Relativistic hydrodynamics(*)

M. L. EKIEL-JEŻEWSKA and Ł. A. TURSKI (WARSZAWA)

PHENOMENOLOGICAL *relativistic* hydrodynamics (RH) is fifty years old. Up to now it was regarded as an interesting although a bit academic topic. This has changed recently due to growing interest in astrophysical and high energy physics applications. This paper does not pretend to be a complete review of the present state to research in the field of relativistic hydrodynamics. We shall concentrate here on discussion of RH as viewed from the physical point of view, and we shall address those points which we found quite confused in the literature. In particular we shall discuss the role of the *observer*, the problem of *physically motivated* choice of the *hydrodynamical variables*, and the invariance of the *constitutive relations*. Most of those problems will be put into a proper perspective by the careful analysis of the *relativistic kinetic Boltzmann equation*.

1. Introduction

RELATIVISTIC HYDRODYNAMICS (RH) is almost fifty years old [1]([1]). In early days motivation for developing RH was most esthetic—any branch of classical physics was supposed to have its relativistic generalization. Ideas about practical use of RH were quite vague. This has changed during last three decades or so. The LANDAU and LIFSHITZ [2] formulation of RH was used by Landau in his theory of high energy proton–proton collisions and now RH is widely used in the theory of heavy ion collisions [3] especially to describe the dynamics of quark–gluon plasma [4], an "exotic" system formed when high energy heavy ions collide. RH is extensively used in astrophysical applications, for example analysis of X- and γ-ray sources [5]. Considerable work has been done on developing relativistic plasma-dynamics [6], particularly in view of fusion plasma analysis.

Foundations and limits of applicability of the non-relativistic hydrodynamics are well understood. We know how to derive hydrodynamics from microscopic theory (both classical and quantum) and we also know what role is played by hydrodynamic concepts in microscopic world. Power law time decay of the microscopic correlation functions is just one of the examples [7]. Unfortunately, no formulation of relativistic statistical mechanics exists which could be compared with contemporary non-relativistic one, thus conclusions reached by Havas almost quarter of the century ago still hold [8]. In formulation of RH we have therefore to relay on phenomenological constructions and/or the only well studied "microscopic" model, namely *relativistic Boltzmann kinetic equation* theory (RBKE) which is widely believed to capture most of the essential features of relativistic theory of *neutral* gases [9].

This lack of firm microscopic foundations has lead to considerable confusion in the literature about RH, epitomized by almost endless discussion about proper relativistic form

([1]) There were several earlier attempts to formulate relativistic hydrodynamics of perfect fluid, for example J. L. SYNGE, Proc. London Math. Soc., **43**, 376, 1937. For those references cf. R. C. TOLMAN, *Relativistic thermodynamics and cosmology*, Clarendon Press, Oxford 1969 or earlier editions of that book, also J. L. SYNGE, *Relativity: The general theory*, North Holland, Amsterdam 1960.

of the heat conduction equation. John MADDOX can of worms contains more in stock when we include RH [10]. It is indeed somewhat disturbing that well known equations of RH, those proposed by Landau and Lifshitz, or by Eckart are indeed parabolic differential equations, thus they allow for infinite speed of signal propagation. Several authors [11, 12, 13, 14, 15, 16, 17], have tried to rectify this be using either non-standard phenomenological approach or proposing entirely new description of relativistic fluids dynamics. That has confused the situation even more, for it is not at all obvious which set of equations is adequate for what physical situations, and whether those equations should be related in any way.

The microscopic analysis based on RBKE does not provide us with a unique way out of the problem. Indeed several different methods of deriving hydrodynamic equation out of the kinetic equation, known from non-relativistic theory, have been generalized for RBKE [9, 18]. Thus we have variants of Chapman–Enskog theory [17, 19], and relativistic extensions of the Grad method [20, 21, 22, 23]. Attempts to formulate generalized hydrodynamics (the set of hydrodynamic-like equations with time and space dependent transport coefficients) were also done [14, 16]. It has been shown that the relativistic generalization of the Chapman–Enskog method, or the Grad procedure are not unique and, therefore, derivation of hydrodynamics from RBKE is a much more subtle problem than in non-relativistic limit. One has to define, in agreement with specific physical situation, what are the measured variables and which of them are *slow* and which are *fast* [17]. This point of view has been recently supported by mathematically rigorous analysis of the RBKE and its relation to hydrodynamics [24].

The plan of this paper, which does not pretend to be a comprehensive review of RH, is as follows. In Sect. 2 we review various phenomenological formulations of the RH. We first discuss the perfect fluid dynamics and later on analyze the problem of dissipative processes in relativistic domain. In Sect. 3 we introduce the concept of the RBKE and discuss its main properties. This is followed by critical appraisal of various derivations of the RH from RBKE. Sect. 4 is devoted to final discussions and conclusions.

2. Phenomenological formulation

In this section we shall discuss various formulations of the phenomenological relativistic hydrodynamics trying to distinguish postulates used in them and to asses their physical relevance. We shall first discuss the perfect fluid approximation and later on address the *question* of the RH, namely the dissipation.

