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Effects of inertia and high-frequency harmonic vibrations on the lift and friction forces in viscoelastic slider-bearing flows

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IN THIS PAPER, being a direct continuation of the previous considerations [1] for low-frequency harmonic vibrations, we take into account the inertia terms in the corresponding equations of motion. Applying a perturbation method, particular solutions are discussed for small-amplitude but high-frequency vibrations superposed on the fundamental flow. A comparison of the results with those obtained for low-frequency vibrations is also presented.

W obecnej pracy, będącej bezpośrednią kontynuacją poprzednich rozważań [1] dla drgań o niskich częstościach, uwzględniono człony inercyjne w odpowiednich równaniach ruchu. Stosując metodę perturbacyjną, przedyskutowano szczegółowe rozwiązania dla drgań o małych amplitudach lecz wysokich częstościach, nałożonych na przepływ podstawowy. Przedstawiono także porównanie wyników z wynikami uzyskanymi poprzednio dla drgań o niskich częstościach.

В настоящей работе, являющейся продолжением работы [1] — о низкочастотных колебаниях, учитываются инерциальные члены в соответствующих уравнениях движения. Применен пертурбационный метод и рассматриваются частные решения для колебаний с малыми амплитудами, но высокой частоты, наложенных на основное течение. Представлено сравнение результатов с ранее полученными для низкочастотных колебаний.

1. Introduction

IN OUR PREVIOUS paper [1] we considered the case of small-amplitude harmonic vibrations superposed on slow steady-state flows in a plane slider bearing (wedge flows). To discuss the behaviour of lift and friction forces acting on the upper or lower part of the bearing, we used the model of incompressible second-order fluid as well as the model of generalized Newtonian fluid with shear-dependent (decreasing) viscosity. We assumed, moreover, that intertia effects in the equations of motion considered could be disregarded for very low frequencies of superposed vibrations. Thus, all the results discussed in paper [1] were valid for suffiently slow, inertialess motions.

In the present paper, being a direct continuation of the previous considerations, we use the same models of fluids and take into account the linearized inertia terms in the equations of motion (cf. [2]). Particular solutions are discussed for small-amplitude but high-frequency harmonic vibrations, or more exactly, for large values of "the frequentative Reynolds number" (cf. Sect. 2.1). On applying a perturbation method, similar to that proposed by JONES and WALTERS [3], the case of moving and vibrating slider is presented in greater detail.

The results concerning the behaviour of the dynamic lift forces caused by normal stresses and shear-dependent (decreasing) viscosities are very similar to those obtained in [1]. On the contrary, the behaviour of dynamic friction forces is quite different from that in the previous case; an evident reduction is observed instead of enhancement. A direct comparison of the results obtained in both cases gives some information on the validity of the methods applied.

2. Basic solutions for small-amplitude harmonic vibrations superposed on steady flows

In the situation shown in Fig. 1, the upper part of the bearing, hereafter called the slider, moves horizontally with velocity $U(1+\varepsilon\exp i\omega t)$, while the lower one, hereafter called the base, remains stationary. U denotes a constant velocity of steady fundamental motion, ω — a constant angular frequency of superposed harmonic vibrations, and

$$\varepsilon = \frac{\alpha\omega}{U}, \tag{2.1}$$

where α is a small amplitude of disturbances. It is assumed, moreover, that $h_0/l \ll 1$ and h_0/h_1 is close to unity for the lubrication approximation to be valid.

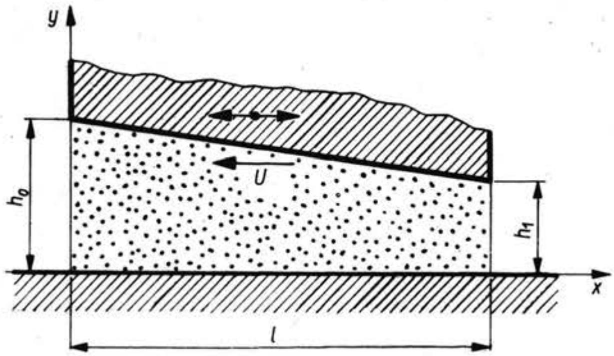

FIG. 1.

2.1. Newtonian fluids

Solutions of the problem considered for steady flows of an incompressible Newtonian fluid were presented elsewhere [1]. In the case of unsteady flows, the equations of motion with linearized inertia terms, under the assumption of lubrication approximation (cf. [4]), viz.

$$\frac{\partial p}{\partial x} - \eta\frac{\partial^2 u}{\partial y^2} = -\varrho\frac{\partial u}{\partial t}, \quad \frac{\partial p}{\partial y} = 0 \tag{2.2}$$

are to be solved with the following boundary conditions:

$$\begin{gathered} u(x,0) = 0, \quad u(x,h) = U(1+\varepsilon\exp i\omega t), \\ p(0,y) = p(l,y) = 0. \end{gathered} \tag{2.3}$$

It is worth noting that the inertia terms in Eqs. (2.2) written in nondimensional forms are proportional to the quantity

$$\mathrm{Re}_v = \frac{\varrho\varepsilon U h_0}{\eta} = \frac{\varrho\alpha\omega h_0}{\eta}, \tag{2.4}$$

which may be called "the vibrational Reynolds number" (cf. [5]). This number is usually small if the amplitude α is sufficiently small and the frequency ω not too high.

