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On the reliability of postbuckling analyses for nonconservative imperfect systems

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CRITICAL STATES of the inverted double pendulum under a partial follower load are thoroughly reconsidered with the aid of a complete nonlinear dynamic analysis which includes both material and geometrical nonlinearities. The material nonlinearity is of quadratic type so that the perfect system loses its stability through an asymmetric branching point. It is found that while for the perfect system the nonlinear static analysis gives the same critical loads as those obtained by the nonlinear dynamic analysis, in the case of the imperfect system appreciable deviations between these loads occur. An attempt is made to explain this phenomenon by clarifying the physical meaning of each of these critical loads.

Stany krytyczne podwójnego wahadła poddanego obciążeniu śledzącemu są w pełni zbadane wraz z uwzględnieniem nieliniowej analizy dynamicznej zarówno opisującej zachowanie materiału jak i nieliniowowść geometryczną. Nieliniowe zachowanie materiału przyjęto w postaci funkcji kwadratowej, zatem układ idealny traci stabilność w asymetrycznym punkcie bifurkacji. W pracy wykazano, że w przypadku układu z imperfekcjami, pojawiają się istotne różnice w wartości obciążeń krytycznych wyznaczonych z analizy statycznej i dynamicznej. Różnic takich nie obserwuje się w układach bez imperfekcji. Zjawisko to wyjaśnia się przez fizyczną interpretację obciążeń krytycznych.

Проанализирована проблема энергетического баланса теплого двигателя, опирающегося на использовании свойств некоторых твердых тел, заключающихця в памяти формы; анализ проведен с точки зрения термомеханики металлов, области знаний, находящихся на грани термомеханики и металлургии. Для примера послузхились моделью, состоящей из двух дисков обхваченных проволокой, изготовленной из сплава сохраняющего память формы. Показано, что некоторое количество энергии, накопленной в материале при повышенной температуре в виде энергии деформации, выделяется на деформацию проволоки и на другие цели, тогда как целый процесс происходит мещду резервуарами тепла с разными температурами. Остальная часть энергии преобразуется в конечный счет в кинетическую энергию, приводящую систему дисков. Доказано, что изменения кривой "напряжение-деформация", вызванные превращением имеющим место в металле, составляют существенный фактор в построении эффективного двигателя этого рода.

1. Introduction

IN THE LAST ten years numerous studies were done on the postbuckling response of nonconservative elastic systems referring to simple frames [1-4] and general discrete systems [5, 6]. The last two analyses constitute an extension of the general nonlinear stability theory of Thompson to nonconservative, nonlinearly elastic, discrete systems of divergence instability. PLAUT [5, 6] discussed the three basic types of bifurcation points including also the case of coincident critical points. His analysis was illustrated by a nonlinear elastic double pendulum of quadratic or cubic type under a partial follower load for which linearized static and dynamic analyses were available [7]. Recently, KOUNADIS and MAHRENHOLTZ [8] showed that nonconservative systems made from nonlinear elastic material may sometimes be analysed by using static methods of analysis.

The results of the foregoing postbuckling analyses, concerning mainly nonconservative systems of limit point instability, have not been compared to those obtained by using a nonlinear dynamic analysis. This, of course, is due to the intractability of the highly nonlinear equations of motion. Therefore, it is not known if the nonlinear dynamic analysis gives the critical loads obtained by a static analysis as this occurs for nonconservative linear (bifurcational) systems.

Interesting nonlinear dynamic analyses of nonconservative systems discussing the post-critical steady-state response [9, 10] and/or the transient response [11] are basically valid in the vicinity of the critical state, the higher order terms being disregarded.

More general and thorough analyses discussing the critical and stability response of gradient and nongradient, one or multiple-parameter, autonomous discrete systems by studying the Jacobian eigenvalues in the vicinity of a critical state were presented by Huseyin and his associates [12-14]. However, the above nonlinear dynamic analyses based on local (linearized) solutions are inadequate to predict certain phenomena (associated mainly with the global response) which may occur in nonlinear multiple-parameter autonomous systems.

