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Optimum hypersonic wings and wave riders(*)

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HYPERSONIC wings and wave riders are considered in small-disturbance theory. Perturbations of flat plate flow are worked out to improve performance. Similarity solutions for power law and exponential shock waves are constructed. An air-foil based on the exponential shock is shown to be optimum. Wave riders are constructed from these solutions and their properties are examined. Perturbations of the simple wave-rider flows are studied, in order to optimize some properties.

1. Introduction

IN THIS PAPER special cases of hypersonic flows will be considered in order to illustrate some general points about the design of efficient lifting surfaces at hypersonic speeds.

The framework will be the inviscid theory of hypersonic small disturbance flows (HSDT). A similarity parameter of such flows is $H = 1/M_\infty^2\delta^2$ where M_∞ = flight Mach number, δ = typical flow deflection. Most of our considerations concern the limit $H \to 0$, since in that case we can have similarity solutions. In this case also we need only consider the flow on the lower surface since pressure on an expansion surface is $O(H)$.

In Sect. 2 HSDT is outlined. In Sect. 3 special coordinates for plane flow are introduced and $H = 0$ is considered. In Sect. 4 perturbations of wedge flow are considered in order to demonstrate that a flat plate airfoil at hypersonic speeds is not an (inviscid) optimum. It is possible to have favorable interference between the body and its shock-wave. The results of Sect. 4 indicate that concave airfoils have some benefit and a special class of these is discussed is Sect. 5. This class is generated by shock waves of exponential shape, and within this class an optimum shape is found. The performance is considerably better than a flat plate airfoil. Finally, in Sect. 6 it is shown how, using the two-dimensional flow behind an exponential shock wave, the geometry of a three-dimensional wave rider is defined. The performance of such a wave rider is then studied.

2. Hypersonic small-disturbance theory (HSDT)

The theory can be constructed as a limit of the steady Euler equations, and shock relations as $\delta \to 0$, $M_\infty \to \infty$ ($x, y = \overline{y}/\delta$, $z = \overline{z}/\delta$, $H = 1/M_\infty^2\delta^2$) fixed.

Here δ — thickness ratio or typical flow deflection, M_∞ — free stream Mach number, x, $\overline{y}$, $\overline{z}$ — physical coordinates, body length is the unit, H — hypersonic similarity parameter.

The asymptotic expansion has the form

$$(2.1)\qquad \frac{\mathbf{q}}{V} = \mathbf{i}\{1 + O(\delta^2)\} + \delta \mathbf{v}_T(x, y, z; H) + \ldots, \qquad \mathbf{v}_T = (v, w), \qquad \mathbf{v}_T \cdot \mathbf{i} = 0,$$

$$(2.2)\qquad \frac{\overline{p} - p_\infty}{\rho_\infty V^2} = \delta^2 p(x, y, z; H) + \ldots,$$

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$$\frac{\rho}{\rho_\infty} = \sigma(x, y, z; H) + \ldots . \tag{2.3}$$

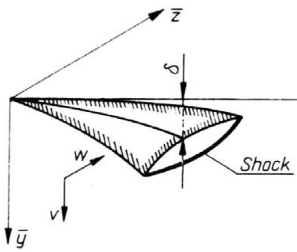

FIG. 1. Hypersonic wing.

See Fig. 1, for details see Ref. [1]. The derivative along a streamline becomes

$$\mathbf{q} \cdot \nabla = \frac{\partial}{\partial x} + \mathbf{v}_T \cdot \nabla_T, \qquad \nabla_T = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{2.4}$$

The Euler equations are simplified to

$$\frac{\partial \sigma}{\partial x} + \nabla_T \cdot (\sigma \mathbf{v}_T) = 0, \quad \text{continuity}, \tag{2.5}$$

$$\frac{\partial \mathbf{v}_T}{\partial x} + \mathbf{v}_T \cdot \nabla_T \mathbf{v}_T = -\frac{1}{\sigma}\nabla_T p, \quad \text{transverse momentum}, \tag{2.6}$$

$$\left(\frac{\partial}{\partial x} + \mathbf{v}_T \cdot \nabla_T\right)\left(\frac{p + H/\gamma}{\sigma^\gamma}\right) = 0, \quad \text{entropy}. \tag{2.7}$$