We seek a solution of the problem in the form

$$u = u_0 + \varepsilon u_1 \exp i\omega t, \quad p = p_0 + \varepsilon p_1 \exp i\omega t, \tag{2.5}$$

where the subscripts 0 refer to steady-state parts of Newtonian solutions. Substituting Eqs. (2.5) into Eqs. (2.2), we obtain a system of differential equations, the solution of which is the following:

$$u_0 = \frac{U}{h} y + \frac{1}{2\eta}\frac{dp_0}{dx}(y-h)y, \quad \frac{dp_0}{dx} = \frac{6\eta U}{h^2}\left(1 - \frac{H}{h}\right), \tag{2.6}$$

where

$$h = h_0 + ax, \quad H = \frac{2h_0 h_1}{h_0 + h_1}, \tag{2.7}$$

and

$$\begin{gathered} u_1 = A\,\mathrm{ch}\,ky + B\,\mathrm{sh}\,ky + \frac{i}{\varrho\omega}\frac{dp_1}{dx}, \\ k = \nu + i\nu, \quad \nu^2 = \frac{\varrho\omega}{2\eta}, \end{gathered} \tag{2.8}$$

where

$$A = -\frac{i}{\varrho\omega}\frac{dp_1}{dx}, \quad B = \frac{U}{\mathrm{sh}\,ky} + \frac{i}{\varrho\omega}\frac{dp_1}{dx}\frac{\mathrm{ch}\,ky - 1}{\mathrm{sh}\,ky}. \tag{2.9}$$

A constant volume-discharge along the slit, viz.

$$Q_1 = \int_0^h u_1 dy = \mathrm{const}, \quad \frac{dQ_1}{dx} = 0, \tag{2.10}$$

leads to the differential equation

$$\frac{d^2 p_1}{dx^2}(kh - 2\,\mathrm{cth}\,kh) + \frac{dp_1}{dx} ak = iak\varrho\omega U \frac{\mathrm{ch}\,kh - 1}{\mathrm{sh}^2 kh}, \tag{2.11}$$

where $a = (h_1 - h_0)/l$. A solution of the homogeneous equation (2.11) can be presented in the form

$$\frac{dp_1}{dx} = C \exp\left(-\int \frac{d(kh)}{kh - 2\,\mathrm{cth}\,kh}\right). \tag{2.12}$$

Further integration, however, leads to very complex expressions which are not necessary for our present considerations.

In what follows we shall be interested in simplified solutions valid for sufficiently high frequencies, i.e. for the cases in which $kh \gg 1$ or $\nu h \gg 1$. Then, shkh, chkh, and cthkh

can be replaced by $\frac{1}{2}\exp kh$ and 1, respectively. The above conditions are equivalent to the assumption that

$$\mathrm{Re}_\omega = \frac{\varrho\omega h_0^2}{\eta} \gg 1, \tag{2.13}$$

or, more exactly, that

$$\left(\frac{1}{2}\,\mathrm{Re}_\omega\right)^{1/2} = \left(\frac{\varrho\omega}{2\eta}\right)^{1/2} h_0 \gg 1, \tag{2.14}$$

where Re_ω may be called "the frequentative Reynolds number". This number is entirely independent of the amplitude α. It can be checked that

$$\mathrm{Re}_\omega \gg 1, \quad \mathrm{Re}_v \ll 1 \quad \text{if} \quad \alpha/h_0 \ll 1. \tag{2.15}$$

Therefore we may consider the case of small-amplitude and high-frequency superposed vibrations if the amplitude of disturbances is much smaller than the distance between both parts of the bearing.

Under the above assumptions Eqs. (2.11) and (2.12) lead to

$$\frac{d^2p_1}{dx^2} + \frac{dp_1}{dx}\frac{a}{h} = 0, \tag{2.16}$$

and

$$\frac{dp_1}{dx} = C\exp\left(-\int\frac{d(kh)}{kh}\right) = C\ln(kh)^{-1}. \tag{2.17}$$

On taking into account the boundary conditions (2.3) we arrive at the approximate solution

$$p_1 = 0, \tag{2.18}$$

$$u_1 = Ue^{-k(h-y)} = Ue^{-\nu(h-y)}[\cos\nu(h-y) - i\sin\nu(h-y)]. \tag{2.19}$$

The corresponding lift force acting on the slider can be obtained by integration of the pressure p, viz.

$$P_N = \int_0^l p\,dx = \int_0^l p_0\,dx = \frac{6\eta U}{a^2}\left[\ln\lambda - 2\,\frac{\lambda-1}{\lambda+1}\right], \tag{2.20}$$

where

$$\lambda = \frac{h_0}{h_1} \quad (\lambda \geq 1). \tag{2.21}$$

In a similar way the mean friction forces acting on the slider and base can be determined by integration of the shear stresses. To his end, however, an averaging process over one cycle of vibrations should be aaplied, viz.

$$\langle\Phi\rangle = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} [\![\Phi]\!]\left(\mathrm{Re}\,u, \mathrm{Re}\,\frac{\partial u}{\partial y}, \ldots\right)dt, \tag{2.22}$$

where $[\![\Phi]\!]$ denotes a function of real parts of the kinematic quantities being variables in Φ (cf. [1]).

Thus we arrive at the following mean values of friction forces:

$$(2.23)\qquad \langle F_{NS}\rangle = \frac{\omega}{2\pi}\int\limits_0^{\frac{2\pi}{\omega}}\int\limits_0^l [\![T_N^{12}]\!]_{y=h}dx\,dt = -\frac{\eta U}{a}\left[4\ln\lambda - 6\frac{\lambda-1}{\lambda+1}\right]$$

for the moving and vibrating slider (*S*), and

$$(2.24)\qquad \langle F_{NB}\rangle = \frac{\omega}{2\pi}\int\limits_0^{\frac{2\pi}{\omega}}\int\limits_0^l [\![T_N^{12}]\!]_{y=0}dx\,dt = \frac{\eta U}{a}\left[-2ln\lambda + 6\frac{\lambda-1}{\lambda+1}\right]$$

for the stationary base (*B*). Since for Newtonian fluids

$$(2.25)\qquad [\![T_N^{12}]\!] = \eta\,\mathrm{Re}\left(\frac{\partial u_0}{\partial y}\right) + \varepsilon\eta\,\mathrm{Re}\left(\frac{\partial u_1}{\partial y}\exp i\omega t\right),$$

the friction forces are identical with those obtained for steady flows (cf. [1]).

