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Optimization of structures subjected to aeroelastic instability phenomena

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IN THIS PAPER problems concerning sensitivity analysis and optimization of aeroelastic stability of structures are considered. In Sect. 1 the problem of maximizing the critical speed of aeroelastic stability of the plate in supersonic gas flow is stated. Sensitivity analysis of continuous fluttering system is developed. The gradients of flutter and divergence speed are first derived and with their use necessary optimality conditions are established. The method of solution is described and numerical results are presented. In Sect. 2 discrete aeroelastic systems are investigated. The implicit expressions of the derivatives of flutter and divergence speed with respect to arbitrary discrete parameters of the system are achieved. A short review of studies made in this field is given.

W pracy rozpatrzono problemy dotyczące analizy podatności i optymalizacji stateczności aerosprężystej konstrukcji. W rozdziale 1 sformułowano zagadnienie maksymalizacji prędkości krytycznej dla aerosprężystej stateczności płyty w opływie naddźwiękowym. Rozwinięto analizę podatności dla układów ciągłych poddanych flatterowi. Wyprowadzono najpierw wzory na gradienty flatteru i prędkość dywergencji i wykorzystano do określenia warunków optymalizacji. Opisano metodę rozwiązania i przedstawiono wyniki numeryczne. W rozdziale drugim omówiono dyskretne układy aerosprężyste. Otrzymano niejawne związki dla pochodnych flatteru i prędkości dywergencji względem dowolnych parametrów dyskretnych rozważanego układu. Zamieszczono również krótki przegląd prac dotyczących tej problematyki.

В данной работе для распределенных и дискретных систем расматриваются вопросы, связанные с анализом чувствительности и оптимизацией характеристик аэроупругой устойчивости. В первом разделе ставится задача максимизации критической скорости потери аэроупругой устойчивости пластинки в сверхзвуковом потоке газа. Производится анализ чувствительности распределенных систем, подверженных флаттеру. Впервые получены выражения градиентов для критических скоростей флаттера и дивергенции и с их использованием выведены необходимые условия экстремума. Дано описание метода решения задачи оптимизации, представлены численные результаты. Во втором разделе исследуются дискретные аэроупругие системы. Получены явные выражения для производных от критических скоростей флаттера и дивергенции по отношению к произвольным дискретным параметрам, характеризующим систему. Дан краткий обзор работ, посвященных оптимизации характеристик аэроупругой устойчивости.

Introduction

AT PRESENT the problems of optimization of structures dealing with phenomena of aeroelastic instability are of great interest to researchers. The first publications in this field appeared about ten years ago. There are now about fifty. Some, though incomplete, information about the papers in this field may be found in [1–16] with corresponding lists of references. As compared with static structural optimization problems, the problems of dynamic stability optimization cause greater difficulties in analytic and numerical studies. This fact is emphasized by many authors. At present, aeroelastic stability optimization problems are developed in two ways. The first considers optimization problems of distributed systems governed by differential equations. See, for example, the works of ASHLEY and MCINTOSH [1], BIRIUK [4], PIERSON [5, 9], WEISSHAAR [6, 11], ARMAND [8], PLAUT [7], VEPA (thesis), SANTINI and others [12], BANICHUK and MIRONOFF [15], SEYRANIAN [16] and others. In the papers [4, 5, 9, 11] the direct methods of mathematical programming for the search of optimal solutions were used.

The second way of developing aeroelastic stability optimization problems considers discrete systems, see, for example, works of BUN'KOV [2], TURNER [3], HAFTKA and others [10], MCINTOSH and ASHLEY [13], CARDANI and MANTEGAZZA [14] and others.

In this paper an attempt is made to develop a common variational approach to study both distributed and discrete systems subjected to aeroelastic instability phenomena. In Sect. 1 this approach is illustrated by the problem of determining the thickness function of solid plate of constant volume having maximal critical speed of aeroelastic stability. The expressions of sensitivity functions — functional gradients of critical flutter and divergence speed with respect to thickness variation — are obtained. The necessary optimality conditions are established and with their use the optimal solution is obtained numerically. In Sect. 2 sensitivity analysis and optimization problems are studied on linear models of discrete systems of general type subjected to aeroelastic instability phenomena.

1. Panel flutter optimal problem

1.1. Statement

Let us consider vibrations of a thin elastic plate of variable thickness in supersonic gas flow. It is assumed that the plate is symmetric with respect to its neutral plane and that the plate span is much greater than its dimension in the direction of the flow (Fig. 1).

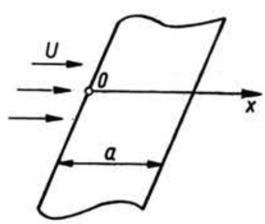

FIG. 1.