Let

$$y_s = G(x, z) \tag{2.8}$$

be the equation of the shock surface on the lower side of a lifting wing in a uniform freestream. Then the shock jump conditions can be written, for $v_s = v(x, G, z)$ etc.

$$v_s = \frac{2}{\gamma + 1}\left(\frac{G_x}{1 + G_z^2}\right) - H\frac{2}{\gamma + 1}\left(\frac{1}{G_x}\right), \tag{2.9}$$

$$p_s = G_x v_s = \frac{2}{\gamma + 1}\left(\frac{G_x^2}{1 + G_z^2}\right) - H\frac{2}{\gamma + 1}, \tag{2.10}$$

$$\frac{1}{\sigma_s} = \frac{\gamma - 1}{\gamma + 1} + H\frac{1 + G_z^2}{G_x^2}. \tag{2.11}$$

Let

$$y_b = F(x, y) \tag{2.12}$$

be the equation of the lower surface of the wing. The boundary condition of tangent flow is

$$v_b = F_x + w_b F_z, \qquad v_b = v(x, F, z). \tag{2.13}$$

In this approximation it is not necessary to consider x-momentum. The system of equations and boundary conditions is exactly analogous to that for unsteady flow in a (y, z)-plane, with x playing the role of t.

3. Two-dimensional flow; limit $H \to 0$ ($M_\infty \to \infty$)

For simplicity, and in order to be able to obtain similarity solutions, we will further restrict considerations to the limit $H \to 0$. For two-dimensional flow:

$$\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma v}{\partial y} = 0, \quad \text{continuity}, \tag{3.1}$$

$$\sigma\left(\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}, \quad \text{momentum}, \tag{3.2}$$

$$\left(\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(\frac{p}{\sigma^\gamma}\right) = 0, \quad \text{entropy}. \tag{3.3}$$

The bow shock conditions are

$$v_s = \frac{2}{\gamma+1}\theta, \quad p_s = \frac{2}{\gamma+1}\theta^2, \quad \frac{1}{\sigma_s} = \frac{\gamma-1}{\gamma+1}, \tag{3.4}$$

where $\theta = G'(x)$ — shock angle (scaled). The tangent flow condition is

$$v_b = F'(x), \quad 0 < x < 1, \quad F(1) = 1. \tag{3.5}$$

In order to use the entropy integral, which shows that the entropy $\sim \log(p/\sigma^\gamma)$ is constant on a streamline, a stream function $\psi(x, y)$ is introduced such that

$$\sigma = \psi_y, \quad \sigma v = -\psi_x. \tag{3.6}$$

Then

$$p/\sigma^\gamma = k(\psi), \tag{3.7}$$

where $k(\psi)$ can be related to shock geometry. Note that ahead of the bow shock

$$\sigma = 1, \quad \text{so that } \psi_s = y = G(x). \tag{3.8}$$

Writing Eqs. (3.1) and (3.2) in (x, ψ) we have

$$\frac{\partial \sigma}{\partial x} + \sigma^2\frac{\partial v}{\partial \psi} = 0, \tag{3.9}$$

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial \psi} = 0. \tag{3.10}$$

A further change of coordinates is useful so that the flow domain between shock and body is mapped into a known region; the unknown shock shape then appears in the equations.