2.2. Second-order viscoelastic fluids

The model of an incompressible second-order fluid can be used either in the case of slow flows or in the case of very short memory effects (slightly non-Newtonian fluids). If the Deborah number defined as

$$(2.26)\qquad \mathrm{De} = -\frac{a_1 U}{\eta h_0},\quad \text{for}\quad \alpha_1 < 0,$$

is sufficiently small, we can use the following constitutive equation:

$$(2.27)\qquad \mathbf{T} = -p\mathbf{1} + \eta\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,\quad \mathrm{tr}\,\mathbf{A}_1 = 0,$$

where α_1 and α_2 are material constants, p denotes a hydrostatic pressure. The Rivlin–Ericksen kinematic tensors are defined by the recurrence formulae

$$(2.28)\qquad \mathbf{A}_1 = \nabla\mathbf{v} + (\nabla\mathbf{v})^T,\quad \mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1\nabla\mathbf{v} + (\nabla\mathbf{v})^T\mathbf{A}_1.$$

If the velocity field $\mathbf{v}$ and the pressure p can be presented in the form

$$(2.29)\qquad \mathbf{v} = \mathbf{v}_N + \mathbf{v}',\quad p = p_N + p',$$

where the subscripts N denote Newtonian quantities and primes refer to second-order terms, the equations of motion with linearized inertia terms lead to

$$(2.30)\qquad \begin{aligned} \nabla p_N - \eta\nabla^2\mathbf{v}_N &= -\varrho\frac{\partial}{\partial t}\mathbf{v}_N,\\ \nabla p' - \eta\nabla^2\mathbf{v}' &= -\varrho\frac{\partial}{\partial t}\mathbf{v}' + \alpha_1\,\mathrm{div}\big(\mathbf{A}_2(\mathbf{v}_N) - \mathbf{A}_1^2(\mathbf{v}_N)\big) + (\alpha_1+\alpha_2)\mathrm{div}\,\mathbf{A}_1^2(\mathbf{v}_N). \end{aligned}$$

For plane isochoric flows with boundary conditions determined in velocities, the Tanner theorem is valid if only linearized inertia effects are involved (cf. [6, 2]). This means that $\mathbf{v}' = \mathbf{0}$ satisfies Eq. $(2.30)_2$, if

$$p' = \frac{\alpha_1}{\eta}\frac{dp_N}{dt} + \left(\frac{3}{2}\alpha_1 + \alpha_2\right)\varkappa^2, \quad \varkappa^2 = \frac{1}{2}\operatorname{tr}\mathbf{A}_1^2(\mathbf{v}_N). \tag{2.31}$$

Thus, under the assumption of lubrication approximation, we have

$$-T^{22} = p_N + \frac{\alpha_1}{\eta}\frac{dp_N}{dt} - \frac{1}{2}\alpha_1\left(\frac{\partial u_N}{\partial y}\right)^2, \tag{2.32}$$

while the shear stresses are exactly the same as those for Newtonian fluids.

2.3. Fluids with shear-dependent viscosity

The model of generalized Newtonian fluid with shear-dependent viscosity (decreasing, if $\eta_2 > 0$) (cf. [1]), viz.

$$\mathbf{T} = -p\mathbf{1} + \left(\eta - \frac{1}{2}\eta_2 \operatorname{tr}\mathbf{A}_1^2\right)\mathbf{A}_1, \quad \operatorname{tr}\mathbf{A}_1 = 0 \tag{2.33}$$

leads to

$$\nabla p_1 - \eta\nabla^2\mathbf{v}' = -\varrho\frac{\partial \mathbf{v}'}{\partial t} - \eta_2 \operatorname{div}\left(\varkappa^2\mathbf{A}_1(\mathbf{v}_N)\right), \quad \varkappa^2 = \frac{1}{2}\operatorname{tr}\mathbf{A}_1^2(\mathbf{v}_N). \tag{2.34}$$

Although in the present case the Tanner theorem is not generally valid, we shall try to integrate Eq. (2.34) in an approximate way, taking into account certain mean values and the lubrication approximation. Thus for $\mathbf{v}' = \mathbf{0}$, we have

$$\frac{\partial p'}{\partial x} = -3\eta_2\left(\frac{\partial u_N}{\partial y}\right)^2\frac{\partial^2 u_N}{\partial y^2}, \quad \frac{\partial p'}{\partial y} = 0, \tag{2.35}$$

and

$$p'_G = -3\eta_2 \int \left[\frac{1}{2}\left(\left.\frac{\partial u_N}{\partial y}\right|_{y=0} + \left.\frac{\partial u_N}{\partial y}\right|_{y=h}\right)\right]^2 \frac{1}{2}\left(\left.\frac{\partial^2 u_N}{\partial y^2}\right|_{y=0} + \left.\frac{\partial^2 u_N}{\partial y^2}\right|_{y=h}\right)dx + C_G \tag{2.36}$$

for the mean value of the shear gradient (G) across the slit, and

$$p'_P = -\frac{3}{2}\eta_2 \int \left[\left(\frac{\partial u_N}{\partial y}\right)^2_{y=0}\left.\frac{\partial^2 u_N}{\partial y^2}\right|_{y=0} + \left(\frac{\partial^2 u_N}{\partial y^2}\right)^2_{y=h}\left.\frac{\partial^2 u_N}{\partial y^2}\right|_{y=h}\right]dx + C_P \tag{2.37}$$

for the mean value of pressure itself (P).(1)

2.4. Perturbation method for superposed vibrations

In what follows we are interested in the changes of lift and friction forces caused by superposed harmonic vibrations. To this end, we apply a perturbation method similar to that proposed by JONES and WALTERS [3] (cf. also [5]).

(1) Integration constants C_G and C_P may be omitted in further considerations because of the boundary conditions (2.3) (cf. Appendix).

The expressions (2.5) can be used as the first terms in the series expansions with ε treated as a small parameter. Because of the boundary conditions (2.3), all the terms of order ε^2 and higher in series for u and p are equal to zero. Substituting Eqs. (2.5) either into Eq. (2.32) in the case of second-order fluids, or into Eqs. (2.36), (2.37) in the case of fluids with shear-dependent viscosity, we can calculate the ε^n-order terms ($n = 0, 1, 2$) of the corresponding lift forces. For $n = 0$, we obtain steady-state solutions, while for $n = 1, 2$, the mean values of dynamic forces can be determined by the averaging process shown in Eq. (2.22)). It is also worth noting that for $n = 1$, all the mean values of forces are equal to zero. Thus, for every lift force P, we have

$$\langle P\rangle = \langle P_0\rangle + \langle P_2\rangle, \tag{2.38}$$

where $<P_0> = P_0$ refers to a steady-state solution, and $<P_2>$ denotes a dynamic change of order ε^2 due to superposed harmonic vibrations.

