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Extended field theory of a rigid conductor of heat versus hydrodynamics of a phonon gas Hilbert iteration(*)

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BY THE EXAMPLE of a phonon gas, it is shown that the field theories obtained previously by cutting the hierarchy of moment equations with the help of the variational procedure of entropy maximization can be, for a suitably chosen class of kinetic equations, derived directly from the kinetic equation by means of a Hilbert iteration. The distribution function which is a solution of the variational problem of entropy maximization is at the same time the leading term of the assumed formal expansion of the solution of the kinetic equation with respect to a small parameter. This method, which results in field theories possessing structural properties of extended thermodynamics, requires very restrictive assumptions concerning the structure of a collision integral.

1. Introduction

IN THE PAPER [1], the structural properties of the extended field theory of a rigid conductor of heat, derived from the variational principle of entropy maximization, were discussed. The system of moment equations corresponding to the sequence of balance laws of extended field theory of a rigid conductor of heat was derived from the kinetic Boltzman–Peierls equation for phonons. Next, from the standard procedure of statistical physics (compare [2]), the phonon distribution function was determined which maximizes the kinetic entropy under the constraints corresponding to the prescribed values of the quantities taken as the primitive fields of the theory. This distribution function, depending on the Lagrange multipliers of the variational problem, was inserted into the corresponding system of moment equations, which led to the field theory with fields of Lagrange multipliers as independent variables. Such an approach is analogous to that applied by Dreyer in the kinetic theory of material gases [3]. According to that approach, a closed system of field equations is obtained as a consequence of the additional postulate (i.e. cutting of the hierarchy of moment equations by means of the variational procedure) which is independent of the kinetic theory itself.

A natural question arises as to whether such a closed system of field equations can be derived directly from the kinetic theory, without making any additional assumptions. In the context of this problem, in the present paper we apply to the kinetic Boltzman–Peierls equation the reasoning motivated by the Hilbert iteration of the kinetic theory of material gases [4,5]. For simplicity, we shall restrict the considerations to a single branch of phonon excitations, neglecting all interactions between different branches. As a result of this reasoning, leading to a slightly modified version of the Hilbert iteration,

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the field equations obtained before by means of the variational procedure of the entropy maximization occur now as a consequence of the necessary condition for the first step of iteration. The distribution function which solves the variational problem is at the same time the leading term of the asymptotic expansion of the solution of the kinetic equation with respect to a small parameter.

However, this method can be applied only under strong restrictions regarding the collision integral of the kinetic equation. Except for the very special case, leading to the equations of the low-temperature phonon gas hydrodynamics of Nielsen and Shklovsky (discussed in [6]), these assumptions are not consistent with the properties of collision rates known from solid state physics. The reasoning presented in this paper can be applied also for derivation of extended field theory of material gases obtained previously by Dreyer by means of the variational principle of entropy maximization. Similarly as in the case of phonon gases, such an approach requires very strong assumptions concerning the structure of a collision integral.

It is well known, that the field theories in which the state of a system is represented by a distribution function that maximizes the entropy, have the structural properties of extended thermodynamics (in the sense that dissipative fluxes can be taken as primitive fields and a system of field equations is consistent with the additional balance law which can be interpreted as the entropy law) [1, 3]. By means of the reasoning presented in this paper, such field theories can be derived from the kinetic theory, but only under very restrictive and unphysical conditions imposed on the form of collision rates. It seems to be likely that these theories can not, in fact, be derived under less restrictive assumptions; however, this must remain, at the moment, an open question.

2. Hilbert iteration

In the Hilbert iteration one expands a solution of a kinetic equation with respect to a small parameter introduced into the collision integral [4,5]. Insertion of this expansion into the kinetic equation and separation of terms multiplied by different powers of small parameter leads to an infinite sequence of linear integral equations. Such an approach has a formal character and the convergence problems related with it has not yet been solved [4,5]. In order to formulate the iteration procedure, one has to specify the explicit form of a collision integral and introduce into it a small parameter. Then the structure of the collision integral determines a system of field equations which must be satisfied as a necessary condition for the first step of iteration. Therefore we shall search for such a form of a collision integral which leads to the system of field equations identical to that obtained previously from the maximization of entropy [1]. Since for our purpose, which is the derivation of the field theories, it is sufficient to consider the first step of iteration, the subsequent steps shall not be discussed.

