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Wave speeds in periodic elastic layers

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THE SPATIALLY periodic system of elastic layers is considered. The displacement u_i in the elementary cell consists of the displacements corresponding to the wave propagating to the left and the wave propagating to the right. The displacement u_{i+1} in the neighbouring cell is defined by u_i and the transition matrix M. It is shown that a parameter φ may be defined leading to the (essential for further calculations) relation $M(\varphi)^n = M(n\varphi)$. This relation allows us to define the phase speed. The phase speed is real for small frequencies, but for large frequencies it may be complex.

Rozpatruje się periodyczny w przestrzeni ośrodek warstwowy. Przemieszczenie u_i w komórce elementarnej jest sumą przemieszczenia odpowiadającego fali propagującej się w prawo i fali propagującej się w lewo. Przemieszczenie u_{i+1} w sąsiedniej komórce elementarnej określone jest przez u_i i macierz przejścia M. Pokazano, że można zdefiniować pewien parametr φ taki, że macierz przejścia $M = M(\varphi)$ ma istotną dla obliczeń własność $M(\varphi)^n = M(n\varphi)$. Ta własność, zupełnie taka sama jak własność liczb zespolonych, pozwala na łatwą interpretację rezultatów oraz na zdefiniowanie prędkości fazowej w układzie warstwowym. Prędkość fazowa w układzie warstwowym dla małych częstości jest rzeczywista, dla innych częstości może być zespolona.

Рассматривается периодическая в пространстве, слоистая среда. Перемещение u_i в элементарной ячейке является суммой перемещения отвечающего волне распространяющейся вправо и волны распространяющейся влево. Перемещение u_{i+1} в соседней элементарной ячейке определено через u_i и матрицу перехода M. Показано, что можно определить некоторый параметр φ, такой, что матрица перехода $M = M(\varphi)$ имеет существенное для расчетов свойство $M(\varphi)^n = M(n\varphi)$. Это свойство, вполне же такое самое как свойство комплексных чисел, позволяет легко интерпретировать результаты и определить фазовую скорость в слоистой системе. Фазовая скорость в слоистой системе для мальх частот является действительной, для других частот может быть комплексной.

THE SYSTEMS of layers were dealt with in many papers, e.g. in the already classical ones [1–9]. In tne present paper essential is the introduction of a new parameter φ and representation of the transition matrix $M(\varphi)$ in the form satisfying the identity $M(\varphi)^n = M(n\varphi)$. This allows us to define the phase speed in the composite.

1. Reflection and transmission

Consider the system of homogeneous elastic layers, Fig. 1. The layer situated between x_k and x_{k+1} is denoted by L_k. The Lamé constants and density of the layer L_k are denoted by λ_k, μ_k, ρ_k, $k = 1, 2, 3,\ldots$. In the direction x

perpendicular to the layers propagates the sinusoidal wave of frequency ω. Due to the reflections, the wave propagating in the opposite direction appears. The total displacement in the layer L_k is

$$u_k = A_k \exp i\omega[t-(x-x_k)/c_k] + B_k \exp i\omega[t+(x-x_k)/c_k], \tag{1.1}$$

where t is time, $x_k \leqslant x \leqslant x_{k+1}$, and c_k is the wave speed in the k-th layer

$$c_k^2 = (\lambda_k + 2\mu_k)/\rho_k. \tag{1.2}$$

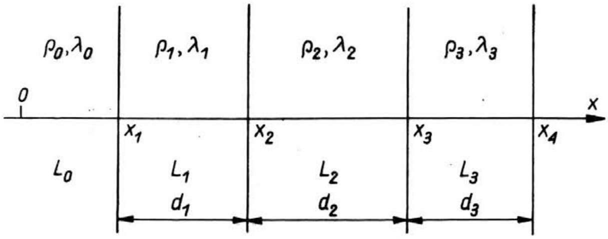

FIG. 1.

The displacement u_k consists of two parts. The first part in Eq. (1.1) represents the wave of amplitude A_k running in the x direction. The second part represents the wave of amplitude B_k running in the -x direction. The displacement u_k satisfies the equation of motion

$$c_k^2 u_{k,xx} = u_{k,tt}. \tag{1.3}$$

The physical displacement is the real part of the complex-valued function $u_k(x,t)$.

At the boundary between the layers both the displacement and the stress vector are continuous. This fact leads to the relations

$$\begin{aligned} A_{k-1}\exp(-i\alpha_k) + B_{k-1}\exp(i\alpha_k) &= A_k + B_k, \\ \varkappa_k[-A_{k-1}\exp(-i\alpha_k) + B_{k-1}\exp(i\alpha_k)] &= -A_k + B_k, \end{aligned} \tag{1.4}$$

where

$$\alpha_k = \omega(x_k - x_{k-1})/c_{k-1}, \quad \varkappa_k = (\rho_{k-1}c_{k-1})/(\rho_k c_k). \tag{1.5}$$

Equation (1.4) may be solved for A_k, B_k to yield

(1.6)
$$\begin{bmatrix} A_k \\ B_k \end{bmatrix} = M_k \begin{bmatrix} A_{k-1} \\ B_{k-1} \end{bmatrix},$$

(1.7)
$$M_k = \frac{1}{2}\begin{bmatrix} (1+\varkappa_k)\exp(-i\alpha_k) & (1-\varkappa_k)\exp(i\alpha_k) \\ (1-\varkappa_k)\exp(-i\alpha_k) & (1+\varkappa_k)\exp(i\alpha_k) \end{bmatrix}.$$

The transition matrix M_k allows us to express A_k, B_k by A_{k-1}, B_{k-1}. The determinant of M_k depends on $\varkappa_k$ but not on α_k,

