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On the motion of gas bubbles in a perfect fluid

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VARIOUS forms have been proposed for the dynamic equation governing the relative motion of a body in a nonuniformly flowing liquid. The correct form is suggested by the recent work of LANDWEBER and MILOH [4]. However, the analysis wherein contains an error. An alternative derivation of their result, avoiding this error, is given here. An application, restricting to the representation by dipoles only, is made to the nonsteady flow of bubbles through a widening pipe.

Zaproponowano różne postacie równań dynamicznych rządzących ruchem względnym ciała w nierównomiernym przepływie cieczy. Ścisła postać tych rówań została zasugerowana w niedawno opublikowanej pracy przez LANDWEBERA i MILOHA w 1980 roku. Przedstawiona tam analiza zawiera jednak pewien błąd. W niniejszej pracy pokazano inną wersję wyprowadzenia tych równań z uniknięciem wymienionego błędu. Wyniki zastosowano (ograniczając się do reprezentacji dipolowej) do nieustalonego przepływu pęcherzyków w rozszerzającej się rurze.

Предложены разные виды динамических уравнений, описывающих относительное движение тела в неравномерном течении жидкости. Точный вид этих уравнений предложен, в недавно опубликованной работе, Ландвебером и Милхом в 1980 году. Представленный там анализ содержит однако некоторую ошибку. В настоящей работе показана другая версия вывода этих уравнений с избежанием упомянутой ошибки. Результаты применены (ограничиваясь дипольным представлением) к неустановившемуся течению пузырьков в расширяющейся трубе.

1. The motion of a sphere in a nonuniform flow

IN A GAS/LIQUID flow there is in general a significant relative motion between the gasphase, bubbles for example, and the fluid phase, water for example. For the prediction of bubbly flows, which occur in a wide variety of situations in engineering, a relation is required which describes the dynamics of this relative motion, see for example VAN WIJNGAARDEN [9]. This presents, as textbooks on fluid dynamics show, no problems as long as the liquid motion is spatially uniform. However, when the liquid motion is nonuniform the situation is much more complicated and controversial relations have been proposed. Here we restrict our attention to the unidirectional motion of a rigid massless sphere in an inviscid, incompressible liquid. It has been assumed that, in analogy with motion in a uniform flow, the relative dynamics are completely determined by local properties of the flow field.

When the liquid density is denoted by ϱ, the local liquid velocity in the x direction by $u(x, t)$, the virtual mass by m (for a sphere $m = 1/2\varrho\tau$), we find in HINZE [2] the relation

$$\frac{D}{Dt}\, m(v-u) = \varrho\tau\,\frac{du}{dt}\,. \tag{1.1}$$

Here v is the velocity of the centre of the sphere and τ its volume, while D/Dt and d/dt

are material derivatives associated with the motion of the sphere and of the liquid respectively,

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}, \tag{1.2}$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}. \tag{1.3}$$

The difference between these operators vanishes of course when u and v are uniform in x, in which case Eq. (1.1) is a well-known result of classical hydrodynamics (see eg. LAMB [3] ch. 6). HINZE [2] quotes a work by TCHEN (1947) in applying Eq. (1.1) to the motion of a sphere in a turbulent flow. PROSPERETTI and VAN WIJNGAARDEN [6] in dealing with critical or choking flow of a bubbly liquid take it that also the fluid acceleration must be related to the moving sphere and use instead of (1.1)

$$\frac{D}{Dt}m(v-u)=\varrho\tau\frac{Du}{Dt}. \tag{1.4}$$

Finally VOINOV, VOINOV and PETROV [7] derive

$$\frac{D}{Dt}mv-\frac{d}{dt}mu=\varrho\tau\frac{du}{dt}, \tag{1.5}$$

which we may, for later reference, write with the help of Eqs. (1.2) and (1.3) and for a rigid sphere as

$$\varrho\tau\frac{Dv}{Dt}-(m+\varrho\tau)\frac{D}{Dt}(v-u)-(m+\varrho\tau)(v-u)\frac{\partial u}{\partial x}=0. \tag{1.6}$$