2.1. Perfect fluid hydrodynamics

The phenomenological description of a relativistic, perfect (non-dissipative) fluid is fairly straightforward. What one has to do is to rewrite Euler equations taking into account basic laws of relativity theory. The LANDAU and LIFSHITZ [2] formulation of the perfect fluid relativistic hydrodynamics (PRH), is generally accepted and does not differ

in an essential way from the others [25]. We shall describe here the simplest example of one component, isotropic neutral, relativistic liquid. Since the fluid is non-dissipative, the energy transfer within the fluid is only due to particle motion (^{2}) and therefore the *energy flux* is parallel to the fluid velocity U^μ.

The energy-momentum tensor $T^{\mu\nu}$ and the particle current vector N^μ assume the form

$$(2.1) \qquad T^{\mu\nu} = T^{\mu\nu}_{eq} = eU^\mu U^\nu - p\Delta^{\mu\nu},$$

$$(2.2) \qquad N^\mu = N^\mu_{eq} = nU^\mu,$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu$ and n, e and p are particle and energy densities, and the pressure measured in the *local rest frame*, respectively. $g^{\mu\nu}$ is the space-time metric tensor with the $(+ - - -)$ signature. The light velocity $c = 1$ in what follows.

The basic *conservation laws* read now

$$(2.3) \qquad \partial_\mu T^{\mu\nu} = 0,$$

$$(2.4) \qquad \partial_\mu N^\mu = 0.$$

To turn conservation laws into *equations of motion* we have to supplement them by the constitutive relation, which in RH reduces to the *equation of state*, expressing pressure as a suitable function of the particle and energy densities, n and e (or n and the temperature T)

$$(2.5) \qquad p = \mathfrak{p}(n, e), \qquad (e = \mathfrak{e}(n, T)),$$

were the bracketed term indicates the possible change of variables.

Having those we obtain from Eqs. (2.3) and (2.4) the set of *five* equations describing the dynamics of a perfect relativistic fluid

$$(2.6) \qquad U^\mu \partial_\mu n + n \partial_\mu U^\mu = 0,$$

$$(2.7) \qquad nU^\mu \left[\partial_\mu \left(\frac{h}{n} \right) - \frac{1}{n} \partial_\mu p \right] = 0,$$

$$(2.8) \qquad hU^\nu \partial_\nu U^\mu - \Delta^{\mu\nu} \partial_\nu p = 0.$$

In the above $h = e + p$ is the fluid enthalpy density.

The basic elements of the above construction of RH are

   i) conservation laws,

   ii) relativity principle,

   iii) consistency with the local equilibrium thermodynamics.

We shall now discuss the role of each of the above points.

Ad i). Conservation laws are expressed in terms of the energy momentum tensor and the particle current vector. For a specific physical system, in the local rest frame, those objects are expressed as a functions of local thermodynamics variables p, n, e related to each other by the proper equation of state.

Ad ii). Relativity principle establishes transformation laws of $T^{\mu\nu}$ and N^μ as those of tensors, and allows us to calculate their values in an arbitrary inertial frame as the function of the local rest frame variables.

(^{2}) We used that term in its continuous media mechanics meaning.

Ad iii). Thermodynamics plays much more subtle role in RH as in its non-relativistic counterpart, for the transformation laws for macroscopic variables are not trivial. In non-dissipative case one can use as the thermodynamic variables those defined in arbitrary inertial frame by $\tilde{n} = N^0 = nU^0$, $\tilde{e} = T^{00} = hU^0U^0$, $\tilde{p} = p$. That does not lead to any problem since we know exactly how to express those variables via those used previously, using well known transformation laws.

In the following section points i)–iii) will serve as guidelines in our analysis of the dissipative hydrodynamics under the proviso of replacing local equilibrium thermodynamics with its non-equilibrium extension.

2.2. Dissipative hydrodynamics

In discussing the dissipative RH (DRH), we shall again restrict ourselves to a simple example of isotropic, one component, neutral fluid. We shall also assume that this fluid is a Newtonian one. We are unaware of any attempts to develop relativistic generalization of the non-Newtonian fluids.

As we have already mentioned there are several competing versions of DRH. We will present here, as an example, the LANDAU and LIFSHITZ [2] DRH, paying particular attention to the guideline points i)–iii) from Sect. 2. 1. Main steps in other formulations of DRH are outlined in Table 1.

Table 1.