Let

$$\xi(\psi) = x,$$

coordinate where a streamline ψ crosses the shock (see Fig. 2). Then, from Eqs. (3.7) and (3.4)

$$\frac{p}{\sigma^\gamma} = \frac{2}{\gamma+1}\frac{\theta^2(\xi)}{\left(\dfrac{\gamma+1}{\gamma-1}\right)^\gamma}. \tag{3.11}$$

Thus, is terms of rescaled variables the basic system (3.9) and (3.10) is (x, ξ) is

$$c^2\frac{\theta(\xi)^{\frac{2}{\gamma}+1}}{p^{*\frac{1}{\gamma}+1}}\frac{\partial p^*}{\partial x} + \frac{\partial v^*}{\partial \xi} = 0, \tag{3.12}$$

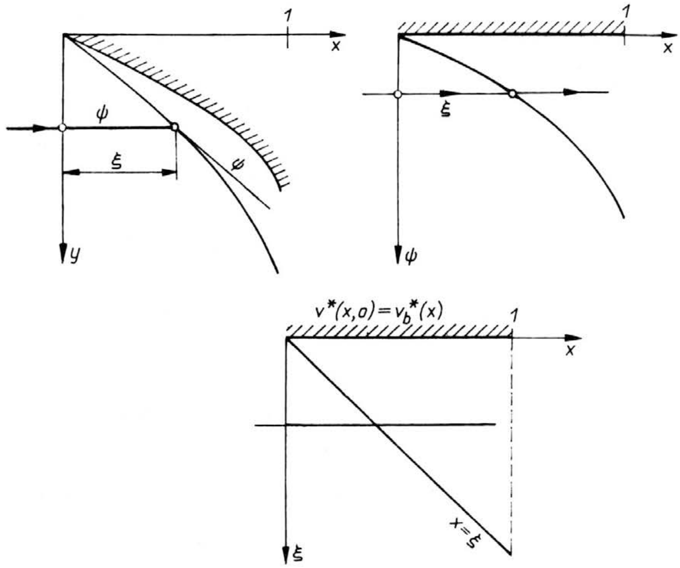

FIG. 2. Airfoil problem in various coordinates.

$$\frac{\partial p^*}{\partial \xi} + \theta(\xi)\frac{\partial v^*}{\partial x} = 0\,, \tag{3.13}$$

where

$$c^2 = \frac{\gamma - 1}{\gamma}, \quad p^* = \frac{\gamma+1}{2}p, \quad v^* = \frac{\gamma+1}{2}v\,.$$

The shock conditions are simply

$$p_s^* = p(\xi,\xi) = \theta^2(\xi), \quad v_s^* = \theta(\xi)\,. \tag{3.14}$$

The shape of the airfoil is given by

$$v^*(x,0) = v_b^*(x) = \frac{\gamma+1}{2}F'(x)\,. \tag{3.15}$$

4. Perturbed wedge flow

A flat surface provides optimum c_L for a given c_D in linearized supersonic theory. In this section we consider as a basic flow that on the flat undersurface of a lifting airfoil

$$p_0^* = \left(\frac{\gamma+1}{2}\right)^2, \quad v_0^* = \frac{\gamma+1}{2}, \quad \theta_0 = \frac{\gamma+1}{2}\,. \tag{4.1}$$

Because of the assumption $H \to 0$ it is not necessary to discuss pressures on the upper (expansion surface). Perturbation of the flow (4.1) are studied in order to find out if the flat surface is still an optimum. Consider the flat surface to be disturbed $O(\epsilon)$ and linearize Eqs. (3.12), (3.13), (3.14), (3.15) according to

$$p^* = p_0^*(1 + \epsilon\widetilde{p}(x,\xi) + \ldots)\,, \tag{4.2}$$

$$v^* = v_0^* = (1 + \epsilon\widetilde{v}(x,\xi) + \ldots)\,, \tag{4.3}$$

$$\theta = \theta_0(1 + \epsilon\widetilde{\theta}(\xi) + \ldots)\,. \tag{4.4}$$

Then

$$c^2\frac{\partial\widetilde{p}}{\partial x}+\frac{\partial\widetilde{v}}{\partial\xi}=0,\qquad \frac{\partial\widetilde{p}}{\partial\xi}+\frac{\partial\widetilde{v}}{\partial x}=0. \tag{4.5}$$

This system is equivalent to the wave equation and has the general solution

$$\widetilde{p}(x,\xi)=f(x-c\xi)+g(x+c\xi), \tag{4.6}$$

$$\widetilde{v}(x,\xi)=cf(x-c\xi)-cg(x+c\xi). \tag{4.7}$$

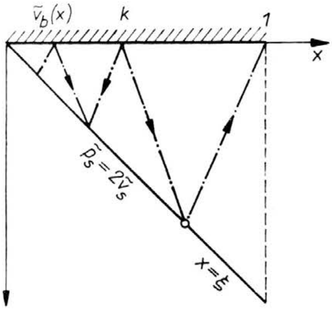

FIG. 3. Geometry of reflected waves.