The deduction of these relations from hydrodynamic principles is in all cases somewhat loose and necessarily approximative. The most convincing is Eq. (1.5) and it is found in Voinov, Voinov and Petrov roughly as follows:

Let ε be the ratio of the sphere's radius a, say to a characteristic length L representative for the distance from other bodies. In a bubbly flow ε could be the ratio between a and the average distance between the bubbles. VOINOV, VOINOV and PETROV [7] show that for small ε there is a Lagrangian for the motion of the sphere moving through a liquid when the motion of other bodies produce a velocity u, given by

$$L=\frac{1}{2}m\{v-u(x,t)\}^2-\varrho\tau p+O(\varepsilon^5), \tag{1.7}$$

where p is the pressure in the liquid.

The Euler equation associated with the Lagrangian L is now

$$\frac{D}{Dt}m(v-u)-m(v-u)\frac{\partial u}{\partial x}+\varrho\tau\frac{\partial p}{\partial x}=0,$$

which is indeed equivalent with Eq. (1.5) and (1.6) because the equation of motion in the liquid is

$$\varrho\frac{du}{dt}=-\frac{\partial p}{\partial x}.$$

A rigorous derivation of the force exerted on a body in an arbitrary potential flow has been attempted recently by LANDWEBER and MILOH [4] for the rather general case in which the flow potential of the moving body can be described with singularities. For the present case, where only one spatial coordinate is involved, their result gives instead of the term

$$-\frac{3}{2}\varrho\tau(v-u)\frac{\partial u}{\partial x}$$

on the left hand side of Eq. (1.6), the (exact) expression

$$-\sum_{s} 4\pi\varrho M_q\left(\frac{\partial^q u}{\partial x^q}\right)_s, \tag{1.8}$$

where M_q is the strength of the singularity of order q and $\partial^q/\partial x^q$ is the derivative of order q of the velocity of the liquid in x_s, the site of the considered singularity. Unfortunately an error is contained in the derivation of the term in LANDWEBER and MILOH [4] corresponding with Eq. (1.8). This error is (Landweber-private communication) due to an unpermissable interchange of integration and differentiation in dealing with certain surface integrals.

Nevertheless the final result in LANDWEBER and MILOH [4] is by the present author believed to be correct because it can be derived in another way which is outlined here. The force on a sphere in a flow field $\mathbf{u}$ is with velocity $\mathbf{v}$ of the sphere and velocity $\mathbf{U}$ of the liquid at the centre of the sphere (BATCHELOR [1])

$$\mathbf{F} = -\int_{\text{sphere}} p\,d\mathbf{A} = \varrho\tau\frac{D\mathbf{v}}{Dt} - \frac{3}{2}\varrho\tau\frac{D}{Dt}(\mathbf{v}-\mathbf{U}) + \frac{1}{2}\varrho\int_{\text{sphere}}(\mathbf{u}\cdot\mathbf{u}-2\mathbf{v}\cdot\mathbf{u})\,d\mathbf{A}. \tag{1.9}$$

The difficulties in the present problem are with the evaluation of the integral in Eq. (1.9) which is zero in case $\mathbf{u}$ is the flow caused by the body in a uniform stream and in the absence of circulation. The first and second terms on the right-hand side of Eq. (1.9) stem from the unsteady term in the pressure, as calculated with Bernoulli's Theorem. With the help of the relation (A being the surface of the sphere)

$$\int_A(\mathbf{u}\cdot\mathbf{v})\,d\mathbf{A} - \int_A \mathbf{u}(\mathbf{v}\cdot d\mathbf{A}) = 0$$

valid due to $\nabla\times\mathbf{u} = 0$ we can write the integral in Eq. (1.9), which we denote with $\mathbf{F}'$ as

$$\frac{\mathbf{F}'}{\varrho} = \int_A \frac{1}{2}(\mathbf{u}\cdot\mathbf{u})\,d\mathbf{A} - \int_A \mathbf{u}(\mathbf{v}\cdot d\mathbf{A}). \tag{1.10}$$

Relation (1.10) follows also directly from a momentum balance applied to a volume enclosing the body.