The physical quantities which are primitive fields of the extended field theory of a rigid conductor of heat are the energy density e, the heat flux $\mathbf{q}$, the flux of the heat flux $\mathbf{Q}$ and a sequence of other quantities which occur in the procedure of taking succeeding extensions of the energy balance (by extension of the balance law we mean here substitution of the constitutive law for the flux by the evolution equation of a divergence form; such a procedure can be repeated many times, see [1]).

In kinetic description of dynamics of a phonon gas these quantities are identified with

the corresponding moments of a phonon distribution function. Let us denote by

$$\overset{(i)}{\mathbf{N}}(\mathbf{k}) = \hbar\omega(\mathbf{k})[\overset{i}{\otimes}\mathbf{v}(\mathbf{k})], \quad i = 0, 1, 2, \ldots \tag{2.1}$$

the Euclidean tensor, which is a product of the phonon dispersion curve $\omega(\mathbf{k})$ and of the i-th tensor power of the phonon group velocity $\mathbf{v}(\mathbf{k}) = \nabla_{\mathbf{k}}\omega(\mathbf{k})$ ($\mathbf{k}$ is the quasi-particle wave-vector, $2\pi\hbar$ denotes Planck's constant and $\nabla_{\mathbf{k}}$ is a gradient with respect to the variable $\mathbf{k}$).

The moments, defined as integrals of the product of the phonon distribution function and of the tensors $\overset{(i)}{\mathbf{N}}(\mathbf{k})$ shall be denoted as

$$\overset{(i)}{\mathbf{Q}}(\mathbf{x}, t) = \int \overset{(i)}{\mathbf{N}}(\mathbf{k}) f(\mathbf{x}, \mathbf{k}, t) \frac{d^3\mathbf{k}}{(2\pi)^3}, \quad i = 0, 1, 2, \ldots, \tag{2.2}$$

where $f(\mathbf{x}, \mathbf{k}, t)$ is the phonon distribution function, $\mathbf{x}$ denotes the spatial variable, t is the time, $d^3\mathbf{k}$ denotes the integration over the First Brillouin Zone and the factor $(2\pi)^{-3}$ is introduced according to the normalization convention for the distribution function which is usually applied in solid state physics [7]. The moments defined by Eq. (2.2) correspond to the primitive fields of the extended field theory of a rigid conductor of heat, where in particular $e = \overset{(0)}{Q}$, $\mathbf{q} = \overset{(1)}{\mathbf{Q}}$ and $\mathbf{Q} = \overset{(2)}{\mathbf{Q}}$.

In general, the collision integral of the kinetic equation which describes the processes in which n quasi-particles transform into p quasi-particles (and vice versa) has the form

$$\begin{aligned}\tilde{J}(f) = \frac{1}{n!}\frac{1}{(p-1)!} \int d^3\mathbf{k}_1 \ldots d^3\mathbf{k}_n d^3\mathbf{k}_{n+1} \ldots \ldots d^3\mathbf{k}_{p+n-1} \\ \times \mathcal{P}(\mathbf{k}_1, \ldots, \mathbf{k}_n; \mathbf{k}_{n+1}, \ldots, \mathbf{k}_{p+n-1}, \mathbf{k}) \\ \times \{f(\mathbf{k}_1) \ldots f(\mathbf{k}_n)[f(\mathbf{k}_{n+1}) + 1] \ldots [f(\mathbf{k}_{p+n-1}) + 1][f(\mathbf{k}) + 1] \\ - f(\mathbf{k}_{n+1} \ldots f(\mathbf{k}_{p+n-1}) f(\mathbf{k})[f(\mathbf{k}_1) + 1] \ldots [f(\mathbf{k}_n) + 1]\} \\ + \frac{1}{p!}\frac{1}{(n-1)!} \int d^3\mathbf{k}_1 \ldots d^3\mathbf{k}_{n-1} d^3\mathbf{k}_{n+1} \ldots d^3\mathbf{k}_{p+n} \mathcal{P}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}, \mathbf{k}; \mathbf{k}_{n+1}, \ldots, \mathbf{k}_{p+n}) \\ \times \{f(\mathbf{k}_{n+1} f(\mathbf{k}_{n+2}) \ldots f(\mathbf{k}_{p+n})[f(\mathbf{k}_1) + 1] \ldots [f(\mathbf{k}_{n-1}) + 1][f(\mathbf{k}) + 1] \\ - f(\mathbf{k}_1) \ldots f(\mathbf{k}_{n-1}) f(\mathbf{k})[f(\mathbf{k}_{n+1}) + 1] \ldots [f(\mathbf{k}_{p+n}) + 1]\},\end{aligned} \tag{2.3}$$

