Arch. Mech., 34, 3, pp. 309–324, Warszawa 1982

Plane channel flows with surface mass transfer and velocity slip on moving walls

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EXACT integrals of the Navier–Stokes equations are used to describe laminar, steady, plane flows of a viscous, incompressible fluid in channels with porous, moving walls. At the walls the Beavers–Joseph slip conditions are assumed. The problem is reduced to find successively two functions fulfilling ordinary differential equations of the fourth and second order and satisfying six boundary conditions. The solution is obtained in the form of power series expansions with an iteration scheme for their coefficients.

Do opisu laminarnych, ustalonych, płaskich przepływów cieczy lepkiej i nieściśliwej w kanałach o porowatych i ruchomych ściankach wykorzystane są ścisłe rozwiązania równań Naviera–Stokes'a. Na ściankach kanału przyjmowane są warunki poślizgu Beaversa–Josepha. Problem jest sprowadzony do kolejnego wyznaczania dwóch funkcji spełniających zwyczajne równania różniczkowe czwartego i drugiego rzędu oraz sześć warunków brzegowych. Rozwiązanie otrzymane jest w formie szeregów potęgowych, z podaniem schematu iteracyjnego dla ich współczynników.

С целью описания ламинарных, установившихся плоских течений вязкой и несжимаемой жидкости в каналах с пористыми и подвижными стенками были использованы точные решения уравнений Навье-Стокса. Для стенок канала были приняты условия скольжения Биверса-Жозефа. Задача сводится к очередному определению двух функций, выполняющих обычные дифференциальные уравнения четвертого и второго порядка, а также шесть граничных условий. Решение получено в виде степенных рядов, с приведением итерационной схемы для их коэффициентов.

Introduction

LAMINAR, steady, plane flows of a viscous, incompressible fluid in an infinite channel with permeable, longitudinally moving walls is considered (Fig. 1). Through the walls the fluid is uniformly filtrated and on its surface the velocity slip may occur, according to the slip condition formulated by G. S. BEAVERS and D. D. JOSEPH [1]. The flow will be described by such solutions of the Navier–Stokes equations, which may be expressed by functions of one variable only. Such solutions are called exact integrals of the Navier–Stokes equations [2].

The study of flows in channels with porous walls has been developed for about 30 years. Also the particular exact integrals of the Navier–Stokes equations are here often applied.

For symmetric flows in channels with permeable and immovable walls A. BERMAN in 1953 [3] presented a solution by means of a function of one variable, fulfilling an ordinary differential equation. The analysis of this solution, mainly computed nummerically,

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was continued by R. M. TERRIL [4, 5] and others. F. WHITE, B. F. BARFIELD and M. J. GOGLIA [6] found for it the power series expansion. The slip velocity conditions on the wall was introduced by E. M. SPARROW, G. S. BEAVERS and L. Y. HUNG [7]. Other particular solutions, with uniform cross-flow in the channel, were presented by A. BERMAN in 1958 [8] for channels with a longitudinal pressure gradient but with immobile walls and by K. R. CRAMER [9] and by C. M. LILLEY [10] for one moving wall but without the pressure gradient. An extension of the first Berman solution for channels with moving permeable walls is presented in [11]. Now the influence of the slip velocity condition is additionally taken into account.

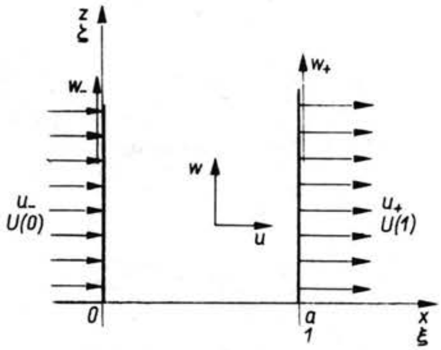

FIG. 1.

The study of flows in channels with porous walls finds various practical applications. It is applied also in the research of two-phase flows through pipes [12]: the condensation of the gaseous phase on the walls is here described by the suction effect through the wall surface. The influence of moving walls and of the velocity slip condition arises in another practical application in the textile industry. Analysing the formation of chemical fibres, the hydrodynamic interaction of multifilament bundle with the surrounding fluid is of practical importance and may be an object of theoretical interest. The bundle of fibres may often be considered as a moving porous wall of a channel [13, 14] and, due to high porosity of the bundle, on its surface the velocity slip condition should be taken into account. This slip condition seems to be particularly important for bundles formed by a row of fibres which may be considered as a porous membrane with very high porosity and with velocity slip on its surface [15].

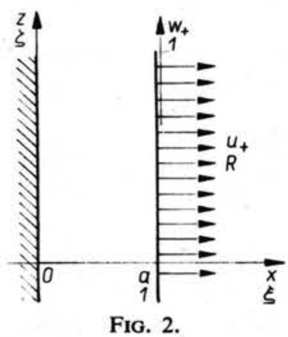

FIG. 2.

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Following A. BERMAN [3], R. M. TERRIL [4, 5] and others, it will be assumed that the cross-flow velocity depends only on the distance from the channel's wall and by means of this assumption exact integrals of the Navier–Stokes equations may be obtained. These integrals are determined by two functions of one variable for which, after [6], the solution in the form of power series expansions will be found.

As an example for the obtained results, the flow in a channel with one impermeable wall at rest will be analysed (Fig. 2).

1. General solution

Let us introduce a Cartesian system of coordinates x, z, with the z axis parallel to the walls of the channel (Fig. 1). By u and w we will denote the coordinates of the fluid velocity. The Navier–Stokes equations of the plane, steady flows are

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,$$

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\varrho\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{1.1}$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\varrho\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right),$$

where p is the pressure and ν is the kinematic coefficient of viscosity. We assume that the density ϱ and the coefficients of viscosity ν and $\mu=\varrho\nu$ are constant.