Since on the body $\mathbf{v}\cdot d\mathbf{A} = \mathbf{u}\cdot d\mathbf{A}$, we may write Eq. (1.10) as

$$\mathbf{F}'/\varrho = \int_A \frac{1}{2}(\mathbf{u}\cdot\mathbf{u})\,d\mathbf{A} - \int_A \mathbf{u}(\mathbf{u}\cdot d\mathbf{A}). \tag{1.11}$$

In a uniform flow there will be only one singularity in the volume, a dipole in the centre (remember that we exclude circulation). In an arbitrary nonuniform flow there will be other singularities, quadrupoles etc., as well. As shown by LANDWEBER and MILOH [4] we can convert the integral in Eq. (1.11) in an integral over the surface A_s enclosing the singularities:

$$\mathbf{F}'/\varrho = \sum \left\{ \int_{A_s} \frac{1}{2} (\mathbf{u} \cdot \mathbf{u}) d\mathbf{A} - \int_{A_s} \mathbf{u}(\mathbf{u} \cdot d\mathbf{A}) \right\}. \tag{1.12}$$

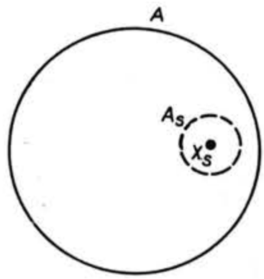

FIG. 1. $\mathbf{x}_s$ is the site of singularity within the sphere with surface A. A_s is a spherical surface surrounding $\mathbf{x}_s$.

Now we write, as in LANDWEBER and MILOH [4] $\mathbf{u}$ as the sum of a regular that is free of vorticity and divergence, field $\mathbf{u}_R$ and a singular part $\mathbf{u}_s$, and introduce this into Eq. (1.12). From this point on, our analysis differs from LANDWEBER and MILOH [4]. The part of $\mathbf{F}'$ containing $\mathbf{u}_R$ only is zero because the equation of motion gives both

$$\frac{\partial \mathbf{u}_R}{\partial t} + \nabla \left(\frac{p}{\varrho} + \frac{1}{2} \mathbf{u}_R^2 \right) = 0$$

and

$$\frac{\partial \mathbf{u}_R}{\partial t} + \nabla \left(\frac{p}{\varrho} + \mathbf{u}_R \mathbf{u}_R \right) = 0,$$

where $\mathbf{u}_R \mathbf{u}_R$ is the dyadic representing momentum flux. For the completely singular part $\mathbf{F}' = 0$ too because a singularity induces no force in itself. Therefore $\mathbf{F}'/\varrho$ comes from the remaining "mixed" terms which we write as

$$\mathbf{F}'/\varrho = \sum_s \left\{ \int_{A_s} (\mathbf{u}_R \cdot \mathbf{u}_s) d\mathbf{A} - \int_{A_s} d\mathbf{A} \cdot (\mathbf{u}_R \mathbf{u}_s) - \int_{A_s} d\mathbf{A} \cdot (\mathbf{u}_s \mathbf{u}_R) \right\},$$

where $\mathbf{u}_R \mathbf{u}_s$ and $\mathbf{u}_s \mathbf{u}_R$ are dyadics.