(1.8)
$$\det M_k = \varkappa_k.$$

2. Periodic layers

Consider now the case when a set of layers is repeated periodically in space. The elementary cell may consist of an arbitrary number of layers. The simplest cell consist of two layers only, Fig. 2. Denote

(2.1)
$$\varkappa = (\rho_a c_a)/(\rho_b c_b), \quad \alpha_a = \omega\, d_a/c_a, \quad \alpha_b = \omega\, d_b/c_b;$$

(2.2)
$$M_a = \frac{1}{2}\begin{bmatrix} (1+\varkappa)\exp(-i\alpha_a) & (1-\varkappa)\exp(i\alpha_a) \\ (1-\varkappa)\exp(-i\alpha_a) & (1+\varkappa)\exp(i\alpha_a) \end{bmatrix},$$
$$M_b = \frac{1}{2}\begin{bmatrix} (1+1/\varkappa)\exp(-i\alpha_b) & (1-1/\varkappa)\exp(i\alpha_b) \\ (1-1/\varkappa)\exp(-i\alpha_b) & (1+1/\varkappa)\exp(i\alpha_b) \end{bmatrix}.$$

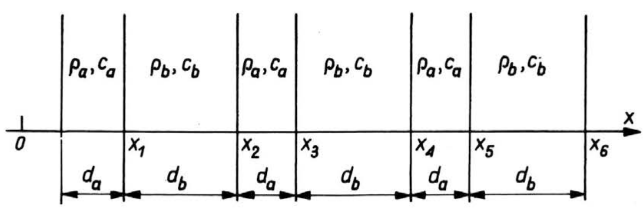

FIG. 2.

Therefore

$$\varkappa_k = \varkappa, \quad c_k = c_a, \quad \alpha_k = \alpha_a, \quad M_k = M_a \quad \text{for } k = 0,\ 2,\ 4,\ldots,$$
$$\varkappa_k = 1/\varkappa \quad c_k = c_b, \quad \alpha_k = \alpha_b, \quad M_k = M_b \quad \text{for } k = 1,\ 2,\ 5,\ldots.$$

In the above formulae M_a is the transition matrix from a to b, and M_b the transition matrix from b to a.

The purpose of the further analysis is to calculate the average wave speed in the set of layers. Concentrate first on the displacements in the layers of type a. In accord with Eq. (1.6), for each intiger n there is

$$\begin{bmatrix} A_{2n} \\ B_{2n} \end{bmatrix} = M^n \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \quad M = M_b\ M_a, \quad n = 1,\ 2,\ 3, \ldots, \tag{2.3}$$

where M is the transition matrix for one cell. From Eq. (2.2) it follows that this transition matrix has the following components:

$$\begin{aligned} 4\ M_{11} &= (2 + \varkappa + 1/\varkappa) \exp i(-\alpha_a - \alpha_b) + (2 - \varkappa - 1/\varkappa) \exp i(-\alpha_a + \alpha_b), \\ 4\ M_{21} &= (\varkappa - 1/\varkappa) \exp i(-\alpha_a - \alpha_b) - (\varkappa - 1/\varkappa) \exp i(-\alpha_a + \alpha_b), \end{aligned} \tag{2.4}$$

$$M_{22} = \overline{M_{11}}, \qquad M_{12} = \overline{M_{21}}. \tag{2.5}$$

Note that the matrix M is non-Hermitean since M_{11} is not real. The matrix with the symmetry (2.5) will be further called W-symmetric. The product of two W-symmetric matrices is W-symmetric. The matrix M is the periodic function of α_a and α_b.

For $\varkappa = 2$, $d_a = 2d_b$ some displacement profiles are given in Figs. 3–6. For $\alpha_a = \alpha_b = 0.1$ (Fig. 3), $\alpha_a = \alpha_b = 0.5$ (Fig. 4) and $\alpha_a = \alpha_b = 1$ (Fig. 5), the

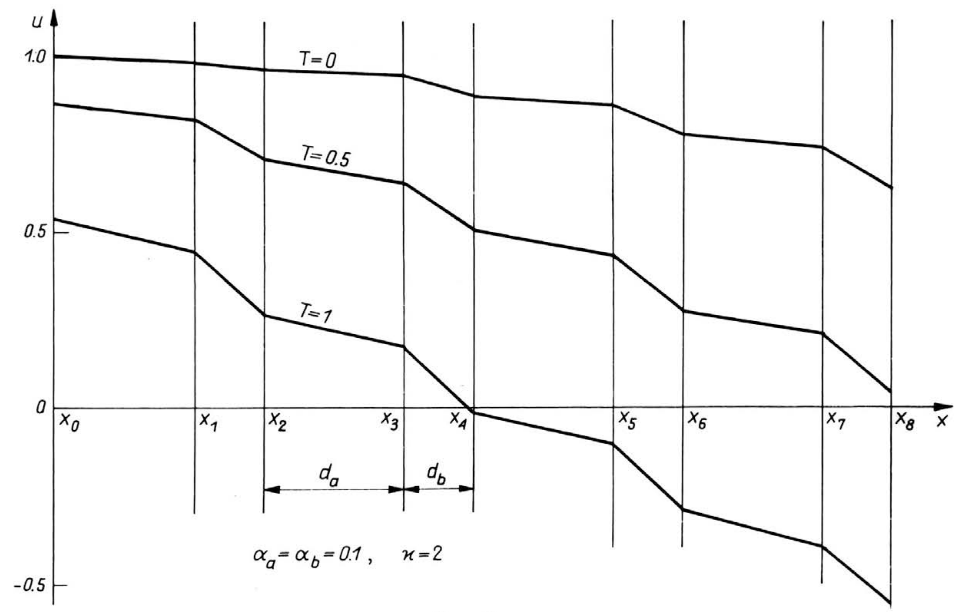

FIG. 3.