For our problem the velocity components u, w should fulfill the following boundary conditions on the walls of the channel:

$$u(0,z)=u_-,\quad u(a,z)=u_+,$$

$$w(0,z)-w_-=\varkappa_- a\frac{\partial w(0,z)}{\partial x},\quad w(a,z)-w_+=-\varkappa_+ a\frac{\partial w(a,z)}{\partial x}, \tag{1.2}$$

where a is the channel's width, u_-, u_+ — the filtration velocities through the walls, w_-, w_+ — the longitudinal wall velocities and $\varkappa_-$, $\varkappa_+$ are the nondimensional constants characterizing the slip conditions [1]. For $\varkappa_-=\varkappa_+=0$, no slip occurs and the velocities of walls are equal to the velocities of the adjacent fluid.

Assuming $u=u(x)$, in a way shown in [2–7, 11], an exact integral of the stated problem may be obtained and it may be presented in an nondimensional form by means of the formulae

$$\xi=\frac{x}{a},\quad \zeta=\frac{z}{a},$$

$$\psi(\xi,\zeta)=\zeta U(\xi)+\Pi'\int W_p(\xi)d\xi-R_-\int W_-(\xi)d\xi-R_+\int W_+(\xi)d\xi,$$

$$\frac{au}{\nu}=\frac{\partial\psi}{\partial\zeta}=U(\xi),\quad \frac{aw}{\nu}=-\frac{\partial\psi}{\partial\xi}=-\zeta U'(\xi)-\Pi'W_p(\xi)+R_-W_-(\xi)-R_+W_+(\xi), \tag{1.3}$$

$$\frac{a^2p}{\mu\nu}=-\frac{1}{2}U^2(\xi)+U'(\xi)+\Pi'\zeta+\frac{1}{2}\Pi''\zeta^2,$$

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where

$$\Pi' = \frac{a^3 \partial p(x,0)}{\mu\nu \partial z}, \quad \Pi'' = \frac{a^4 \partial^2 p(x,z)}{\mu\nu \partial z^2},$$

(1.4) $$R_- = \frac{aw_-}{\nu}, \quad R_+ = \frac{aw_+}{\nu}, \quad R_s = \frac{a(u_+ + u_-)}{\nu}, \quad R_D = \frac{a(u_+ - u_-)}{\nu}$$

are constants, $\psi(\xi, \zeta)$ is a nondimensional stream function, $U(\xi)$ fulfills a nonlinear differential equation

(1.5) $$U''' - UU'' + (U')^2 = -\Pi''$$

with four boundary conditions

(1.6) $$U(0) = \frac{R_s - R_D}{2}, \quad U'(0) - \varkappa_- U''(0) = 0,$$
$$U(1) = \frac{R_s + R_D}{2}, \quad U'(1) + \varkappa_+ U''(1) = 0;$$

after finding $U(\xi)$, the functions $W_-(\xi)$ and $W_+(\xi)$ fulfill the homogeneous linear equation

(1.7) $$W''_\pm - UW'_\pm + U'W_\pm = 0$$

with the unhomogeneous boundary conditions

(1.8) $$W_-(0) - \varkappa_- W'_-(0) = 1, \quad W_-(1) + \varkappa_+ W'_+(1) = 0,$$
$$W_+(0) - \varkappa_- W'_+(0) = 0, \quad W_+(1) + \varkappa_+ W'_+(1) = 1,$$

and $W_p(\xi)$ fulfills the unhomogeneous linear equation with the homogeneous boundary conditions

(1.9) $$W''_p - UW'_p + U'W_p = -1,$$
$$W_p(0) - \varkappa_- W'_p(0) = W_p(1) + \varkappa_+ W'_p(1) = 0.$$

It may be verified, by introducing Eq. (1.3) into the Navier–Stokes equations (1.1) and into the boundary conditions (1.2), that this solution is really an exact integral of the stated problem.

The longitudinal velocity w and its mean value $\overline{w}$ may be divided into four parts (1.3). Only the first part $\zeta[U(0) - U(1)]$ of $a\overline{w}/\nu$ changes with ζ, the remaining parts are on ζ independent and contain, as factors, the mean values $\overline{W}_\pm$, $\overline{W}_p$ defined by the formula

(1.10) $$\overline{W} = \int_0^1 W(\xi) d\xi.$$

If $\Pi'' \neq 0$, the constant Π' may be eliminated by choosing the origin $z = 0$ in the point, where $\partial p(x, 0)/\partial z = 0$. Also the function $W_p(\xi)$ and its mean value $\overline{W}_p$ may be obtained by means of the simple formulae

(1.11) $$\frac{W_p(\xi)}{\overline{W}_p} = \frac{U'(\xi)}{R_D}, \quad \overline{W}_p = \frac{R_D}{\Pi''}, \quad \Pi'' \neq 0.$$

In Eqs. (1.3) the first term $-\zeta U'(\xi)$ and the second term $-\Pi' W_p(\xi)$ of aw/ν are strictly correlated: their sum does not depend on the choice of $z = 0$.

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Since four boundary conditions (1.6) are not needed to be satisfied by $U(\xi)$ fulfilling the differential equation (1.5) of the third order only, the pressure constant

$$-\Pi'' = U''' - UU'' + (U')^2 = U'''(0) - U(0)U''(0) + [U'(0)]^2 \tag{1.12}$$

may be not arbitrary and it should depend on the solution $U(\xi)$. Differentiating Eq. (1.5), the constant Π'' may be eliminated and to determine $U(\xi)$ we obtain the fourth order equation

$$U'''' - UU''' + U'U'' = 0, \tag{1.13}$$

with four boundary conditions (1.6).

Now our problem has been reduced to be solved in two steps. In the first step the solution $U(\xi)$ of the fourth order nonlinear ordinary differential equation (1.13) with four boundary conditions (1.6) should be found. This solution describes the flow in a channel with permeable but not moving walls. In the second step the influence of the motion of walls is taken into account by means of two functions $W_-(\xi)$ and $W_+(\xi)$ which fulfill the homogeneous linear equation (1.7) and unhomogeneous boundary conditions (1.8).