The boundary conditions at the shock can be linearized to read (Fig. 3)

$$\widetilde{p}_s=2\widetilde{v}_s, \tag{4.8}$$

while the body shape can be described by

$$\widetilde{v}(x,0)=\widetilde{v}_b(x)=\widetilde{F}'(x),\qquad \widetilde{F}(1)=0\ \text{(say)}. \tag{4.9}$$

The reflection at the shock (4.8) shows that

$$g((1+c)\xi)=-\frac{1-2c}{1+2c}f((1-c)\xi). \tag{4.10}$$

A compression wave thus reflects from the shock as an expansion wave with the reflection coefficient λ

$$\lambda=\frac{1-2c}{1+2c},=0.139\quad\text{if}\quad\gamma=7/5. \tag{4.11}$$

The geometry of the reflections is given by

$$k=\frac{1-c}{1+c},=0.451\quad\text{if}\quad\gamma=7/5. \tag{4.12}$$

A disturbance arriving at x_0 on the airfoil left the airfoil at kx_0, struck the shock and reflected. The boundary condition (4.9) now shows that

$$\widetilde{v}_b(x)=f(x)+\lambda f(kx),\qquad 0<x<1. \tag{4.13}$$

This functional equation for f can be solved by iteration (Ref. [2, 3])

$$f(x)=\frac{1}{c}\sum_{m=0}^{\infty}(-\lambda)^m\widetilde{v}_b(k^m x),\qquad 0<x<1. \tag{4.14}$$

The resulting pressure distribution is

$$\widetilde{p}_b(x) = \widetilde{p}(x,0) = \frac{1}{c}\widetilde{v}_b(x) + \frac{2}{c}\sum_{n=1}^{\infty}(-\lambda)^n \widetilde{v}_b(k^n x). \tag{4.15}$$

This formula represents the pressure at a point as a local effect and the sum of single, double, etc. reflections from the shock wave. Lift L and drag D follow from

$$L = \rho_\infty U^2 \delta^2 \frac{\gamma+1}{2}\left\{1 + \epsilon\int_0^1 \widetilde{p}(x,0)\,dx\right\}, \tag{4.16}$$

$$D = \rho_\infty U^2 \delta^3 \frac{\gamma+1}{2}\left\{1 + \epsilon\int_0^1 (\widetilde{p}(x,0) + \widetilde{v}_a(x,0))\,dx\right\}. \tag{4.17}$$

A figure of merit, independent of the angle of attack δ, is to $O(\epsilon)$,

$$\frac{c_L^{3/2}}{c_D} = (\gamma+1)^{1/2}\left\{1 + \epsilon\left(\frac{1}{2}\int_0^1 \widetilde{p}_b(x)\,dx - \int_0^1 \widetilde{v}_b(x)\,dx\right)\right\}. \tag{4.18}$$

By virtual of (4.9) this quantity can be expressed in terms of the ordinates of the airfoil at discrete points which reflect to $x = 1$.

$$\frac{c_L^{3/2}}{c_D} = (\gamma+1)^{1/2}\left\{1 + \epsilon\sum_{n=1}^{\infty}\left(\frac{-\lambda}{k}\right)^n \widetilde{F}(k^n)\right\}. \tag{4.19}$$

In view of the alternating signs in the sum, $c_L^{3/2}/c_D$ can be increased and a type of optimum can be found if $\widetilde{F}(k^n)$ alternates sign, to produce a corrugated airfoil (see Fig. 4). For example,

$$\widetilde{F} = (-k)^n, \quad \frac{c_L^{3/2}}{c_D} = (\gamma+1)^{1/2}\left\{1 + \epsilon\frac{\lambda}{1-\lambda}\right\}, \quad \gamma = 7/5, \quad (\gamma+1)^{1/2} = 1.5492. \tag{4.20}$$

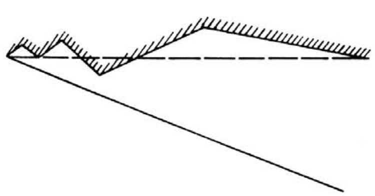

FIG. 4. An optimum airfoil.