The main objective of this investigation is, a) — to establish critical loads mainly of divergence instability by studying the global response associated with the original nonlinear equations of motion. To this end large amplitude motion and large time solutions are considered, b) — to compare static critical loads of the imperfect system with those obtained by using a nonlinear dynamic analysis and c) — to shed some light on the actual nature of the follower type nonconservative loading.

2. Mathematical analysis

Consider the partially fixed imperfect model shown in Fig. 1 consisting of two (weightless) pin-jointed rigid links of equal length l, which carry the concentrated masses m_1 and m_2 at B and A, respectively. The model is acted upon by a partial follower load P applied

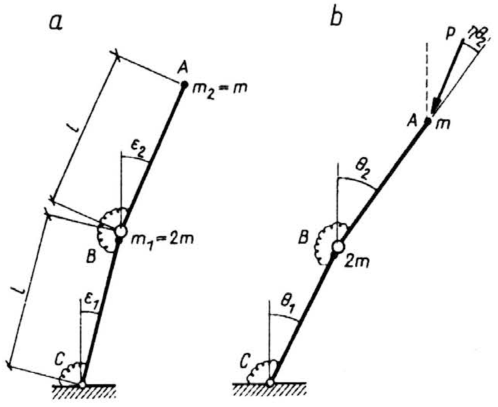

FIG. 1. Unstressed (a) and stressed (b) state of a double pendulum under partial follower load.

at its tip A at the angle $\eta\vartheta_2$ with respect to the deformed axis of the upper link. Such a load becomes tangential for $\eta = 0$ and constant directional (conservative) for $\eta = 1$; namely, for $\eta \neq 1$ it is nonconservative. The bending moments at C and B obeying a nonlinear elastic quadratic law are given by

$$M_C = k\psi_C(1 + \delta_1\psi_C), \quad M_B = k\psi_B(1 + \delta_2\psi_B), \tag{2.1}$$

respectively, where $k > 0$ is the linear spring constant and δ_1 and δ_2 are the nonlinear spring constants, while ψ_C and ψ_B are the angles of rotation at C and B measured from the unstrained configuration. This is identified by the initial imperfections $\vartheta_1 = \varepsilon_1$ and $\vartheta_2 = \varepsilon_2$; therefore in the deformed state we have $\psi_C = \vartheta_1 - \varepsilon_1$ and $\psi_B = (\vartheta_2 - \varepsilon_2) - (\vartheta_1 - \varepsilon_1)$. If $\delta_1 = \delta_2 = 0$, both springs are elastic (Hookean material). When δ_1 and δ_2 are both negative, the nonlinear elastic springs are of "soft" type, otherwise (i.e. if $\delta_1, \delta_2 > 0$) they are "hard" type.

PLAUT [5] has treated only the case $\eta \geq 4/9$ (corresponding to divergence instability), while HERRMANN and BUNGAY [7] investigated the flutter and divergence instability of the linear perfect ($\varepsilon_1 = \varepsilon_2 = 0$) model for $-2 \leq \eta \leq 2$ (or $-1 \leq \alpha = 1 - \eta \leq 3$).

3. Nonlinear static analysis

Application of the principle of virtual work for arbitrary variations $\delta\vartheta_1$ and $\delta\vartheta_2$ yields

$$\frac{\partial U}{\partial \vartheta_i} - Q_i = 0 \quad (i = 1, 2)\,, \tag{3.1}$$

where the strain energy U and the generalized (nonpotential) loads Q_i (due to P) are given by

$$\begin{gathered}\frac{U}{k} = \frac{1}{2}(\vartheta_1 - \varepsilon_1)^2 + \frac{1}{3}\delta_1(\vartheta_1 - \varepsilon_1)^3 + \frac{1}{2}(\vartheta_2 - \varepsilon_2 - \vartheta_1 + \varepsilon_1)^2 + \frac{1}{3}\delta_2(\vartheta_2 - \varepsilon_2 - \vartheta_1 + \varepsilon_1)^3,\\ Q_1 = p\sin[\vartheta_1 + (\eta - 1)\vartheta_2], \quad Q_2 = p_2 \sin \eta\vartheta_2, \quad p = p_l/k\,.\end{gathered} \tag{3.2}$$

The critical bifurcational load $p^c(\varepsilon_1 = \varepsilon_2 = 0)$ is equal to [6]

$$p^c = \frac{1}{2}(3 \pm \sqrt{9 - \frac{4}{\eta}}) \quad \text{for} \quad \eta = 4/9 \quad \text{or} \quad \eta < 0\,. \tag{3.3}$$

For $\eta = 4/9$ we have a coincident (double) critical point at $p^c = 1.5$. This point, as shown by KOUNADIS [15], and KOUNADIS and MAHRENHOLTZ [8], is always associated with a jump in the critical load.