A property of invariancy of Eq. (1.13) with respect to the transformation

$$U(\xi) = \lambda \mathcal{U}(\lambda \xi) \tag{1.14}$$

should be mentioned. If $\mathcal{U}(\xi)$ fulfills Eq. (1.13) and λ is an arbitrary constant, it is easy to verify that $U(\xi)$ defined by Eq. (1.14) should also fulfill Eq. (1.13). This property of invariancy is useful to obtain some particular solutions $U(\xi)$.

It is easy to verify that introducing

$$W(\xi) = U''(\xi), \qquad \overline{W} = U'(1) - U'(0), \tag{1.15}$$

into Eq. (1.7) we obtain Eq. (1.13), what means that $W = U''$ is a particular solution of Eq. (1.7). This solution, in linear combination with another linearly independent solution of Eq. (1.7), could be used to obtain $W_\pm$ fulfilling Eq. (1.7) and satisfying the boundary conditions (1.8). However, by using power series expansion, the application of the solution (1.15) will be rather not necessary.

2. Power series expansion

We will assume the solution $U(\xi)$ in the form of the power series expansion

$$U(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \qquad a_n = \alpha_n / n! \tag{2.1}$$

Introducing it into Eq. (1.13) and comparing the like powers of ξ, we find for higher order coefficients the recurrent formulae

$$\alpha_n = n! \sum_{k=0}^{n=1} \frac{k(k-1)(2k-n-1)}{n(n-1)(n-2)(n-3)} a_k a_{n-k-1} = \tag{2.2}$$

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$$\text{(2.2) [cont.]} \quad = \alpha_0\alpha_{n-1} + \sum_{k=1}^{\operatorname{int}\frac{n-2}{2}} \left[\binom{n-1}{k} - 4\binom{n-3}{k-1}\right]\alpha_k\alpha_{n-k-1} - \frac{1-(-1)^n}{2}\,\frac{2}{n-3}\binom{n-3}{\frac{n-5}{2}}\alpha^2_{\frac{n-1}{2}}$$

$$n = 4, 5, 6, \ldots.$$

taking into account that the first coefficients, a_0, a_1, a_2, a_3, should be considered as four arbitrary constants. These constants should be determined by four boundary conditions (1.6).

Not all coefficients a_n must be different from zero. We may obtain some particular solutions $U(\xi)$ of Eq. (1.13) with the following, not vanishing, coefficients a_n.

(0) Only two coefficient a_0, a_1 may not vanish. The obtained linear function $U = a_0 + a_1\xi$ will not be the main object of our interest here.

(1) Not vanishing odd coefficients

$$\text{(2.3)} \quad \alpha_{2n-1} = \sum_{k=1}^{\operatorname{int}\frac{n-1}{2}} \left[\binom{2n-2}{2k-1} - 4\binom{2n-4}{2k-2}\right]\alpha_{2k-1}\alpha_{2n-2k-1} - \frac{1+(-1)^n}{2}\,\frac{2}{n-3}\binom{2n-5}{n-4}\alpha^2_{n-1},$$

$$n = 3, 4, 5, \ldots$$

with α_1, α_3 arbitrary give a solution $U(\xi)$, which describes the flow symmetric with respect to the ζ-axis.

(2) Not vanishing coefficients

$$\text{(2.4)} \quad \alpha_{3n-1} = \sum_{k=1}^{\operatorname{int}\frac{n-1}{2}} \left[\binom{3n-2}{3k-1} - 4\binom{3n-4}{3k-2}\right]\alpha_{3k-1}\alpha_{3n-3k-1} - \frac{1+(-1)^n}{2}\,\frac{2}{3n-4}\binom{3n-4}{\frac{3n-6}{2}}\alpha^2_{\frac{3n-2}{2}}$$

$$n = 2, 3, 4, \ldots$$

with only one arbitrary constant α_2 and with $\alpha_0 = \alpha_1 = \alpha_3 = 0$ give a solution $U(\xi)$, for which $-\Pi'' = \alpha_3 - \alpha_0\alpha_2 + \alpha_1^2 = 0$ and $U(0) = U'(0) = 0$ (left impermeable wall).

(3) Not vanishing coefficients

$$\text{(2.5)} \quad \alpha_{4n-1} = \sum_{k=1}^{\operatorname{int}\frac{n-1}{2}} \left[\binom{4n-2}{4k-1} - 4\binom{4n-4}{4k-2}\right]\alpha_{4k-1}\alpha_{4n-4k-1} - \frac{1+(-1)^n}{2}\,\frac{2}{2n-3}\binom{4n-5}{2n-4}\alpha^2_{2n-1},$$

$$n = 2, 3, 4, \ldots$$

with only one arbitrary constant α_3 and with $\alpha_0 = \alpha_1 = \alpha_2 = 0$. The obtained particular solution $U(\xi)$ fulfills also the boundary conditions $U(0) = U'(0) = 0$ on the left wall and the symmetry to the ζ-axis.