For the description of aerodynamic forces we use the linearized piston theory formula of A. A. Ilyushin. The equation of vibration has the form [17]

$$\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 w}{\partial x^2}\right)+2H(x)\varrho\frac{\partial^2 w}{\partial t^2}+g=0,$$

(1.1)

$$g=\frac{2p_0\varkappa}{c_0}\left(\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}\right),\qquad D(x)=\frac{2EH^3(x)}{3(1-\nu^2)}.$$

Here $D(x)$, $2H(x)$, $w(x, t)$ are, respectively, bending stiffness, thickness and deflection functions of the plate. The parameters ϱ, E, ν, U, $\varkappa$, p_0, c_0 denote density of the plate material, Young's modulus, the Poisson's ratio, speed of undisturbed flow, polytropic exponent, pressure and sound speed of a gas at infinity.

Consider solutions of plate deflection expressed by the formula

(1.2) $$w(x, t) = u(x)e^{st},$$

where $u(x)$ is a complex function of the real variable x and s is a complex number. With the use of Eq. (1.2) the vibrational equation (1.1) in nondimensional variables takes the form

(1.3) $$(h^3u'')'' + \sigma^2 hu + \Lambda\sigma u + \beta u' = 0.$$

Here the nondimensional quantities are used:

$$\tilde{x} = x/a, \quad \tilde{u}(\tilde{x}) = u(\tilde{x})/a, \quad h(\tilde{x}) = H(\tilde{x})/a,$$

$$\sigma = s\sqrt{\frac{3\varrho a^2(1-\nu^2)}{E}}, \quad \Lambda = \frac{p_0\varkappa}{c_0}\sqrt{\frac{3(1-\nu^2)}{E\varrho}},$$

$$\beta = \frac{3(1-\nu^2)p_0\varkappa}{Ec_0}U.$$

The primes denote differentiation with respect to x, the sign tilda over the symbols in Eq. (1.3) and below is dropped.

When $h(x)$, Λ and β are given, Eq. (1.3) with the boundary conditions determines the non-selfadjoint eigenvalue problem in which σ is an eigenvalue. The equilibrium form $u \equiv 0$ of the plate in gas flow is stable when all eigenvalues σ belong to the left half of the complex plane, i.e. $\mathrm{Re}\,\sigma < 0$. When the quantities $h(x)$ and Λ are fixed, the mentioned equilibrium form may become unstable for some values of nondimensional speed β. Critical divergence speed β_d is determined by the condition $\sigma = 0$ and critical flutter speed β_f is characterized by the relations $\mathrm{Re}\,\sigma = 0$, $\mathrm{Im}\,\sigma = \omega \neq 0$ [17].

The nondimensional volume of the plate is

(1.4) $$V = \int_0^1 h(x)dx.$$

Now we state the optimization problem: it is necessary to find the thickness function $h_0(x)$ satisfying the constant volume condition $V(h_0) = V_0$ and maximizing the minima of critical speeds β_d, β_f.

The quantities Λ and V_0 are the problem parameters. The mathematical formulation of the stated problem is described by the relations

(1.5) $$\max_{h\in\Omega}\min[\beta_f(h), \beta_d(h)] = \min[\beta_f(h_0), \beta_d(h_0)],$$

$$\Omega = \left\{h(x)\colon V(h) = \int_0^1 h\,dx = V_0,\ h(x) \geqslant 0\right\}.$$

1.2. Variations of critical speeds

Now we calculate the variations of flutter and divergence critical speeds with respect to thickness variation $\delta h(x)$. Consider the vibrational equation at flutter assuming in Eq. (1.3) $\sigma = i\omega$, $\beta = \beta_f$

$$Lu \equiv (h^3u'')'' - \omega^2 hu + i\omega\Lambda u + \beta_f u' = 0, \tag{1.6}$$

where i is imaginary unity $i = \sqrt{-1}$ and ω is flutter frequency. To calculate the increment of critical flutter speed, we include the equation in variations taking to the functions h, u the variations δh, $\delta u = \delta u_1 + i\delta u_2$, the frequency and the critical flutter speed — variations $\delta\omega$, $\delta\beta_f$.

Then we multiply the equation in variations by the arbitrary complex function $v(x)$ and integrate from 0 to 1. Using integration by parts we get

$$\begin{aligned}\int_0^1 (3h^2u''v'' - \omega^2 uv)\delta h dx + \delta\omega\Big(-2\omega\int_0^1 huv dx + i\Lambda\int_0^1 uv dx\Big) \\ + \int_0^1 [(h^3v'')'' - \omega^2 hv + i\Lambda\omega v - \beta_f v']\delta u dx + \delta\beta_f\int_0^1 u'v dx + [v\delta(h^3u'')' \\ - v'\delta(h^3u'') + (h^3v'')\delta u' - \big((h^3v'')' - \beta_f v\big)\delta u]_0^1 = 0.\end{aligned} \tag{1.7}$$

For concrete definition we consider, for instance, a clamped plate which corresponds to the boundary conditions

$$u(0) = u'(0) = 0, \quad u(1) = u'(1) = 0. \tag{1.8}$$

In this case we imply on the function v the next conditions:

$$L^*v \equiv (h^3v'')'' - \omega^2 hv + i\Lambda\omega v - \beta_f v' = 0, \tag{1.9}$$

$$v(0) = v'(0) = 0, \quad v(1) = v'(1) = 0. \tag{1.10}$$

Equation (1.9) with the boundary conditions represents the adjoint eigenvalue problem for $v(x)$ with respect to the problem (1.7), (1.8). It is known that the eigenvalues and their multiplicity in adjoint problems are the same [18]. Note that the relations (1.9) and (1.10) describe flutter instability of the same plate with inverse direction of the flow.

