Arch. Mech., 34, 3, pp. 351–358, Warszawa 1982

Shock wave in piecewise linear elastic material

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THE SIX-DIMENSIONAL deformation space is divided into a number of regions. For each deformation belonging to a fixed region there holds the same linear stress-strain relation. The shock wave propagates in this piecewise-linear elastic material. The general algebraic propagation equations are given and three special cases are discussed.

Sześciowymiarowa przestrzeń odkształceń jest podzielona na pewną liczbę rozłącznych obszarów. Każdemu obszarowi odpowiada liniowy związek naprężenie–odkształcenie. W takim odcinkami liniowym materiale sprężystym propaguje się fala silnej nieciągłości. Zakładając, że jest to fala adiabatyczna, podaje się algebraiczne warunki propagacji dla przypadku ogólnego i dyskutuje się przypadki szczególne.

Шестимерное пространство деформаций разделено на определенное количество разединенных областей. Каждой области соответствует линейное соотношение напряжение–деформация. В таком интервалами линейном упругом материале распространяется ударная волна. Предполагая, что эта волна адиабатическая, даются алгебраические условия распространения для общего случая и обсуждаются частные случаи.

Introduction

THE PIECEWISE linear elastic material as the approximation of the nonlinear elastic material was discussed in [1]. In the present paper the strong discontinuity wave propagating in such a material will be considered. The corresponding solution for the nonlinear material was given in [2] for the one-dimensional case, and in [3] for the three-dimensional case.

1. Piecewise linear elastic material

Denote by u_i the displacement vector, and by ε_{ij} the (linear) strain tensor. In the Cartesian coordinate system $\{x^i\}$ we have

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}), \tag{1.1}$$

where the comma denotes the partial differentiation.

The symmetric tensors ε_{ij} are the points of the six-dimensional linear space V. Divide V into a number of disjoint regions $V_1, V_2, V_3, \ldots, V_K, V_L, \ldots$ and assume that to each region there corresponds the linear stress-strain relation. To different regions there correspond in general different stress-strain relations. Denoting by σ the stored energy, by τ^{ij}

the stress tensor, by T the temperature and by ϱ the mass density, we have for ε_{ij} belonging to the region V_K the following relations:

$$\sigma = \frac{1}{2}\overset{K}{c}{}^{ijrs}\varepsilon_{ij}\varepsilon_{rs}+\overset{K}{c}{}^{ij}\varepsilon_{ij}+\overset{K}{c}+\frac{1}{2}A\eta^2+B\eta+d^{ij}\varepsilon_{ij}\eta,$$

(1.2)

$$\frac{1}{\varrho}\tau^{ij} = \overset{K}{c}{}^{ijrs}\varepsilon_{rs}+\overset{K}{c}{}^{ij}+d^{ij}\eta,$$

$$T = A\eta+d^{ij}\varepsilon_{ij}+B,$$

(1.3)

$$\overset{K}{c}{}^{ijrs} = \overset{K}{c}{}^{rsij} = \overset{K}{c}{}^{jirs}, \quad \overset{K}{c}{}^{ij} = \overset{K}{c}{}^{ji}, \quad d^{ij} = d^{ji},$$

where $\overset{K}{c}{}^{ijrs}$, $\overset{K}{c}{}^{ij}$, $\overset{K}{c}$ are constants. The constans describing the thermal behaviour d^{ij}, A and B are assumed to be region-independent, but the generalization is straightforward. The material defined by Eq. (1.2) was discussed in [1] and [2].