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After finding $U(\xi)$, we may seek the second unknown function $W_{\pm}(\xi)$. We will assume $W_{\pm}(\xi)$ also in the form of the power series expansion

$$W_{\pm}(\xi) = \sum_{k=0}^{\infty} b_{\pm n}\xi^{n}. \tag{2.6}$$

Introducing it into Eq. (1.7) and comparing like powers of ξ, we obtain for higher order coefficients the recurrent formula

$$b_{\pm n} = \sum_{k=0}^{n-1} \frac{2k-n+1}{n(n-1)} a_{n-k-1} b_{\pm k}, \quad n = 2, 3, 4, \ldots \tag{2.7}$$

with $b_{\pm 0}$, $b_{\pm 1}$, as arbitrary constants which should be determined by the boundary conditions (1.8). The functions $W_{\pm}(\xi)$ describe the relative distribution of the velocity components, due to the motion of walls. Their mean values (1.10) are defined by the formula

$$\overline{W}_{\pm} = \int_{0}^{1} W_{\pm}(\xi)d\xi = \sum_{k=0}^{\infty} \frac{b_{\pm k}}{k+1}. \tag{2.8}$$

Introducing the power series expansions (2.1) and (2.6), we assumed tacitly that these series are convergent. Their convergence has not been proved as yet in general, and it was the object of interest in a particular case only. For the case of symmetrical flow $R_s = 0$, without velocity slip $\varkappa_- = \varkappa_+ = 0$, F. M. White proved the convergence of the series expansion with odd coefficients (2.3). For uniform cross flow $R_D = 0$ and $U = \text{const}$, the exact solutions $W_{\pm}(\xi)$, $W_p(\xi)$ in closed form may be obtained (cf. [8, 9, 10], and 3.2), and the convergence of their power series expansions is evident. It seems that for moderate values of R_D these series should be convergent, too.

3. Particular cases

3.1. No cross flow $U = 0$

The first step solution $U = 0$ fulfills obviously Eq. (1.13) and the boundary conditions (1.6). In the second step Eqs. (1.7) and (1.9) are reduced to the simple form $W_{\pm}'' = 0$, $W_p'' = -1$, and their solutions, fulfilling the boundary conditions (1.8), are

$$\begin{gathered} \underset{0}{W}_- = \frac{1-\xi+\varkappa_+}{1+\varkappa_-+\varkappa_+}, \quad \underset{0}{W}_+ = \frac{\xi+\varkappa_-}{1+\varkappa_-+\varkappa_+}, \quad \underset{0}{\overline{W}}_{\pm} = \frac{\frac{1}{2}+\varkappa_{\mp}}{1+\varkappa_-+\varkappa_+}, \\ \underset{0}{W}_p = \frac{\left(\frac{1}{2}+\varkappa_+\right)(\xi+\varkappa_-)}{1+\varkappa_-+\varkappa_+} - \frac{\xi^2}{2}, \quad \underset{0}{\overline{W}}_p = \frac{\left(\frac{1}{2}+\varkappa_+\right)\left(\frac{1}{2}+\varkappa_-\right)}{1+\varkappa_-+\varkappa_+} - \frac{1}{6}. \end{gathered} \tag{3.1}$$

We will denote these particular solutions by the index zero.

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These solutions describe for $\varkappa_- = \varkappa_+ = 0$ the well-known "classical" plane Couette and Poiseuille flows. For not vanishing $\varkappa_-, \varkappa_+$ they take into account the influence of slip conditions at the walls.

3.2. Uniform cross flow $U = \text{const}$

For $R_D = 0$, we find immediately the first step solution in the closed form

$$U = \frac{R_s}{2} = \text{const}, \quad \Pi'' = 0. \tag{3.2}$$

This solution describes the uniform cross flow with the constant pressure gradient $\partial p(x, z)/\partial z = (\mu\nu/a^3)\Pi' = \text{const}$. Then, the second step solutions $W_\pm(\xi)$, $W_p(\xi)$ may be expressed by one auxiliary function

$$\frac{W_\pm(\xi) - \underset{0}{W}_\pm(\xi)}{\overline{W}_\pm - \underset{0}{\overline{W}}_\pm} = \frac{W_p(\xi)}{\overline{W}_p} = V(\xi), \tag{3.3}$$

$$V(\xi) = \frac{e^{\frac{R_s}{2}\xi} - 1 - (e^{\frac{R_s}{2}} - 1)\underset{0}{W}_+(\xi) - \frac{R_s}{2}[e^{\frac{R_s}{2}}\varkappa_+ \cdot \underset{0}{W}_+(\xi) - \varkappa_- \cdot \underset{0}{W}_-(\xi)]}{\left(\frac{2}{R_s} - \underset{0}{\overline{W}}_+\right)(e^{\frac{R_s}{2}} - 1) - 1 - \frac{R_s}{2}(e^{\frac{R_s}{2}}\varkappa_+ \cdot \underset{0}{\overline{W}}_+ - \varkappa_- \cdot \underset{0}{\overline{W}}_-)}.$$

The mean values $\overline{W}_\pm$, $\overline{W}_p$ are here determined by the formulae

$$\overline{W}_\pm = \frac{\pm\frac{2}{R_s}(e^{\pm\frac{R_s}{2}} - 1) - 1 \pm \frac{R_s}{2}\varkappa_\mp}{e^{\pm\frac{R_s}{2}} - 1 \pm \frac{Rs}{2}(\varkappa_\pm e^{\pm\frac{R_s}{2}} + \varkappa_\mp)},$$

$$\overline{W}_p = \frac{2}{Rs}\,\frac{\frac{2}{R_s}(e^{\frac{R_s}{2}} - 1)(1 + \varkappa_- + \varkappa_+) - \left[e^{\frac{R_s}{2}}\left(\frac{1}{2} + \varkappa_-\right) + \left(\frac{1}{2} + \varkappa_+\right)\right] - \frac{R_s}{2}\left[e^{\frac{R_s}{2}}\left(\frac{1}{2} + \varkappa_-\right)\varkappa_+ - \left(\frac{1}{2} + \varkappa_+\right)\varkappa_-\right]}{-(e^{\frac{R_s}{2}} - 1) - \frac{R_s}{2}(e^{\frac{R_s}{2}}\varkappa_+ + \varkappa_-)}. \tag{3.4}$$

The diagrams of $\overline{W}_+$, $\overline{W}_p$ versus R_s are shown in Fig. 3 and in Fig. 4 some examples are given of longitudinal velocity distributions $V(\xi)$, $W_+(\xi)$.