This result shows that a flat bottom is not optimum. Since the main effect comes from the aft sections of the airfoil, it can be asked if a smooth cambered airfoil could perform better. This question is studied in the next section.

5. Similarity solution, exponential shock waves

The problem of the airfoil shapes generated by exponential shock waves was considered in Ref. [4]. The results are summarized and will be used later to construct three-dimensional wings. Solutions are sought for the system (3.12), (3.13) with

$$\theta(\xi) = Ae^{a\xi}, \quad G(\xi) = \frac{A}{a}(c^{a\xi} - 1), \tag{5.1}$$

$a > 0$ yields concave airfoils. Correspondingly,

$$p^*(x, \xi) = A^2 e^{2a\xi} P(x), \quad X = x - \xi, \tag{5.2}$$

$$v^*(x, \xi) = Ae^{a\xi} V(x). \tag{5.3}$$

The shock is located at $x = 0$ with the shock conditions

$$P_s = P(0) = 1, \quad V_s = V(0) = 1. \tag{5.4}$$

The zero streamline ($\xi = 0$) can be considered to define the airfoil shape

$$y = F(x) = \frac{2}{\gamma + 1} A \int\limits_0^x V(x)\, dx. \tag{5.5}$$

The normalization

$$F(1) = 1 \tag{5.6}$$

shows that $A = A(a)$, so that we have a one parameter family of solutions. The basic system (3.12), (3.13) becomes

$$\frac{c^2}{P^{\frac{1}{\gamma}+1}} \frac{dP}{dX} + aV - \frac{dV}{dX} = 0, \tag{5.7}$$

$$2aP - \frac{dP}{dX} + \frac{dV}{dX} = 0. \tag{5.8}$$

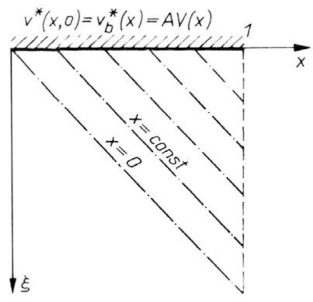

FIG. 5. Similarity solution exponential shock.

The boundary value problem is shown in Fig. 5. From Eqs. (5.7), (5.8)

$$\frac{dP}{dV} = \frac{V + 2P}{V + 2c^2 P^{-\frac{1}{\gamma}}} \tag{5.9}$$

and

$$a dX = \left(1 - \frac{c^2}{P^{\frac{1}{\gamma}+1}}\right) \frac{dP}{V + 2P}. \tag{5.10}$$

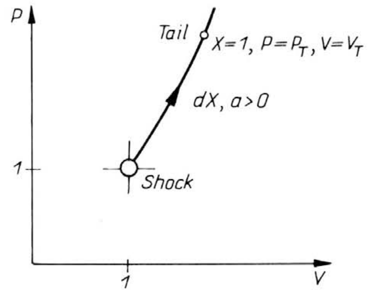

FIG. 6. Phase plane for exponential shock.