In a similar way the changes of friction forces can be calculated.

3. Lift forces under superposed vibrations

For second-order fluids the perturbation method outlined in Sect. 2.4 leads to the following value of lift force acting on the slider in steady flows ($n = 0$):

$$P_{0S} = \int_0^l -T_0^{22}|_{y=h}\,dx = P_N - \frac{1}{2}\alpha_1 \int_0^l \left(\frac{\partial u_0}{\partial y}\right)^2_{y=h} dx = P_N + \frac{2\alpha_1 U^2}{ah_0}\left[3\,\frac{\lambda^3-1}{(\lambda+1)^2} - 2(\lambda-1)\right], \tag{3.1}$$

where P_N and λ are defined in Eqs. (2.20) and (2.21), respectively.

The mean value of the dynamic lift force caused by superposed vibrations ($n = 2$) amounts to

$$\langle P_{2S}\rangle = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\int_0^l [\![-T_2^{22}]\!]_{y=h}\,dx\,dt = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} -\frac{1}{2}\varepsilon^2\alpha_1\int_0^l \mathrm{Re}\left(\frac{\partial u_1}{\partial y}\exp i\omega t\right)_{y=h}\mathrm{Re}\left(\frac{\partial u_1}{\partial y}\exp i\omega t\right)_{y=h} dx\,dt = -\frac{1}{4}\varepsilon^2\alpha_1\int_0^l \left|\frac{\partial u_1}{\partial y}\right|^2_{y=h} dx, \tag{3.2}$$

where we used the relationship

$$\mathrm{Re}A\,\mathrm{Re}A = \frac{1}{2}[\mathrm{Re}A^2 + |A|^2], \tag{3.3}$$

with $|A|$ denoting the modulus of the complex function A. Substituting from Eq. (2.19) and assuming that, without essential loss of generality, $\operatorname{Im}\frac{\partial u_1}{\partial y} = 0$, we finally arrive at

$$(3.4) \qquad \langle P_{2S}\rangle = -\frac{1}{4}\varepsilon^2\alpha_1 U^2 \frac{\varrho\omega l}{\eta}.$$

For fluids with shear-dependent viscosity, we obtain the following values of lift forces in steady flows ($n = 0$):

$$(3.5) \qquad P_{0V/G} = \int_0^l p_{0G}\,dx = P_N - 3\eta_2 \int_0^l \int \left[\frac{1}{2}\left(\frac{\partial u_0}{\partial y}\bigg|_{y=0} + \frac{\partial u_0}{\partial y}\bigg|_{y=h}\right)\right]^2 \frac{\partial^2 u_0}{\partial y^2}\,dx\,dx$$

$$= P_N + 3\frac{\eta_2 U^3}{a^2 h_0^2}\left[\frac{\lambda^3-1}{\lambda+1} - (\lambda^2-1)\right],$$

calculated at the mean shear gradient (G), and

$$(3.6) \qquad P_{0V/P} = \int_0^l p_{0P}\,dx = P_N - \frac{3}{2}\eta_2 \int_0^l \int \left[\left(\frac{\partial u_0}{\partial y}\right)^2_{y=0} + \left(\frac{\partial u_0}{\partial y}\right)^2_{y=h}\right] \frac{\partial^2 u_0}{\partial y^2}\,dx\,dx$$

$$= P_N + 3\frac{\eta_2 U^3}{a^2 h_0^2}\left[\frac{9}{5}(\lambda^5-1)\left(\frac{2}{\lambda+1}\right)^3 - \frac{81}{10}(\lambda^4-1)\left(\frac{2}{\lambda+1}\right)^2 + 14(\lambda^3-1)\frac{2}{\lambda+1} - 10(\lambda^2-1)\right],$$

calculated at the mean pressure (P), respectively.

The mean values of the dynamic lift forces caused by superposed vibrations ($n = 2$) amount to

$$(3.7) \qquad \langle P_{2V/G}\rangle = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \int_0^l [\![p'_{2G}]\!]\,dx\,dt$$

$$= -\frac{3}{2}\varepsilon^2\eta_2 \int_0^l \int \Bigg\{\left|\frac{1}{2}\left(\frac{\partial u_1}{\partial y}\bigg|_{y=0} + \frac{\partial u_1}{\partial y}\bigg|_{y=h}\right)\right|^2 \operatorname{Re}\left(\frac{\partial^2 u_0}{\partial y^2}\right) + \operatorname{Re}\left[\frac{1}{2}\left(\frac{\partial u_1}{\partial y}\bigg|_{y=0} + \frac{\partial u_1}{\partial y}\bigg|_{y=h}\right)\right] \operatorname{Re}\left[\frac{1}{2}\left(\frac{\partial^2 u_1}{\partial y^2}\bigg|_{y=0} + \frac{\partial^2 u_1}{\partial y^2}\bigg|_{y=h}\right)\right] \operatorname{Re}\left[\frac{1}{2}\left(\frac{\partial u_0}{\partial y}\bigg|_{y=0} + \frac{\partial u_0}{\partial y}\bigg|_{y=h}\right)\right]$$

$$+ \operatorname{Im}\left[\frac{1}{2}\left(\frac{\partial u_1}{\partial y}\bigg|_{y=0} + \frac{\partial u_1}{\partial y}\bigg|_{y=h}\right)\right] \operatorname{Im}\left[\frac{1}{2}\left(\frac{\partial^2 u_1}{\partial y^2}\bigg|_{y=0} + \frac{\partial^2 u_1}{\partial y^2}\bigg|_{y=h}\right)\right] \times \operatorname{Re}\left[\frac{1}{2}\left(\frac{\partial u_0}{\partial y}\bigg|_{y=0} + \frac{\partial u_0}{\partial y}\bigg|_{y=h}\right)\right]\Bigg\}\,dx\,dx$$

