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Pseudo-similarity analysis of long rods in the transient creep range

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PSEUDO-SIMILARITY solutions are determined for thin long rods that operate in the creep range of temperatures and which are subjected to time-dependent loadings. A time -hardening creep expression is used as the material constitutive relationship. The similarity variable is introduced in the non-similar description, and the method of quasilinearization is used to solve the nonlinear partial differential equation. When time is zero, the non-similar solution degenerates to a similarity solution.

Wyznaczono rozwiązania pseudopodobne dla długich prętów w zakresie niestacjonarnego pełzania pod działaniem temperatury i zmiennych w czasie obciążeń. W związku konstytutywnym uwzględniono człon odpowiadający pełzaniu. Wprowadzono zmienną podobieństwa i zastosowano metodę quasi-linearyzacji do rozwiązania nieliniowego równania różniczkowego cząstkowego. Dla zerowych czasów rozwiązanie problemu degeneruje się do rozwiązania podobieństwa.

Определены решения псевдоподобия для длинных стержней в области нестационарной ползучести под действием температуры и переменных во времени нагружений. В определяющем соотношении учтен член отвечающий ползучести. Введена переменная подобия и применен метод квазилинеаризации для решения нелинейного дифференциального уравнения в частных производных. Для нулевых времен решение задачи вырождается в решение подобия.

Notations

- x coordinate of the axis of the rod,
- t time,
- $\sigma(x, t)$ nominal compressive stress; compressive stress is assumed to be positive,
- $v(x, t)$ particle velocity,
- $\varepsilon(x, t)$ nominal compressive strain; compressive strain is assumed to be positive,
- m, n, A material constants describing first stage of creep,
- ζ similarity variable,
- ρ density of the rod material

1. Introduction

SIMILARITY representation is obtainable for a boundary-value problem provided the governing partial differential equations and the auxiliary conditions are invariant under a group of transformations [1]. A similarity transformation essentially reduces the number of independent variables in partial differential systems. Consolidation of auxiliary conditions is required, and the resulting solutions are therefore domain and boundary condition limited. If any of the equations or auxiliary conditions is not invariant under a group of transformations, then the problem description becomes non-similar. For such problems the methods of 1) superposition of similarity solutions, 2) fundamental solutions and 3) pseudo-similarity analysis have been used [2]. For nonlinear problem descriptions, however, the method of pseudo-similarity analysis is perhaps the most promising.

In this paper, pseudo-similarity solutions are determined for long rods that operate in the creep range of temperatures, and which are subjected to time-dependent loadings. The first stage of creep relationship based on the time-hardening formulation [3] is appropriate for time-dependent load applications. It is assumed that the creep deformations are large compared to the elastic deformations (Fig. 1). The solution essentially describes how the stresses or displacements at diferent locations in the rod change with time for the variable loadings. The results are useful for evaluation of creep-damage and related design considerations.

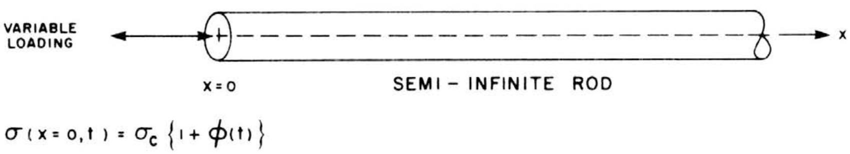

FIG. 1. Impact of a semi-infinite rod.

2. Governing equations

The time-hardening relationship for the first stage of creep (Fig. 2) can be expressed as

$$\frac{\partial \varepsilon}{\partial t} = At^m\sigma^n, \tag{2.1}$$

A, m and n are material parameters that depend on the temperature. For commercial aluminum $A = 4.18\times 10^{-17}$, $m = -0.8$ and $n = 3.5$.

