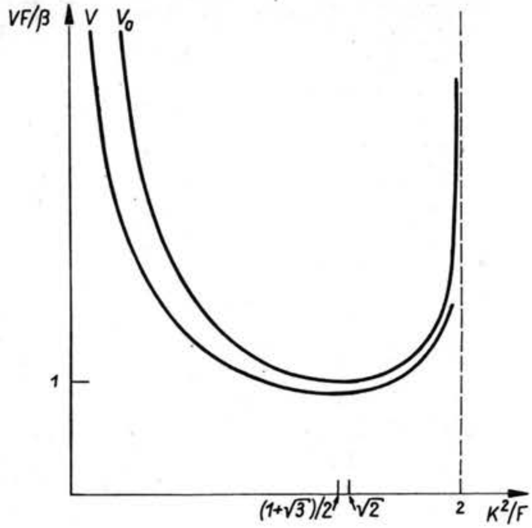

FIG. 2. Marginal stability curves for the nonviscous (V_0, $R = 0$) and the viscous-limit model (V, $R\downarrow 0$).

(Note that in both cases a large β raises the minimum of the curve and so has a stabilizing influence. Furthermore, in all cases instability is impossible for $k^2 > 2F$, corresponding to the Eady-(short wave) cut-off).

A „physical" explanation of the behaviour described above runs as follows (NEWELL [5], HART [3]): „HOLOPAINEN [4] has noted that the limit $E^{1/2} \to 0$ is singular in that long waves are more unstable than the inviscid theory would suggest. This comes from the constribution of the $r \cdot \beta$ term (cf. Eq. (2.11) here). The presence of friction nullifies the stabilizing influence of β at large wavelengths by allowing the fluid to move more easily across geostrophic (2Ω/depth)-contours. When $\beta = 0$, the frictional influence is non-singular".

In the next section we will give a mathematical explanation and establish the relation between the limit- and the nonviscous case.

4. Complex polynomials and the two-layer model

We consider the dispersion relation (2.10). When $r = 0$, this is a second-degree polynomial with real coefficients. The corresponding curve $\mathrm{Im}\,\omega = 0$ is Eq. (3.7) in the preceding section. When $r > 0$, $r \ll 1$ we can interpret this as a perturbation into the complex plane of the real coefficients of Eq. (2.10) with $r = 0$. This is what causes the jump in Fig. 2.

In general it is sufficient to consider polynomials of the form

$$z^2 + a = 0 \quad a \in \mathscr{R}, \tag{4.1}$$

where we have marginal stability for $a = 0$.

The perturbed polynomial then looks like

$$z^2 + ibz + a + ic = 0, \quad |b|, |c| \ll 1. \tag{4.2}$$

To have a root $z_1 \in \mathscr{R}$ of Eq. (4.2) we must have

$$ibz_1+ic = 0 \tag{4.3}$$

so $z_1 = -\frac{c}{b}$ and

$$a = -\frac{c^2}{b^2} \tag{4.4}$$

and this gives us the marginal stability curve. It is obvious from Eqs. (4.1), (4.2) and (4.4) that we find the unperturbed polynomial as b, $c \to 0$, but not necessarily the corresponding marginal stability curve $a = 0$. This only happens in special cases, e.g. $c = 0$ or $b^2 = c$.

So the mathematical mechanism that produces the difference between the nonviscous and the viscous-limit model is really very simple. It amounts to the statement that for polynomials with complex coefficients, the treshold Im(root) = 0 does not depend in a continuous way on the imaginary parts of the coefficients. When the imaginary parts of the coefficients tend to zero, the curve representing the imaginary part of the root is squeezed towards the axis, however, the intersection of the curve with the axis barely moves.