where $\mathcal{P}(\mathbf{k}_1, \ldots, \mathbf{k}_n; \mathbf{k}_{n+1}, \ldots, \mathbf{k}_{p+n})$ and $\mathcal{P}(\mathbf{k}_{n+1}, \ldots, \mathbf{k}_{p+n}; \mathbf{k}_1, \ldots, \mathbf{k}_n)$ are the collision rates of both processes. The collision rates are non-negative functions which do not depend on the permutations of variables belonging to each set of arguments separated by a semicolon. We assume that

$$\mathcal{P}(\mathbf{k}_1, \ldots, \mathbf{k}_n; \mathbf{k}_{n+1}, \ldots, \mathbf{k}_{n+p}) = \mathcal{P}(\mathbf{k}_{n+1}, \ldots, \mathbf{k}_{n+p}; \mathbf{k}_1, \ldots, \mathbf{k}_n). \tag{2.4}$$

In general, the collision integral $J(f)$ of the kinetic equation for phonons is a sum of the terms of the form (2.3). It can be easily checked that for such collision integrals the Boltzman H-theorem holds (problems related to the construction of collision integrals for phonons and technical details involved in the proof of H-theorem are discussed, for example, in [8]). By a summational invariant of the collision integral $J(f)$ we mean such a function $\mathbf{M}(\mathbf{k})$, defined on a set of wave-vectors and with values in the space of Euclidean tensors, that for every distribution function f the following identity holds

$$\int \mathbf{M}(\mathbf{k}) J(f) \frac{d^3\mathbf{k}}{(2\pi)^3} = 0. \tag{2.5}$$

In kinetic theory of phonons the law of conservation of energy is taken into account by the requirement that the phonon energy

$$\hbar\omega(\mathbf{k}) = \overset{(0)}{N}(\mathbf{k}) \tag{2.6}$$

is a summational invariant for all collision integrals.

As it was already mentioned in Introduction, the set of primitive fields of a macroscopic (hydrodynamical) description depends on the structural properties of a collision integral. Therefore, in order to obtain a macroscopic description with a prescribed set of primitive fields, we have to specify the appropriate class of collision integrals. In order to do that, we shall assume that in the collision integral $J(f)$, which is a sum of the terms of the form (2.3), a distinguished term $\overline{J}(f)$ appears possessing, besides the phonon energy, also $l-1$ additional collision invariants $\overset{(i)}{\mathbf{N}}(\mathbf{k})$ for $i = 2, 3, \ldots, l-1$ (collision invariants of $\overline{J}(f)$ correspond to the prescribed set of primitive fields $e, \mathbf{q}, \mathbf{Q}, \ldots, \overset{(l-1)}{\mathbf{Q}}$). The form of the term $\overline{J}(f)$ is given by the expression (2.3) with the additional restriction that the quantities $\overset{(i)}{\mathbf{N}}(\mathbf{k})$, $i = 1, 2, \ldots, l-1$ are conserved during a collision. This restriction can be taken into account by introducing into Eq. (2.3) the corresponding products of Dirac's functions, denoted by δ_1^l and δ_2^l:

$$\begin{aligned}
\delta_1^l &= \prod_{i=0}^{l-1} \delta(\overset{(i)}{\mathbf{N}}(\mathbf{k}_1) + \ldots + \overset{(i)}{\mathbf{N}}(\mathbf{k}_n) - \overset{(i)}{\mathbf{N}}(\mathbf{k}_{n+1}) - \ldots - \overset{(i)}{\mathbf{N}}(\mathbf{k}_{n+p-1}) + \overset{(i)}{\mathbf{N}}(\mathbf{k}))\,, \\
\delta_2^l &= \prod_{i=0}^{l-1} \delta(\overset{(i)}{\mathbf{N}}(\mathbf{k}_1) + \ldots + \overset{(i)}{\mathbf{N}}(\mathbf{k}_{n-1}) - \overset{(i)}{\mathbf{N}}(k) - \overset{(i)}{\mathbf{N}}(\mathbf{k}_{n+1}) - \ldots - \overset{(i)}{\mathbf{N}}(\mathbf{k}_{n+p}))\,,
\end{aligned} \tag{2.7}$$

where $l < n$ and $l < p$.