Theory	I Conservation laws	II Frame of reference	III Primary variables	IV Conditions of fit
1a L&L	$\partial_\mu T^{\mu\nu} = 0$ $\partial_\mu N^\mu = 0$	energy local rest frame	n_E, e_E, U_E^μ $T^{\mu\nu}U_{E\nu} = e_E U_E^\mu$ $n_E = N^\mu U_{E\mu}$	$n = n_E$ $U^\mu = U_E^\mu$ $e = e_E$
1b E	$N^\mu = N_{eq}^\mu + \nu^\mu$	particle local frame	n_N, e_N, U_N^μ $N^\mu = n_N U_N^\mu$ $U_{N\mu} U_N^\mu = 1$ $e_N = T^{\mu\nu}U_{N\nu}U_{N\mu}$	$n = n_N$ $U^\mu = U_N^\mu$ $e = e_N$
2 vK	$T^{\mu\nu} = T_{eq}^{\mu\nu} + \Pi^{\mu\nu}$	any inertial observer	$\tilde{n}, \tilde{e}, T^{0k}$ $\tilde{e} = T^{00}, \tilde{n} = N^0$	n, e, U^μ $(e + P(n,e))U^0U^k = T_{eq}^{0k} = T^{0k}$ $nU^0 = N_{eq}^0 = \tilde{n}$ $e(U^0)^2 + P(n,e)U^2 = T_{eq}^{00} = \tilde{e}$
3a I&S		energy local rest frame	as in 1a	as in 1a
3b I&S	Warning: Cf, IV	particle local rest frame	as in 1b	as in 1b
4 L&M&R	$T_{eq}^{\mu\nu}, N_{eq}^\mu$ may have various definition.	particle local rest frame	$T^{\mu\nu}, N^\mu$ "extended thermodynamics"	as in 1b

L&L: Laudau-Lifshitz, Ref.[2]; E: Eckart, Ref.[1];
vK: van Kampen, Ref.[17]; I&S: Israel-Stewart, Ref.[23];
L&M&R: Liu-Müller-Ruggeri, Ref.[15]

	V	VI	VII	VIII
	Equation of state	Non-equilibrium entropy flux	Constitutive relations	Transformation of variables
1a	$P_E = P(u_E, e_E)$ $e_E = e(n_E, T_E)$	linear in ν_μ	$\Pi^{\mu\nu} = \Pi_E^{\mu\nu}(\mathcal{Z}_E),$ $\nu^\mu = \nu_E^\mu(\mathcal{Z}_E)$ $U_{E\mu}\Pi^{\mu\nu} = 0.$ $\mathcal{Z}_E = (n_E, e_E, U_E^\alpha, \Delta_{E\varrho}^\lambda \partial_\lambda n_E,$ $\Delta_{E\varrho}^\lambda \partial_\lambda e_E, \Delta_{E\varrho}^\lambda \partial_\lambda U_E^\alpha)$ (∗ A ∗)	Lorentz covariant
1b	$P_N = \check{P}(u_N, e_N)$ $e_N = \check{e}(n_N, T_N)$	linear in $U_\mu \Pi^{\mu\nu}$	$\Pi^{\mu\nu} = \Pi_N^{\mu\nu}(\mathcal{Z}_N)$ $n^\mu = 0.$ $\mathcal{Z}_N = (n_N, e_N, U_N^\alpha, \Delta_{E\varrho}^\lambda \partial_\lambda n_N,$ $\Delta_{E\varrho}^\lambda \partial_\lambda e_N, \Delta_{E\varrho}^\lambda \partial_\lambda U_N^\alpha)$ (∗ A ∗)	Lorentz covariant
2	$P = \check{P}[n(V), e(V)]$ $e(V) = \check{e}[n(V), \check{T}]$ $V = (\check{n}, \check{e}, T^{0k})$	linear in $\check{\Pi}^{kl}$ and $\check{\nu}^k$	$\check{\Pi}^{kl} = \check{\Pi}^{kl}(\check{\mathcal{Z}}), \ \nu^k = \check{\nu}^k(\check{\mathcal{Z}})$ $\check{\mathcal{Z}} = (\check{n}, \check{e}, T^{0k}, \partial_m \check{n},$ $\partial_m \check{e}, \partial_m T^{0k})$ (∗ B ∗)	Measurment dependent. No Lorentz covariance. Relativity principle holds.
3a	as in 1a	quadratic in $(\Pi, \pi^{\mu\nu}, q^\mu)$	$\Pi^{\mu\nu} = \Pi_E^{\mu\nu} + \Pi_{E2}^{\mu\nu}(\mathcal{Y})$ $\nu^\mu = \nu_E^\mu + \nu_{E2}^\mu(\mathcal{Y})$ $\mathcal{Y} = (\partial_\alpha \pi^{\mu\nu}, \partial_\alpha q^\beta, \partial\Pi)$ $\Pi = \frac{1}{2}\Pi_\mu^\mu, \ \pi^{\mu\nu} = \Pi^{\mu\nu} - \Pi\Delta^{\mu\nu}$ $q^\mu = (n^\mu h_E)/n_E.$ (∗ C ∗)	
3b	as in 1b		$\Pi^{\mu\nu} = \Pi_N^{\mu\nu} + \Pi_{N2}^{\mu\nu}(\mathcal{Y})$ $\mathcal{Y}$ as in 3a with $q^\mu = \Pi^{\mu\nu}U_{N\nu}.$ (∗ C ∗)	Lorentz covariant
4	as in 1b		$A^{\alpha\beta\gamma} = a^{\alpha\beta\gamma}(N^\mu, T^{\mu\nu})$ $I^{\alpha\beta} = i^{\alpha\beta}(N^\mu, T^{\mu\nu})$ $\partial_\gamma A^{\alpha\beta\gamma} = I^{\alpha\beta}.$ (∗ D ∗)	

LEGEND

(∗ A ∗) $\Pi^{\mu\nu}$ and nn^μ are linear functions of the local rest frame gradients of primary variables.