When the creep deformations are relatively large compared to elastic deformation, then Eq. (2.1) represents the physical situation quite well. The one-dimensional deformation of thin rods can be described by the following equations:

Equilibrium

$$\frac{\partial \sigma}{\partial x} = -\rho\frac{\partial v}{\partial t}. \tag{2.2}$$

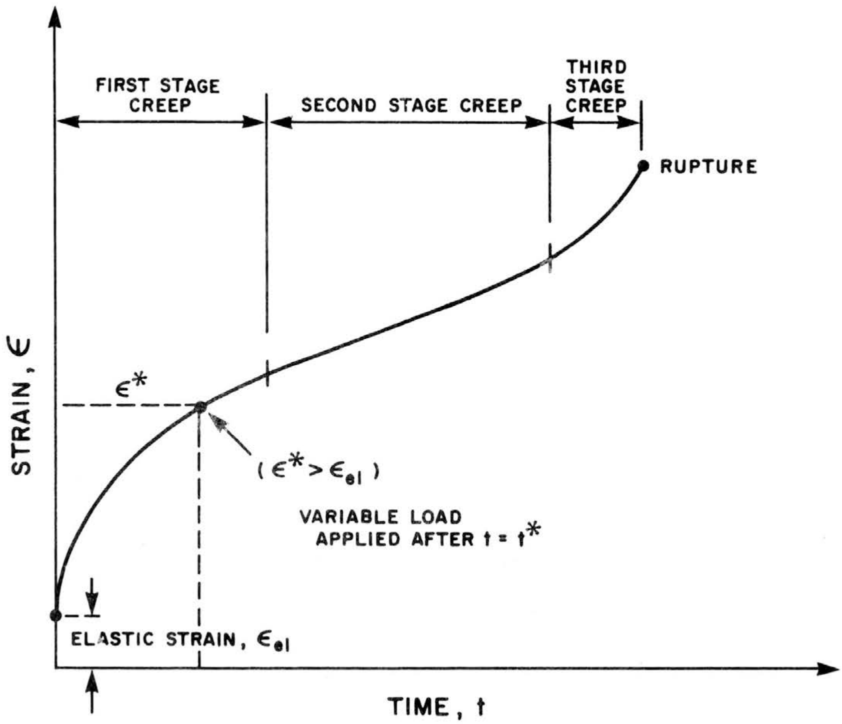

FIG. 2. Uniaxial creep curve.

Compatibility

(2.3) $$\frac{\partial \varepsilon}{\partial t} = -\frac{\partial v}{\partial x}.$$

Constitutive relationship

(2.4) $$\frac{\partial \varepsilon}{\partial t} = At^m \sigma^n$$

The auxiliary condition describing the time-dependent application of loading is given by:

(2.5) $$\sigma(x=0;\ t) = \sigma_c\{1+\phi(t)\},\quad t \geqq 0,$$

σ_c is a constant that corresponds to stress at $t = 0$. Here $\phi(t)$ is the variation in stress with time that is caused by external action. The following $\phi(t)$ variations are considered in this paper:

$$\phi(t) = t,$$
$$\phi(t) = t^2,$$
$$\phi(t) = 0.5\ \sin\ (0.5\ t).$$

The other auxiliary conditions are

$$\sigma(x \to \infty,\ t) = 0 \tag{2.6}$$

and

$$\sigma(x,\ t = 0) = 0. \tag{2.7}$$

Equations (2.1), (2.2) and (2.3) can be combined to give

$$\frac{\partial^2 \sigma}{\partial x^2} - n\sigma A\, t^m \sigma^{n-1} \frac{\partial \sigma}{\partial t} - m\rho A\ \ t^{m-1} \sigma^n = 0. \tag{2.8}$$

Equation (2.8) is quasilinear parabolic. The system of Eqs. (2.5) to (2.8) is solved using "pseudo-similarity analysis".