Hence, the explicit form of $\overline{J}(f)$ reads

$$\begin{aligned}
\overline{J}(f) = \frac{1}{n!}\frac{1}{(p-1)!} \int d^3\mathbf{k}_1 \ldots d^3\mathbf{k}_n d^3\mathbf{k}_{n+1} \ldots d^3\mathbf{k}_{p+n-1} \mathcal{P}(\mathbf{k}_1, \ldots, \mathbf{k}_n; \mathbf{k}_{n+1}, \ldots \\
\ldots, \mathbf{k}_{n+p-1}, \mathbf{k}) \delta_1^l \{f(\mathbf{k}_1) \ldots f(\mathbf{k}_n)[f(\mathbf{k}_{n+1}) + 1] \ldots [f(\mathbf{k}_{p+n-1}) + 1][f(\mathbf{k}) + 1] \\
- f(\mathbf{k}_{n+1}) \ldots f(\mathbf{k}_{p+n-1}) f(\mathbf{k})[f(\mathbf{k}_1) + 1] \ldots [f(\mathbf{k}_n) + 1]\} \\
+ \frac{1}{p!}\frac{1}{(n-1)!} \int d^3\mathbf{k}_1 \ldots d^3\mathbf{k}_{n-1} d^3\mathbf{k}_{n+1} \ldots d^3\mathbf{k}_{p+n} \mathcal{P}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}, \mathbf{k}; \mathbf{k}_{n+1}, \ldots, \mathbf{k}_{p+n}) \delta_2^l \\
\times \{f(\mathbf{k}_{n+1}) f(\mathbf{k}_{n+2}) \ldots f(\mathbf{k}_{n+p})[f(\mathbf{k}_1) + 1] \ldots [f(\mathbf{k}_{n-1}) + 1][f(\mathbf{k}) + 1] \\
- f(\mathbf{k}_1) \ldots f(\mathbf{k}_{n-1}) f(\mathbf{k})[f(\mathbf{k}_{n+1}) + 1] \ldots [f(\mathbf{k}_{n+p}) + 1]\}\,.
\end{aligned} \tag{2.8}$$

By a direct computation it can easily be checked that the quantities $\overset{(0)}{N}(\mathbf{k}), \overset{(1)}{\mathbf{N}}(\mathbf{k}), \ldots, \overset{(l-1)}{\mathbf{N}}(\mathbf{k})$ are summational invariants of the collision integral (2.8):

$$\int \overset{(i)}{\mathbf{N}}(\mathbf{k}) \overline{J}(f) \frac{d^3\mathbf{k}}{(2\pi)^3} = 0\,, \qquad i = 0, 1, \ldots, l-1\,. \tag{2.9}$$

We shall formally introduce into the collision integral (2.8) the factor $1/\varepsilon$. Our kinetic equation takes then a form

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} = J_1(f) + \frac{1}{\varepsilon} J_2(f)\,, \tag{2.10}$$

where

$$\frac{1}{\varepsilon} J_2(f) = \overline{J}(f), \tag{2.11}$$

and $J_1(f)$ denotes a sum of all the remaining terms of the form (2.3).