Let us introduce the notations

$$\begin{aligned} A &= 3h^2u''v'' - \omega^2 uv, \\ B &= \int_0^1 \Big(v\frac{\partial L}{\partial\omega}u\Big)dx = -2\omega\int_0^1 huv dx + i\Lambda\int_0^1 uv dx, \\ C &= \int_0^1 \Big(v\frac{\partial L}{\partial\beta_f}u\Big)dx = \int_0^1 u'v dx. \end{aligned} \tag{1.11}$$

Note that B and C are complex constants and the function A is a complex function. With the use of Eqs. (1.8)–(1.11) the expression (1.7) becomes (1.12)

$$\int_0^1 A\delta h dx + B\delta\omega + C\delta\beta_f = 0. \tag{1.12}$$

Now we multiply Eq. (1.12) by the complex — conjugate to B quantity $\bar{B}$ and take the imaginary part. Because $\mathrm{Im}(B\bar{B}) = 0$ and $\delta\beta_f$, $\delta\omega$ are real quantities, we obtain the expression for variation:

$$(1.13) \qquad \delta\beta_f = \int_0^1 g\delta h dx, \quad g = -\frac{\mathrm{Im}(A\bar{B})}{\mathrm{Im}(C\bar{B})}.$$

So the function g is the gradient of the functional of critical flutter speed with respect to the control function $h(x)$.

Similarly, from Eq. (1.12) the variation of the flutter frequency may be achieved:

$$(1.14) \qquad \delta\omega = \int_0^1 t\delta h dx, \quad t = -\frac{\mathrm{Im}(A\bar{C})}{\mathrm{Im}(B\bar{C})}.$$

Thus, to determine the gradients g and t it is required to solve main and adjoint problems of flutter instability (1.6), (1.8); (1.9), (1.10) and calculate the complex functions $u(x)$, $b(x)$ and real quantities β_f, ω. Then, according to Eq. (1.11) the complex constants B, C and complex function A can be determined and thus the gradients g and t can be found from Eqs. (1.13) and (1.14). Note that the eigenfunctions u and v are determined up to an arbitrary complex constant because eigenvalue problems of flutter instability are homogeneous problems. Nevertheless, it is easy to see that the gradients g and t are not changed when the functions u and v are multiplied by arbitrary complex constants.

The plate may violate its stability by the static form (divergence). Let us determine the variation of critical divergence speed. Taking in Eqs. (1.6), (1.7) and (1.9) $\omega = 0$, $\delta\omega = 0$ and repeating the calculations given above, we get

$$(1.15) \qquad \delta\beta_d = \int_0^1 e\delta h dx, \quad e = -\frac{3h^2u''v''}{\int_0^1 u'v dx}.$$

The function e represents the gradient of the critical divergence speed. The eigenfunctions u and v in this case are real quantities.

Knowing the sensitivity functions — the gradients of the critical flutter and divergence speed — we can improve the dynamic stability characteristics of the system by a rational way.

REMARK. When the boundary conditions (1.8) for the function u are changed, then only the boundary conditions for the adjoint function v are to be replaced. All the remaining relations required for the calculation of the gradients g, e and t are the same.

1.3. Optimality conditions

Now we derive the necessary optimality conditions in the stated problem (1.5). Since the gradient of the plate volume functional is equal to 1, we obtain the necessary conditions of optimality of the function $h_0(x)$ [20, 21]:

$$(1.16) \qquad \lambda e(x) + (1-\lambda)g(x) + \mu = 0,$$

$$\begin{array}{lll} \lambda = 0 & \text{if} & \beta_f(h_0) < \beta_d(h_0), \\ \lambda = 1 & \text{if} & \beta_d(h_0) < \beta_f(h_0), \\ 0 \leqslant \lambda \leqslant 1 & \text{if} & \beta_d(h_0) = \beta_f(h_0). \end{array} \tag{1.17}$$

The Lagrange's multipliers λ and μ are determined by the isoperimetric condition $V(h) = V_0$ and the conditions (1.17). The cases $\lambda = 0$ and $\lambda = 1$ corresponds respectively to the problems of maximization of flutter and divergence speed. The last case $0 \leqslant \leqslant \lambda \leqslant 1$ corresponds to the equality condition of critical speeds at optimal solution.