If δ_1 and δ_2 are not both zero, the critical state for $\eta \geq 4/9$ and $\eta < 0$ corresponds to an asymmetric branching point [6], while for $\delta_1 = \delta_2 = 0$ the critical point becomes a stable symmetric branching point. Indeed, as was shown by KOUNADIS [16, 17], the critical state of the perfect linear (Hookean) system is stable.

The critical state $(p_{\rm cr}, \vartheta_1^{\rm cr}, \vartheta_2^{\rm cr})$ of the imperfect system is determined by solving Eqs. (3.1) subject to the condition (of vanishing of the stability determinant)

$$|D_{ij}| = 0, \quad D_{ij} \neq D_{ji}\,, \tag{3.4}$$

where $D_{ij} = (\partial^2 U/\partial\vartheta_i \partial\vartheta_j - \partial Q_i/\partial\vartheta_j$. Whether or not the system of Eqs. (3.1) and (3.4) has a solution, depends on the values of ε_1 and ε_2. If there is no solution the model exhibits a rising (stable) equilibrium path.

Extending the foregoing analysis by PLAUT [5] to the case $\eta < 4/9$ one can prove the existence of adjacent equilibrium states for that range of values of η. The model may be either stable or loses its stability through a limit point depending on the values of ε_1 and ε_2; therefore, in contrast with all previous linearized analyses, the static methods are also applicable for the range $\eta < 4/9$ or $\alpha = 1 - \eta > 5/9$ [7, 18]. For instance, if $\eta = 0.2$, $\delta_1 = -2.5$, $\delta_2 = -0.75$, $\varepsilon_1 = 0.05$ and $\varepsilon_2 = -0.05$ one can find $p_{\rm cr} = 0.7357$.

Hence, the important conclusion is deduced that the applicability of static methods of analysis for systems which in their ideal state are associated with an asymmetric branching point may depend on the presence of imperfections. For systems which in their ideal state are associated with a symmetric branching point the applicability of static methods may depend on the presence of material nonlinearity [8, 16, 17, 19]. A detailed analysis for the effect of material nonlinearity in buckling of conservative elastic systems is presented by VARELLIS and KOUNADIS [20].

Whether or not the limit point loads (divergence instability) coincide with the dynamic critical loads will be discussed below with the aid of a nonlinear dynamic analysis. The integration of equations of motion is achieved numerically using the Runge–Kutta's fourth-order scheme and, in some cases, an approximate but efficient analytic technique [21, 22]. Considering the stability of motion in the large, in the sense of Lagrange (boundedness of solution), dynamic buckling is defined as that state for which an escaped (leading to an unbounded) motion is initiated. The smallest load corresponding to that state is defined as the dynamic (either divergence of flutter) instability load.

4. Nonlinear dynamic analysis

Lagrange's equations of motion may be written as follows [23]

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\vartheta}_i}\right) - \frac{\partial K}{\partial \vartheta_i} + \frac{\partial U}{\partial \vartheta_i} - Q_i = 0, \quad i = 1,2, \tag{4.1}$$

where the dot denotes differentation with respect to t, while the kinetic energy K is given by [16, 17]

$$K = \frac{1}{2}m_1 l^2 \dot{\vartheta}_1^2 + \frac{1}{2}m_2 l^2[\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\dot{\vartheta}_1\dot{\vartheta}_2\cos(\vartheta_1 - \vartheta_2)]. \tag{4.2}$$