(∗ B ∗) $\check{\Pi}^{kl}$ and $\check{n}^k$ are linear functions of the observer frame gradients of primary variables.

(∗ C ∗) Π, q^β and $\pi^{\mu\nu}$ solve a set of first order linear partial differential equations.

(∗ D ∗) $a^{\alpha\beta\gamma}$ and $i^{\alpha\beta}$ are linear functions of $\Pi^{\mu\nu}; n^\mu = 0.$

The conservation laws are the same in both PRH and DRH. Thus the energy momentum tensor and the particle current vector obey Eqs. (2.3) and (2.4). Obviously both $T^{\mu\nu}$ and N^μ are now different than those given by Eqs. (2.1) and (2.2). In the Newtonian fluid limit both those objects can be written as

(2.9) $$T^{\mu\nu} = T_{eq}^{\mu\nu} + \Pi^{\mu\nu},$$

(2.10) $$N^\mu = N_{eq}^\mu + \nu^\mu,$$

where $\Pi^{\mu\nu}$ and ν^μ are "small" corrections.

We immediately encounter out two main problems. The first one—called the *constitutive relation* problem—is, what are the basic physical variables describing the system, and how the dissipative terms of the energy momentum tensor and the particle current vector

are expressed in terms of those *primary variables*. The second one, which goes under the name of *conditions of fit*, or *matching conditions*, refers to how the primary variables are matched with the local equilibrium quantities n, e, U^μ. The choice of primary variables is determined by the kind of measurements the *observer* is doing in order to describe his system and therefore is related to the choice of the particular frame of reference. Thus, following out guidelines, we shall now discuss the choice of the frame of reference.

In PRH primary variables and hydrodynamical equations are first defined in the local rest frame and then Lorentz transformed to other frames. In DRH the energy-momentum tensor and particle current vector are expressed via constitutive relations holding in a certain rest frame. It is by no means obvious if that (approximate from microscopic viewpoint) procedure should be interchangeable with the Lorentz transformation [17]. The decision which rest frame to use cannot then be reached without detailed analysis of a given physical situation. As a matter of fact in the dissipative case even the concept of the local rest frame is *not* uniquely defined, since the energy transport not necessarily follows the particle motion. For that reason the particle current and the energy current might have different characteristic velocities and therefore one can define different rest frames depending on which velocity is considered to be the proper one. In the Landau and Lifshitz formulation it is assumed that the local rest frame is that one in which energy current vanishes $T^{0i} = 0$ (the energy rest frame).

The choice of primary variables is determined by what kind of measurements one can perform on the fluid. In the Landau and Lifshitz theory one assumes that the primary variables are

the velocity of the energy local rest frame U_E^μ,

the energy density e_B,

the particle density n_E.

They are defined as

$$(2.11) \qquad\qquad T^{\mu\nu} U_{E\nu} = e_E U_E^\mu,$$

$$(2.12) \qquad\qquad e_E = T^{\mu\nu} U_{E\nu} U_{E\mu},$$

$$(2.13) \qquad\qquad n_E = N^\mu U_{E\mu}.$$

Having defined the local rest frame and the set of primary variables we have to address the issue of what are the proper conditions of fit. In other words we have to tell how our primary variables n_E, e_E, U_E^μ are related to the local equilibrium quantities appearing in the non-dissipative part of the Eqs. (2.10), (2.11). In the Landau and Lifshitz theory those conditions read

$$(2.14) \qquad\qquad \begin{aligned} n_E &= n, \\ e_E &= e, \\ U_E^\mu &= U^\mu. \end{aligned}$$

The remaining difficult point is the constitutive relation for DRH. Since classical hydrodynamics should be applicable to situations close to the equilibrium and we discuss the Newtonian fluids, it is correct to assume that the equation of state is actually the same as in PRH (see Sect. 2.1.) but now the pressure in the function of density and energy density related to the primary variables via the conditions of fit. Now, we have to provide expressions for dissipative terms in Eqs. (2.9) and (2.10). In order to do so we again shall analyze the specific physical model. Indeed the constitutive relation has to contain

detailed information about the model we are discussing. In case of the isotropic and neutral liquid neither uniform translational motion nor rigid rotation can cause dissipation. Furthermore the second law of thermodynamics requires that the non-equilibrium entropy flux must be specified. In the Landau and Lifshitz theory that current is assumed to be linear in non-equilibrium particle current ν^μ. The entropy production is then written as

$$(2.15) \qquad \partial_\mu S^\mu = \frac{1}{T} \Pi^{\mu\nu} \partial_\mu U_\nu - \frac{1}{nT} n^\mu (\partial_\mu p - \frac{h}{T} \partial_\mu T).$$