From Eqs. (2.1), (2.2), (2.9) and (2.11) it follows, that the kinetic equation (2.10) implies the following infinite sequence of moment equations

$$\begin{aligned}
&\frac{\partial}{\partial t} \overset{(0)}{Q} + \operatorname{div} \overset{(1)}{\mathbf{Q}} = 0, \\
&\frac{\partial}{\partial t} \overset{(1)}{\mathbf{Q}} + \operatorname{div} \overset{(2)}{\mathbf{Q}} = \int \overset{(1)}{\mathbf{N}}(\mathbf{k}) J_1(f) \frac{d^3\mathbf{k}}{(2\pi)^3}, \\
&\qquad \vdots \\
&\frac{\partial}{\partial t} \overset{(l-1)}{\mathbf{Q}} + \operatorname{div} \overset{(l)}{\mathbf{Q}} = \int \overset{(l-1)}{\mathbf{N}}(\mathbf{k}) J_1(f) \frac{d^3\mathbf{k}}{(2\pi)^3}, \\
&\frac{\partial}{\partial t} \overset{(l)}{\mathbf{Q}} + \operatorname{div} \overset{(l+1)}{\mathbf{Q}} = \int \overset{(l)}{\mathbf{N}}(\mathbf{k}) J_1(f) \frac{d^3\mathbf{k}}{(2\pi)^3} + \frac{1}{\varepsilon} \int \overset{(l)}{\mathbf{N}}(\mathbf{k}) J_2(f) \frac{d^3\mathbf{k}}{(2\pi)^3}, \\
&\qquad \vdots
\end{aligned} \tag{2.12}$$

where div denotes the operator of Euclidean divergence.

For $\varepsilon \ll 1$ the productions in the first l equations of the sequence (2.12) become smaller than the productions in all the remaining equations of this sequence.

In the kinetic theory of material gases the moment equations with vanishing productions correspond to the conservation laws of continuum mechanics. In the system (2.12) only the first equation has a vanishing production and it corresponds to the balance of energy. The production terms of the next $l-1$ equations are smaller than productions in all the remaining equations. Hence, after introducing a small parameter ε, the moments from the first l equations of the sequence (2.12) can be interpreted as quasiconservative quantities. These quasiconservative quantities are the energy, the heat flux, the flux of the heat flux,... etc., including the quantity which is a tensor of the order $l-1$.

Motivated by the procedure of Hilbert iteration, we shall take the parameter ε in the kinetic equation (2.10) as a small parameter of a formal asymptotic expansion. The distribution function f shall be written as a sum of terms, multiplied by different powers of ε;

$$f = \overset{(0)}{f} + \varepsilon \overset{(1)}{f} + \varepsilon^2 \overset{(2)}{f} + \ldots + \varepsilon^n \overset{(n)}{f} + \ldots. \tag{2.13}$$

The kinetic equation (2.10) can be written in the form

$$\varepsilon(Df - J_1(f)) = J_2(f), \tag{2.14}$$

where

$$D = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j}. \tag{2.15}$$

From the structure of the collision term (2.3) it follows that $J_1(f)$ and $J_2(f)$ are combinations of multilinear functions of f. Hence, after inserting Eq. (2.13) into $J_1(f)$ and $J_2(f)$, they shall become the sums of terms, which are multiplied by different powers of ε

$$J_1(f) = \sum_{j=0}^{\infty} \varepsilon^j J_1^{(j)}, \tag{2.16}$$

$$J_2(f) = \sum_{j=0}^{\infty} \varepsilon^j J_2^{(j)}, \tag{2.17}$$

where $J_1^{(j)}$ and $J_2^{(j)}$ depend only on $\overset{(0)}{f}, \overset{(1)}{f}, \ldots, \overset{(j)}{f}$.

After inserting Eqs. (2.16) and (2.17) into Eqs. (2.14), we obtain

$$\sum_{j=1}^{\infty} \varepsilon^j \left\{ D \overset{(j-1)}{f} - \overset{(j-1)}{J_1}(\overset{(0)}{f}, \overset{(1)}{f}, \ldots, \overset{(j-1)}{f}) \right\} = \sum_{j=0}^{\infty} \varepsilon^j \overset{(j)}{J_2}(\overset{(0)}{f}, \overset{(1)}{f}, \ldots, \overset{(j)}{f}). \tag{2.18}$$

In Eq. (2.18) only one term is proportional to ε, what implies that

$$J_2(\overset{(0)}{f}) = 0. \tag{2.19}$$

It can be checked that $J_2(f)$ vanishes after inserting into it the distribution function of the form

$$f_l(\overset{(0)}{\Lambda}, \ldots, \overset{(l-1)}{\Lambda}) = \left\{ \exp\left[\sum_{i=0}^{l-1} \overset{(i)}{\Lambda} \cdot \overset{(i)}{\mathbf{N}}(\mathbf{k}) \right] - 1 \right\}^{-1}. \tag{2.20}$$