Note that the optimal problem of panel flutter similar to Eq. (1.5) was considered by TURNER in [3]. The problem of maximization of bending-torsion flutter critical speed was considered by Vepa in his thesis. Note that the necessary optimality conditions obtained in these works do not agree with the strict relations presented above. The reason of this discord lies in the fact that at the derivation of optimality conditions flutter frequency was not varied. Besides, real and complex quantities were not distinguished in the thesis of Vepa.

1.4. Symmetry in boundary conditions

In this section we shall establish some properties of the gradients g, t and e and the optimal solution $h_0(x)$ in the case of symmetrical boundary conditions applied to the mode of vibration $u(x)$.

Consider simply supported or clamped boundary conditions. Let us do the transformation of the argument $\xi = 1-x$. It is easy to see that in this case Eq. (1.6) for the function $u(x)$ is replaced by Eq. (1.9) and Eq. (1.9) for the function $v(x)$ is replaced by Eq. (1.6). The boundary conditions for the functions $u(x)$ and $v(x)$ are the same because of the symmetry of the boundary conditions. Note further that the quantities A and B in Eq. (1.11) are symmetrical with respect to u and v and do not change when the transformation $\xi = 1-x$ is made. As to C it is transformed as follows:

$$C = \int_0^1 \frac{du}{dx} v dx = \int_0^1 \frac{dv}{d\xi} u d\xi.$$

Here the integration by parts was used. Using the expressions (1.13)–(1.15), we conclude that the gradients $g(\xi)$, $t(\xi)$, $e(\xi)$ differ from the respective relations (1.13)–(1.15) only by the notations $u \to v$, $v \to u$. So the gradients g, t and e in the case of symmetry of the boundary conditions applied to the function u are invariant with respect to transformation $\xi = 1-x$.

Let us analyse the necessary optimality conditions. Because of proved invariance of the gradients with respect to transformation $\xi = 1-x$ the next assertions are valid:

1. If $h_0(x)$, $u(x)$, $v(x)$ is the solution of the system of the necessary conditions (1.6), (1.9), (1.16) and (1.17) with the symmetrical boundary conditions, then the functions

$$h_0(\xi) = h_0(1-x), \quad av(\xi) = av(1-x), \quad bu(\xi) = bu(1-x)$$

are also the solution of this system, u and v being either vibrational modes of flutter or modes of divergence; a and b are arbitrary complex constants.

2. If there exists a unique solution which realizes the maximal value of critical speed of dynamic stability at symmetrical boundary conditions, then

$$h_0(\xi) = h_0(1-x) = h_0(x), \quad u(\xi) = u(1-x) = av(x), \quad v(\xi) = v(1-x) = bu(x), \qquad \text{or} \qquad h_0(x) = h_0(1-x), \quad v(x) = cu(1-x),$$

where c is an arbitrary complex constant.

The symmetrical optimal solutions obtained numerically for simply supported and clamped plates are presented in the works of PIERSON [5, 9], WEISSHAAR [6, 11], SANTINI and others [12].

1.5. Optimal cantilever plate

Consider now a cantilever plate clamped at $x = 1$ and free at $x = 0$. In this case aeroelastic stability of the optimal plate is violated by the divergence. To prove this fact, it is necessary to solve the problem of the plate having maximal critical speed of divergence β_d and calculate for it critical flutter speed β_f.

If $\beta_f > \beta_d$, then the obtained solution realizes the maximal value of critical speed at which aeroelastic stability is violated.

Thus we consider the optimal problem of divergence instability. The solution of this problem is denoted by $h_d(x)$. The necessary optimality conditions (1.16), (1.17) with (1.15) include the case under study. But under the considered boundary conditions some simplifications are possible; this is connected with the decrease of the order of Eqs. (1.6) and (1.9) and the simplification of the boundary conditions. We take in Eq. (1.6) $\omega = 0$, replace β_f by β_d and use the notation $u'(x) = \varphi(x)$. As a result, the boundary value problem describing divergence of the plate in the considered case of boundary conditions takes the form

$$(1.18) \qquad \begin{gathered} (h^3\varphi')'' + \beta_d\varphi = 0, \\ (h^3\varphi')_{x=0} = (h^3\varphi')'_{x=0} = 0, \quad \varphi(1) = 0. \end{gathered}$$

The function u is determined by φ by the integral

$$u(x) = -\int_x^1 \varphi(x)dx.$$

The adjoint to Eq. (1.18) eigenvalue problem is described by the relations [18]:

$$(1.19) \qquad \begin{gathered} (h^3\psi'')' - \beta_d\psi = 0, \\ \psi(1) = \psi'(1) = 0, \quad (h^3\psi'')_{x=0} = 0. \end{gathered}$$

Now we multiply Eq. (1.18) by the adjoint function $\psi(x)$ and integrate it twice by parts taking into account the boundary conditions for φ and ψ. We obtain

$$(1.20) \qquad \beta_d = -\int_0^1 h^3\varphi'\psi''dx\Big[\int_0^1 \varphi\psi dx\Big]^{-1}.$$