Second law requires that this must be positive. This is indeed the case if we assume that $\Pi^{\mu\nu}$ equals

$$(2.16) \qquad \Pi^{\mu\nu} = \eta(\Delta^{\mu\sigma} \Delta^{\nu\tau} \partial_\sigma U_\tau - \tfrac{2}{3} \Delta^{\mu\nu} \Delta^{\sigma\tau} \partial_\sigma U_c) + \zeta \Delta^{\mu\nu} \partial_\lambda U^\lambda,$$

and

$$(2.17) \qquad \nu^\mu = \frac{n}{h} \kappa \Delta^{\mu\nu} (\partial_\nu T - \frac{T}{h} \partial_\nu p),$$

where the transport coefficients η, ζ (shear and bulk viscosities) and κ (heat conductivity) are all positive. Note that we have expressed particle current ν^μ in terms of temperature T rather than energy density. This is consistent in view of the assumed conditions of fit, and implicitly assumed (one-to-one) relation $e = e(n, T)$. Recall also that $A^{(\mu\nu)} \equiv (A^{\mu\nu} + A^{\nu\mu})/2$.

On using Eqs. (2.9), (2.10), (2.16) and (2.17) we obtain from Eqs. (2.1) and (2.2) the set of equations of motion for the Landau and Lifshitz DRH. By inspection one sees that those equations are written in manifestly Lorentz covariant form. Note, however, that this is not necessary from the viewpoint of the relativistic invariance. The canonical formulation of both PRH and relativistic plasmadynamics is most conveniently written in the non-covariant form [26, 27, 28]. That symplectic formulation can be extended in plasmadynamics to full metriplectic description of a class of dissipative processes [29].

In DRH the problem of manifestly covariant versus non-covariant but relativistic invariant description was source of much of the controversy, until VAN KAMPEN [30] pointed out that the problem of transformation of the thermodynamic variables cannot be discussed independently of the thorough analysis of how the system interacts with its surroundings. In fact even the seemingly covariant expressions, Eqs. (2.9) and (2.10) are de facto valid in those frames of references in which our physical system *appears* to the observer as being close to local equilibrium. In general there is no guarantee that if the system is close to local equilibrium for one observer it will be so for any other. The *linear* response like expressions Eqs. (2.9) and (2.10) cannot be then generally valid.

We have found it convenient to summarize several known formulations of the DRH in the form of the Table 1.

For sake of completeness we have included in the Appendix A explicit expressions for the energy-momentum tensor and particle current in each of the entries from the Table 1.

For unaccustomed reader the situation in DRH might look pretty strange at that stage. So many different formulations, so many different ingredients put into a theory which in the non-relativistic limit is conceptually (but not technically) simple. He might start to wonder whether those different formulations are indeed different, and if one cannot relate them to each other by some sort of "simple" operation. It turns out that conventional formulations of DRH are consistent. For example the Landau and Lifshitz and Eckart theories are both assumed to be valid close to the local equilibrium. It turns out that

differences between both theories (stemming from difference in the choice of the primary variables etc) are actually of the second order in deviations from local equilibrium. Thus if someone analyzes phenomena which are linear in those deviations he, or she, is free to choose whichever of those two theories. The results will be the same. Of course one would like to have hydrodynamics which allows to go "further out" of the equilibrium. The Israel and Stewart and Liu, Müller and Ruggeri theories actually attempted to do so. In the linear regime again all those theories coincide with the Landau and Lifshitz or Eckart. Difficulties start immediately when we want to go beyond linear theory. Recently van Kampen questioned attempts to guarantee manifestly covariant structure of the theory [17]. His approach does not suffer from several difficulties of other theories for he uses different definition, of what is meant by small deviation from equilibrium.

Consider a simple example of a diffusion type equation. Let us have, in some frame of reference, a scalar variable ρ which obeys an equation $\partial_t \rho = \mathcal{D}\nabla^2\rho$ leading to the well known dispersion relation $i\omega = \mathcal{D}\mathbf{k}^2$. Suppose we transform that dispersion relation to another frame; one finds that this "predicts" now existence of two hydrodynamical modes, one of which is unstable. This of course is lacking any physical meaning, for there is no physical instability embedded in the phenomenon going on and that instability results from erroneous attempt to be "Lorentz covariant". In our simple example as in more complex cases the observer making *his measurements* and determining what are *his primary* variables induces *his separation* between space and time variables. He sets up *initial* or *boundary* problem he wants to solve. Now, when one performs the Lorentz transformation from one frame to another one not only has to transform equations but also the initial (boundary) conditions, and what was the initial condition for given observer it becomes a complicated mixed initial and boundary problem for the other. Unless this concoction is explicitly taken into account, one runs into confusion trying to compare predictions obtained by one or the other observer.