As in the theory of generalized functions we can regard $\mathbf{u}_R$ as a "good function" (in the sense of LIGHTHILL [5]) regularizing the integrals. Applying the divergence theorem on the integrals above we obtain, bearing in mind that $\nabla \cdot \mathbf{u}_R = 0$

$$\frac{\mathbf{F}'}{\varrho} = \sum_s - \int_{\tau_s} \mathbf{u}_R (\nabla \cdot \mathbf{u}_s) d\tau, \tag{1.13}$$

where the integration extends over the volume τ_s of a small sphere surrounding the singularity. If the latter is a source with strength Q, say, the integral gives

$$\mathbf{F}'_Q = -\varrho Q \mathbf{u}_R(\mathbf{x}_s). \tag{1.14}$$

For a dipole we add the contributions of a sink in $\mathbf{x}_s - \delta\mathbf{x}_1/2$ and a source in $\mathbf{x}_s + \delta\mathbf{x}_1/2$ obtaining

$$(1.15) \quad \mathbf{F}'_D = \mathbf{F}'_Q(\sin k) + \mathbf{F}'_Q(\text{source}) = \varrho Q \mathbf{u}_R(\mathbf{x}_s - \delta\mathbf{x}_1/2) - \varrho Q \mathbf{u}_R(\mathbf{x}_s + \delta\mathbf{x}_1/2) = -\varrho Q(\delta\mathbf{x}_1 \cdot \nabla)\mathbf{u}_R(\mathbf{x}_s) = -4\pi\varrho(\mathbf{m} \cdot \nabla)\mathbf{u}_R(\mathbf{x}_s).$$

Here the dipole strength $\delta\mathbf{x}_1 Q/4\pi$ is indicated with $\mathbf{m}$. In the same way we obtain for a quadrupole

$$\mathbf{F}'_{\text{quadr}} = -\varrho Q(\delta\mathbf{x}_1 \cdot \nabla)(\delta\mathbf{x}_2 \cdot \nabla)\mathbf{u}_R(\mathbf{x}_s).$$

When, as in our case, all singularities are directed along the x axis, we obtain

$$(1.16) \quad F' = -4\pi\varrho \sum_s M_q \frac{\partial^q}{\partial x^q} \{u_R(x_s)\},$$

which agrees, cf. Eq. (1.8), with the result obtained by LANDWEBER and MILOH [4].

If we now substitute Eq. (1.16) into Eq. (1.9), it follows upon comparison with Eqs. (1.4)–(1.6) that Voinov, Voinov and Petrov's result (1.6) is correct if the regular part of the fluid velocity varies slowly enough to permit the neglect of singularities of order larger than one.

2. Application to flow in a widening pipe

In this section we apply the result obtained above to the following two-phase flow problem, (see Fig. 2): A long straight tube has a narrow section I followed by a gentle transition to a wide section II. The tube is filled with an incompressible inviscid liquid with density ϱ in which small massless spheres (bubbles) with constant volume are homogeneously distributed at time $t < 0$. Gravity is ignored and we consider what happens with the bubbles when the liquid, at rest for times $t < 0$, is at $t = 0$ instantaneously accelerated to a constant velocity $u(x)$ which in the narrow section of the tube is u_I and which

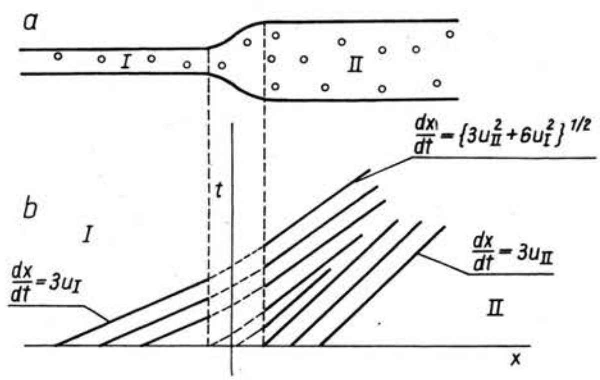

FIG. 2. a) Widening tube in which (bubbly) flow is instantaneously generated; b) x—t plane with characteristics.

is u_{II} in the wide section. It is readily concluded from Eq. (1.5) that at $t = 0^+$ the bubble velocity

$$v(x, 0) = 3u(x), \quad t = 0^+, \tag{2.1}$$

when we bear in mind that $m = 1/2\varrho\tau$. Note that also from all other proposed relations such as Eq. (1.1) or Eq. (1.4) the result (2.1) follows. In the approximation used here $u(x)$ is the liquid velocity in the absence of particles. It was shown in VAN WIJNGAARDEN [8] that Eq. (2.1) holds at $t = 0^+$ also when the motion of neighbouring particles is accounted for.