The solutions $W_\pm(\xi)$ describe a generalized "Couette-type" flow, due to the motion of walls. For $\varkappa_- = \varkappa_+ = 0$ they were found in 1959 by K. R. CRAMER [9] and C. M. LILLEY [10]. The solution $W_p(\xi)$ describes a generalized "Poiseuille-type" flow, due to the constant pressure gradient Π'. For $\varkappa_- = \varkappa_+ = 0$ it was found in 1958 by BERMAN [8]. The influence of the slip condition on the right hand side wall was taken into account in [14].

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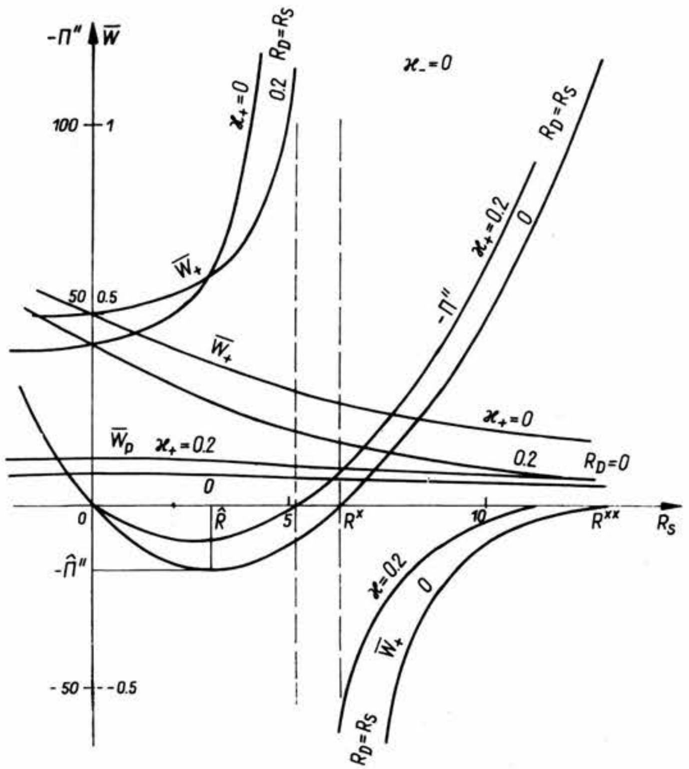

FIG. 3.

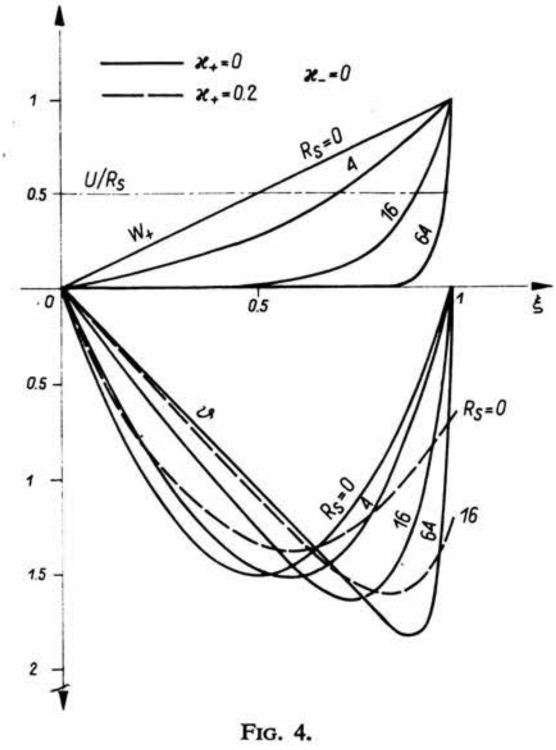

FIG. 4.

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3.3. One impermeable wall

In previous cases the same result could be obtained by means of the power series expansions (2.1) and (2.6), but it was possible to avoid it. Let us consider now a particular case of channel flow with the left wall $\xi = 0$ impermeable $U(0) = 0$, fixed $R_- = 0$ and without the velocity slip $\varkappa_- = 0$ (Fig. 2). For given $R = R_D = R_s$ and $\varkappa_+$, we will assume $U(\xi)$ and $W_+(\xi)$ in the form of the power series (2.1) and (2.6), and we should find the arbitrary constants $a_0, a_1, a_2, a_3, b_{+0}, b_{+1}$, from the boundary conditions (1.6) and (1.8).

From two conditions (1.6) $U(0) = U'(0) = 0$ and from the conditions (1.8) $W_+(0) = 0$ we find immediately

$$a_0 = a_1 = b_{+0} = 0. \tag{3.5}$$

The remaining conditions (1.6) give the relations

$$\sum_{n=2}^{\infty} n[1+(n-1)\varkappa_+]a_n = 0, \quad R = \sum_{n=2}^{\infty} a_n, \tag{3.6}$$

for two arbitrary constants $a_2 = \alpha_2/2$, $a_3 = \alpha_3/6$. The constants α_2, α_3 allow to determine from Eq. (2.2) the following coefficients:

$$\begin{array}{lll}
\alpha_0 = \alpha_1 = \alpha_4 = 0, & & \\
\alpha_5 = -\alpha_2^2, & \alpha_6 = -2\alpha_2\alpha_3, & \alpha_7 = -2\alpha_3^2, \\
\alpha_8 = -\alpha_2^3, & \alpha_9 = -4\alpha_2^2\alpha_3, & \alpha_{10} = -16\alpha_2\alpha_3^2, \\
\alpha_{11} = -27\alpha_2^4-16\alpha_3^3, & \alpha_{12} = -181\alpha_2^3\alpha_3, & \alpha_{13} = -840\alpha_2^2\alpha_3^2, \\
\ldots & \ldots & \ldots
\end{array}$$