for the mean value of the shear gradient (G), and

$$(3.8) \qquad \langle P_{2V/P}\rangle \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \int_0^l [\![p'_{2P}]\!]\,dx\,dt = -\frac{3}{4}\varepsilon^2\eta_2 \int_0^l \int \left\{\left[\left|\frac{\partial u_1}{\partial y}\right|^2 \operatorname{Re}\left(\frac{\partial^2 u_0}{\partial y^2}\right)\right.\right.$$

$$\text{(3.8) [cont.]}\quad +\mathrm{Re}\left(\frac{\partial u_1}{\partial y}\right)\mathrm{Re}\left(\frac{\partial^2 u_1}{\partial y^2}\right]\mathrm{Re}\left(\frac{\partial u_0}{\partial y}\right)\Bigg]_{y=0}+\Bigg[\left|\frac{\partial u_1}{\partial y}\right|^2\mathrm{Re}\left(\frac{\partial^2 u_0}{\partial y^2}\right)+\mathrm{Re}\left(\frac{\partial u_1}{\partial y}\right)\mathrm{Re}\left(\frac{\partial^2 u_1}{\partial y^2}\right)$$
$$+\mathrm{Re}\left(\frac{\partial u_0}{\partial y}\right)\Bigg]_{y=h}+\left[\mathrm{Im}\left(\frac{\partial u_1}{\partial y}\right)\mathrm{Im}\left(\frac{\partial^2 u_1}{\partial y^2}\right)\right]\mathrm{Re}\left(\frac{\partial u_0}{\partial y}\right)\Bigg]_{y=0}$$
$$+\left[\mathrm{Im}\left(\frac{\partial u_1}{\partial y}\right)\mathrm{Im}\frac{\partial^2 u_1}{\partial y^2}\right)\mathrm{Re}\left(\frac{\partial u_0}{\partial y}\right)\Bigg]_{y=h}\Bigg\}dx\,dx$$

for the mean value of pressure (P), respectively. In deriving Eqs. (3.7) and (3.8), we used the fact that

$$\text{(3.9)}\qquad \mathrm{Re}A\,\mathrm{Re}B\,\mathrm{Re}C=\frac{1}{4}\left[\mathrm{Re}(ABC)+\mathrm{Re}(\bar{A}BC)+\mathrm{Re}(A\bar{B}C)+\mathrm{Re}(AB\bar{C})\right],$$

where overbars denote the conjugate complex functions. Substituting from Eqs. (2.6) and (2.19), and assuming that $\mathrm{Im}\frac{\partial u_1}{\partial y}=\mathrm{Im}\frac{\partial^2 u_1}{\partial y^2}=0$, we obtain finally

$$\text{(3.10)}\qquad \langle P_{2V/G}\rangle=-\frac{9}{4}\,\varepsilon^2\eta_2 U^3\frac{\varrho\omega}{\eta a^2}\left[\ln\lambda-\frac{\lambda-1}{\lambda+1}\right],$$

$$\text{(3.11)}\qquad \langle P_{2V/P}\rangle=2\langle P_{2V/G}\rangle.$$

The relative increase or decrease of the mean dynamic lift forces as compared with the lift forces acting in steady flows can be characterized by the following ratios (cf. [1]):

$$\text{(3.12)}\qquad I_{LN}=\frac{\langle P_{2S}\rangle+P_{0S}-P_{0S}}{P_{0S}}=\frac{\langle P_{2S}\rangle}{P_{0S}},$$

and

$$\text{(3.13)}\qquad I_{LV/G}=\frac{\langle P_{2V/G}\rangle}{P_{0V/G}},\qquad I_{LV/P}=\frac{\langle P_{2V/P}\rangle}{P_{0V/P}},$$

where the subscripts LN, LV/G and LV/P mean "the lift force caused by normal stresses", "the lift force caused by variable viscosity, calculated at mean shear gradients" and "the lift force caused by variable viscosity, calculated at mean pressures", respectively.

Thus, Eqs. (3.12) and (3.13) lead to

$$\text{(3.14)}\qquad I_{LN}=\frac{\varepsilon^2\mathrm{Re}_\omega \mathrm{De}\,s_0(\lambda^2-1)^2}{24\lambda^2(\lambda+1)[(\lambda+1)\ln\lambda-2(\lambda-1)]+8\,\mathrm{De}\,s_0\,\lambda[3(\lambda^3-1)(\lambda-1)-2(\lambda-1)^2(\lambda+1)^2]},$$

where $s_0=h_0/l$, and to

$$\text{(3.15)}\qquad I_{LV/G}=\frac{-\varepsilon^2\mathrm{Re}_\omega V[(\lambda+1)\ln\lambda-(\lambda-1)]}{8[(\lambda+1)\ln\lambda-2(\lambda-1)]-4V\lambda(\lambda-1)},$$

$$\text{(3.16)}\qquad I_{LV/P}=\frac{-5\varepsilon^2\mathrm{Re}_\omega\cdot V(\lambda+1)^2[(\lambda+1)\ln\lambda-(\lambda-1)]}{20(\lambda+1)^2[(\lambda+1)\ln\lambda-2(\lambda-1)]+V[144(\lambda^5-1)-324(\lambda^4-1)\times(\lambda+1)+280(\lambda^3-1)(\lambda+1)^2-100(\lambda^2-1)(\lambda+1)^3]}$$

where

$$V = \frac{\eta_2 U^2}{\eta h_0^2} \tag{3.17}$$

is the dimensionless number characterizing a decrease of shear-dependent viscosity.