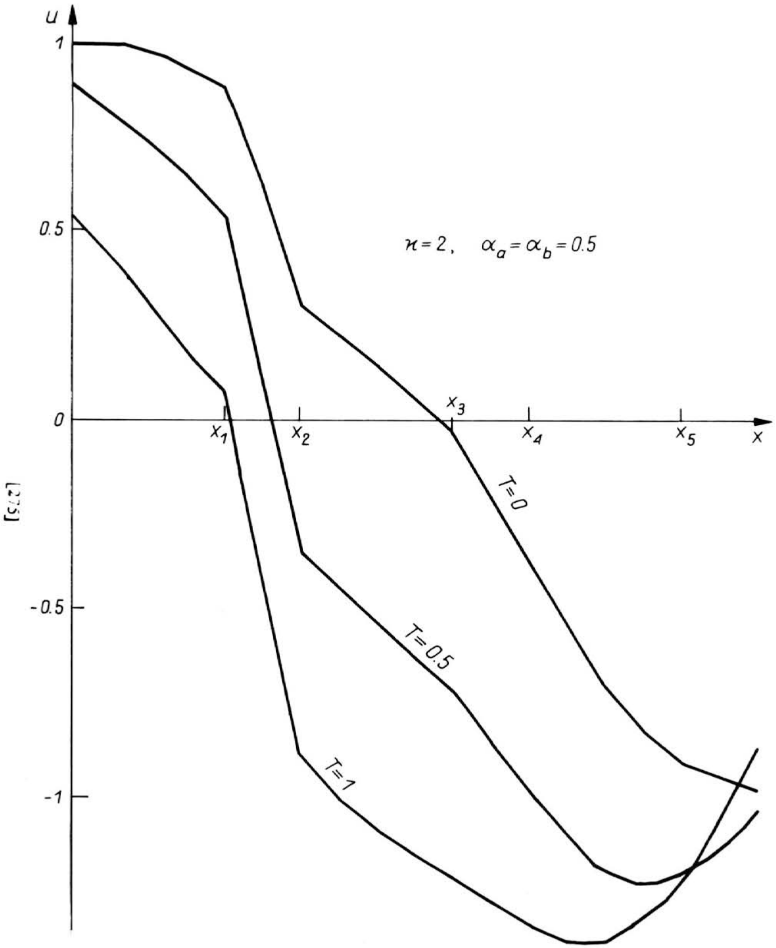
u
1
0.5
0
-0.5
-1
x
x1
x2
x3
x4
x5
ϰ = 2, α_a = α_b = 0.5
T=0
T=0.5
T=1

FIG. 4.

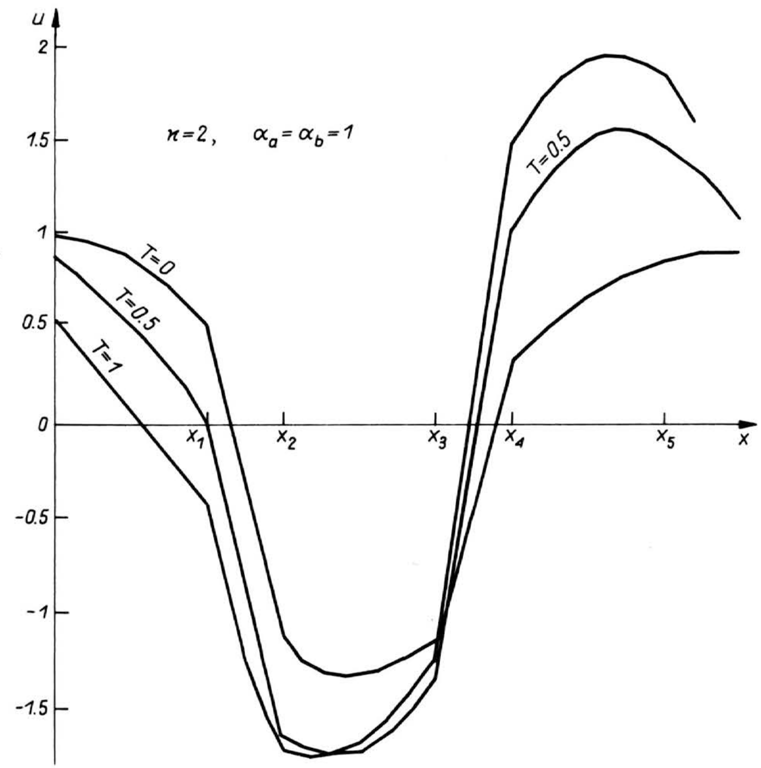
u
2
1.5
1
0.5
0
-0.5
-1
-1.5
x
x1
x2
x3
x4
x5
ϰ = 2, α_a = α_b = 1
T=0
T=0.5
T=1
T=0.5

FIG. 5.

displacement remains small in each cell. For $\alpha_a = \alpha_b = 1.4$ (Fig. 6) the displacement grows exponentially with n.