Denote by V_{KL} the boundary between V_K and V_L. Assuming that τ^{ij} is a continuous function of ε_{ij}, we have

$$(\overset{K}{c}{}^{ijrs}-\overset{L}{c}{}^{ijrs})\varepsilon_{rs}+(\overset{K}{c}{}^{ij}-\overset{L}{c}{}^{ij}) = 0 \tag{1.4}$$

for $\varepsilon_{ij}\in V_{KL}$. Because Eq. (1.4) is linear in ε_{ij}, the boundary V_{KL} is a hyperplane. Since it divides two 6-dimensional regions, it is a 5-dimensional hyperplane. From this fact it follows that the coefficients in Eq. (1.4) must have the following form:

(1.5)

$$\overset{K}{c}{}^{ijrs}-\overset{L}{c}{}^{ijrs} = 2\overset{KL}{K}\,\overset{KL}{m}{}^{ij}\,\overset{KL}{m}{}^{rs},$$

$$\overset{K}{c}{}^{ij}-\overset{L}{c}{}^{ij} = 2\overset{KL}{N}\,\overset{KL}{m}{}^{ij},$$

where $\overset{KL}{m}{}^{ij}$ is a symmetric tensor, and $\overset{KL}{K}$, $\overset{KL}{N}$ two constants. Not restricting the generality assume $\overset{KL}{K} = +1$ or $\overset{KL}{K} = -1$. For $\overset{KL}{K} = +1$ the region V_L is softer than the region V_K. No additional restrictions have to be imposed on m^{ij} if the material has no additional symmetry. The points ε_{ij} situated on the boundary V_{KL} satisfy the equation

$$\overset{KL}{K}\,\overset{KL}{m}{}^{rs}\varepsilon_{rs}+\overset{KL}{N} = 0. \tag{1.6}$$

At the boundary V_{KL} not only τ^{ij}, but also σ must be a continuous function of ε_{ij}. From Eqs. (1.2) and (1.5) it follows that

$$\overset{K}{c}-\overset{L}{c} = \overset{KL}{N}{}^2/\overset{KL}{K}. \tag{1.7}$$

For two neighbouring regions V_1 and V_2, we repeat here the relations (1.2) slightly changing the notation

$$\sigma = \frac{1}{2}(c^{ijrs}\pm Km^{ij}m^{rs})\varepsilon_{ij}\varepsilon_{rs}+(c^{ij}\pm Nm^{ij})\varepsilon_{ij}+\left(c\pm\frac{1}{2}\frac{N^2}{K}\right)+\frac{1}{2}A\eta^2+B\eta+d^{ij}\varepsilon_{ij}\eta,$$

(1.8)

$$\frac{1}{\varrho}\tau^{ij} = (c^{ijrs}\pm Km^{ij}m^{rs})\varepsilon_{rs}+(c^{ij}\pm Nm^{ij})+d^{ij}\eta;$$

$$c^{ijrs} = c^{jirs} = c^{rsij}, \quad c^{ij} = c^{ji}, \quad m^{ij} = m^{ji}, \tag{1.9}$$

where the upper sign holds for the region V_1 and the lower sign for the region V_2. The equation of the dividing surface V_{12} reads (cf. Eq. (1.6))

$$Km^{rs}\varepsilon_{rs}+N = 0. \tag{1.10}$$

2. Conservation equations

At the front $\mathscr{S}$ of the shock wave the displacement gradient $u_{i,k}$ is discontinuous. Denote by U the propagation speed of $\mathscr{S}$ and by n_i the unit vector orthogonal to $\mathscr{S}$. Since the displacement u_i is continuous at $\mathscr{S}$, there hold the compatibility equations (cf. [3])

$$\begin{aligned} [\![u_{i,k}]\!] &= pH_i n_k, \\ [\![\dot{u}_i]\!] &= -pH_i U, \end{aligned} \tag{2.1}$$

$$H_i H^i = 1, \quad n_i n^i = 1. \tag{2.2}$$

The double brackets denote the jump at $\mathscr{S}$. In terms of the values $(\cdot)^B$, $(\cdot)^F$ at the rear and front sides $\mathscr{S}^B$, $\mathscr{S}^F$ of $\mathscr{S}$, we have

$$[\![\cdot]\!] = (\cdot)^B-(\cdot)^F.$$

The vector pH_i is the amplitude of the shock wave. Due to Eq. (2.2) the parameter p has the meaning of the intensity of the shock wave.