In VAN WIJNGAARDEN [8] the motion of a mixture as in Fig. 2 was investigated including the effect of one particle on the motion of the others. However, only the situation at $t = 0^+$ was considered. Here we consider the subsequent motion though only in the lowest approximation in terms of the concentration. For $t > 0$ u is steady and the equation for v is obtained from Eq. (1.6), by taking $\partial u/\partial t = 0$ and $m = 1/2\varrho\tau$,

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 3u\frac{\partial u}{\partial x}. \tag{2.2}$$

In order to determine a velocity distribution v which satisfies Eq. (2.2) and which is for $t = 0$ given by Eq (2.1), we trace the characteristics given by

$$\frac{dx}{dt} = v(x, t). \tag{2.3}$$

Equations (2.1)–(2.3) say that along these line

$$2\frac{dv}{dt} = \frac{1}{3}\frac{1}{v}\frac{d}{dt}v^2(x, 0),$$

or upon integration

$$v^2(x, t) = \frac{1}{3}v^2(x, 0) + C, \quad \text{along} \quad v = \frac{dx}{dt}. \tag{2.4}$$

In this relation C is a constant which for each characteristic follows from the initial condition (2.1). Let x_0 be the value of x where a characteristic cuts the x-axis. Then we have from Eqs. (2.1) and (2.4)

$$v^2(x_0) = \frac{1}{3}v^2(x_0) + C, \quad \text{whence} \quad C = \frac{2}{3}v^2(x_0).$$

Therefore the solution is

$$v^2(x, t) = \frac{1}{3}v^2(x, 0) + \frac{2}{3}v^2(x_0), \quad \text{on} \quad \frac{dx}{dt} = v. \tag{2.5}$$

In Fig. 2b some characteristics are drawn. At the left characteristics start as straght linesi in region I in the x, t plane, in the direction $dx/dt = 3u_I$, because there $v(x, 0) = v(x_0) = 3u_I$. Upon entering the transition to the wider section these characteristics are bent upward because $v(x, 0)$ decreases. In region II they become straight again in a direction V which is, with $v(x, 0) = 3u_{II}$ and $v(x_0) = 3u_I$, given by

$$\frac{dx}{dt} = V = \{3u_{II}^2 + 6u_I^2\}^{1/2}. \tag{2.6}$$

Clearly

(2.7) $$3u_{II} < V < 3u_I.$$

Characteristics starting at $t = 0$ in region *II* remain straight lines in the direction $dx/dt = 3u_{II}$. Therefore bubbles which start to move at $t = 0$ in region I are overtaking at some later instant bubbles which started out in region II.

Physically this means the formation of clusters of bubbles; moreover, the possibility of coalescence arises. This may generate large bubbles as this is the case for instance in fluid beds.

Mathematically speaking, the solution of Eqs. (2.1)–(2.2) becomes multivalued when characteristics intersect, like in the theory of shock waves. This means that other physical effects such as viscosity have to be introduced to prevent the solution to break down, a situation well known in gasdynamics and in the theory of water waves of finite amplitude.

So here again we have an example of how in quite different parts of fluid dynamics the same fundamental features arise.

Professor W. FISZDON, as we know him, is an expert in recognizing such features and has been in his professional life a rigorous promotor of fluid dynamics, as both his papers and the well-known biannual conferences organized by him in Poland testify. With pleasure I dedicate this paper to him on the occasion of his 70th birthday and I wish him many years to come amidst his family and friends all over the world.

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