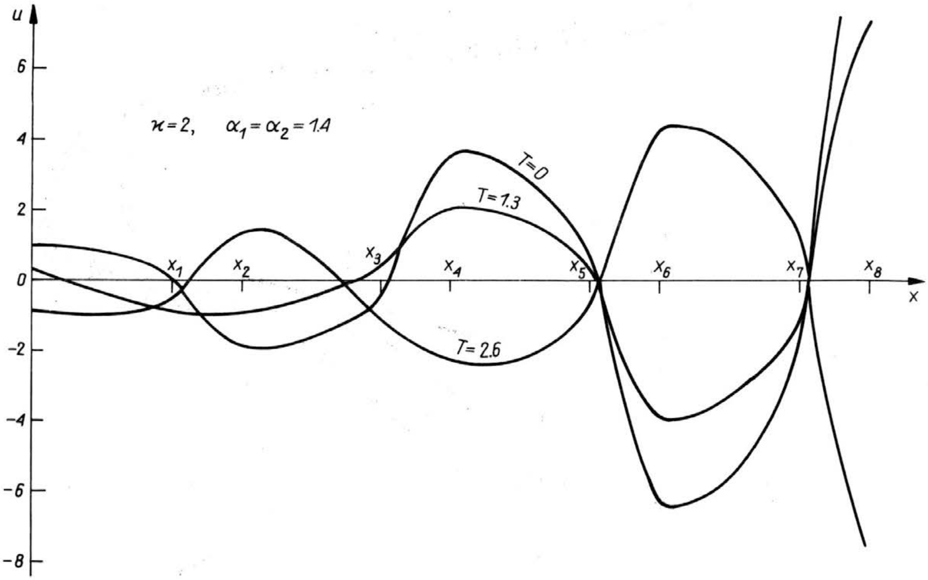

FIG. 6.

Consider first the case when ω is sufficiently small. Then Re $M_{11} < 1$ and, in accord with the derivation given in the Appendix, the matrix M may be written in the form

$$M = \begin{bmatrix} \cos\varphi - iE\sin\varphi & (C+iD)\sin\varphi \\ (C-iD)\sin\varphi & \cos\varphi + iE\sin\varphi \end{bmatrix}, \tag{2.6}$$

$$\varphi = \arccos H, \qquad H = \operatorname{Re} M_{11}, \tag{2.7}$$

$$E^2 - C^2 - D^2 = 1, \tag{2.8}$$

where φ, E, C, D are real parameters. The important identity holds

$$M^n = \begin{bmatrix} \cos n\varphi - iE\sin n\varphi & (C+iD)\sin n\varphi \\ (C-iD)\sin n\varphi & \cos n\varphi + iE\sin n\varphi \end{bmatrix}, \qquad n = 1, 2, \ldots \tag{2.9}$$

Figure 7 shows, for fixed $\varkappa$, the values of α_a, α_b for which $-1 < H < 1$ and φ is real. For fixed $\varkappa$ the region between the curves corresponds to $H <$ -1.

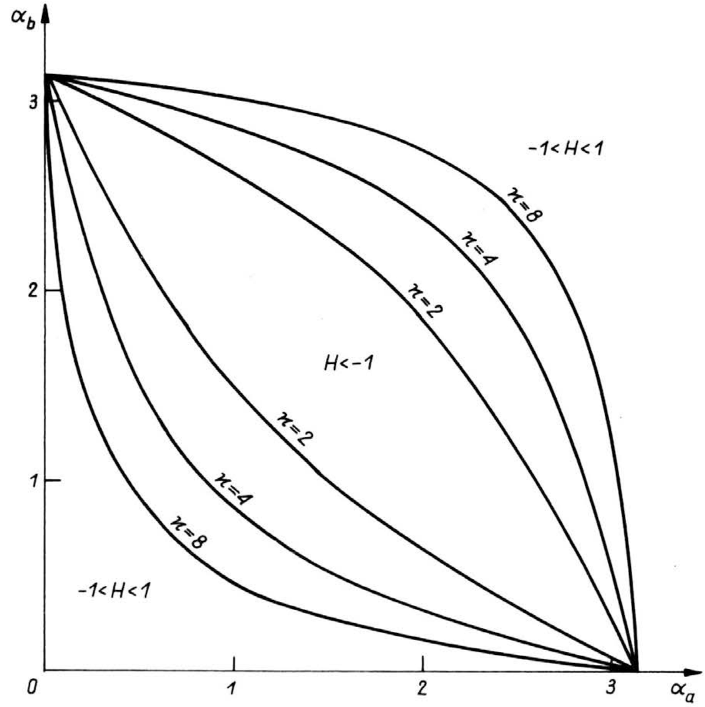

FIG. 7.

The smaller is $\varkappa$, the larger is the region for which φ is real. Because of the periodicity of M, the picture repeats for larger values of α_α, α_b. For $\varkappa = 2$ the regions are shown in Fig. 8.

Calculate the displacement at the beginning of each layer of the kind a, therefore at the discrete set of points

$$x_n = n(d_a + d_b), \quad n = 1, 2, 3, \ldots \tag{2.10}$$

In accord with Eqs. (1.1), (2.3) and (2.9), we obtain

$$u(x_n, t) = \{[M_{11}(n\varphi) + M_{21}(n\varphi)]\, A_0 + [M_{12}(n\varphi) + M_{22}(n\varphi)]\, B_0\} \exp i\omega t \tag{2.11}$$

or, in terms of the parameters φ, C, D, E,

$$u(x_n, t) = \{[\cos n\varphi + (C - iE - iD)\sin n\varphi]\, A_0 + [\cos n\varphi + (C + iE + iD)\sin n\varphi]\, B_0\} \exp i\omega t. \tag{2.12}$$