From the definition of ε_{ij} and the relation $(2.1)_1$, it follows:

$$2[\![\varepsilon_{ij}]\!] = pH_i n_j+pH_j n_i. \tag{2.3}$$

Both the jump of the deformation tensor ε_{ij} and of the entropy η are in general not equal to zero. To simplify the notation denote the entropy jump by S

$$S = [\![n]\!]. \tag{2.4}$$

Since ε_{ij} and η are discontinuous at $\mathscr{S}$, also the stress τ^{ij} and stored energy σ are discontinuous at $\mathscr{S}$, cf. Eq. (1.8). The conservation laws for momentum and energy must therefore be written in the integral form, from which there result the Cotchine equations, cf. [4] and [5]

$$[\![\tau^{ij}]\!]n_j+\varrho U[\![\dot{u}^i]\!] = 0, \tag{2.5}$$

$$\varrho U\left[\!\!\left[\sigma+\frac{1}{2}\dot{u}^i\dot{u}_i\right]\!\!\right]+[\![\tau^{ij}\dot{u}_j]\!]n_i = 0. \tag{2.6}$$

Equations (1.8), (2.1), (2.5) and (2.6) are the governing equations for the shock wave propagation in the piecewise linear elastic material. This set of equations must be complemented by the entropy inequality. We confine our considerations here to the adiabatic process. In this case the heat flux equals zero and

$$[\![\eta]\!] \geqslant 0. \tag{2.7}$$

3. Single region

The deformation state of the material points situated at the front side $\mathscr{S}^F$ of the discontinuity surface $\mathscr{S}$ differs from that of the points situated at the rear side $\mathscr{S}^B$. These two states belong either to a single region or to two neighbouring regions V_K, and V_L, or to two regions V_K, V_L having no common boundary. Consider first the case, when both states belong to the same region V_K. The jumps of the stored energy and stress are

$$\tag{3.1} \begin{gathered} [\![\sigma]\!] = \frac{1}{2} c^{ijrs}[\![\varepsilon_{ij}\varepsilon_{rs}]\!] + c^{ij}[\![\varepsilon_{ij}]\!] + \frac{1}{2} A[\![\eta^2]\!] + B[\![\eta]\!] + d^{ij}[\![\varepsilon_{ij}\eta]\!], \\ \frac{1}{\varrho}[\![\tau^{ij}]\!] = c^{ijrs}[\![\varepsilon_{rs}]\!] + d^{ij}[\![\eta]\!]. \end{gathered}$$

Note that for the jump of the product of two quantities a, b the following formula holds:

$$\tag{3.2} [\![ab]\!] = a^B b^B - a^F b^F = [\![a]\!]\langle b\rangle + \langle a\rangle[\![b]\!],$$

where $\langle \cdot \rangle$ denotes the arithmetic average of a and b

$$\tag{3.3} 2\langle a\rangle = (\cdot)^B + (\cdot)^F.$$

Taking into account the symmetries of the coefficients (cf. Eq. (1.9)) we have

$$\tag{3.4} \begin{gathered} [\![\sigma]\!] = c^{ijrs}\langle\varepsilon_{ij}\rangle pH_r n_s + c^{ij}pH_i n_j + AS + 2B\langle\eta\rangle S + d^{ij}\langle\varepsilon_{ij}\rangle S + d^{ij}\langle\eta\rangle pH_i n_j, \\ \frac{1}{\varrho}[\![\tau^{ij}]\!] = c^{ijrs}pH_r n_s + d^{ij}S. \end{gathered}$$

It remains to calculate the jumps of the two products in Eq. (2.6). Basing on Eqs. (3.2) and (2.1), we have

$$\tag{3.5} \begin{gathered} \left[\!\!\left[\frac{1}{2}\dot{u}_i\dot{u}^i\right]\!\!\right] = -\langle\dot{u}_i\rangle pH^i U, \\ \frac{1}{\varrho}[\![\tau^{ij}\dot{u}_j]\!] = -(c^{ijrs}\langle\varepsilon_{rs}\rangle + c^{ij} + d^{ij}\langle\eta\rangle)pH_j + (c^{ijrs}pH_r n_s + d^{ij}S)\langle\dot{u}_j\rangle. \end{gathered}$$