The valid question at this point is the following. Since all those difficulties with relativistic generalization of the hydrodynamics are related to choice of primary variables and proper constitutive relation, is it possible at all to describe the relativistic fluid dynamics by means of differential equations for few macroscopic fields? Perhaps we shall simply give up and look for some other description. The answer to that question should come from the microscopic analysis. Unfortunately, as already mentioned, there is no microscopic relativistic statistical mechanics which goes beyond Boltzmann-like kinetic theory. In the following section we shall try to shed some light on RH, using relativistic generalization of the Boltzmann kinetic theory.

3. Microscopic foundations of the relativistic hydrodynamics

In this section we shall outline the only well established microscopic approach to the derivation of the RH, namely that based on RBKE. We will begin our analysis by recalling rudimentary informations about RBKE.

3.1. The relativistic Boltzmann kinetic equation

As in all proceeding sections we consider one-component system of neutral, point-like, massive particles in the absence of any external forces. We assume that the only interactions between those particles are point collisions and that the state of such a system is

completely determined by the one-particle distribution function $F(\mathbf{r}, \mathbf{p}, t)$, where $\mathbf{r}$ denotes particle position in three-dimensional space, $\mathbf{p}$ is its *kinetic* momentum ([3]). Furthermore we assume that the evolution of the distribution function is governed by the generalization of the Boltzmann equation [9, 18]

$$(3.1) \qquad \frac{\partial F}{\partial t} + \frac{\mathbf{p}}{p_0} \cdot \frac{\partial F}{\partial \mathbf{r}} = \frac{2}{m} \int d^3\mathbf{p}_1 d\Omega \left(\frac{gm\sqrt{s}}{4p_{10}p_0} \right) \sigma(g, \theta)[F'F_1' - FF_1],$$

where we follow the standard convection of primed variables and $\sqrt{s} = |p_1 + p|$ is the total energy, $2g = |p_1 - p|$ the relative momentum, and $\cos(\theta) = 1 - 2(p_\mu - p_{1\mu})(p^\mu - p'^\mu)/(4m^2 - s)$ defines the scattering angle. $d\Omega = \sin(\theta)d\theta d\phi$ and $\sigma(g, \theta)$ denotes the scattering cross-section. Note that all the above variables are defined in the center-of-mass (CM) frame.

Equation (3.1) is analogous to the classical Boltzmann equation with three main differences: 1) the length of the relativistic velocity $|\mathbf{p}/p_0|$ is bounded from above by the velocity of light (in our units = 1), 2) the relative momentum $2g$ depends in a different way on the momenta $\mathbf{p}$, $\mathbf{p}_1$, 3) the extra factor $m\sqrt{s}/4p_0 p_{10}$ resulting from the relativistic transformation law between the CM and the moving frame. Also note, that the cross-section $\sigma(g, \theta)$ in Eq. (3.1) is not related to the interaction potential between gas particle in a fashion known from the non-relativistic classical mechanics. The functional form of $\sigma(g, \theta)$ has serious repercussions on global properties of the RBKE [31]. Detailed discussion of the RBKE is given in [9].

3.2. Derivation of the RH from the RBKE

We shall now outline main approximation schemes used to derive the RH equations from the RBKE.

3.2.1. The relativistic Chapman-Enskog method. The relativistic Chapman–Enskog procedure, similarly as its non-relativistic predecessor, is based on separation between *fast* and *slow* variables. It is assumed that the system is close to the local equilibrium and that its distribution function can be decomposed into local equilibrium part and *small* correction $\delta F(\mathbf{r}, \mathbf{p}, t)$ depending on time through the slow, local equilibrium variables only. The time evolution of slow quantities is assumed to be given by the relativistic Euler equations. What follows is that the non-equilibrium correction $\delta F(\mathbf{r}, \mathbf{p}, t)$ is expressed completely in terms of the space gradients of local equilibrium variables, and that leads to the dissipative hydrodynamics equation with specific expressions for transport coefficients. Unfortunately, unlike the non-relativistic version, the Chapman–Enskog procedure is not unique. The separation between slow and fast variables depends on the choice of the frame of reference. Thus we can derive either Landau and Lifshitz or Eckart hydrodynamics using the Chapman–Enskog procedure, depending on whether the separation between slow and fast variables is done in energy or particle frame, respectively [18].

Recently VAN KAMPEN [17] has argued that the proper approach should relay on a different analysis. Every observer should use his own separation of time scales and therefore his own concept of local equilibrium. This observer should then derive his own

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([3]) The difference between use of kinetic or canonical momentum becomes crucial in analysis of the charged particle systems, cf. Ref. [27, 28].

set of hydrodynamic equations, which will be not manifestly covariant but will still obey the relativity principle.

3.2.2. The relativistic version of the Grad method.

The application of the Grad moment expansion method to the RBKE consists in writing the distribution function F as a linear combination of the polynomials $(1, p^\mu, p^\alpha p^\beta)$ with coefficients which are some functions of local equilibrium quantities. It is not clear, from the discussion one can find in the literature, what should be the physical reason for that assumption.

$$(3.2) \qquad F \approx f_0[a + a_\mu p^\mu + a_{\alpha\beta} p^\alpha p^\beta].$$

On substituting this equation into the RBKE Eq. (3.1) and multiplying it by subsequent relativistic Hermite polynomials and performing integration over the momenta, we obtain a close set of equations for coefficients a, a_μ and $a_{\alpha\beta}$ regarded as the hydrodynamic equations [12].