The quantities $\overset{(i)}{\Lambda}$, $i = 0, 1, \ldots, l-1$ in Eq. (2.20) are totally symmetric tensors and a dot denotes their full contraction with the tensors $\overset{(i)}{\mathbf{N}}(\mathbf{k})$ given by Eq. (2.1). We shall assume that $J_2(f)$ vanishes only on the functions of the form (2.20). Such an assumption can be motivated by considering the case of a kinetic equation with the collision term given by $J_2(f)$. The quantities $\overset{(i)}{\mathbf{N}}(\mathbf{k})$, $i = 0, 1, \ldots, l-1$ are then conserved in a collision and the distribution function (2.20) describes equilibrium states — as a solution of the variational problem of entropy maximization with constraints corresponding to the fixed values of conserved quantities. Our assumption means that in such a case dissipation vanishes only in equilibrium states.

Hence, the solution of (2.19) is given by

$$\overset{(0)}{f} = f_l(\overset{(0)}{\Lambda}, \ldots, \overset{(l-1)}{\Lambda}). \tag{2.21}$$

For $j \geq 1$ the expression $J_2^{(j)}$ can be split into two parts. One of them, $J_{2,1}^{(j)}(\overset{(0)}{f}, \overset{(1)}{f}, \ldots, \overset{(j-1)}{f})$, is a sum of these terms from $J_2^{(j)}$ which do not depend on $\overset{(j)}{f}$. The second part, $J_{2,2}^{(j)}$, is composed of these terms from $J_2^{(j)}$ which depend on $\overset{(j)}{f}$. It can be easily observed that $J_{2,2}^{(j)}$ does not depend upon $\overset{(0)}{f}, \ldots, \overset{(j-1)}{f}$. Therefore we can write

$$J_2^{(j)}(\overset{(0)}{f}, \overset{(1)}{f}, \ldots, \overset{(j)}{f}) = J_{2,1}^{(j)}(\overset{(0)}{f}, \overset{(1)}{f}, \ldots, \overset{(j-1)}{f}) + J_{2,2}^{(j)}(\overset{(j)}{f}), \tag{2.22}$$

where, in particular, $J_{2,2}^{(1)}$ is equal to zero.

From Eq. (2.21) it follows that $J_{2,2}^{(j)}$ is equal to the result of insertion of $\overset{(j)}{f}$ into the collision integral $J_2(f)$ linearized around $f_l(\overset{(0)}{\Lambda}, \ldots, \overset{(l-1)}{\Lambda})$. For simplicity, in the sequel of this paper the distribution function $f_l(\overset{(0)}{\Lambda}, \ldots, \overset{(l-1)}{\Lambda})$ shall be denoted by f_l, and the linear operator obtained by linearization of the collision term J_2 around f_l shall be denoted $\mathbf{A}_{f_l}$.

From

$$J_{2,2}^{(j)}(\overset{(j)}{f}) = \mathbf{A}_{f_l}(\overset{(j)}{f}) \tag{2.23}$$

it follows that the equation corresponding to these terms of Eq. (2.18) which are proportional to ε reads

$$Df_l - J_1(f_l) = \mathbf{A}_{f_l}(\overset{(1)}{f}). \tag{2.24}$$

Equations corresponding to higher powers of ε have the following general form:

$$D\overset{(j-1)}{f} - J_1^{(j-1)}(\overset{(0)}{f}, \overset{(1)}{f}, \ldots, \overset{(j-1)}{f}) - J_{2,1}^{(j)}(\overset{(0)}{f}, \ldots, \overset{(j-1)}{f}) = \mathbf{A}_{f_l}(\overset{(j)}{f}). \tag{2.25}$$

In Eqs. (2.24) and (2.25) the terms of the expansion (2.13) corresponding to the highest powers of ε occur as arguments of the linear operator $\mathbf{A}_{f_l}$. This observation suggests a procedure of solving the kinetic equation (2.10) from the infinite sequence of linear integral equations.