With the aid of relation (4.2) and the dimensionless quantities

$$\theta_i(\tau) = \vartheta_i(\tau), \quad \tau = t\left(\frac{k}{m_2 l^2}\right)^{\frac{1}{2}}, \quad m = \frac{m_1}{m_2}, \quad i = 1,2 \tag{4.3}$$

Eqs. (4.1) become

$$\begin{aligned}(1+m)\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2) + 2(\theta_1-\varepsilon_1) - (\theta_2-\varepsilon_2) + \delta_1(\theta_1-\varepsilon_1)^2 \\ - \delta_2(\theta_1-\varepsilon_1-\theta_2+\varepsilon_2)^2 - p\sin[\theta_1+(\eta-1)\theta_2] = 0, \\ \ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_1^2\sin(\theta_1-\theta_2) - (\theta_1-\varepsilon_1) + \theta_2 - \varepsilon_2 + \delta_2(\theta_1-\varepsilon_1-\theta_2+\varepsilon_2)^2 - p\sin\eta\theta_2 = 0.\end{aligned} \tag{4.4}$$

These equations, after linearization and setting $m = 2$ and $\varepsilon_1 = \varepsilon_2 = 0$, coincide with the corresponding equations presented by HERRMANN and BUNGAY [7]. The system of Eqs (4.4) subject to the appropriate initial conditions is solved numerically by using the fourth-order Runge–Kutta scheme with a step size 0.01. These results in some cases have been checked by employing an approximate analytic technique [21, 22].

For the perfect system ($\varepsilon_1 = \varepsilon_2 = 0$) Eqs. (4.4) subject e.g. to the initial ($t = 0$) conditions

$$\dot{\theta}_2 = 0, \quad \dot{\theta}_1 \simeq 0 \quad (\text{e.g. } \dot{\theta}_1 = 10^{-8} \text{ or } \dot{\theta}_1 = 10^{-10}) \tag{4.5}$$

yield for $\eta \geq 4/9$ and $\eta < 0$ dynamic critical loads p_{cr}^D tending asymptotically to the corresponding static ones given by Eq. (3.3).

For the imperfect system the associated initial ($t = 0$) conditions are

$$\theta_1 = \varepsilon_1, \quad \theta_2 = \varepsilon_2, \quad \dot{\theta}_1 = \dot{\theta}_2 = 0\,. \tag{4.6}$$

PLAUT [6], considering the case $\eta = \frac{4}{9}$, $\varepsilon_1 = 0$ and $\varepsilon_2 = -0.02$, has found that the limit point load is $p^M = 1.20$ for $\delta_1 = -0.40$ and $\delta_2 = -0.20$, while for $\delta_1 = 1.5$ and $\delta_2 = -0.75$ this load becomes $p^M = 1.06$. Using a nonlinear dynamic analysis the corresponding loads are 1.178 and 1.038, respectively; it means that the maximum difference between the static and dynamic critical load is 2 %. Although the mass ratio may have a considerable effect on the dynamic critical load [24], in this case a change of the mass ratio from $m = 2$ to $m = 1$ does not practically affect the foregoing percentage. Indeed, for $m = 1$ the dynamic critical load of the last case becomes 1.0386; that is 0.05% higher than that corresponding to $m = 2$.