3.2.3. Other methods.

There are several other methods of deriving the hydrodynamic equations from the RBKE. We already mentioned the projection operator approach which leads to the generalized hydrodynamics (time and space dependent transport coefficients) [14, 16]. The variational approach should also be mentioned [32, 33]. None of those derivations, similarly to Chapman–Enskog or Grad methods, is mathematically rigorous.

One can obtain a deeper insight into the meaning of relativistic hydrodynamics and its relation to the RBKE by analyzing, in a mathematically rigorous way, the solutions of the later. In Ref. [24] the detailed analysis of the Cauchy problem for the linearized RBKE was given. Working entirely in $\mathcal{L}^2(\mathbf{r}, \mathbf{p})$ function space it was shown that the asymptotic in time (times larger than the mean free path) solution of the RBKE converges to its *hydrodynamic* part, that is the one whose time dependence is determined completely by set of partial differential equations for slow variables. Those equations are very close to the linearized van Kampen hydrodynamic equations.

4. Conclusions

The dissipative relativistic hydrodynamics is far from being well understood. Various derivations of that macroscopic theory suffer from lack of direct relation to the experimental situation and ambiguities related to the choices of primary variables, and separation between fast and slow motions. Rigorous mathematical analysis seems to point out to the van Kampen-like approach, as that one close to the correct macroscopic description. But adopting van Kampen point of view accounts to saying that one has to perform detailed analysis of each concrete problem one wants to analyze before engaging oneself into dispute about which set of equations should be used. In particular one should bear in mind what kind of experiments one can perform on the system, how it interacts with environment and whether theoretical description really requires frequently spurious and obscuring manifestly covariant formulation. In the "laboratory" application of the RH, namely the heavy ion collision theory, [3, 34] the Landau and Lifshitz formulation is picked up by the analysis of the experiments. A lot more work is needed on the RBKE before one will be able to establish the *standard* way of deriving the DRH. We shall gain more by working on the kinetic theory than by trying to develop formal theories remotely related to real physical applications.

5. Appendix A

In this Appendix we have summarized basic equations of motion from various formulations of DRH listed in the Table 1. If not mentioned separately, equations of motion are obtained by substituting the given expressions for the energy momentum tensor and particle current vector into the conservation laws.

1a. Landau and Lifshitz [2]

$$(A.1) \quad T_E^{\mu\nu} = e_E U_E^\mu U_E^\nu - p_E \Delta_E^{\mu\nu} + \zeta_E \Delta_E^{\mu\nu} \partial_\rho U_E^\rho$$
$$+ 2\eta_E \left(\partial^\rho U_E^{(\nu} \Delta_{E\sigma}^{\mu)} - \frac{1}{3} \Delta_E^{\mu\nu} \partial_\rho U_E^\rho \right),$$

$$(A.2) \quad N_E^\mu = n_E U_E^\mu - \frac{\kappa_E n_E}{h_E} \Delta_E^{\mu\rho} \left(\partial_\rho T_E - \frac{T_E}{h_E} \partial_\rho p \right),$$

where ζ_E, η_E and κ_E are positive functions of n_E, e_E and $A^{(\mu\nu)} = (A^{\mu\nu} + A^{\nu\mu})/2$.

1b. Eckart [1]

$$(A.3) \quad T_N^{\mu\nu} = e_N U_N^\mu U_N^\nu - p_N \Delta^{\mu\nu} + \zeta_N \Delta_N^{\mu\nu} \partial_\rho U_N^\rho$$
$$+ 2\eta_N \left(\partial^\sigma U_N^{(\nu} \Delta_{N\sigma}^{\mu)} - \frac{1}{3} \Delta_N^{\mu\nu} \partial_\rho U_N^\rho \right) + 2\kappa_N n_N \Delta_N^{\rho\nu} U_N^\mu \left(\partial_\rho T_N - \frac{T_N}{h_N} \partial_\rho p \right),$$

$$(A.4) \quad N_N^\mu = n_N U_N^\mu,$$

where transport coefficients are now functions of u_N, e_N.

2. van Kampen [17]

$$(A.5) \quad \partial_t \tilde{n} = -\partial_i \tilde{N}^i,$$
$$(A.6) \quad \partial_t T^{0i} = -\partial_j \tilde{T}^{ij},$$
$$(A.7) \quad \partial_t \tilde{e} = -\partial_i T^{0i},$$

where $i, j = 1, 2, 3$ and

$$(A.8) \quad \tilde{N}^i = n(V)U^i(V) + \tilde{n}^i(\tilde{\mathcal{Z}}),$$
$$(A.9) \quad \tilde{T}^{ij} = h(V)U^i(V)U^j(V) - g^{ij}p(V) + \tilde{\Pi}^{ij}(\tilde{\mathcal{Z}}).$$

Variables V, $\tilde{\mathcal{Z}}$ have been defined in the Table 1. The tilde terms in Eq. (2.24) and (2.25) are uniquely defined as linear functions of the observer frame gradients of the primary variables. There are technical difficulties in obtaining explicit expressions for those terms.