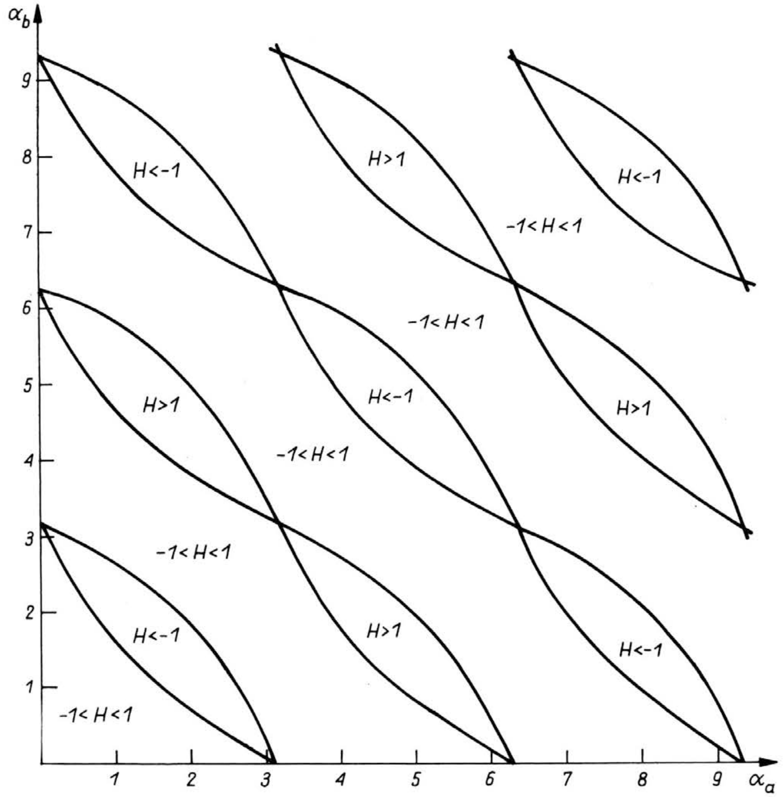

FIG. 8.

The physical displacement is the real part of u as given by the Eq. (2.12)

$$\text{Re}(u) = (A_0 + B_0)(\cos n\varphi + C\sin n\varphi)\cos\omega t + (A_0 - B_0)(E + D)\sin n\varphi \sin\omega t. \tag{2.13}$$

Simple trigonometric transformations lead to the formula

$$\begin{aligned} \text{Re}(u) &= w_R + w_L, \\ 2w_R &= A_0[(1 + E + D)\cos(\omega t - n\varphi) - C\sin(\omega t - n\varphi)] \\ &\quad + B_0[(1 - E - D)\cos(\omega t - n\varphi) - C\sin(\omega t - n\varphi)], \\ 2w_L &= A_0[(1 - E - D)\cos(\omega t + n\varphi) + C\sin(\omega t + n\varphi)] \\ &\quad + B_0[(1 + E + D)\cos(\omega t + n\varphi) + C\sin(\omega t + n\varphi)]. \end{aligned} \tag{2.14}$$

The waves with the phase $(\omega t - n\varphi)$ run to the right, and the waves with the phase $(\omega t + n\varphi)$ run to the left. Note that the amplitudes of all waves are constant and do not depend on n, in contrast to the amplitudes A_k, B_k

in Eq. (2.3) which are functions of n. The arguments of the trigonometric functions (phase) do depend on n. Using Eq. (2.10) we obtain the formulae for phases f_R and f_L of the waves running to the right and the left

$$f_R = \omega t - \varphi x_n(d_a + d_b)^{-1}, \quad f_L = \omega t + \varphi x_n(d_a + d_b)^{-1}, \tag{2.15}$$

or, in accord with Eq. (2.1), the formulae

$$\begin{aligned} f_R &= \omega [t - \varphi x_n(\alpha_a c_a + \alpha_b c_b)^{-1}], \\ f_L &= \omega[t + \varphi x_n(\alpha_a c_a + \alpha_b c_b)^{-1}]. \end{aligned} \tag{2.16}$$

Both formulae hold for the discrete set of points $x = n(d_a + d_b)$ and arbitrary t. It follows that the phase speed c of the waves w_R and w_L is given by the formula

$$c = (\alpha_a c_a + \alpha_b c_b)/\varphi, \tag{2.17}$$

or, using Eq. (2.8), in the explicit form

$$c = (\alpha_a c_a + \alpha_b c_b)/\arccos(\mathrm{Re}\, M_{11}). \tag{2.18}$$

It is seen from Eq. (2.4) that $\varphi = \varphi(\omega)$. Therefore, in accord with the above formula, we obtain $c = c(\omega)$ and the system of layers is dispersive. By differentiation of c with respect to ω the group speed $c_g = dc/d\omega$ may be obtained. Note that for the homogeneous system $\varkappa = 1$ and

$$\mathrm{Re}\, M_{11} = \cos(\alpha_a + \alpha_b), \quad \varphi = \alpha_a + \alpha_b, \quad c = \frac{\alpha_a c_a + \alpha_b c_b}{\alpha_a + \alpha_b}. \tag{2.19}$$

In this case the speed c does not depend on ω and the system is non-dispersive. The curves $c\ (\omega)$ for some speed ratios will be shown in the next Section.

In the above calculations only the points $x = x_{2k}$ were taken into account. The question arises what speed is obtained for other points. We shall show that Eq. (2.18) holds for other points too. Take $x = x_{2k} + p$, where p is fixed, $0 < p < d_a$, and consider the periodic system consisting of three layers, Fig. 9. The three layers are introduced purely formally. Physically this system does not differ from the two-layered system considered above. The layers of thickness p and $d_a - p$ are of the material a and the layer of thickness d_b of the material b. Denoting

$$\gamma = \omega \frac{p}{c_a}, \tag{2.20}$$

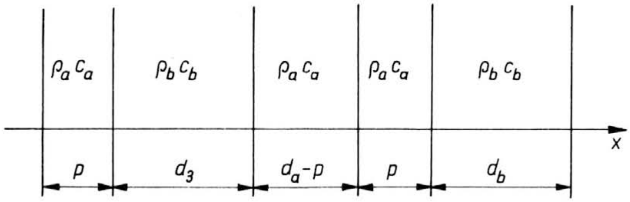

FIG. 9.