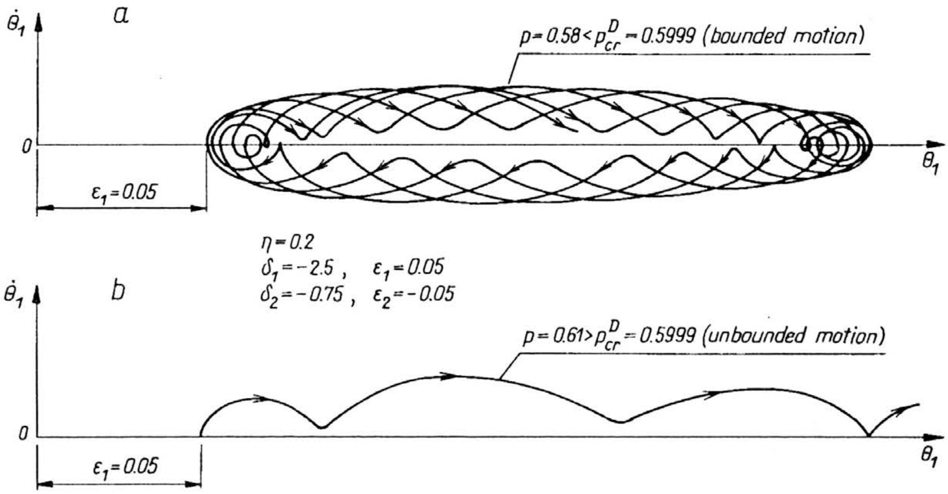

FIG. 2. Bounded (a) and unbounded (b) motion in the phase plane $-(\dot{\theta}_1, \theta_2)$ of an imperfect model ($\varepsilon_1 = 0.05, \varepsilon_2 = -0.05$) with $\eta = 0.2$, $\delta_1 = -2.5$ and $\delta_2 = -0.75$.

For the numerical data used in the previous section ($\eta = 0.2$, $\delta_1 = -2.5$, $\delta_2 = -0.75$, $\varepsilon_1 = 0.05$, and $\varepsilon_2 = -0.05$) Eqs. (4.4) and (4.6) give a dynamic critical load p_{cr}^D equal to 0.5999; that is different than the corresponding static one $p_{\text{cr}} = 0.7257$. Figure 2 shows the motions in the $(\theta_1, \dot{\theta}_1)$-phase plane for $p = 0.58 < p_{\text{cr}}^D$ (bounded amplitude) and for $p = 0.61 > p_{\text{cr}}^D$ (unbounded amplitude). Note that if $m = 1$ (instead of $m = 2$), the dynamic critical load becomes 0.6025; that is 0.4% higher. The difference between the static (divergence) buckling load and the dynamic instability load is also observed for other values of the nonconservativeness parameter η. Different types of motion occur if only η changes. Figures 3a, b show bounded (for $p = 2.275 < p_{\text{cr}}^D$) and unbounded (for $p = 2.280 > p_{\text{cr}}^D$) motions in the $(\theta_1, \dot{\theta}_1)$-phase plane if $\eta = 1.4$, $\delta_1 = -2.5$, $\delta_2 = -0.75$, $\varepsilon_1 = 0.05$, and $\varepsilon_2 = -0.05$.

From Fig. 4 one can see the effect of the imperfection parameter ε_2 on the static and dynamic critical load at the coincident critical point $\eta = 4/9$ for $\varepsilon_1 = 0.02$, $\delta_1 = -2.5$ and $\delta_2 = -0.75$. It is clear that the difference between the dynamic critical load and the corresponding static one remains practically constant, being equal to 7.6%. For $\eta = -1$, $\delta_1 = -2.5$, $\delta_2 = -0.75$, $\varepsilon_1 = 0.05$ and $\varepsilon_2 = -0.05$, the dynamic critical load is equal to

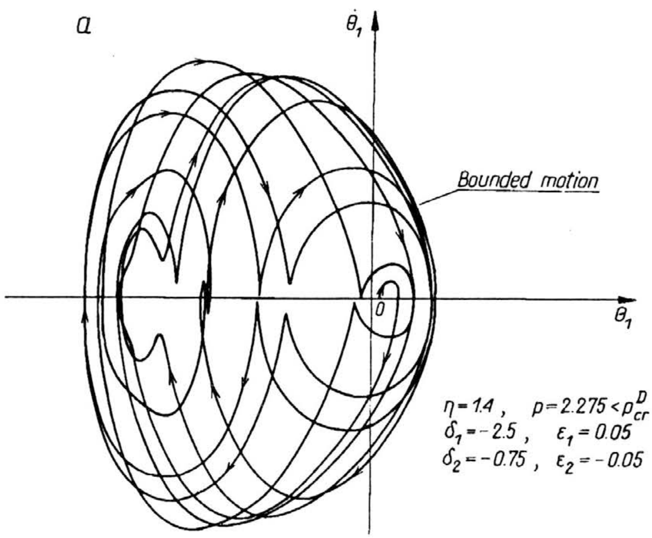

FIG. 3 a) Bounded motion in the phase plane —$(\dot{\theta}_1, \theta_1)$ of an imperfect model ($\varepsilon_1 = 0.05, \varepsilon_2 = -0.05$) with $\eta = 1.4$, $\delta_1 = -2.5$ and $\delta_2 = -0.75$.