This functional is stationary with respect to φ and ψ on the functions defined by Eqs. (1.18) and (1.19). The validity of this assertion may be verified by the immediate varying of the functional (1.20) with respect to φ and ψ. Using this property, we get the expression for variation:

$$\delta\beta_d = \int_0^1 e\delta h dx, \quad e = -\frac{3h^2\varphi'\psi''}{\int_0^1 \varphi\psi dx}. \tag{1.21}$$

The necessary condition of β_d, which is maximum at the constant volume $V(h) = V_0$, leads to the relation

$$h^2\varphi'\psi'' = \mu = \text{const}. \tag{1.22}$$

The equations and boundary conditions (1.18), (1.19) and (1.22) with the isoperimetric condition $V(h) = V_0$ permit to determine the functions $h_d(x)$, $\varphi(x)$, $\psi(x)$ and the multiplier μ that realizes the extremal solution of the optimal problem of plate divergence in gas flow. Note that the solution h_d may be expressed in the form

$$h_d(x) = V_0 h_*(x) \tag{1.23}$$

because the functionals (1.4) and (1.20) are homogeneous with respect to h [24]. In Eq. (1.23) $h_*(x)$ designates the solution of the optimal problem of divergence at the isoperimetric condition $V(h) = V_0 = 1$. In this way, the parameter V_0 is excluded from the consideration.

1.5.1. Asymptotics. Let us investigate the asymptotic behaviour of the functions $h_*(x)$, $\varphi(x)$, $\psi(x)$ near the boundary $x = 0$. For convenience we use the notation $\alpha_* = \beta_d V_0^{-3}$ and rewrite Eqs. (1.18) and (1.19) in the form

$$\begin{aligned} (h_*^3\varphi')'' &= -\alpha_*\varphi, \\ (h_*^3\psi'')' &= \alpha_*\psi. \end{aligned} \tag{1.24}$$

The functions $\varphi(x)$, $\psi(x)$ are supposed to be normalized quantities $\varphi(0) = \psi(0) = 1$. Then we take $\varphi(x) = 1+o(1)$, $\psi(x) = 1+o(1)$, where x belongs to the neighbourhood of the point $x = 0$. Substituting these relations in the right sides of Eqs. (1.24) and integrating them with the use of boundary conditions, we achieve

$$\begin{aligned} h_*^3(x)\varphi'(x) &= -1/2\alpha_* x^2 + o(x^2), \\ h_*^3(x)\psi''(x) &= \alpha_* x + o(x). \end{aligned} \tag{1.25}$$

Multiplying these equations by each other and using the optimality conditions (1.22), we obtain $h_*(x) = c_1 x^{3/4} + o(x^{3/4})$, $c_1 = (-\alpha_*^2 V_0^2/2\mu)^{1/4}$. According to Eq. (1.25) the expansions for $h_*(x)$, $\varphi(x)$, $\psi(x)$ near $x = 0$ can be found in the form

$$\begin{aligned} h_*(x) &= c_1 x^{3/4} + c_2 x + \ldots, \\ \varphi(x) &= 1 + a_1 x^{3/4} + a_2 x + \ldots, \\ \psi(x) &= 1 + b_1 x^{3/4} + b_2 x + \ldots. \end{aligned} \tag{1.26}$$

Substitution of these expressions in Eq. (1.24), (1.22) permits to find the relations between the expansional coefficients

$$c_2 = 0, \quad a_2 = 0, \quad 3a_1 c_1^3 = -2, \quad 3b_1 c_1^3 = -16.$$

The obtained asymptotics are required for the further numerical solution of Eqs. (1.24) and (1.22) with the respective boundary conditions and isoperimetric condition

$$V(h) = 1.$$

1.5.2. Numerical results. A numerical solution of the optimal problem was realized by the gradient method in the space of control function $h(x)$ with the use of the asymptotics (1.26). At every step of the gradient procedure in $h(x)$ the next integral equations were solved:

$$\varphi(x) = \alpha_* \int_x^1 h^{-3}(\zeta)d\zeta \int_0^\zeta (\zeta-\eta)\varphi(\eta)d\eta,$$

$$\psi(x) = \alpha_* \int_x^1 h^{-3}(\eta)(\eta-x)d\eta \int_0^\eta \psi(\zeta)d\zeta.$$

These equations are equivalent to the eigenvalue problems above. Variations of $h(x)$ were made according to the formula

$$\delta h^{(n)}(x) = h^{(n+1)}(x) - h^{(n)}(x) = \gamma\left[\frac{e^{(n)}(x)}{\langle e^{(n)}\rangle} - 1\right],$$

where $e^{(n)}(x)$ is the functional gradient of the critical divergence speed (1.21); γ is the positive number chosen by the researcher (step of gradient): $\langle e^{(n)}\rangle$ is the constant defined by the constant volume condition $\delta V(h) = 0$

$$\langle e^{(n)}\rangle = \int_0^1 e^{(n)}(x)dx.$$

It is easy to see that for sufficiently small γ this algorithm increases divergence speed at every step and satisfies the constant volume condition. The computations were stopped when Eq. (1.22) was satisfied within fixed computational error. The details of the computations are described in [22].