We now study in detail the behaviour of the growth-rate for small viscosity. Using the substitutions (4.5), Eq. (2.10) is transformed into Eq. (4.6) corresponding to Eq. (4.2):

$$\begin{aligned} k^2 &= cF, \qquad && c \in (0,2), \\ z &= \frac{F\omega}{\beta k_x}, \qquad && S = \frac{FV}{\beta}, \\ R &= crF/(k_x\beta), \end{aligned} \tag{4.5}$$

$$w^2+ib_2Rw+c_1(R)-\frac{b_1^2}{4}+(c_2-b_1b_2/2)Ri = 0 \tag{4.6}$$

with

$$\begin{aligned} c(c+2)b_1 &= -c(c+2)S+2(c+1), \qquad && c(c+2)b_2 = 2(c+1), \\ c(c+2)c_1(R) &= -(c+1)S-cS^2-1+R^2, \qquad && c(c+2)c_2 = -(c+1)S+2, \\ & && w = z+b_1/2. \end{aligned} \tag{4.7}$$

Note that Eq. (4.4) now reads

$$c_1(R)-b_1^2/4 = \frac{(c_2-b_1b_2/2)}{b_2^2}\frac{R^2}{R^2}$$

and substituting the relation (4.7) indeed gives the marginal stability curve $V(\beta, r, k)$, Eq. (3.3).

For $c = \sqrt{2}$, the most unstable wave in the nonviscous case (3.2), we computed the growth-rate for decreasing values of R, as a function of the shear S. The result is shown in Fig. 3a, b.

The limit curve Im c_0 is the inviscid growth-rate (3.8). For $R = 0.1$ the growth-rate is linear in $\Delta = V - V(\beta, r, \sqrt{2}F)$ (cf. Eq. (3.5)), although instability already sets in below