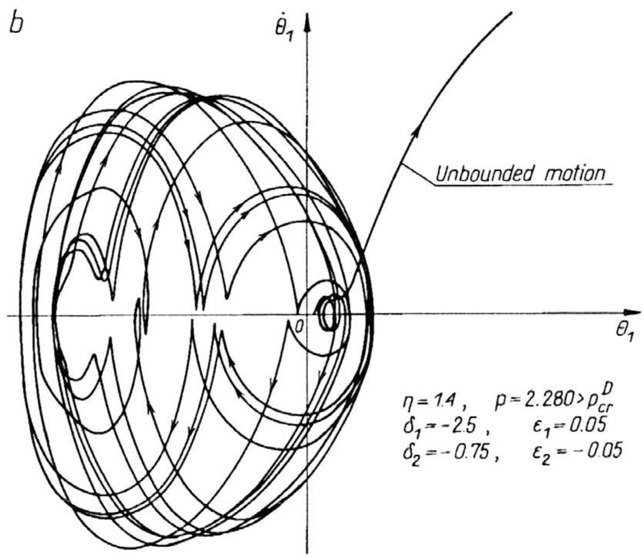

FIG. 3 b) Unbounded motion in the phase plane —$(\dot{\theta}_1, \theta_1)$ of an imperfect model ($\varepsilon_1 = 0.05, \varepsilon_2 = -0.05$) with $\eta = 1.4$, $\delta_1 = -2.5$ and $\delta_2 = -0.75$.

1.555, while the corresponding static load is 1.763, that is greater than the former one by about 13.5%. A few similar results for another nonconservative model were reported by SOTIROPOULOS and KOUNADIS [19].

In bifurcational systems under follower loading with a trivial fundamental path, in which flutter does not occur before divergence, the static critical loads coincide with the dynamic critical loads regardless of wether or not the ratio of the concentrated

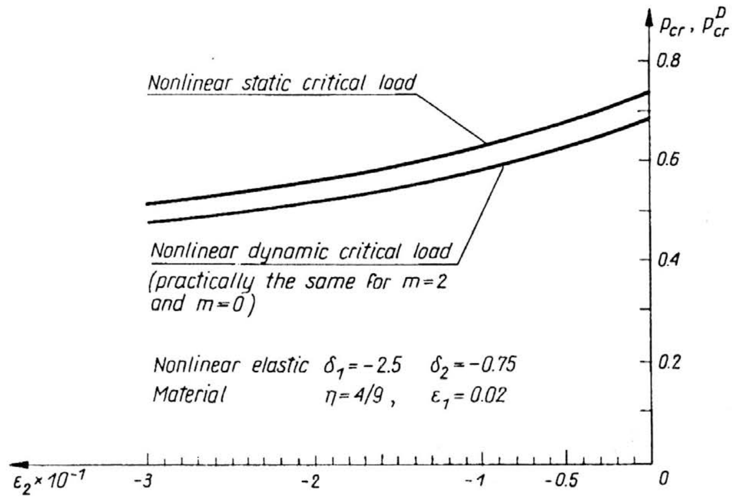

FIG. 4. Static and dynamic critical load (p_{cr}, p_{cr}^D) versus ε_2 for a nonlinear elastic system associated with a coincident critical point.

masses varies. However, this is not so far limit point systems. In this case the static critical load is, in general, higher than the corresponding dynamic one [16, 17]. This is also valid regardless of wether or not the mass ratio may affect the dynamic critical load. Then the following question is automatically raised: which is the physical meaning of each of these critical loads? Before answering the question it is interesting to consider a similar case in conservative systems under a suddenly applied load of infinite duration [25–28]. Indeed, in this case the static critical load coincides with the corresponding dynamic load only in bifurcational systems. According to this important finding the kinetic criterion for establishing critical loads of conservative or nonconservative divergence instability systems is restricted to bifurcational systems.