we obtain the transition matrix M^* in the form of a product of three matrices

(2.21)
$$M^* = \begin{bmatrix} \exp i(-\alpha_a+\gamma) & 0 \\ 0 & \exp i(\alpha_a+\gamma) \end{bmatrix},$$

$$\begin{bmatrix} (1+1/\varkappa)\exp(-i\alpha_b) & (1-1/\varkappa)\exp(i\alpha_b) \\ (1-1/\varkappa)\exp(-i\alpha_b) & (1+1/\varkappa)\exp(i\alpha_b) \end{bmatrix} \begin{bmatrix} (1+\varkappa)\exp(-i\gamma) & (1-\varkappa)\exp(i\gamma) \\ (1-\varkappa)\exp(-i\gamma) & (1+\varkappa)\exp(i\gamma) \end{bmatrix}.$$

After calculating the product we obtain $M^*_{11} = M_{11}$ what leads to

(2.22)
$$\varphi^* = \varphi.$$

It follows that the phase speed c^* equals c for each point situated in the layer a. On the other hand, Re M_{11} as given by Eq (2.4), is invariant with respect to the interchange $a \to b$. Summarizing: the phase speed c is the same for each point of the elementary cell.

Above we have assumed $-1 < \mathrm{Re}\, M_{11} < 1$. If these inequalities do not hold, we face other cases described in the Appendix. Then in the formula (2.12), instead of the trigonometric functions $\cos\varphi$, $\sin\varphi$, the hyperbolic functions $\mathrm{ch}\,\varphi$, $\mathrm{sh}\,\varphi$ appear. This leads to the exponential growth or exponential decay of the displacements. In this case the phase speed c is complex-valued. The frequency ω_1 for which Re $M_{11} = 1$ or -1 will be called critical. For small ω the speed c is always real.

3. Average speeds

Before analyzing the formula (2.18) for the phase speed, let us define two other speeds. The wave travelling with speed c_a in the layer L_a and speed c_b in the layer L_b covers the distance $d_a + d_b$ in the time interval $d_a/c_a + d_b/c_b$. The first average speed c_1 is defined by the relation

(3.1)
$$(d_a + d_b)/c_1 = d_a/c_a + d_b/c_b.$$

From Eq. (2.1) it follows

$$c_1 = (\alpha_a c_a + \alpha_b c_b)/(\alpha_a + \alpha_b). \tag{3.2}$$

Define next the second average speed c_2. Denote by ρ and E the density and Young modulus of the hypothetic homogeneous material possessing the same mass and rigidity as the system of layers. Due to a tensile stress σ in the x-direction, the unit cell and the homogeneous material have the same elongation. Therefore ρ and E are defined by the relations (cf. Fig. 10)

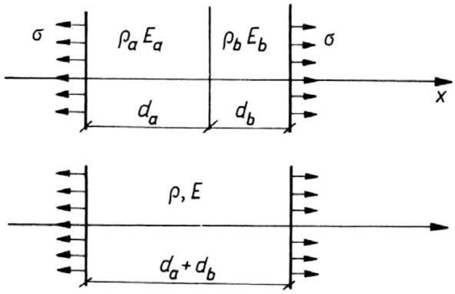

FIG. 10.

$$\begin{aligned} \rho(d_a + d_b) &= \rho_a d_a + \rho_b d_b, \\ (d_a + d_b)\sigma/E &= d_a \sigma/E_a + d_b \sigma/E_b, \end{aligned} \tag{3.3}$$

which lead to

$$\rho = (\rho_a d_a + \rho_b d_b)/(d_a + d_b), \tag{3.4}$$

$$E = (d_a + d_b)/(d_a/E_a + d_b/E_b). \tag{3.5}$$

In the homogeneous material the Young modulus is the product of the density and the squared propagation speed. Define the squared speed c_2 as the ratio E/ρ. Then

$$c_2^2 = \frac{(d_a + d_b)^2}{(d_a/E_a + d_b/E_b)\,(\rho_a d_a + \rho_b d_b)}. \tag{3.6}$$

Basing on Eq. (2.1) this formula may be transformed to the two equivalent formulae

$$c_2^2 = \frac{(\alpha_a c_a + \alpha_b c_b)^2}{\alpha_a^2 + \alpha_b^2 + (\varkappa + 1/\varkappa)\,\alpha_a \alpha_b}, \tag{3.7}$$

$$c_2^2 = c_1^2 \frac{(\alpha_a + \alpha_b)^2}{\alpha_a^2 + \alpha_b^2 + (\varkappa + 1/\varkappa)\,\alpha_a\alpha_b}. \tag{3.8}$$

Since for each $\varkappa$ we have $(\varkappa + 1/\varkappa) \geqslant 2$, it follows from Eq. (3.8) that $c_2 \leqslant c_1$.

Analyze now the formula (2.17) for the phase speed c. Calculate first the speed c for small α_a, α_b, i.e. for small frequency ω. In accord with Eq. (2.4) we have

$$\begin{aligned} M_{11} &= [(2 + \varkappa + 1/\varkappa)\cos\omega\,(d_a/c_a + d_b/c_b) \\ &\qquad + (2 - \varkappa - 1/\varkappa)\cos\omega\,(d_a/c_a - d_b/c_b)]/4 \\ &\approx 1 - \frac{\omega^2}{2}\left[\left(\frac{d_a}{c_a}\right)^2 + \left(\frac{d_b}{c_b}\right)^2 + \left(\varkappa + \frac{1}{2}\right)\frac{d_a d_b}{c_a c_b}\right], \\ \varphi &\approx \left[\alpha_a^2 + \alpha_b^2 + \left(\varkappa + \frac{1}{\varkappa}\right)\alpha_a\alpha_b\right]^{1/2}. \end{aligned} \tag{3.9}$$

Substitution of the last result into Eq. (2.17) yields the approximate relation

$$c \approx (\alpha_a c_a + \alpha_b c_b)\left[\alpha_a^2 + \alpha_b^2 + \left(\varkappa + \frac{1}{\varkappa}\right)\alpha_a\alpha_b\right]^{-1/2}. \tag{3.10}$$

Comparison with Eq. (3.7) leads to the conclusion that for $\omega \Rightarrow 0$ also $c \Rightarrow 0$. In view of the periodicity of M, the same result holds for other ω, provided $\alpha_a \Rightarrow 2\pi n_1$, $\alpha_b \Rightarrow 2\pi n_2$, where n_1, n_2 are integers.