3. Pseudo-similarity analysis

The following non-dimensional quantities can now be defined:

$$\bar{x} = \frac{x}{x_0},\ \ \bar{t} = \frac{t}{t_0}\ \ \text{and}\ \ \bar{\sigma} = \frac{\sigma}{\sigma_0}, \tag{3.1}$$

x_0, t_0 and σ_0 are arbitrary reference quantities which will be determined by invoking "invariance" of the governing equation and auxiliary conditions such that a "minimum parametric description" occurs [1]. Substituting Eqs. (3.1) into (2.8), the non-dimensional problem description can be written as

$$\bar{\sigma} = f(\bar{x}, \bar{t};\ \pi_e,\ \pi_b), \tag{3.2}$$

where

$$\pi_e = \rho A t_0{}^{m-1} \sigma_0{}^{n-1} x_0{}^2$$

and

$$\pi_b = \frac{\sigma_0}{\sigma_c},$$

π_e and π_b are non-dimensional parametres that have been extracted from the equation and boundary conditions.

To obtain minimum parametric description, Eq. (2.8) and the auxiliary conditions (2.5) to (2.7) are rendered invariant, i.e.

(3.3)
$$\pi_e = 1,$$
$$\pi_b = 1.$$

Therefore, the mathematical description becomes

(3.4)
$$\bar{\sigma} = f(\bar{x}, \bar{t}).$$

Setting $\pi_e = 1$,

$$\rho A t_0{}^{m-1} \sigma_0{}^{n-1} x_0{}^2 = 1,$$

(3.5)
$$x_0 = \frac{\sigma_0^{\frac{1-n}{2}} t_0^{\frac{1-m}{2}}}{\sqrt{\rho A}}.$$

Since x_0 does not occur in the original problem description, it can be suitably eliminated. Therefore, the similarity variable is

(3.6)
$$\zeta = \frac{kx}{t^\theta},$$

where

$$\theta = \frac{1-m}{2},$$

and

$$k = \sqrt{\rho A \sigma_0^{n-1}}.$$

Since $\sigma_0 = \sigma_c$ from $\pi_b = 1$, the similarity tranformation can be written as

$$\sigma(x,t) = \sigma_c f(\zeta),$$

where

$$\zeta = \frac{kx}{t^\theta}.$$

Substitution of Eq. (3.7) into Eq. (2.8) leads to the nonlinear ordinary differential equation

$$f'' + \frac{n(1-m)}{2}\zeta f^{n-1} f' - m f^n = 0. \tag{3.8}$$

Similarity solutions exist for specific forms of $\phi(\tau)$. In this case, $\phi(\tau) = 0$. Therefore, auxiliary conditions become

$$\begin{aligned} f(0) &= 1, \\ f(\infty) &= 0. \end{aligned} \tag{3.9}$$

If $\phi(t)$ is not invariant under a suitable group of transformations, then a non-similar description would result, which can be written as

$$\zeta = \frac{kx}{t^\theta}, \quad \tau = t, \qquad \sigma(x,t) = F(\zeta, \tau). \tag{3.10}$$

The transformed Eq. (2.8), which would still be a partial differential equation is now given by

$$\frac{\partial^2 F}{\partial \zeta^2} + \frac{n(1-m)}{2}\zeta F^{n-1}\frac{\partial F}{\partial \zeta} - m F^n = n\tau F^{n-1}\frac{\partial F}{\partial \tau} \tag{3.11}$$

with conditions

$$\begin{aligned} F(0,\tau) &= 1 + \phi(\tau), \\ F(\infty, \tau) &= 0. \end{aligned} \tag{3.12}$$

When $\tau = 0$, similarity form, Eq. (3.8) would result.

4. Numerical solutions

4.1. Similarity solutions

When $\phi(\tau) = 0$, Eqs. (3.8) and (3.9) constitute the similarity representation. Rewriting them in a first order form,

$$f' = u,$$
$$u' + \frac{n(1-m)}{2}\zeta f^{n-1} u - m f^n = 0$$

with boundary conditions

$$
\begin{aligned}
f(0) &= 1,\\
f(\zeta_m) &= 0.
\end{aligned}
\tag{4.2}
$$

The method of quasi-linearization is used to solve the above system. ζ_m is initially chosen as an arbitrary number. Successive "estimates" of ζ_m that would lead to minimal change in the slope, $f'(\zeta_m)$, would lead to the condition $f(\zeta_m \to \infty) = 0$.