Diagrams illustrating dependence of I_{LN} and $I_{LV/G}$, $I_{LV/P}$ (divided by $\varepsilon^2 \mathrm{Re}_\omega$) on $\mathrm{De} \cdot s_0$ and V, respectively, are shown in Figs. 2 and 3. It is seen from Fig. 2 that the relative increase of the dynamic lift force caused by normal stress is always positive and tends to 0.25 for increasing Deborah numbers. The effect of enhancement, however, is weaker for larger λ. On the other hand, Fig. 3 shows that the relative increase of the dynamic forces caused by shear-dependent viscosity is negative. The reduction of forces is

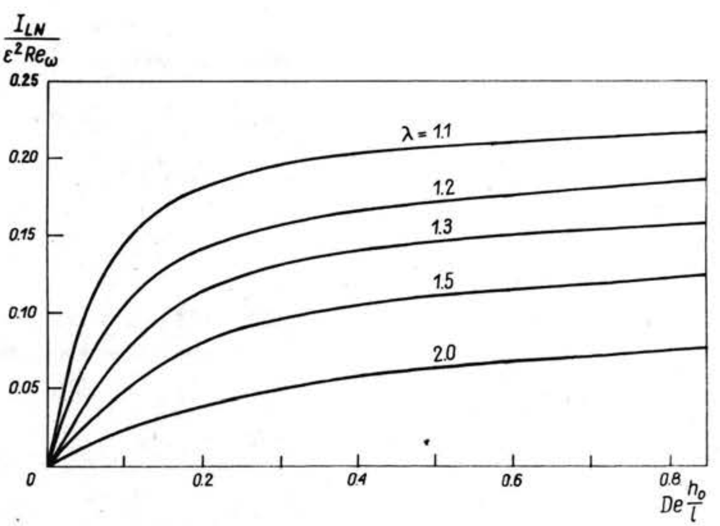

FIG. 2.

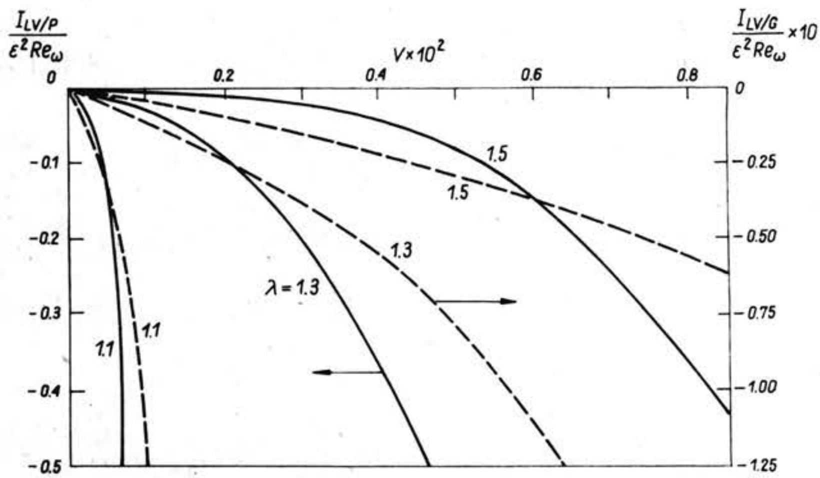

FIG. 3.

weaker for increasing λ. Essential differences are clearly visible between the results calculated approximately at the mean shear gradient (G) and the mean pressure (P).

We must bear in mind that all the diagrams considered are reliable only for λ little differing from unity (lubrication approximation).

4. Friction forces under superposed vibrations

For fluids with shear-dependent viscosity we obtain the following values of friction forces in steady flows ($n = 0$):

$$(4.1)\quad F_{0S} = \int_0^l T_0^{12}|_{y=h}\,dx = \langle F_{NS}\rangle - \eta_2 \int_0^l \left(\frac{\partial u_0}{\partial y}\right)^3_{y=h} dx$$
$$= \langle F_{NS}\rangle - \frac{\eta_2 U^3}{a h_0^2}\left[4(\lambda^2-1) - 24\,\frac{\lambda^3-1}{\lambda+1} + 54\,\frac{\lambda^4-1}{(\lambda+1)^2} - \frac{216}{5}\,\frac{\lambda^5-1}{(\lambda+1)^3}\right],$$

in the case of moving slider (S), and

$$(4.2)\quad F_{0B} = \int_0^l T_0^{12}|_{y=0}\,dx = \langle F_{NB}\rangle - \eta_2 \int_0^l \left(\frac{\partial u_0}{\partial y}\right)^3_{y=0} dx$$
$$= \langle F_{NB}\rangle - \frac{\eta_2 U^3}{a h_0^2}\left[\frac{216}{5}\,\frac{\lambda^5-1}{(\lambda+1)^3} - \frac{27}{4}\,\frac{(\lambda^2-1)(3\lambda^2+13)}{(\lambda+1)^2} + 96\,\frac{\lambda^3-1}{\lambda+1} - 32(\lambda^2-1)\right],$$

in the case of the stationary base (B). The Newtonian forces $\langle F_{NS}\rangle$ and $\langle F_{NB}\rangle$ are defined in Eqs. (2.23) and (2.24).

The mean values of the dynamic friction forces caused by superposed vibrations ($n = 2$) are

$$(4.3)\quad \langle F_{2S}\rangle = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\int_0^l [\![T_2^{12}]\!]_{y=h}\,dx\,dt$$
$$= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} -3\eta_2\varepsilon^2 \int_0^l \mathrm{Re}\left(\frac{\partial u_1}{\partial y}\exp i\omega t\right)_{y=h} \mathrm{Re}\left(\frac{\partial u_1}{\partial y}\exp i\omega t\right)_{y=h} \mathrm{Re}\left(\frac{\partial u_0}{\partial y}\right)_{y=h} dx\,dt$$
$$= -\frac{3}{2}\varepsilon^2\eta_2 \int_0^l \left|\frac{\partial u_1}{\partial y}\right|^2_{y=h} \mathrm{Re}\left(\frac{\partial u_0}{\partial y}\right)_{y=h} dx,$$

for the moving and vibrating slider (S), and

$$(4.4)\quad \langle F_{2B}\rangle = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\int_0^l [\![T_2^{12}]\!]_{y=0}\,dx\,dt$$

$$= \frac{\omega}{2\pi}\int\limits_0^{\frac{2\pi}{\omega}} -3\eta_2\varepsilon^2\int\limits_0^l \mathrm{Re}\left(\frac{\partial u_1}{\partial y}\exp i\omega t\right)_{y=0}\mathrm{Re}\left(\frac{\partial u_1}{\partial y}\exp i\omega t\right)_{y=0}\mathrm{Re}\left(\frac{\partial u_0}{\partial y}\right)_{y=0} dx\,dt$$

(4.4) [cont.]

$$= -\frac{3}{2}\varepsilon^2\eta_2\int\limits_0^l \left|\frac{\partial u_1}{\partial y}\right|^2_{y=0}\mathrm{Re}\left(\frac{\partial u_0}{\partial y}\right)_{y=0} dx,$$

for the stationary base (*B*).