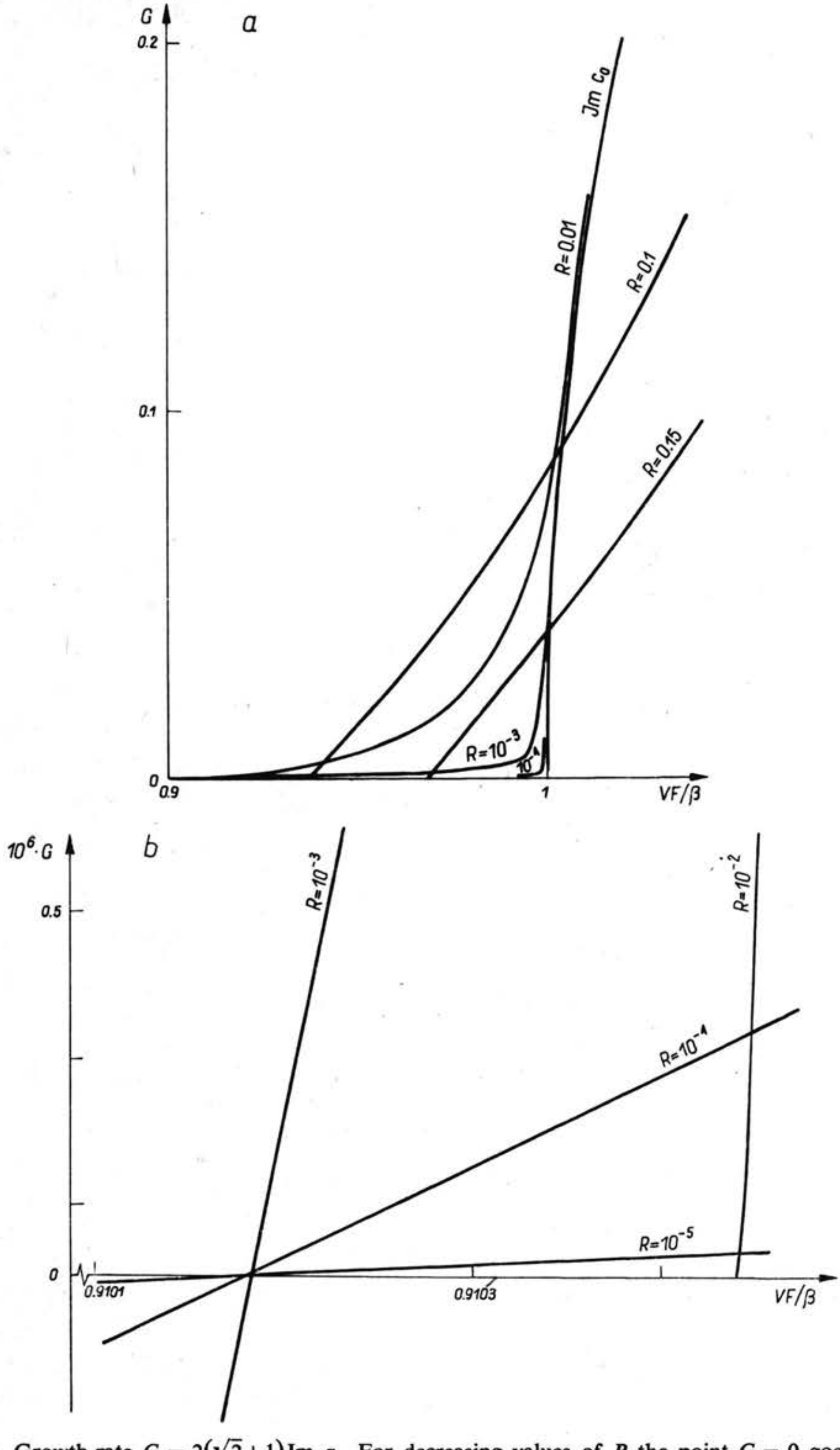

FIG. 3.a. Growth-rate $G = 2(\sqrt{2}+1)\,\mathrm{Im}\, z$. For decreasing values of R the point $G = 0$ goes to $S = VF/\beta \approx 0.91$, while for $S < 1$ G tends to zero. b. Growth-rate $10^6 \cdot G$, to illustrate the behaviour of $G = 0$ for decreasing values of R.

the minimum of V_0. For smaller R, instability always sets in at $S \approx 0.91$ (cf. Sect. 3), but becomes less and less pronounced as R tends to zero.

So for $0.91 < S < 1$ there is always instability but the growth-rate tends to zero in this whole region as $R\downarrow 0$. The adjustment to the square-root-like behaviour of $\operatorname{Im} c_0$ becomes more abrupt as $R\downarrow 0$. For $S > 1$, the inviscid growth-rate is found, determined by $\Delta = V - V_0$.

The nonlinear analysis for $0.91 < S < 1$ is given in ROMEA [9]. Instead of the amplitude oscillation found in PEDLOSKY [6], for slow time $T = \Delta^{1/2}t \to \infty$, a steady state is found for $T = \Delta t \to \infty$. For the details we refer to the two papers mentioned.

5. Nonspherical models

In the nonspherical models, i.e. $\beta = 0$, we find nothing analogous to what was found in the previous sections. The two cases are

$$\beta = 0, \quad r \neq 0, \tag{5.1}$$

$$\beta = 0, \quad r = 0. \tag{5.2}$$

No difference exists between the case (3.1) with $\beta\downarrow 0$ and Eq. (5.1) because the imaginary parts of the coefficients are not involved. We might expect a singularity in r, when letting $r\downarrow 0$ in Eq. (5.1) but there it turns out that we have one of the special cases mentioned in Sect. 4: $C = 0$ in Eq. (4.4).

Proof: with coefficients Eqs. (4.5) and (4.7) modified as follows:

$$z = \frac{\omega F}{k_x}, \quad S = FV, \quad R = \frac{rF}{k_x}c, \tag{5.3}$$

$$\begin{aligned} b_1 &= -S, & b_2 &= 2\frac{c+1}{c(c+2)}, \\ c_1(R) &= \frac{cS^2 - R^2}{c(c+2)}, & c_2 &= -S\frac{c+1}{c(c+2)}, \end{aligned} \tag{5.4}$$

we find again Eq. (4.6).

Now form Eq. (4.4)

$$c_1(R) - \frac{b_1^2}{4} = -\frac{\left(c_2 - \dfrac{b_1 b_2}{2}\right)^2}{b_2^2}\,\frac{R^2}{R^2} = -\left(\frac{-S(c+1)}{c(c+2)} - \frac{-2S(c+1)}{2c(c+2)}\right)\Big/ b_2^2 = 0. \tag{5.5}$$

So the marginal stability curves in Eqs. (5.1) ($\beta = 0$, $r\downarrow 0$) and (5.2) ($\beta = 0$, $r = 0$) will coincide.

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Received October 10, 1981.