Expressing Eq. (4.1) in terms of finite differences,

$$(f_j - f_{j-1}) - h_j(u_j - u_{j-1}) = 0, \tag{4.3}$$

$$(u_j - u_{j-1}) + h_j \frac{n(1-m)}{2} \zeta_{j-\frac{1}{2}} f_{j-\frac{1}{2}}^{n-1} u_{j-\frac{1}{2}} - mh_j f_{j-\frac{1}{2}}^{n} = 0, \tag{4.4}$$

where $h_j = \zeta_j - \zeta_{j-1}$, and the subscript j denotes the increment level for ζ. Linearization of Eqs. (4.3) and (4.4) is achieved by means of the relationships

$$
\begin{aligned}
f_j^{(\nu+1)} &= f_j^{(\nu)} + \delta f_j,\\
u_j^{(\nu+1)} &= u_j^{(\nu)} + \delta u_j;
\end{aligned}
\tag{4.5}
$$

using the approximation

$$(f_{j-\frac{1}{2}} + \delta f_{j-\frac{1}{2}})^a = f_{j-\frac{1}{2}}^a + a f_{j-\frac{1}{2}}^{a-1} \delta f_{j-\frac{1}{2}},$$

the following equations can be obtained:

$$
\begin{aligned}
&(\delta f_j^{(\nu)} - \delta f_{j-1}^{(\nu)}) - \frac{h_j}{2}(\delta u_j^{(\nu)} + \delta u_{j-1}^{(\nu)}) = (f_{j-1}^{(\nu)} - f_j^{(\nu)}) + \frac{h_j}{2}(u_j^{(\nu)} + u_{j-1}^{(\nu)} = 0,\\
&(\delta u_j^{(\nu)} - \delta u_{j-1}^{(\nu)}) + \frac{n(1-m)}{2} h_j \zeta_{j-\frac{1}{2}} \{ (f_{j-\frac{1}{2}}^{(\nu)} + (n-1) f_{j-\frac{1}{2}}^{(\nu)\,n-2} \}\\
&\qquad \times \{ u_{j-\frac{1}{2}} + \delta u_{j-\frac{1}{2}} \} - mh_j \{ f_{j-\frac{1}{2}}^{(\nu)\,n} + n f_{j-\frac{1}{2}}^{(\nu)} \delta f_{j-\frac{1}{2}}^{(\nu)} \} = u_{j-1}^{(\nu)} - u_j^{(\nu)}.
\end{aligned}
\tag{4.6}
$$

The superscript ν is the iteration number. Although iterations are implied hereafter, ν is dropped for convenience.

Equations (4.6) can be further simplified as follows:

$$
\begin{aligned}
(\delta f_j - \delta f_{j-1}) - \frac{h_j}{2}(\delta u_j + \delta u_{j-1}) &= (r_1)_{j-\frac{1}{2}},\\
(\beta_1)_j \delta f_{j-1} + (\beta_2)_j \delta f_j + (\beta_3)_j \delta u_{j-1} + (\beta_4)_j \delta u_j &= (r_2)_{j-\frac{1}{2}},
\end{aligned}
\tag{4.7}
$$

with boundary conditions

$$\delta f_1 = 0.$$

and

(4.8) $$\delta f_J = 0.$$

In the above,

$$(\beta_1)_j = (\beta_2)_j = \frac{n(1-m)}{4} h_j \zeta_{j-\frac{1}{2}} (n-1) f_{j-\frac{1}{2}}^{n-2} - \frac{mn}{2} h_j f_{j-\frac{1}{2}}^{n-1},$$

$$(\beta_3)_j = -1 + \frac{n(1+m)}{4} h_j \zeta_{j-\frac{1}{2}} f_{j-\frac{1}{2}}^{n-1}$$

and

$$(\beta_4)_j = (\beta_3)_j + 2.$$

The iteration is started after specifying an initial approximate solution

(4.9) $$f(\zeta) = 1 - \frac{\zeta}{\zeta_m}.$$

The boundary conditions, Eqs. (4.2), are satisfied. The solution is improved on the basis of subsequent iterations using Eqs. (4.7) and (4.8).