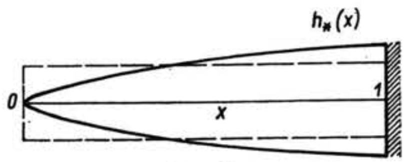

FIG. 2.

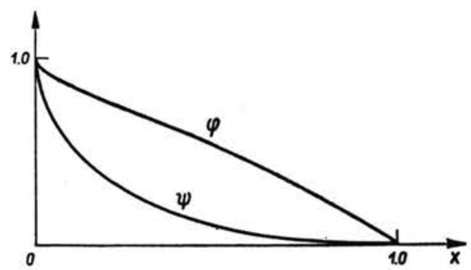

FIG. 3.

The functions $h_*(x)$, $\varphi(x)$, $\psi(x)$ obtained numerically are shown in Figs. 2 and 3. Using the method of extremum investigation described in [23], we can prove that the extremals h_*, φ, ψ realize the strong maximum of the functional of divergence speed at given constraints.

The value of critical divergence speed for a plate of constant thickness ($h(x) = 1$) shown in Fig. 2 by a dashed line is equal to $\beta_d = 6.33$ [17].

The value β_d computed for the plate with the thickness function $h_*(x)$ equals 11.8. So, critical divergence speed for the optimal plate exceeds this speed for the constant thickness plate 1.86 times.

1.5.3. Aeroelastic stability study. Investigate now the dependence of aeroelastic stability of the plate with the thickness $h_d(x) = V_0 h_*(x)$ on the speed parameter $\alpha = \beta/V_0^3$. For this purpose the finite difference method was applied. At computations the segment [0, 1] was divided into $N = 20, 30$ equal parts, derivatives in Eq. (1.6) and the boundary conditions were replaced by finite difference relations. Computations were made both for $\alpha > 0$ and for $\alpha < 0$ (this case corresponds to the inverse direction of the flow speed).

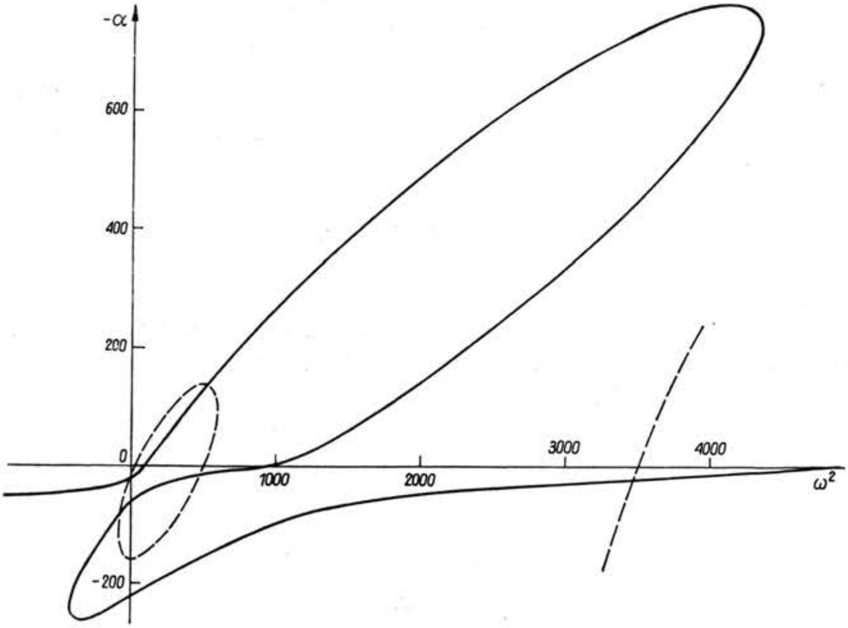

FIG. 4.

In Fig. 4 the dependence of frequency of vibrations ω on parameter α is presented. At computations in Eq. (1.6) $\sigma = i\omega$ and $\Lambda = 0$ (absence of damping) were taken. The dashed line shows the behaviour of frequencies of the plate with $h(x) = 1$. It should be noted that these two curves essentially differ from each other.

From the numerical results it follows that aeroelastic stability of the plate with $h_d(x)$ is violated at $\alpha > 0$ by the divergence, $\beta_f > \beta_d$. Hence at $\alpha > 0$ the plate with the thickness $h_d(x)$ is the solution of the optimal problem (1.5), $h_0(x) \equiv h_d(x) = V_0 h_*(x)$.

At inverse direction of the speed of the flow ($\alpha < 0$) the stability of this plate is violated by flutter $\alpha_f = -780$, Fig. 4. Thus the region of stability of the optimal plate with zero damping lies in the region $-780 < \alpha < 11.8$. Note that the region of stability of the plate with the thickness $h(x) = 1$ is defined by the inequality $-123 < \alpha < 6.33$ [17]. The obtained results show that variation of the plate thickness essentially act son the distribution of frequencies and, respectively, on the region of aeroelastic stability of the plate.