Substitute now Eqs. (3.4) and (3.5) into the energy conservation equation (2.6) and the momentum conservation equation (2.5) to obtain

$$\tag{3.6} \varrho U(A\langle\eta\rangle + B + d^{ij}\langle\varepsilon_{ij}\rangle)S - \varrho U^2\langle\dot{u}_i\rangle pH^i + c^{ijrs}pH_r n_s n_i\langle\dot{u}_j\rangle + d^{ij}\langle\dot{u}_j\rangle n_i S = 0,$$

$$\tag{3.7} \varrho c^{ijrs}pH_r n_s n_j - \varrho U^2 pH^i + d^{ij}n_j S = 0.$$

Multiply Eq. (3.7) by $\langle\dot{u}_i\rangle$ and subtract the resulting equation from Eq. (3.6). Taking into account the fact that the expression in the brackets in Eq. (3.6) equals the average temperature T (cf. Eq. $(1.2)_3$)

$$\tag{3.8} A\langle\eta\rangle + B + d^{ik}\langle\varepsilon_{ik}\rangle = \langle T\rangle,$$

the resulting equation reduces to the equation

$$\tag{3.9} S\langle T\rangle = 0.$$

It follows that $S = 0$ and Eq. (3.7) reduces to the propagation condition

$$\tag{3.10} (c^{ijrs}n_j n_s - U^2\delta^{ir})H_r = 0.$$

The normalized amplitude H_i is the proper vector, and the squared speed U^2 is the proper number of the tensor $c^{ijrs}n_j n_s$. The propagation speed U does not depend on the intensity of the wave.

4. Two adjoining regions

Assume that the state at the rear side $\mathcal{S}^B$ of the discontinuity surface $\mathcal{S}$ coresponds to the region V_1, and the state at the front side $\mathcal{S}^F$ to the region V_2. Both regions V_1 and V_2 have a common boundary V_{12}. The formulae for the stored energy σ and the stress tensor τ^{ij} are given by Eq. (1.8), and the equation of V_{12} is given by Eq. (1.10).

The jump of the displacement gradient $u_{i,k}$ is given by Eq. (2.1) where pH_i is a vector to be calculated. Instead of using one single parameter p, we shall use two parameters p_B, p_F defined by the following relations:

$$p = p_B + p_F, \quad p_B \geqslant 0, \quad p_F \geqslant 0, \tag{4.1}$$

$$\begin{aligned} (u_{i,k})^B &= a_{ik} + p_B H_i n_k, \\ (u_{i,k})^F &= a_{ik} - p_F H_i n_k, \\ K m^{ik} a_{ik} + N &= 0. \end{aligned} \tag{4.2}$$

Provided $(u_{i,k})^B$ and $(u_{i,k})^F$ are given and $m^{ik}H_i n_k \neq 0$, this system may be solved for p_B, p_F and a_{ik}. Note that a_{ik} is situated on V_{12}.

Define now two additional parameters a_i and a by the relations

$$\begin{aligned} (\dot{u}_i)^B &= a_i + p_B H_i U, \\ (\dot{u}_i)^F &= a_i - p_F H_i U, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \eta^B &= a + p_B S, \\ \eta^F &= a - p_F S. \end{aligned} \tag{4.4}$$

Basing on the relations (4.2)–(4.4) and (1.10), we calculate the jumps of the strain tensor, specific energy and of the products appearing in the energy conservation equation (2.6). The calculations lead to the following formulae:

$$\begin{aligned} [\![\sigma]\!] = (p_B + p_F) c^{ijrs} a_{ij} H_r n_s + \frac{1}{2}(p_B^2 - p_F^2) c^{ijrs} H_i H_r n_j n_s \\ + \frac{1}{2}(p_B^2 + p_F^2) K (m^{ij} H_i n_j)^2 + (p_B^2 - p_F^2)\left(d^{ij} H_i n_j S + \frac{1}{2} A S^2\right) \\ + (p_B + p_F)\,(c^{ij} H_i n_j + A a S + B S + a d^{ij} H_i n_j + d^{ij} a_{ij} S); \end{aligned} \tag{4.5}$$