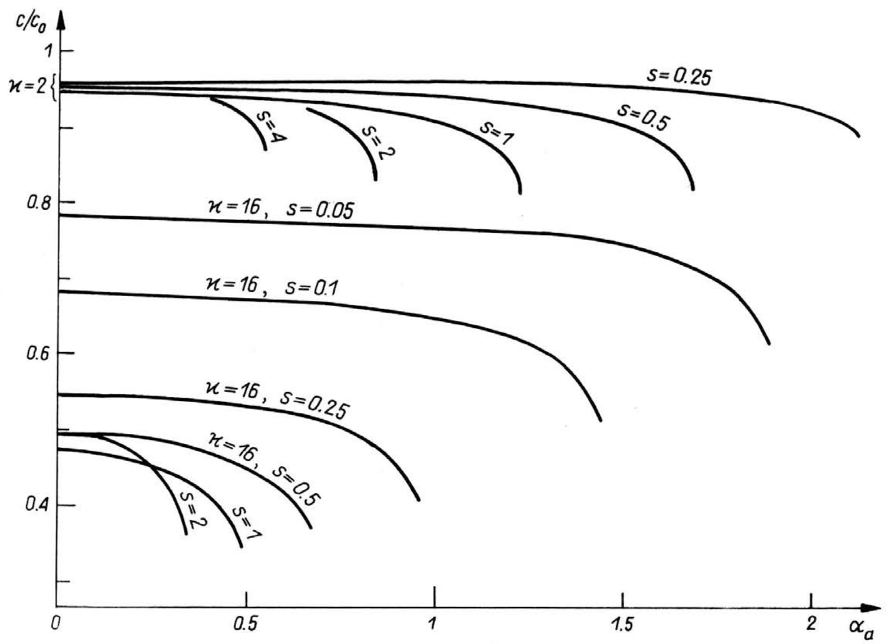

FIG. 11.

Figure 11 gives the functions c/c_0 for $\varkappa = 2$ and $\varkappa = 16$ for several ratios $s = \alpha_b/\alpha_a$. For each s, ratio c/c_o is a decreasing function of ω. At $\alpha_a = 0$ there is $c = c_2$. The function being monotonically decreasing for each frequency ω we have

$$c \leqslant c_2. \tag{3.11}$$

Due to the periodicity it was assumed that $\varphi \leqslant 2\pi$.

The above analysis concerned the case when the elementary cell consisted of two layers only. It is straightforward to generalize the results to any number N of layers in the primitive cell. The transition matrices for the layers are then

$$M_{a_k} = \frac{1}{2}\begin{bmatrix} (1+\varkappa_{a_k})\exp(-i\alpha_{a_k}) & (1-\varkappa_{a_k})\exp(i\alpha_{a_k}) \\ (1-\varkappa_{a_k})\exp(-i\alpha_{a_k}) & (1+\varkappa_{a_k})\exp(i\alpha_{a_k}) \end{bmatrix}, \tag{3.12}$$

$$\det M_{a_k} = \varkappa_{a_k}, \qquad k = 1, 2, 3, \ldots, N-1, \tag{3.13}$$

$$M_{a_N} = \frac{1}{2}\begin{bmatrix} (1+\xi)\exp(-i\alpha_{a_N}) & (1-\xi)\exp(i\alpha_{a_N}) \\ (1-\xi)\exp(-i\alpha_{a_N}) & (1+\xi)\exp(i\alpha_{a_N}) \end{bmatrix}, \tag{3.14}$$

$$\begin{gathered} \xi = 1/(\varkappa_{a_1}\varkappa_{a_2}\ldots\varkappa_{a_{N-1}}), \\ \det M_{a_N} = 1/(\varkappa_{a_1}\varkappa_{a_2}\ldots\varkappa_{a_{N-1}}). \end{gathered} \tag{3.15}$$

The transition matrix (3.14) has a special form because it describes the transition back to the first material. Note that each of the above matrices is W-symmetric, therefore their product is W-symmetric. Moreover, their determinant equals 1 due to Eqs. (3.15). It follows that the product of the N matrices (312), (3.14) satisfies Eq. (2.5), therefore the parameter φ may be introduced. The qualitative results obtained for two layers in the primitive cell hold for arbitrary number of layers in this cell.

Appendix

Consider the 2×2 complex-valued matrix M satisfying the relations

$$M_{21} = \overline{M_{12}}, \qquad M_{22} = \overline{M_{11}}, \tag{A.1}$$

$$\det M = 1. \tag{A.2}$$

The matrix with symmetry (A.1) will be called W-symmetric. The product of two W-symmetric matrices is W-symmetric. In general, the W-symmetric matrix is non-Hermitean.