2. Discrete systems

Consider now systems with finite degrees of freedom subjected to aeroelastic instability phenomena such as flutter and divergence. It is assumed that the system is characterized by the parameters m_i, $i = 1, 2, \ldots, N$ that may be varied by the designer. As the "control" parameters m_i various mass and stiffness characteristics may be considered: plate thicknesses, geometrical dimensions of various elements, concentrated masses, etc. In this section we deduce the expressions for partial derivatives of critical flutter and divergence speeds with respect to parameters m_i in the general case of the systems described by linear algebraic equations.

The governing equation for flutter has the form

$$[K(\mathbf{m})-\omega^2 M(\mathbf{m})+A(\omega, V_f)]\boldsymbol{\xi} = 0. \tag{2.1}$$

In this equation K is the stiffness matrix, M is the inertial matrix, A is the complex aerodynamic matrix, $\boldsymbol{\xi}$ is the complex vector of generalized coordinates, V_f is flutter speed, ω is flutter frequency. It is assumed that the matrices K, M, A possess the dimension $n\times n$, the vector $\boldsymbol{\xi}$ has the dimension n and the vector $\mathbf{m}$ the dimension N. The matrix A in general may depend on $\mathbf{m}$.

Find the derivatives of critical flutter speed with respect to parameters m_j, $j = 1, 2, \ldots, \ldots, N$. For convenience the matrix L is introduced:

$$L = K(\mathbf{m})-\omega^2 M(\mathbf{m})+A(\omega, V_f). \tag{2.2}$$

Take variation δm_i of parameter m_i. Then the quantities V_f, ω, $\boldsymbol{\xi}$ yield the variations δV_f, $\delta\omega$, $\delta\boldsymbol{\xi}$.

The equation in variations similar to Eq. (1.7) may be written as:

$$\frac{\partial L}{\partial m_i}\boldsymbol{\xi}\delta m_i+\frac{\partial L}{\partial V_f}\boldsymbol{\xi}\delta V_f+\frac{\partial L}{\partial \omega}\boldsymbol{\xi}\delta\omega+L\delta\boldsymbol{\xi} = 0, \tag{2.3}$$

where the derivatives of matrix L, according to Eq. (2.2), are defined by the relations

$$\frac{\partial L}{\partial m_i} = \frac{\partial K}{\partial m_i}-\omega^2\frac{\partial M}{\partial m_i}, \quad \frac{\partial L}{\partial V_f} = \frac{\partial A}{\partial V_f}, \qquad \frac{\partial L}{\partial \omega} = \frac{\partial A}{\partial \omega}-2\omega M. \tag{2.4}$$

Note that the aerodynamic matrix is generally given in the form $A = A(k, \mathfrak{M})$ where k is the reduced frequency and M is the Mach number [13, 14]

$$k = \frac{\omega c}{V_f}, \quad \mathfrak{M} = \frac{V_f}{a}.$$

Here c is a linear dimension of the system in the direction of the flow, a is sound speed at a given height of flight, V_f and ω are flutter speed and frequency. In this case the derivatives are defined by the formulas

$$\frac{\partial L}{\partial V_f} = \frac{\partial A}{\partial k}\frac{\partial k}{\partial V_f} + \frac{\partial A}{\partial \mathfrak{M}}\frac{\partial \mathfrak{M}}{\partial V_f}, \qquad \frac{\partial L}{\partial \omega} = \frac{\partial A}{\partial k}\frac{\partial k}{\partial \omega} 2\omega M.$$

The final result is

$$\frac{\partial L}{\partial V_f} = \frac{1}{a}\frac{\partial A}{\partial \mathfrak{M}} - \frac{\omega c}{V_f^2}\frac{\partial A}{\partial k}, \qquad \frac{\partial L}{\partial \omega} = \frac{c}{V_f}\frac{\partial A}{\partial k} 2\omega M.$$

Multiply Eq. (2.3) by the transposed vector $\mathbf{q}^T$ where the vector $\mathbf{q}$ is determined from the equation

$$L^T\mathbf{q} = [K^T - \omega^2 M^T + A^T]\mathbf{q} = 0. \tag{2.5}$$

As a result, we get the expression

$$\mathbf{q}^T \frac{\partial L}{\partial m_l}\boldsymbol{\xi}\delta m_l + \mathbf{q}^T\frac{\partial L}{\partial V_f}\boldsymbol{\xi}\delta V_f + \mathbf{q}^T\frac{\partial L}{\partial \omega}\boldsymbol{\xi}\delta\omega = 0 \tag{2.6}$$

since according to Eq. (2.5),

$$\mathbf{q}^T L\delta\boldsymbol{\xi} = \delta\boldsymbol{\xi}^T L^T \mathbf{q} = 0.$$