After finding a_2, a_3, a , ... the coefficients $b_{+1}, b_{+2}, b_{+3}, \ldots$ of the $W_+(\xi)$ development are obtained by means of the formulae

$$\frac{1}{b_{+1}} = \sum_{n=1}^{\infty} (1+n\varkappa_+)\beta_n, \quad b_{+n} = \beta_n b_{+1}, \quad n = 1, 2, 3, \ldots \tag{3.7}$$

where, taking into account Eq. (3.5), the ratios $\beta_n = b_{+n}/b_{+1}$ are determined by the recurrent formulae

$$\beta_n = \sum_{k=1}^{n-3} \frac{2k-n+1}{n(n-1)} a_{n-k-1}\beta_k, \quad n = 4, 5, 6, \ldots, \quad \beta_1 = 1, \quad \beta_2 = \beta_3 = 0. \tag{3.8}$$

Since computing the explicit formulae (3.8) and (3.7) for the $W_+(\xi)$ coefficients does not present essential difficulties, the main problem consists now in finding, for given R and $\varkappa_+$ values, the coefficients $a_2, a_3, a_4, \ldots$ of $U(\xi)$ development. These coefficients fulfill an infinite system of equations: the recurrent formulae (2.2) and the boundary conditions (3.6), and they are included there in an implicit form. Computing such a system of equations presents some difficulties and to avoid them we will here apply an inverse method. This method consists in making a convenient choice of necessary arbitrary constants and in finding afterwards the corresponding values of R and $\varkappa_+$.

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Applying the inverse method, let us introduce an auxiliary function

$$\mathscr{U}(\xi) = \sum_{n=0}^{\infty} \bar{a}_n \xi^n, \tag{3.9}$$

fulfilling Eq. (1.13), with $\bar{a}_0 = \bar{a}_1 = 0$ and $\bar{a}_2, \bar{a}_3$ chosen arbitrarily. By means of the recurrent formulae (2.2) we find the higher order coefficients $\bar{a}_4, \bar{a}_5, \bar{a}_6, \ldots$, which are needed to determine $\mathscr{U}(\xi)$. The function $\mathscr{U}(\xi)$ does not satisfy all homogeneous boundary conditions (1.6) but, according to Eq. (1.14), we may introduce a solution $U(\xi) = \lambda\mathscr{U}(\lambda\xi)$, which already should satisfy all homogeneous conditions (1.6) and mainly the condition $U'(1)+\varkappa_- U''(1) = 0$. In consequence, λ should fulfill the equation

$$\mathscr{U}'(\lambda)+\bar{\varkappa}\mathscr{U}''(\lambda) = 0, \quad \bar{\varkappa} = \lambda\varkappa_+, \tag{3.10}$$

with the constant $\bar{\varkappa}$ given. The coefficients $a_1, a_3, a_4, \ldots$ of the series expansion of $U(\xi)$ are now found from

$$a_n = \lambda^{n+1}\bar{a}_n, \quad n = 2, 3, \ldots. \tag{3.11}$$

So, for the conveniently chosen constants $\bar{a}_2, \bar{a}_3$ and $\bar{\varkappa}$, we may find consecutively: $\mathscr{U}(\xi)$ from Eq. (2.2), λ and $\varkappa_+ = \bar{\varkappa}/\lambda$ from Eq. (3.1) and $U(\xi)$ from Eq. (3.11). According to Eqs. (3.6) and (1.12), the Reynolds number R and the pressure constant $-\Pi''$ are now

$$R = \sum_{n=2}^{\infty} a_n, \quad -\Pi'' = 6a_3. \tag{3.12}$$

In computing procedures the developments used here are, obviously, truncated to N terms, in dependence of the accuracy needed. After finding $U(\xi)$, we obtain from Eqs. (3.8) and (3.7) all needed coefficients $b_{+1}, b_{+2}, b_{+3}, \ldots$ of the series expansion $W_+(\xi)$ also with the required accuracy. The mean value $\overline{W}_+^*$ is determined by the formula (2.8). Afterwards we may find all other quantities characterizing the flows in the considered channel.

To explain the method introduced, let us consider an example with no slip condition $\varkappa_- = \varkappa_+ = 0$ and with the assumed coefficients $\bar{\alpha}_0 = \bar{\alpha}_1 = \bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = 1$. For this case we may apply the series expansion (2.1) with the not vanishing coefficients $\bar{\alpha}_3, \bar{\alpha}_7, \ldots$ $\ldots \bar{\alpha}_{4N-1}$ determined by Eq. (2.5). The first, obtained by Eq. (2.5), coefficients $\bar{a}_{4n-1} = \bar{\alpha}_{4n-1}/(4n-1)!$ up to $N = 15$, are given in Table 1. In this way we determine the truncated function

$$\mathscr{U}'(\xi) \approx \sum_{n=1}^{N} (4n-1)\bar{a}_{4n-1}\xi^{4n-2},$$

then from the condition (3.10) $\mathscr{U}'(\lambda^{**}) = 0$ we find $\lambda^{**} = 3.3314172$ and, from Eq. (3.11), we obtain the coefficients a_{4n-1}^{**} of the solution $U^{**}(\xi)$ of Eq. (1.13), satisfying the boundary conditions $U^{**}(0) = U^{**\prime}(0) = U^{**\prime\prime}(0) = U^{**\prime}(1) = 0$. The quantities characterizing the considered example are here denoted by two stars. The first not vanishing coefficients $a_3^{**}, a_7^{**}, \ldots a_{4N-1}^{**}$ are also given in Table 1. From Eq. (3.12) we find the constants $R^{**} = 13.1190807$ and $(-\Pi'')^{**} = 123.173166$.