4.2. Non-similar solutions

If Eqs. (2.5) to (2.8) are not invariant under a group of transformations, then a non-similar description would result. Equations (3.11) and (3.12) can be discretized by letting

(4.10) $$\frac{\partial F}{\partial \tau} = \frac{F_i - F_{i-1}}{\Delta \tau}, \qquad F_{i-\frac{1}{2}} = \frac{F_i + F_{i-1}}{2}$$

and

$$\alpha_{i-\frac{1}{2}} = \frac{2\tau_{i-\frac{1}{2}}}{\Delta \tau}.$$

Although partial derivatives are involved in the non-similar description, for convenience, we define

$$F' \equiv \frac{\partial F}{\partial \zeta} \text{ etc.}$$

i refers to F at time τ, and $i-1$ to F at time, $\tau - \Delta\tau$.

Equations (3.11) and (3.12) can be written as

$$\frac{1}{2}\Big[\Big\{F_i'' + \frac{n(1-m)}{2}\zeta F_i^{n-1} F_i' - mF_i^n\Big\} + \Big\{F_{i-1}'' + \frac{n(1-m)}{2}\zeta F_{i-1}^{n-1}(F_{i-1}') - mF_{i-1}^n\Big\}\Big] = n\tau_{i-\frac{1}{2}}(F_{i-\frac{1}{2}})^{n-1}\left\{\frac{F_i - F_{i-1}}{\Delta\tau}\right\} \tag{4.11}$$

subject to boundary conditions

$$F_i(0,\tau) = 1 + \phi(\tau),$$

$$F_i(\infty,\tau) = 0.$$

Equation (4.11) can be rewritten as follows:

$$F_i'' + \frac{n(1-m)}{2}\zeta F_i^{n-1} F_i' - mF_i^n - \gamma F_i = S_{i-1}, \tag{4.12}$$

where

$$S_{i-1} = -\Big\{F_{i-1}'' + \frac{n(1-m)}{2}\zeta F_{i-1}^{n-1} F_{i-1}' - mF_{i-1}^n\Big\} - n\alpha_{i-\frac{1}{2}}(F_{i-\frac{1}{2}})^{n-1} F_{i-},$$

and

$$\gamma = n\alpha_{i-\frac{1}{2}} F_{i-\frac{1}{2}}^{n-1}.$$

Equation (4.12) is reduced to a first-order system

$$F_i' = u_i,$$

$$u_i' + \frac{n(1-m)}{2}\zeta F_i^{n-1} u_i - mF_i^n - \gamma F_i = S_{i-1}, \tag{4.13}$$

with

$$F_{i0} = 1 + \phi(\tau),$$
$$F_{im} = 0.$$

Dropping the i and discretizing with respect to ζ, we obtain

$$F_j - F_{j-1} - \frac{h_j}{2}(u_j + u_{j-1}) = 0$$

and

$$(4.14)\quad u_j - u_{j-1} + \frac{h_j}{2} n(1-m)\zeta_{j-\frac{1}{2}} f^{n-1}_{j-\frac{1}{2}} - mf^n_{j-\frac{1}{2}} h_j - \gamma h_j f_{j-\frac{1}{2}} = (S_{i-1,j-\frac{1}{2}}) h_j,$$

with

$$F_0 = 1 + \phi(\tau),$$
$$F_m = 0.$$

Linearization performed in a similar fashion as in the previous case yields

$$F_j^{(\nu+1)} = F_j^{(\nu)} + \delta F_j,$$
$$u_j^{(\nu+1)} = u_j^{(\nu)} + \delta u_j, \text{ etc.}$$