After inserting Eq. (2.13) into Eq. (2.8), separating terms multiplied by different powers of ε and making use of (2.9) we arrive at the following identity

$$\int \overset{(i)}{\mathbf{N}}(\mathbf{k})\mathbf{A}_{f_l}(h)\frac{d^3\mathbf{k}}{(2\pi)^3} = 0 \tag{2.26}$$

satisfied for every function $h(\mathbf{k})$ and for $i = 0, 1, \ldots, l-1$.

For every $i = 0, 1, \ldots, l-1$ the tensor identity (2.26) can be written as a set of equations for components

$$\int \psi_{i,\gamma(i)}(\mathbf{k})\mathbf{A}_{f_l}(h)\frac{d^3\mathbf{k}}{(2\pi)^3} = 0, \tag{2.27}$$

where the index $\gamma(i)$ parametrizes the independent components of the tensor $\overset{(i)}{\mathbf{N}}(\mathbf{k})$, $i = 0, 1, \ldots, l-1$. After multiplying Eq. (2.27) by arbitrary numbers $\alpha_{i,\gamma(i)}$ and adding the results we obtain

$$\int \Big\{ \sum_{i,\gamma(i)} \alpha_{i,\gamma(i)}\psi_{i,\gamma(i)}(\mathbf{k}) \Big\} \mathbf{A}_{f_l}(h)\frac{d^3\mathbf{k}}{(2\pi)^3} = 0. \tag{2.28}$$

The identity (2.28) means, that the sum in parentheses belongs to the kernel of the linear operator adjoint to $\mathbf{A}_{f_l}$. We shall assume that all vectors belonging to this kernel are of that form. Such an assumption means that for the collision operator $J_2(f)$ linearization does not change the number of the collision invariants.

In the first step of iteration one has to solve the equation

$$Df_l - J_1(f_l) = \mathbf{A}_{f_l}(\overset{(1)}{f}). \tag{2.29}$$

It is well known that the necessary condition for solving a linear nonhomogeneous equation is the orthogonality of the nonhomogeneity to the kernel of the adjoint operator. In our case, this condition leads to the system of equations

$$\int \psi_{i,\gamma(i)}(\mathbf{k})\{Df_l - J_1(f_l)\}\frac{d^3\mathbf{k}}{(2\pi)^3} = 0. \tag{2.30}$$

Since the functions $\psi_{i,\gamma(i)}$ denote the independent components of the tensors $\overset{(i)}{\mathbf{N}}(\mathbf{k})$, then from Eq. (2.30) it follows that

$$\int \overset{(i)}{\mathbf{N}}(\mathbf{k})\{Df_l - J_1(f_l)\}\frac{d^3\mathbf{k}}{(2\pi)^3} = 0, \quad i = 0, 1, \ldots, l-1. \tag{2.31}$$

In our considerations we shall restrict ourselves to a discussion of equations (2.31) which are satisfied by the leading term of the expansion (2.13).

After taking into account Eq. (2.15), we can write Eq. (2.31) in the form of the following identities

$$\frac{\partial}{\partial t}\int \overset{(i)}{\mathbf{N}}(\mathbf{k}) f_l(\overset{(0)}{\Lambda},\ldots,\overset{(l-1)}{\Lambda})\frac{d^3\mathbf{k}}{(2\pi)^3} + \operatorname{div}\int \overset{(i)}{\mathbf{N}}(\mathbf{k})\otimes\mathbf{v}(\mathbf{k}) f_l(\overset{(0)}{\Lambda},\ldots,\overset{(l-1)}{\Lambda})\frac{d^3\mathbf{k}}{(2\pi)^3} = \int \overset{(i)}{\mathbf{N}}(\mathbf{k}) J_1(f_l(\overset{(0)}{\Lambda},\ldots,\overset{(l-1)}{\Lambda}))\frac{d^3\mathbf{k}}{(2\pi)^3} \tag{2.32}$$

relating the quantities which depend on the tensor fields $\overset{(i)}{\Lambda}(\mathbf{x},t)$, $i = 0,1,\ldots,l-1$.