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Table 1.

n	$4n-1$	$\bar{a}_{4n-1}$	a^{**}_{4n-1}	b^{**}_{4n-3}
1	3	$1/3!$	20.528861	−0.304111
2	7	$-2/7!$	−6.020488	0.624310
3	11	$-16/11!$	−0.749053	0.203433
4	15	$-16\cdot 133/15!$	−0.374573	0.194210
5	19	$-256\cdot 2819/19!$	−0.168198	0.142025
6	23	$-512\cdot 978931/23!$	−0.067706	0.084585
7	27	$-5.340360\cdot 10^{-17}$	−0.022971	0.039814
8	31	$-1.141956\cdot 10^{-19}$	−0.006050	0.013892
9	35	$-1.486414\cdot 10^{-22}$	−0.000970	0.002850
10	39	$8.701190\cdot 10^{-26}$	0.000070	−0.000256
11	43	$1.137304\cdot 10^{-27}$	0.000113	−0.000502
12	47	$3.376016\cdot 10^{-30}$	0.000041	−0.000220
13	51	$4.693083\cdot 10^{-33}$	0.000007	−0.000044
14	55	$-4.472999\cdot 10^{-36}$	−0.000001	0.000006
15	59	$-4.446887\cdot 10^{-38}$	−0.000001	0.000009

To obtain the function $W^{**}_{+}(\xi)$ describing the influence of the motion of the right wall, we may apply the formulae (3.8) and (3.7), or, more convenient for this particular example $W^{**}_{+}(0) = U^{**''}(0) = 0$, the formula (1.15) giving

$$W^{**}_{+}(\xi) = C^{**}U^{**''}(\xi), \quad b^{**}_{+(4n-3)} = (4n-1)(4n-2)a^{**}_{4n-1}C^{**}, \quad n = 1, 2, \ldots$$

with $C^{**} = -2.468973\cdot 10^{-3}$ found from the condition $W^{**}_{+}(1) = 0$. The coefficients $b^{**}_{+1}, b^{**}_{+2}, \ldots b^{**}_{+(4N-3)}$ are also given in Table 1. The velocity distributions $U^{**}(\xi)/R$, $-U^{**'}(\xi)/R$, $W^{**}_{+}(\xi)$ are presented in Fig. 5.

In an analogous way, by choosing conveniently different values of $\bar{a}_2$, $\bar{a}_3$ and $\bar{\varkappa}$, we may obtain different values of R, $-\Pi''$, $\overline{W}_{+}$ and different solutions $U(\xi)$, $W_{+}(\xi)$. The obtained diagrams of $-\Pi''$ and $\overline{W}_{+}$ versus R for $\varkappa_{+} = 0$ and $\varkappa_{+} = 0.2$ are given in Fig. 3. Some illustrative examples of the velocity distributions $U(\xi)/R$, $-U'(\xi)/R$, $W_{+}(\xi)$ are presented in Fig. 5 for $\varkappa_{+} = 0$ and in Fig. 6 for $\varkappa_{+} = 0.1$. For $\Pi'' = 0$ we may introduce $\bar{\alpha}_2 = 1, \bar{\alpha}_3 = 0$, and for $\varkappa_{-} = \varkappa_{+} = 0$, we obtain $\lambda^* = 2.71001918$, $R^* = 6.30387476$. The coefficients $\bar{a}_{3n-1} = \bar{\alpha}_{3n-1}/(3n-1)!$, $a^{*}_{3n-1} = \lambda^{*3n}\bar{a}_{3n-1} (n = 1, 2, 3 \ldots 11)$ are presented in Table 2.

By means of the velocity distribution $U'(\xi)/R$ we describe the generalized "Poiseuille-type" flow and by $W_{+}(\xi)$ the generalized "Couette-type" flow. The relative distribution of transversal velocities is given by $U(\xi)/R$. Neglecting slip effects $\varkappa_{-} = \varkappa_{+} = 0$, we obtain in the limiting case $R \to 0$ the well-known Couette and Poiseuille velocity distributions $W_{+}(\xi) \to \xi$, $U'(\xi)/R \to \xi(1-\xi)/2$.

It should be emphasized that by increasing R the pressure conconstant Π'' goes through its extremal value $\hat{\Pi}''$ and then, for $R = R^*$, Π'' changes its sign. For $R \to R^*$, the mean value $\overline{W}_{+}$ (Fig. 3) of the velocity distribution $W_{+}(\xi)$ (Figs. 5 and 6) tends to infinity, what provokes losing stability of flow.

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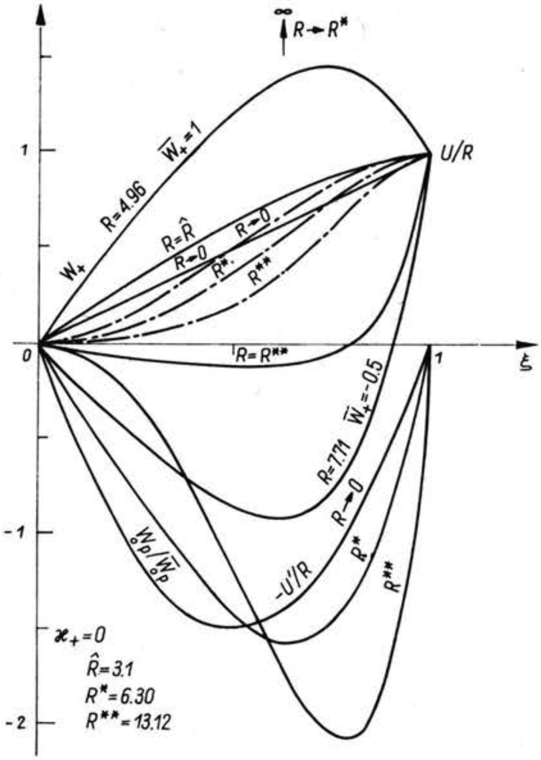

FIG. 5.