Again dropping the iteration index ν and simplifying, we obtain

$$(4.15)\quad \begin{aligned} (\delta F_j - \delta F_{j-1}) - \frac{h_j}{2}(\delta u_j + \delta u_{j-1}) &= (R_1)_{j-\frac{1}{2}},\\ (\beta_5)_j \delta F_{j-1} + (\beta_6)_j \delta F_j + (\beta_7)_j \delta u_{j-1} + (\beta_8)_j \delta u_j &= (R_2)_{j-\frac{1}{2}}, \end{aligned}$$

with boundary conditions

$$\delta F_1 = 0,$$
$$\delta F_m = 0,$$

where

$$(\beta_5)_j = (\beta_6)_j = \frac{n(1-m)}{4} h_j \zeta_{j-\frac{1}{2}} (n-1) F^{n-2}_{j-\frac{1}{2}} u_{j-\frac{1}{2}} - \frac{mn}{2} h_j F^{n-1}_{j-\frac{1}{2}} - \frac{\gamma h_j}{2},$$

$$(\beta_7)_j = -1 + \frac{n(1-m)}{4} h_j \zeta_{j-\frac{1}{2}} F^{n-1}_{j-\frac{1}{2}},$$

$$(\beta_8)_j = (\beta_7)_j + 2,$$

$$(R_1)_{j-\frac{1}{2}} = F_{j-1} - F_j + \frac{h_j}{2}(u_j + u_{j-1}),$$

$$(R_2)_{j-\frac{1}{2}} = (u_{j-1} - u_j) - \frac{n(1-m)}{2} h_j \zeta_{j-\frac{1}{2}} F^{n-1}_{j-\frac{1}{2}} u_{j-\frac{1}{2}} + m h_j F^n_{j-\frac{1}{2}} + \gamma h_j F_{j-\frac{1}{2}} + (S_{i-1,j-\frac{1}{2}}) h_j,$$

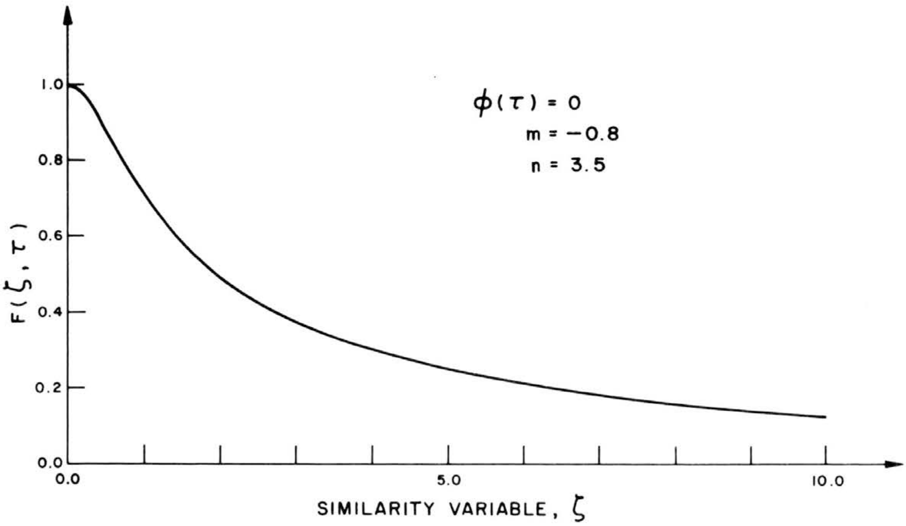

FIG. 3. Similarity solution, $F(\zeta)$ versus ζ.