In view of the above observations one could consider that both critical loads are meaningful as referring to different response states. This, of course, presupposes that it is possible to apply a follower load statically. If the latter case is not feasible, only the dynamic instability load is physically acceptable. Moreover, it is concluded that the stability of a nonconservative system must always be investigated by a dynamic analysis, regardless of wether or not a static (divergence) instability analysis may be employed. Certainly, this is particularly necessary when the effects of masses and damping must be included.

From Fig. 3a, b, one can also observe that the above nonlinear, multiple-parameter, autonomous system ($\delta_1 = -2.50$, $\delta_2 = -0.75$, $\varepsilon_1 = 0.05$, $\varepsilon_2 = -0.05$) for $\eta = 1.4$ exhibits a rather nonperiodic oscillation, looking like a chaotic motion. The same phenomenon occurs for $\eta = 1.6$, while the other parameters remain constant. Indeed, as one can see from Fig. 5a, a sudden leap in the response of the system occurs after a long period of large amplitude, quasi-chaotic, oscillations [29]. For a load less than the buckling load the bounded motion is similarly quasi-chaotic (Fig. 5b). Such a phenomenon is due to sensitivity to initial conditions and not to a strange attractor, since without damping there is no contraction of the phase-space volumes [30]. Indeed, the relative change of the

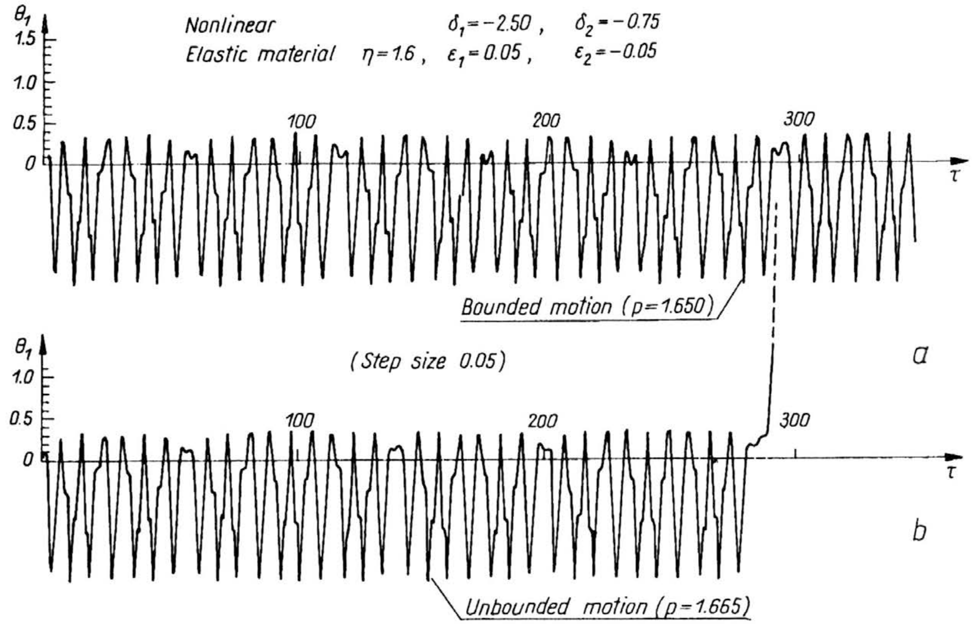

FIG. 5. Quasi-chaotic motion (θ_1 versus τ) bounded (a) and unbounded (b) leading to dynamic buckling after a long period of time.

phase-space volume for this system being equal to [17]

$$\frac{(\dot{\theta}_2 - \dot{\theta}_1)\sin 2(\theta_1 - \theta_2)}{m + \sin^2(\theta_1 - \theta_2)}, \tag{4.7}$$

is not always negative (as in the case of an attractor). This quantity changes sign with varying time, becoming positive (expansion of phase-space volume) and negative (contraction of phase-space volume). The presence of this quantity is due to the nonlinear terms $\dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$ and $\dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$ in Eqs. (4.4) which are looking like damping terms. As it was found, these terms (which govern the global behavior of the model) are also responsible for the sensitivity to initial conditions. From the time series shown in Fig. 6 one can see two motions with slightly different initial conditions which gradually diverge with varying time. Note that chaotic phenomena due to sensitivity to initial conditions have been also observed in Hamiltonian systems [29, 31]. A pertinent plot corresponding to a conservative load is shown in Fig. 7 [32]. In this case conservation of energy enables us to check the accuracy of numerical solution when the large time response is required. Before closing this section the following observation is worth making:

Local solutions in the vicinity of an appropriate equilibrium state by considering small or large amplitude motions about it provide in many cases valuable information regarding the response of a system; however, in case of imperfect, multiple-parameter, autonomous systems such, quite often cumbersome, solutions may be proved inadequate to predict some dynamic bifurcations and instability phenomena associated with non-periodic motions that may occur at large time [17, 28, 32]. On the other hand, an asymptotic analysis for large time solutions may lead to unreliable results. A general dynamic analysis leading to global, large time and amplitude oscillation, solutions based on the original nonlinear

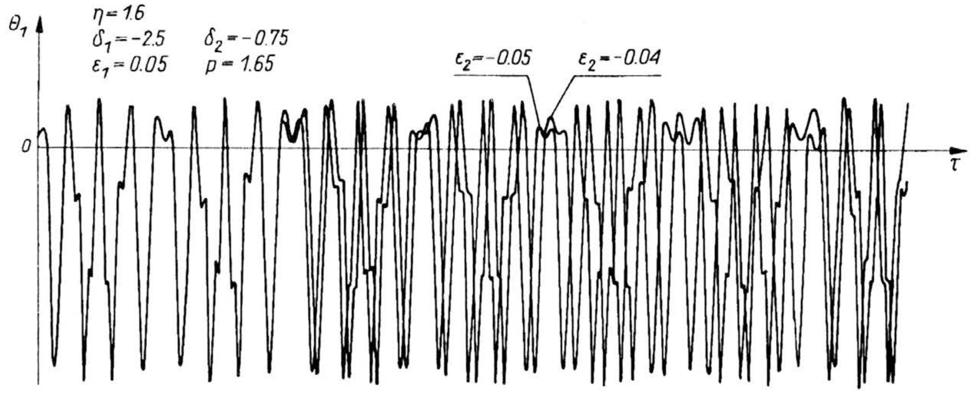

FIG. 6. Two motions corresponding to slightly different initial conditions which diverge with varying time.

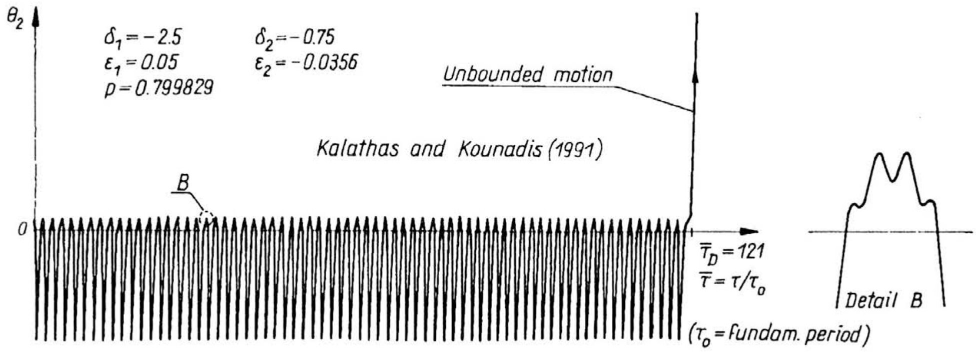

FIG. 7. Unbounded quasi-chaotic motion (θ_2 versus $\overline{\tau}$) corresponding to a conservative loading ($\eta = 1$) leading to dynamic buckling after a long period of time.

differential equations, is the only safe way for predicting the actual behavior of the above systems. Whatever is the approximation technique for establishing such a response, there are cases where a verification via numerical simulation is necessary to be done. Such a numerical procedure can be efficiently applied to discrete systems associated with ordinary differential equations with the aid of high speed computers and modern computational techniques whose accuracy should always be checked if pertinent criteria exist.

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