The integration takes place in the (P, V) phase plane (Fig. 6) and only that $P(V)$ integral curve is needed that passes through (1,1) for a given a. The integration of Eq. (5.10) proceeds from $X = 0$ to $X = 1$, the tail where $P(1) = P_T$, $V(1) = V_T$. Numerical calculations have been carried out for various values of a. Lift and drag now follow from

$$L = \rho_\infty U^2 \delta^2 A^2 \frac{2}{\gamma + 1} \int_0^1 P(X)\, dX\,, \tag{5.11}$$

$$D = \rho_\infty U^2 \delta^3 A^3 \frac{2}{\gamma + 1} \int_0^1 P(X) V(X)\, dX\,. \tag{5.12}$$

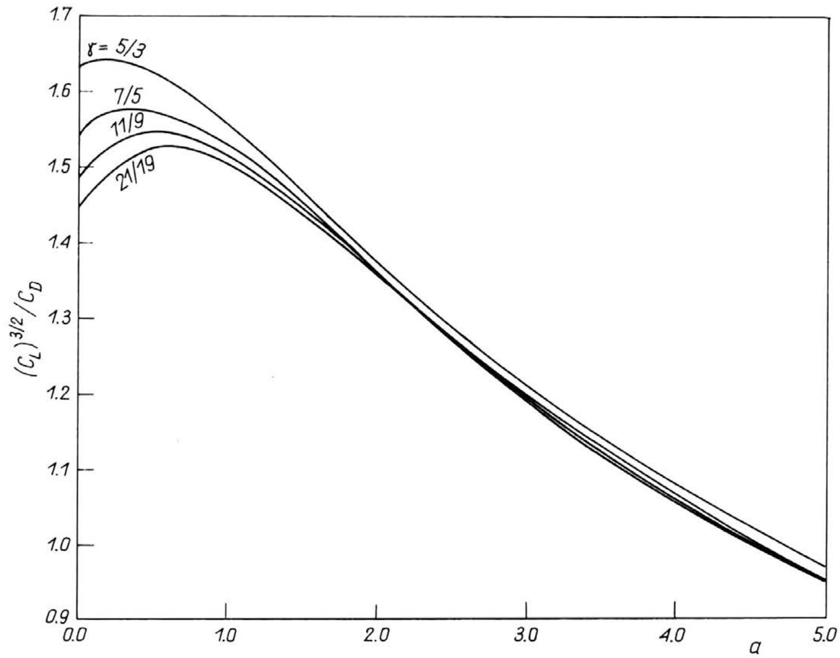

FIG. 7. Curves showing optimum a.

These integrals can be expressed in terms of tail values

$$(5.13) \qquad \frac{c_L^{3/2}}{c_D} = \frac{3}{2}\left(\frac{\gamma+1}{2}\right)^{1/2} \frac{(P_T - V_T)^{3/2}}{a\{P_T V_T - \frac{1}{2}V_T^2 - \frac{1}{2}P_T^{\frac{\gamma-1}{\gamma}}\}}.$$

The dependence of this figure of merit on a is shown in Fig. 7 from which it is seen that an optimum value of a exists

$$(5.14) \qquad a = 0.368\ldots, \qquad \left(\frac{c_L^{3/2}}{c_D}\right)_{OPT} = 1.58\ldots,$$

a value considerably better than that of the flat bottom 1.542. The airfoil shows a slight camber.

Another class of similarity solutions starts with power law bodies and power law shock waves $F(x) = x^k$, $0 < x < 1$. These solutions for concave lower surfaces are studied in (Ref. [5, 6]). In addition to the original airfoil surface ($\xi = 0$) any streamline $\xi = \text{const}$ can be chosen as the lower surface of an airfoil. These stream-surface are two-dimensional wave riders. A two parameter family (κ, ξ) of such flows is thus generated. The optimum $c_L^{3/2}/c_D$ was sought in this family of flows. It was shown that the optimum approaches that generated by the exponential shock wave just discussed.

6. Three-dimensional wave riders

The original wave rider concept was introduced by NONWEILER (Ref. [7]) and the ideas have been generalized by many authors. The basic idea is to fit a solid surface along a stream surface of a known flow. In most examples the basic flow is the uniform flow behind a plane shock and the three-dimensional wave rider supports a two-dimensional flow. The caret wing of Fig. 8 is Nonweiler's simplest example. According to HSDT this flow has all the properties of the two-dimensional flow so that

$$(6.1) \qquad \frac{c_L^{3/2}}{c_D} = \left(\frac{c_L^{3/2}}{c_D}\right)_{2D}.$$