Assuming, as previously, that $\mathrm{Im}\frac{\partial u_1}{\partial y} = 0$, and substituting from Eq. (2.19), we finally arrive at

$$\langle F_{2S}\rangle = 3\varepsilon^2\eta_2 U^3\frac{\varrho\omega}{\eta a}\left[2\ln\lambda - 3\frac{\lambda-1}{\lambda+1}\right], \quad \langle F_{2B}\rangle \approx 0. \tag{4.5}$$

Introducing the ratios characterizing the relative increase or decrease of the mean dynamic friction forces as compared with the friction forces acting in steady-state flows (cf. [1]), viz.

$$\tilde{\bar{I}}_{FS} = \frac{\langle F_{2S}\rangle}{F_{0S}}, \quad \tilde{\bar{I}}_{FB} = \frac{\langle F_{2B}\rangle}{F_{0B}} \approx 0, \tag{4.6}$$

where the subscripts *FS* and *FB* mean "the friction force on the slider" and "the friction force on the base", respectively. Thus Eq. $(4.6)_1$ gives

$$\tilde{\bar{I}}_{FS} = \frac{-30\varepsilon^2\mathrm{Re}_\omega V(\lambda+1)^2[4(\lambda+1)\ln\lambda - 6(\lambda-1)]}{20(\lambda+1)^2[4(\lambda+1)\ln\lambda-6(\lambda-1)] + V[864(\lambda^5-1)-135(\lambda^2-1)\times(3\lambda^2+13)(\lambda+1)+1920(\lambda^3-1)(\lambda+1)^2-640(\lambda^2-1)(\lambda+1)^3]} \tag{4.7}$$

where V is defined in Eq. (3.17)

A diagram illustrating the dependence of $\tilde{\bar{I}}_{FS}$ (divided by $\varepsilon^2\mathrm{Re}_\omega$) on V is shown in Fig. 4. It is seen that the relative decrease of the dynamic friction force on the moving and vibrating slider is diminished for increasing λ. If V is greater than 0.1, the reduction of friction forces practically does not depend on values of V.

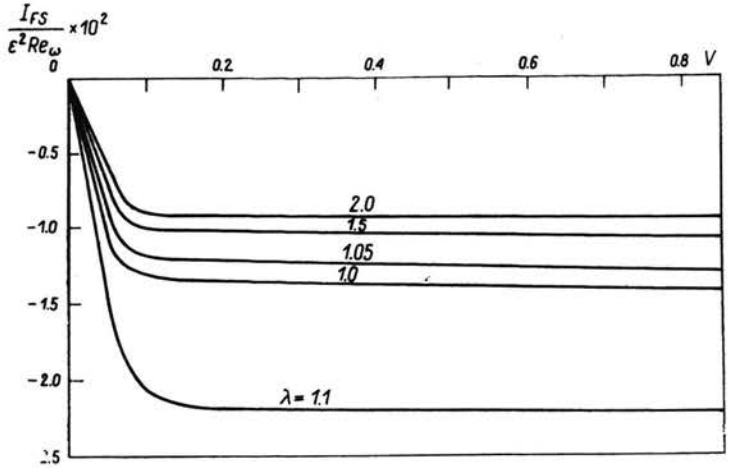

FIG. 4.

In our previous paper [1] we analysed the case in which the lower part of the bearing (base) is moving and vibrating, while the upper one (slider) remains stationary. In a similar way we may conclude that in the present problem

$$\underset{\sim}{\underline{I}}_{FS} = \tilde{\bar{I}}_{FB} \approx 0, \quad \underset{\sim}{\underline{I}}_{FB} = \tilde{\bar{I}}_{FS} < 0, \tag{4.8}$$

where overbars and underbars show which part of the bearing (upper or lower) moves with constant velocity U. Moreover, upper and lower tildas show to which part of the bearing vibrations are applied.

We can also discuss the mixed cases in which vibrations are superposed on the parts of the bearing remaining stationary in the fundamental flow. Using similar notations, we have

$$\underset{\sim}{\bar{I}}_{FS} = -\tilde{\bar{I}}_{FS} > 0, \quad \underset{\sim}{\bar{I}}_{FB} = -\tilde{\bar{I}}_{FB} \approx 0, \tag{4.9}$$

for the moving slider and vibrating base, and

$$\tilde{\underline{I}}_{FS} = -\underset{\sim}{\underline{I}}_{FS} \approx 0, \quad \tilde{\underline{I}}_{FB} = -\underset{\sim}{\underline{I}}_{FB} > 0, \tag{4.10}$$

for the moving base and vibrating slider. Thus, rather an enhancement of the friction forces is observed in the mixed cases under consideration.

5. Discussion

Many results obtained in this paper can be compared with those presented previously in the paper [1].

The general effects of superposed small-amplitude harmonic vibrations on the lift forces caused by normal stresses and shear-dependent (decreasing) viscosity are very similar in the case of low-frequency vibrations (inertialess solutions) and in the case of high-frequency vibrations. Apart from quantitative differences, in both cases the lift forces caused by normal stresses (for second-order fluids) are enhanced while the lift forces caused by shear-dependent (decreasing) viscosity (for generalized Newtonian fluids) are seriously reduced. One may expect that for real viscoelastic fluids (polymer solutions and additives) the lift forces will take mean values between those presented for two limit cases.