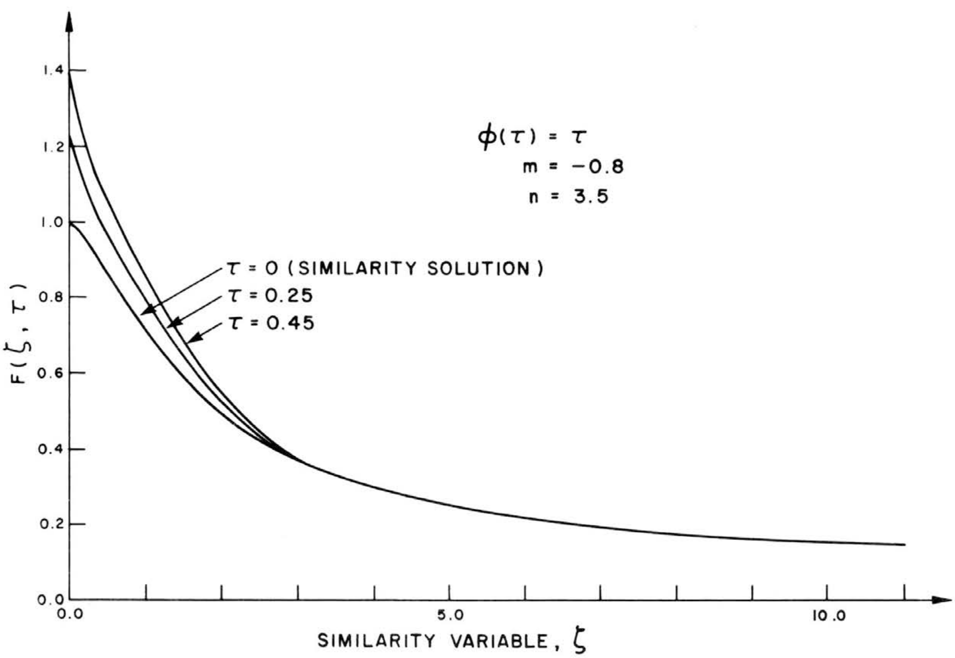

FIG. 4. Non-similar solutions, $F(\zeta,\tau)$ versus ζ for different τ.

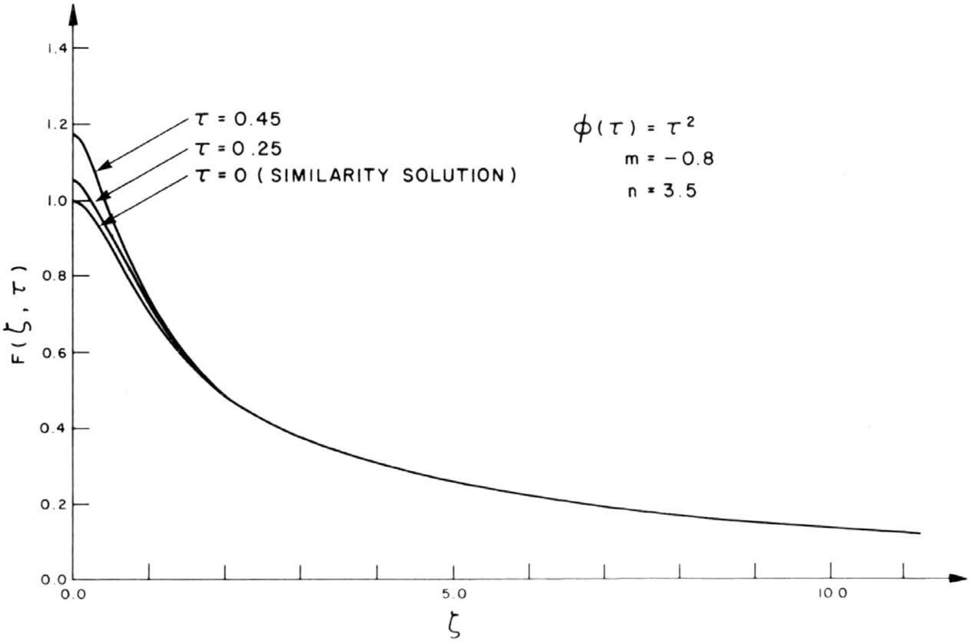

FIG. 5 Non-similar solutions, $F(\zeta,\tau)$ versus ζ for different τ.