For $H = 0$

$$(6.2) \qquad \frac{c_L^{3/2}}{c_D} = (\gamma+1)^{1/2} = 1.5492\ldots, \qquad \gamma = \frac{7}{5}.$$

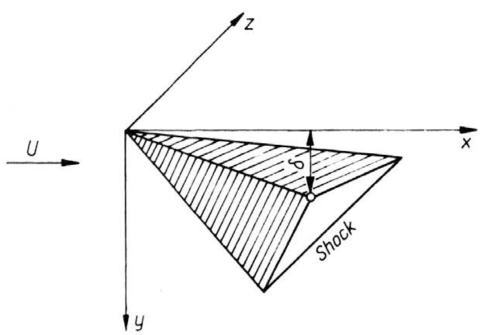

FIG. 8. Nonweiler wave rider.

Since the two-dimensional flow generated by the exponential shock wave had a better performance that the wedge flow, it can be expected that a three-dimensional wave-rider based on the exponential shock wave would have better performance that a flat caret wing. It is now shown how to generate the surface of such a wave rider and how to calculate its performance.

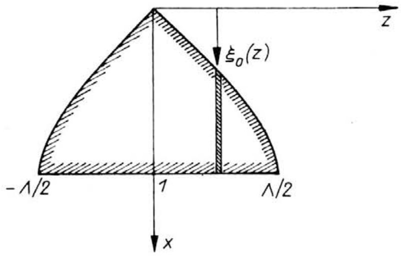

FIG. 9. Planform of similarity wave riders.

The planform can be defined by its projection into the plane $y = 0$ (see Fig. 9)

$$\xi = \xi_0(z) \quad \text{or} \quad z = \frac{\Lambda}{2} f(\xi), \quad f(0) = 0, \quad f(1) = 1, \tag{6.3}$$

Λ — wing span. The wing area is given by

$$\text{wing area} = \delta \int_{-\Lambda/2}^{\Lambda/2} (1 - \xi_0(z))\, dz. \tag{6.4}$$

The streamlines of the flow behind the exponential shock wave follow, generalizing Eq. (5.5) to arbitrary ξ, from

$$y(x, \xi) = \frac{A}{a}(e^{a\xi} - 1) + \frac{2}{\gamma + 1} A e^{a\xi} \int_{\xi}^{x} V(\overline{x} - \xi)\, d\overline{x}, \tag{6.5}$$

$$= \frac{A}{a}(e^{a\xi} - 1) + \frac{2}{\gamma + 1}\frac{A}{a} e^{a\xi} \{V(x - \xi) - 1 + \gamma c^2 (P(x - \xi) - 1)\}. \tag{6.6}$$

Thus the wave rider surface is defined by

$$y = F(x, z) = \frac{A}{a}(a^{a\xi_0(z)} - 1) + \frac{2}{\gamma + 1} e^{a\xi_0(z)} \{V(x - \xi_0(z)) - 1 + \gamma c^2 [P(x - \xi_0(z)) - 1]\}. \tag{6.7}$$

The function $P(X)$, $V(X)$ are of course defined as in Sect. 5. Expressions for c_L, c_D follow and we have

$$\frac{c_L^{3/2}}{c_D} = \frac{2(\gamma + 1)^{1/2}}{(\int_0^1 f d\xi)^{1/2}} \frac{(\int_0^1 f(\xi) e^{2a\xi} V(1 - \xi) d\xi)^{3/2}}{\int_0^1 f(\xi) e^{3a\xi} \{V^2(1 - \xi) - P^{\frac{\gamma - 1}{\gamma}}(1 - \xi)\} d\xi}. \tag{6.8}$$

From this optimum can be sought for various planform $f(\xi)$ and values of a. No general results are available but it has been shown (Ref. [6]) that the limit of special planforms has

$$\frac{c_L^{3/2}}{c_D} \to 1.579\ldots, \qquad \text{with} \qquad a = .369\,. \tag{6.9}$$

Thus the three-dimensional wave rider can approach the two-dimensional optimum value. For example, consider $f(\xi) = \xi^M$ as $M \to 0$.

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