After denoting by $\overset{(i)}{\tilde{\mathbf{Q}}}\overset{(k)}{(\Lambda)}$ the tensor fields obtained by inserting the distribution function f_l (given by Eq. (2.20)) into the expressions (2.2)

$$\overset{(i)}{\tilde{\mathbf{Q}}}\overset{(k)}{(\Lambda)} = \int \overset{(i)}{\mathbf{N}}(\mathbf{k})\Big\{\exp\Big[\sum_{j=0}^{l-1}\overset{(j)}{\Lambda}\cdot\overset{(j)}{\mathbf{N}}(\mathbf{k})\Big]-1\Big\}^{-1}\frac{d^3\mathbf{k}}{(2\pi)^3}, \tag{2.33}$$

we can write the system (2.32) in the form

$$\frac{\partial}{\partial t}\overset{(i)}{\tilde{\mathbf{Q}}}\overset{(k)}{(\Lambda)} + \operatorname{div}\overset{(i+1)}{\tilde{\mathbf{Q}}}\overset{(k)}{(\Lambda)} = \overset{(i)}{\tilde{\mathbf{P}}}\overset{(k)}{(\Lambda)}, \qquad i = 0,1,2,\ldots,l-1, \tag{2.34}$$

where

$$\overset{(i)}{\tilde{\mathbf{P}}}\overset{(k)}{(\Lambda)} = \int \overset{(i)}{\mathbf{N}}(\mathbf{k}) J_1\Big(\Big\{\exp\Big[\sum_{j=0}^{l-1}\overset{(j)}{\Lambda}\cdot\overset{(j)}{\mathbf{N}}(\mathbf{k})\Big]-1\Big\}^{-1}\Big)\frac{d^3\mathbf{k}}{(2\pi)^3}. \tag{2.35}$$

In the case when the domain of the variable $\mathbf{k}$ is the First Brillouin Zone, the compactness of the Brillouin Zone ensures the convergence of integrals in Eq. (2.33) and (2.35). On the contrary, when as the domain of the variable $\mathbf{k}$ the whole three-dimensional vector space is taken (what corresponds to the approximation which is frequently applied for the description of low-temperature effects), the requirement of the convergence of Eqs. (2.33) and (2.35) imposes restrictions on the shape of a dispersion curve.

It can be seen that the system of equations (2.34) which forms a closed system with respect to the fields $\overset{(0)}{\Lambda},\overset{(1)}{\Lambda},\ldots,\overset{(l-1)}{\Lambda}$ is identical with the field theory that can be obtained by applying the variational procedure of entropy maximization to the corresponding system of moment equations derived from the kinetic equation of the form

$$\frac{\partial}{\partial t}f + v_j\frac{\partial}{\partial x_j}f = J_1(f). \tag{2.36}$$

(The variational procedure of entropy maximization and the structural properties of the system (2.34) are discussed in detail in the paper [1]). It should be stressed that the starting point of the derivation of the discussed system of field equation (2.34) by means of Hilbert iteration was not the kinetic equation (2.36) but the kinetic equation (2.10), the collision integral of which contains an additional term $1/\varepsilon J_2(f)$. The properties of this additional term were crucial for the possibility of deriving field theories by means of Hilbert iteration, but collision rates forming this term do not occur explicitly in the field equations.

A collision integral describing phonon interactions at low temperatures naturally has a structure assumed in the iteration procedure; it is given as the sum of two terms representing Normal processes and Umklapp processes, correspondingly. The Normal processes

are dominating what can be modelled by multiplying their collision rates by the inverse of a small parameter. In this case, the iteration procedure leads to the field equations of low-temperature phonon gas hydrodynamics of Nielsen and Shklovsky (discussed in [1]). For a phonon gas, only the low-temperature interactions have properties which justify the iteration procedure; in general, assumptions necessary for this kind of iteration are not consistent with the microscopic information about dissipation processes.

In our approach we have been able to choose such a very special class of dissipation processes, described by the kinetic equation of the form (2.10), that from the iteration procedure it was possible to derive field equations obtained previously by means of the variational procedure of entropy maximization. Our discussion remains valid also for the case of material gases and leads then to "variational" field theories discussed previously by DREYER [3]. Hence, theories of this kind, possessing a structure of Extended Thermodynamics, can be derived from a microscopic theory but at the price of taking a very specific model of dissipation.

The interesting question is whether variational theories can describe also more realistic models of dissipation; we would rather suggest the negative answer. However, this problem cannot be solved by means of the iteration procedure discussed in this paper and shall not be not discussed here.

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