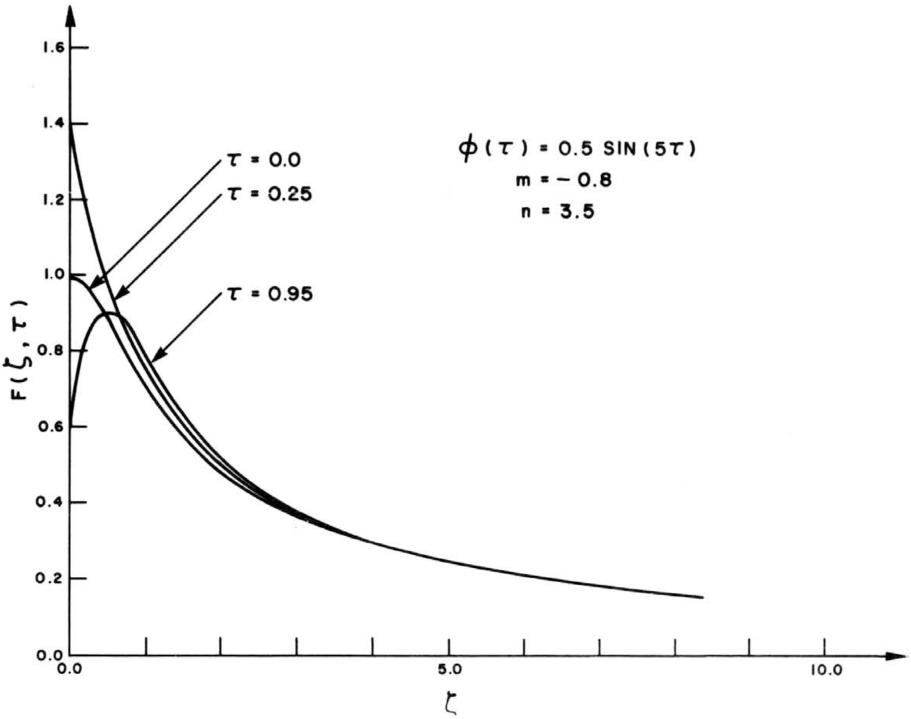

FIG. 6 Non-similar solutions, $F(\zeta,\tau)$ versus ζ for different τ.

$$S_{i-1,j-\frac{1}{2}} = -\left\{\frac{u_{i-1,j} - u_{i-1,j-1}}{h_j} + \frac{n(1-m)}{2}\zeta_{j-\frac{1}{2}} \times F^{n-1}_{i-1,j-\frac{1}{2}} u_{i-1,j-\frac{1}{2}} - mF^{n}_{i-j,j-\frac{1}{2}}\right\} - n\alpha_{i-\frac{1}{2}}(F^{n-1}_{j-\frac{1}{2}}) F_{i-1,j-\frac{1}{2}}.$$

Similarity solutions are generated using the method of quasi-linearization. For $n = 3.5$ and $m = -0.8$, the variation of $f(\zeta)$ versus ζ is plotted in Fig. 3. The non-similar solutions are shown in Fig. 4 for $\phi(\tau) = \tau$, in Fig. 5 for $\phi(\tau) = \tau^2$ and in Fig. 6 for $\phi(\tau) = 0.5 \sin(5\tau)$. In principle, other variantions of $\phi(\tau)$ can be easily used. It is interesting to note that when $\tau = 0$, the non-similar solution degenerates to the similarity solution.

Conclusions

The method of pseudo-similarity analysis has been used to obtain solutions to the problem of a thin long rod operating in the transient creep range that is subjected to time-dependent loading. One of the major limitations of similarity methods, despite its powerful mathematical basis, is that it is restricted to invariant descriptions. Pseudo-similarity analysis is a method that deals with non-invariant equations and auxiliary conditions, and in this sense, extends similarity methods to less specific mathematical descriptions both linear and nonlinear. It should be possible to extend the research efforts pertaining to fluid mechanics and heat transfer, that are abundant in self-similar solutions, to the more useful non-similar solutions.

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