$$\frac{1}{\varrho}[\![\tau^{ij}]\!] = (p_B + p_F)\,(c^{ijrs} H_r n_s + d^{ij} S) + (p_B - p_F) K m^{ij} m^{rs} H_r n_s, \tag{4.6}$$

$$\frac{1}{2}[\![\dot{u}_i \dot{u}^i]\!] = -(p_B + p_F) a^i H_i U + \frac{1}{2}(p_B^2 - p_F^2) U^2, \tag{4.7}$$

$$\begin{aligned} \frac{1}{\varrho}[\![\tau^{ij}\dot{u}_i]\!] = (p_B + p_F)\,(-c^{ijrs} a_{rs} H_i U + c^{ijrs} H_r n_s a_i - c^{ij} H_i U - d^{ij} H_i a U + d^{ij} a_i S) \\ - (p_B^2 - p_F^2)\,(c^{ijrs} H_i H_r n_s U + d^{ij} H_i U S) - (p_B^2 + p_F^2) m^{ij} m^{rs} H_i H_r n_s K U. \end{aligned} \tag{4.8}$$

The expressions (4.5)–(4.8) after substituting into the momentum and energy conservation equations (2.5), (2.6) lead to the following system of algebraic equations:

$$(4.9)\qquad (p_B+p_F)\,(c^{ijrs}H_r n_s n_j+d^{ij}n_j S-H^i U^2)+(p_B-p_F)Km^{ij}m^{rs}H_r n_s n_j = 0,$$

$$(4.10)\qquad -\frac{1}{2}(p_B^2-p_F^2)\,(c^{ijrs}H_i H_r n_j n_s-U^2-S^2A)-\frac{1}{2}(p_B^2+p_F^2)Km^{ij}m^{rs}H_i H_r n_j n_s$$
$$+(p_B+p_F)S(Aa+B+d^{ij}a_{ij}) = 0.$$

In this system of equations the values $(u_{i,k})^F$ and n_k are given. Because p_F may be expressed by H_i using Eq. (4.2), the unknowns are S, U, H_i and p_B. Taking into account the fact that $H_i H^i = 1$, we have 4 equations for 5 unkowns. The set of equations (4.9) and (4.10) posseses therefore a one-parameter family of solutions. As the parameter may be taken, e.g. p, then the propagation speed U, entropy jump S etc. depend on p.

For the discussion of the results the symmetric approach is more convenient, when neither of the regions V_1, V_2 is distinguished. Assume that a_{ik} and the ratio

$$(4.11)\qquad m = p_F/p_B$$

are given, and the unknowns are S, U, H, p_B and p_F. As the parameter will serve the sum $p = p_B+p_F$ measuring the intensity of the wave. The equations (2.2), (4.1), (4.9)–(4.11) constitute the set of 7 algebraic equations for 7 unknowns: S, U, H, p_B and p_F. Note that Eq. (4.10) is of the fourth order, therefore in the general case the solution can not be given in the analytical form.

5. Special solutions

In the special cases $m = 0$, $m = 1$ and $m = \infty$ not only the numerical, but also analytic solutions are available.

The case $m = 0$ corresponds to $p_F = 0$, and in accord with Eq. (4.2) to

$$(5.1)\qquad \begin{aligned}(u_{i,k})^F &= a_{ik},\\ (u_{i,k})^B &= a_{ik}+p_B H_i n_k.\end{aligned}$$

The front state $(\varepsilon_{ij})^F$ is situated on the boundary V_{12}. In this situation either of the relations (1.8) may be used. Therefore for both states hold the relations appropriate for the region V_1, in particular

$$(5.2)\qquad \frac{1}{\varrho}\tau^{ij} = (c^{ijrs}+Km^{ij}m^{rs})\,\varepsilon_{rs}+c^{ij}+Nm^{ij}+d^{ij}\eta,$$

and the results of Sect. 3 hold true if it is assumed

$$(5.3)\qquad \begin{aligned}c^{ijrs} &\to c^{ijrs}+Km^{ij}m^{rs},\\ c^{ij} &\to c^{ij}+Nm^{ij}.\end{aligned}$$