On the contrary, the effects of superposed small-amplitude vibrations on the friction forces acting either on the slider or the base are quite different for small-frequency (inertialess solutions) and high-frequency vibrations. In the first case, a strong enhancement of the friction forces acting on the moving and vibrating slider is observed, while, in the second case, these forces may be reduced considerably.

Apart from the deficiency of the model considered and less important numerical differences, a question arises connected with domains of validity of the present solutions for friction forces as compared with those discussed elsewhere [1]. To answer partly the above question, we can formally equate the absolute value of the ratio (4.7) with the corresponding ratio in the paper [1] (Eq. (4.11)). In such a way we obtain "the frequentative Reynolds number" Re_ω defined in Eq. (2.13) for which positive and negative changes of the friction forces acting on the slider, derived in both cases, are mutually cancelled. Thus we have the following critical values:

$$(5.1) \qquad (\mathrm{Re}_\omega)_c = \frac{864(\lambda^5-1)-135(\lambda^2-1)(3\lambda^3+13)(\lambda+1) + 1920(\lambda^3-1)(\lambda+1)^2-640(\lambda^2-1)(\lambda+1)^3}{20(\lambda+1)^2[4(\lambda+1)\ln\lambda-6(\lambda-1)]},$$

where $\lambda = h_0/h_1$ characterizes inclination of the slider. In Fig. 5, $(\mathrm{Re}_\omega)_c$ is plotted versus λ for λ ranging from 1 to 2. It is worth noting that $(\mathrm{Re}_\omega)_c$ takes the values from 53 for $\lambda = 1$ (horizontal slider) to less than 100 for λ little differing from unity.

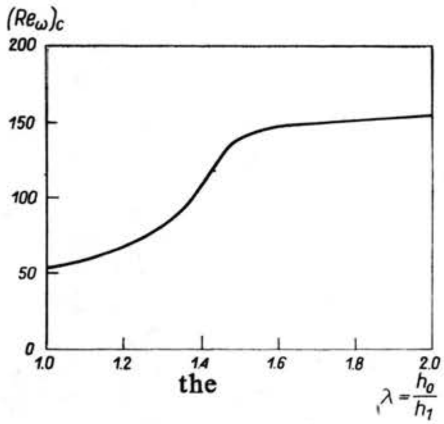

FIG. 5.

Therefore we may conclude that for frequencies for which $\mathrm{Re}_\omega \ll (\mathrm{Re}_\omega)_c$, the results based on the inertialess solutions (cf. [1]) are qualitatively reliable, while for frequencies for which $\mathrm{Re}_\omega \gg (\mathrm{Re}_\omega)_c$, the dynamic solutions for small-amplitude but high-frequency vibrations are more useful.

Appendix

Because of the boundary conditions (2.3), the integration constants C_G and C_P in Eqs. (2.36) and (2.37) may be equal to zero if also: $p'_G(0, y) = p'_G(l, y) = 0$, $p'_P(0, y) = p'_P(l, y) = 0$. Substituting Eq. (2.6) into Eq. (2.36), we obtain for steady flows ($n = 2$)

$$(\mathrm{A.1}) \qquad p'_{0G}(x) = -\frac{3\eta_2 U^2}{\eta}\int \frac{1}{h^2}\frac{dp_0}{dx}\,dx \approx -\frac{3\eta_2 U^2}{\eta h_m^2}\,p_0(x),$$

where approximate integration is performed for h replaced by $h_m = \frac{1}{2}(h_0+h_1)$ or $h_m = H$. Since $p_0(0) = p_0(l) = 0$, we also have $p'_{0G} \approx 0$. Similarly, on the basis of Eq. (2.36), we arrive at

$$(\mathrm{A.2}) \qquad p'_{0P}(x) = -3\eta_2 \int \left[\frac{U^2}{h^2} + \frac{h^2}{4\eta^2}\left(\frac{dp_0}{dx}\right)^2\right]\frac{1}{\eta}\frac{dp_0}{dx}\,dx \approx$$

(A.2) [cont.]

$$\approx -\frac{3\eta_2 U^2}{\eta h_m^2} p_0(x) - \frac{3\eta_2 h_m^2}{4\eta^3} \int \left(\frac{dp_0}{dx}\right)^3 dx$$

$$= -\frac{3n_2 U^2}{\eta h_m^2} p_0(x) - \frac{3\eta_2 h_m^2}{4\eta^3} \left[\left(\frac{dp_0}{dx}\right)^2 p_0(x) - \frac{d^2 p_0}{dx^2} p_0^2(x) + \int p_0^2(x) \frac{d^3 p_0}{dx^3} dx\right].$$

Since, moreover, $(d^3 p_0)/(dx^3)$ has very small values in the interval $[0, l]$ (for parabolic approximation of the pressure profile, it is exactly equal to zero), and $p_0(x)$ is equal to zero for $x = 0$ and $x = l$, the integrand on the right-hand side of Eq. (A.2) is also very small for each $x \in [0, l]$. Thus the boundary conditions for steady pressures are satisfied approximately.

Substituting Eqs. (2.6) and (2.19) into Eqs. (2.36) and (2.37), we obtain for dynamic flows ($n = 2$) the following mean values of pressures (cf. also Eqs. (3.7) and (3.8)):

$$\text{(A.3)} \qquad \langle p'_{2G} \rangle = -\frac{3\eta_2 \varepsilon^2 U^2 \nu^2}{4\eta} \int \frac{dp_0}{dx} dx = -\frac{3\eta_2 \varepsilon^2 U^2 \nu^2}{4\eta} p_0(x)$$

and

$$\text{(A.4)} \qquad \langle p'_{2P} \rangle = -\frac{3\eta_2 \varepsilon^2 U^2 \nu^2}{2\eta} \int \frac{dp_0}{dx} dx = -\frac{3\eta_2 \varepsilon^2 U^2 \nu^2}{2\eta} p_0(x).$$

Since $p_0(0) = p_0(l) = 0$, we also have $<p'_{2G}> = <p'_{2P}> = 0$ for $x = 0$ and $x = l$. Thus the boundary conditions for the mean dynamic pressures are satisfied exactly.

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