Three cases are possible: either $-1 < \mathrm{Re}M_{11} < 1$, or $\mathrm{Re}M_{11} \geqslant 1$, or $\mathrm{Re}M_{11} \leqslant -1$. Consider first

$$-1 < \mathrm{Re}M_{11} < 1. \tag{A.3}$$

Without loosing the generality assume the range $0 < \varphi < \pi$ and write the matrix M in the following form:

$$M = \begin{bmatrix} \cos\varphi - iE\sin\varphi & (C+iD)\sin\varphi \\ (C-iD)\sin\varphi & \cos\varphi + iE\sin\varphi \end{bmatrix}, \tag{A.4}$$

where the real parameters φ, E, C, D are uniquely determined by the relations

$$\varphi = \arccos(\mathrm{Re}\,M_{11}), \tag{A.5}$$

$$E\sin\varphi = \mathrm{Im}\,M_{22}, \quad C\sin\varphi = \mathrm{Re}\,M_{12}, \quad D\sin\varphi = \mathrm{Im}\,M_{12}. \tag{A.6}$$

The relation (A.2) leads to

$$E^2 - C^2 - D^2 = 1. \tag{A.7}$$

By mathematical induction we prove now the formula

$$M^n = \begin{bmatrix} \cos n\varphi - iE\sin n\varphi & (C+iD)\sin n\varphi \\ (C-iD)\sin n\varphi & \cos n\varphi + iE\sin n\varphi \end{bmatrix}. \tag{A.8}$$

Multiplying by M we get

$$\begin{gathered} M^{n+1}_{11} = \cos n\varphi\cos\varphi - (E^2 - C^2 - D^2)\sin n\varphi\sin\varphi - iE\sin(n+1)\varphi, \\ M^{n+1}_{21} = (C-iD)\sin(n+1)\varphi, \quad M^{n+1}_{22} = \overline{M^{n+1}_{11}}, \quad M^{n+1}_{12} = \overline{M^{n+1}_{21}}. \end{gathered} \tag{A.9}$$

Taking now into account Eqs. (A.7), we obtain

$$M^{n+1} = \begin{bmatrix} \cos(n+1)\varphi - iE\sin(n+1)\varphi & C+iD)\sin(n+1)\varphi \\ (C-iD)\sin(n+1)\varphi & \cos(n+1)\varphi + iE\sin(n+1)\varphi \end{bmatrix}, \tag{A.10}$$

what is exactly the formula (A.8) for the power $(n+1)$. The fact that (A.8) holds for $n = 1$ completes the proof.

In the cases $\mathrm{Re}\,M_{11} > 1$ and $\mathrm{Re}\,M_{11} < -1$ the above results may be used provided we allow for complex-valued φ. In the practical calculations, however, it is more convenient to introduce the hyperbolic functions and real parameter ψ, and to re-define the other constants.

Consider first

$$\text{Re}\, M_{11} > 1. \tag{A.11}$$

Defining

$$\psi = \text{Arch}\,(\text{Re}\, M_{11}), \tag{A.12}$$

we can represent M in the form

$$M = \begin{bmatrix} \text{ch}\psi - iE\,\text{sh}\psi & (C+iD)\,\text{sh}\psi \\ (C-iD)\,\text{sh}\psi & \text{ch}\psi + iE\,\text{sh}\psi \end{bmatrix}, \tag{A.13}$$

where the constants E, C, D (other than in the trigonometric case) are defined by the relations

$$E\,\text{sh}\psi = \text{Im}\, M_{22}, \quad C\,\text{sh}\psi = \text{Re}\, M_{12}, \quad D\,\text{sh}\psi = \text{Im}\, M_{12}. \tag{A.14}$$

The condition det $M = 1$ leads to

$$-E^2 + C^2 + D^2 = 1. \tag{A.15}$$

By the mathematical induction, exactly in the same manner as in the trigonometric case, it may be shown that

$$M^n = \begin{bmatrix} \text{ch}n\psi - iE\,\text{sh}n\psi & (C+iD)\,\text{sh}n\psi \\ (C-iD)\,\text{sh}n\psi & \text{ch}n\psi + iE\,\text{sh}n\psi \end{bmatrix}. \tag{A.16}$$

For

$$\text{Re}\, M_{11} < -1, \tag{A.17}$$

$$\psi = \text{Arch}\,(-M_{11}), \tag{A.18}$$

we have the following form of the matrix M:

$$M = \begin{bmatrix} \text{ch}\psi - iE\,\text{sh}\psi & (C+iD)\,\text{sh}\psi \\ (C-iD)\,\text{sh}\psi & \text{ch}\psi + iE\,\text{sh}\psi \end{bmatrix}, \tag{A.19}$$

where the real parameters E, C, D are uniquely defined by the relations

$$-E\,\text{sh}\psi = \text{Im}\, M_{22}, \quad -C\,\text{sh}\psi = \text{Re}\, M_{12}, \quad -D\,\text{sh}\psi = \text{Im}\, M_{12}. \tag{A.20}$$

From det $M = 1$ it follows that

$$-E^2 + C^2 + D^2 = 1, \tag{A.21}$$

$$M = (-1)^n \begin{bmatrix} \mathrm{ch} n\psi - iE\,\mathrm{sh} n\psi & (C+iD)\,\mathrm{sh} n\psi \\ (C-iD)\,\mathrm{sh} n\psi & \mathrm{ch} n\psi + iE\,\mathrm{sh} n\psi \end{bmatrix}. \tag{A.22}$$

The cases $\mathrm{Re}\, M_{11} = 1$ and Re $M_{11} <$ -1 are not included in the formula (A.3) because then E tends to infinity. Elementary calculations show that for Re $M_{11} = 1$ we obtain

$$M = \begin{bmatrix} 1 - iE & C + iD \\ C - iD & 1 + iE \end{bmatrix}, \quad M^n = \begin{bmatrix} 1 - inE & n(C + D) \\ n(C - iD) & 1 + inE \end{bmatrix}, \tag{A.23}$$

and for $\mathrm{Re}\, M_{11} = -1$

$$M = -\begin{bmatrix} 1 - iE & C + iD \\ C - iD & 1 + iE \end{bmatrix}, \quad M^n = (-1)^n \begin{bmatrix} 1 - inE & n(C + D) \\ n(C - iD) & 1 + inE \end{bmatrix}. \tag{A.24}$$

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