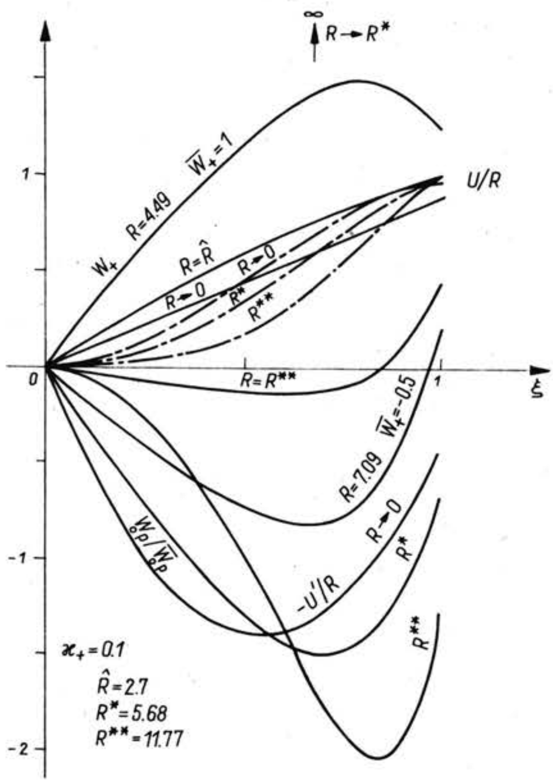

FIG. 6.

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Table 2.

n	$3n-1$	$\bar{a}_{3n-1}$	a^*_{3n-1}
1	2	1/2!	9.951467
2	5	−1/5!	−3.301056
3	8	−1/8!	−0.195538
4	11	−27/11!	−0.106139
5	14	−951/14!	−0.034069
6	17	−51465/17!	−0.008994
7	20	−3355857/20!	−0.001706
8	23	$-5.874818 \cdot 10^{-15}$	−0.000145
9	26	$7.435121 \cdot 10^{-17}$	0.000036
10	29	$1.730927 \cdot 10^{-18}$	0.000017
11	32	$1.311331 \cdot 10^{-20}$	0.000003

4. Final remarks

The flow in a channel with a uniform surface mass transfer on moving walls is mainly determined by the cross flow effects. Two Reynolds numbers R_D and R_s, related to filtration velocities across the walls, are the main constants characterizing the flow properties. Together with $\varkappa_-$ and $\varkappa_+$ they determine the transversal velocity component $U(\xi)$, the longitudinal velocity components: of the generalized "Poiseuille-type" flow

$$-[\zeta U'(\xi)+\Pi' W_p(\xi)] = \begin{cases} -\Pi' W_p(\xi), & R_D = 0, \\ -\left(\zeta + \dfrac{\Pi'}{\Pi''}\right) U'(\xi), & \Pi'' \neq 0, \end{cases}$$

and of the generalized "Couette-type" flow $W_\pm(\xi)$, and the second derivative of pressure Π''. The superposition of all these effects according to Eq. (1.3) describes the resulting flow field. The functions $U(\xi)$, $W_p(\xi)$, $W_\pm(\xi)$, and the pressure constant Π'' depend on R_D, R_s, $\varkappa_-$, $\varkappa_+$, only, the constants R_-, R_+, Π' enter in Eq. (1.3) only as multiplicative factors.

The exact solution presented by Eq. (1.3) is a particular integral of the stated problem. The functions $U(\xi)$, $W_p(\xi)$, $W_\pm(\xi)$ entering in Eq. (1.3) are here found either in closed form (cf. 3.1 and 3.2) or in the form of power series expansions (cf. 2). They could be also found ny numerical integration. As we consider only a particular solution, the following questions may arise. Does this solution exist and is it unique for all values of R_D, R_s, $\varkappa_-$, $\varkappa_+$? Is it stable in respect to possible unsteady perturbations of flow? Are the introduced power series convergent? Answering these quesitons is not our aim here. However, already here it may be expected that not for all values of R_D is the considered problem correctly posed.

Even for $R_D = R_s = \varkappa_- = \varkappa_+ = 0$, when we obtain the well-known Couette and Poiseuille laminar flows, their stability depends on the Reynolds number $\mathrm{Re} = |\Pi'|/12$. Although the mathematical problem for the steady, plane flow is here correctly posed, for higher Reynolds numbers Re the flow becomes unstable and the turbulence appears.

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For $R_D = 0$ the solution in closed form (3.2), (3.3) exists and its uniqueness does not seem to be questionable. This solution describes laminar flows with parallel stream lines. It could be presented in the form of convergent power series expansions also. But the problem to determine the conditions for losing stability of flow remains open for further investigation.

The mathematical correctness of the stated problem seems to be also valid for moderate values of R_D, where viscous effects are predominant. For higher values of R_D, where the influence of inertia forces becomes important, this correctness seems to be rather doubtful. By increasing R_D he pressure constant Π'' reaches for $R = \hat{R}$ the extremal value $\hat{\Pi}''$, then, for $R = R^*$, where Π'' goes through zero, the amplitude of $W_{\pm}(\xi)$ tends to infinity (Fig. 3) and the mathematical problem is incorrectly posed. As for $\Pi'' = 0$ and $R_D = R_D^* \neq 0$ the function $W(\xi) = U'(\xi)$ fulfills the homogeneous equation (1.7) with the homogeneous boundary conditions (1.9), so the solutions $W_{\pm}(\xi)$, $W_p(\xi)$ should be not unique. In consequence, in the vicinity of $\Pi'' \approx 0$, $R_D \approx R_D^*$ small changes of the pressure gradient Π' or of the wall velocities R_-, R_+ should provoke very large changes in the velocity field, what means that an instability of flow may here be expected.

The main aim of this work was to find some exact solutions describing the laminar flows in channels with moving walls. It seems that these solutions may be approximately valid for flows with the moderate Reynolds number R_D. The problems of existence, uniqueness and stability of these solutions were not the object of interest here and they are open for further investigations.

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Received November 19, 1981.

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