With the use of notations

$$H_l = \mathbf{q}^T\frac{\partial L}{\partial m_l}\boldsymbol{\xi}, \quad B = \mathbf{q}^T\frac{\partial L}{\partial \omega}\boldsymbol{\xi}, \quad C = \mathbf{q}^T\frac{\partial L}{\partial V_f}\boldsymbol{\xi}, \tag{2.7}$$

Eq. (2.6) takes the form

$$H_l\delta m_l + B\delta\omega + C\delta V_f = 0. \tag{2.8}$$

Multiplying this equation by the complex conjugate quantities $\bar{B}$ and $\bar{C}$ and taking imaginary part, we find relations similar to Eqs. (1.13) and (1.14):

$$\frac{\partial V_f}{\partial m_l} = -\frac{\mathrm{Im}(H_l\bar{B})}{\mathrm{Im}(C\bar{B})}, \qquad \frac{\partial\omega}{\partial m_l} = -\frac{\mathrm{Im}(H_l\bar{C})}{\mathrm{Im}(B\bar{C})}. \tag{2.9}$$

The obtained relations with the use of Eqs. (2.7) and (2.4) are expressed through the derivatives of the matrices K, M, A and the vectors $\boldsymbol{\xi}$ and $\mathbf{q}$.

Note that the vectors $\boldsymbol{\xi}$ and $\mathbf{q}$ are defined up to arbitrary complex multipliers because of homogeneity of the problems (2.1) and (2.5). Hence the normalization condition can be applied:

$$\mathrm{Re}B = \mathrm{Re}\left(\mathbf{q}^T\frac{\partial L}{\partial\omega}\boldsymbol{\xi}\right) = 0. \tag{2.10}$$

With the use of this condition the derivatives of flutter speed, according to Eq. (2.8), become

$$\frac{\partial V_f}{\partial m_l} = -\frac{\mathrm{Re}H_l}{\mathrm{Re}C} = -\frac{\mathrm{Re}\left(\mathbf{q}^T\frac{\partial L}{\partial m_l}\boldsymbol{\xi}\right)}{\mathrm{Re}\left(\mathbf{q}^T\frac{\partial L}{\partial V_f}\boldsymbol{\xi}\right)}. \tag{2.11}$$

In the expressions (2.10) and (2.11) the symbol Re can be replaced by Im. The expression (2.9) is equivalent to Eq. (2.11) with the normalization condition (2.10).

Consider now the case of aeroelastic instability such as divergence. The divergence equation can be achieved taking in Eq. (2.1) $\omega = 0$ and replacing V_f by V_d:

$$[K(m)+A(V_d)]\boldsymbol{\xi}_d = 0. \tag{2.12}$$

Here V_d is divergence speed, $\boldsymbol{\xi}_d$ is the vector of generalized coordinates at divergence. Consider the adjoint system

$$[K^T(m)+A^T(V_d)]\mathbf{q}_d = 0. \tag{2.13}$$

Similarly to the calculation of the derivatives of flutter speed, we find

$$\frac{\partial V_d}{\partial m_l} = -\frac{\mathbf{q}_d^T \dfrac{\partial K}{\partial m_l}\boldsymbol{\xi}_d}{\mathbf{q}_d^T \dfrac{\partial A}{\partial V_d}\boldsymbol{\xi}_d}. \tag{2.14}$$

The obtained formulas (2.9) and (2.14) can be used both for the derivation of necessary optimality conditions and for numerical solving of various optimization problems of the systems subjected to aeroelastic instability phenomena.

Note that the formulas for derivatives of flutter characteristics in some special case were obtained by V. G. BUN'KOV [2]. The method of calculating derivatives of critical flutter speed, frequency and modes of flutter without use of the adjoint eigenvector is described in [14]. But we doubt that this method can be effective for solving structural optimization problems because calculation of the gradient of flutter speed with respect to N structural parameters involves solving of N systems of linear equations of $2n+2$ order. It should be also noted that iteration formulas suggested in [10, 13] do not satisfy necessary optimality conditions even in the case of their convergence.

3. Conclusion

In this paper, for continuous and discrete systems the expressions of sensitivity characteristics of flutter and divergence speed with respect to distributed and discrete parameters defining aeroelastic behaviour are first derived. It is shown that the calculation of the gradients of critical speeds involves the solution of the so-called adjoint problem which is similar to the main problem of flutter or divergence. The variational method of sensitivity analysis of critical speeds is the most effective when many finite (or infinite) numbers N of defining structural parameters are considered. This is so since only one-fold solving of adjoint problems is required for the calculation of gradient vector of critical speed. Nevertheless, numerical differentiation of flutter or divergence speed as the function of N independent variables requires for gradient calculation not less than $N+1$ — fold solving of the flutter (divergence) problem.

As to optimization problems, our opinion is that the most effective methods for solving structural optimization problems which take into account aeroelastic instability phenomena both for discrete and continuous systems are the methods of mathematical programming using implicit expressions of gradients of flutter and divergence speed such as Eqs. (1.13), (1.15); (2.9) and (2.14).

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