We face an analogous situation if $m = \infty$. In this case $p_B = 0$ and the rear state is situated on the boundary V_{12}. For both states the formulae appropriate for the region V_2 may be used. The results of Sect. 3 hold true if it is assumed

$$(5.4)\qquad \begin{aligned}c^{ijrs} &\to c^{ijrs}-Km^{ij}m^{rs},\\ c^{ij} &\to c^{ij}-Nm^{ij}.\end{aligned}$$

Consider now the case $m = 1$. We now have $p_B = p_F$ and the set of equations (4.9) and (4.10) reduces to

$$(5.5) \qquad c^{ijrs}H_r n_s n_j + d^{ij}n_j S - U^2 H_i = 0,$$

$$(5.6) \qquad -\frac{1}{2}p_B K m^{ij}m^{rs}H_i H_r n_j n_s + S(Aa + B + d^{ij}a_{ij}) = 0.$$

The expression in the brackets in Eq. (5.6) equals the temperature T^* corresponding to the state $u_{i,k} = a_{i,k}$, $\eta = a$. Therefore

$$(5.7) \qquad S = \frac{1}{2}p_B K(m^{ij}H_i n_j)^2/T^* = \frac{1}{4}pK(m^{ij}H_i n_j)^2 T^*.$$

It was assumed above that either $K = 1$, or $K = -1$. From Eq. (5.7) it follows that in the case considered here $K = +1$, because both s and p are nonnegative. From Eq. (1.8) we read now the result: the shock wave moves in the direction of lower elastic moduli. If $\overset{F}{c}{}^{ijkl}$ denotes the moduli corresponding to the front state, and $\overset{B}{c}{}^{ijkl}$ the moduli corresponding to the rear state, then

$$(5.8) \qquad \overset{B}{c}{}^{ijkl} = \overset{F}{c}{}^{ijkl} + 2m^{ij}m^{rs}.$$

From this relation it follows that the sound speed (acceleration wave speed) behind the shock wave is larger than the sound speed in front of the shock wave.

From Eq. (5.7) it follows that S and H_i are functions of p, $S = S(p)$, $H_i = H_i(p)$. We have

$$(5.9) \qquad S(0) = 0, \quad \left.\frac{dS}{dp}\right|_{p=0} \geqslant 0.$$

Substitution of Eq. (5.7) into Eq. (5.5) leads to the propagation condition of the shock wave

$$(5.10) \qquad (c^{ijrs}n_j n_s - U^2\delta^{ir})H_r + S(p)\,d^{ij}n_j = 0.$$

From Eq. (5.8) it follows that for $p \to 0$ the propagation condition (5.10) reduces to the condition

$$(5.11) \qquad (c^{ijrs}n_j n_s - U^2\delta^{ir})H_r = 0,$$

and coincides with the propagation condition of the acceleration wave for the elastic moduli equal to the average value of that of the medium V_1 and the medium V_2. It should be stressed that the acceleration wave cannot propagate at all because on $\mathcal{S}$ the derivative of the stored energy with respect to the displacement gradient is not defined.

Differentiate now Eq. (5.10) with respect to p and substitute $p = 0$. There results the equation

$$(5.12) \qquad (c^{ijrs}n_j n_s - U^2(0)\,\delta^{ij})\left.\frac{dHr}{dp}\right|_{p=0} - 2U(0)\left.\frac{dU}{dp}\right|_{p=0} H^i(0) + d^{ij}n_j\left.\frac{dS}{dp}\right|_{p=0} = 0.$$

Multiplying both sides of Eq. (5.12) by $H_i(0)$ we obtain

$$(5.13) \qquad \left.\frac{dU}{dp}\right|_{p=0} = \frac{1}{2U(0)}\,d^{ij}n_j H_i(0)\left.\frac{dS}{dp}\right|_{p=0}.$$

The value of the left-hand side may be larger of, equal to, or smaller than zero. However, independently of the sign of $dU/dp|_{p=0}$, the sound speed behind the shock wave is larger than the sound speed in front of the shock wave, cf. the remark after Eq. (5.8).

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Received December 30, 1981.