3. Israel and Stewart [12, 35]

There are two versions of this particular formulation, one in energy frame and the other in particle frame. In energy frame we have

$$(A.10) \quad \partial_\mu (T_E^{\mu\nu} + \Pi_{E2}^{\mu\nu}) = 0,$$
$$\partial_\mu (N_E^\mu + \nu_{E2}^\mu) = 0.$$

$(A.11)$ $\Pi_{E2}^{\mu\nu} = \zeta_E \Delta_E^{\mu\nu}(\beta_0 \dot{\Pi} + \alpha_0 \partial_\rho q^\rho)$

$$+2\eta_E[\beta_2 \dot{\pi}^{\mu\nu} + \alpha_1(\Delta_E^{\alpha<\mu}\Delta_E^{\nu>\beta}\partial_\beta q_\alpha - \frac{1}{3}\Delta_E^{\mu\nu}\Delta_E^{\alpha\beta}\partial_\beta q_\alpha)],$$

$(A.12)$ $$\nu_{E2}^\mu = -\kappa_E \Delta_E^{\mu\rho}\frac{n_E}{h_E}T_E(\alpha_1 \partial_\nu \pi_\rho^\nu - \alpha_0 \partial_\rho \Pi - \beta_1 \dot{q}_\rho),$$

where Π and q^μ are defined in the Table 1, α_0, α_1 and β_0, β_1, β_2 are functions of n_E, e_E. $\dot{A} \equiv U^\mu \partial_\mu A$.

In the particle frame we have

$(A.13)$ $$\partial_\mu(T_N^{\mu\nu} + \Pi_{N2}^{\mu\nu}) = 0,$$
$$\partial_\mu N_N^\mu = 0,$$

where

$(A.14)$ $\Pi_{N2}^{\mu\nu} = \zeta_N \Delta_N^{\mu\nu}(\beta_0 \dot{\Pi} + \overline{\alpha}_0 \partial_\rho q^\rho)$

$$+2\eta_N\left[\beta_2 \dot{\pi}^{\mu\nu} + \overline{\alpha}_1\left(\Delta_N^{\alpha(\mu}\Delta_N^{\nu)\beta}\partial_\beta q_\alpha - \frac{1}{3}\Delta_N^{\mu\nu}\Delta_N^{\alpha\beta}\partial_\beta q_\alpha\right)\right] + 2U_N^{(\mu}q_2^{\nu)},$$

$(A.15)$ $$q_2^\mu = -\kappa_N \Delta_N^{\mu\rho}T_N(\overline{\alpha}_1 \partial_\nu \pi_\rho^\nu - \overline{\alpha}_0 \partial_\rho \Pi - \overline{\beta}_1 \dot{q}_\rho).$$

In the above all the coefficients are functions of (particle frame) particle and energy densities.

4. Liu, Müller and Ruggeri [15]

In this formulation there is an additional "conservation" law

$(A.16)$ $$\partial_\mu A^{\lambda\nu\mu} = I^{\lambda\nu},$$

where

$(A.17)$ $A^{\lambda\nu\mu} = (\gamma_1 + \gamma_2 \Pi)U_N^\lambda U_N^\nu U_N^\mu - \frac{1}{2}\left(\gamma_1 + \gamma_2 \Pi - \frac{n}{3}\right)g^{(\lambda\nu}U_N^{\mu)}$

$$+\gamma_3(g^{(\lambda\nu}q^{\mu)} - \sigma U_N^{(\lambda}U_N^\nu U_N^{\mu)}q^\mu)) + \gamma_4 \pi^{(\lambda\nu}U_N^{\mu)},$$

$(A.18)$ $$I^{\lambda\nu} = \beta_1 \Pi(g^{\lambda\nu} - 4U_N^\lambda U_N^\nu) + \beta_2 \pi^{\lambda\nu} + \beta_3 q^{(\lambda}U_N^{\nu)},$$

where $(\alpha\beta\gamma)$ denotes total symmetrization of tensorial indices, $\Pi = \Pi_\mu^\mu/3$, $\pi^{\mu\nu} = \Pi^{\mu\nu} - \Pi \Delta_N^{\mu\nu}$, and $q^\mu = \Pi^{\mu\nu}U_\nu$. The coefficients γ depend on particle and energy density via uniquely defined function of those variables—the chemical potential. β's are non-negative functions of n_N, e_N.

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POLISH ACADEMY OF SCIENCES
INSTITUTE FOR THEORETICAL PHYSICS, WARSZAWA.

Note Added in Proof:

This paper was submitted and accepted for publication in 1989. For that reason it contains no references to recent works. We regret the delay in the publication of our paper and we apologize for incompleteness of our review.