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Differential forms and fluid dynamics

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THE EXTERIOR derivative replaces in many cases the notion of covariant derivative in tensor calculus, but it is more natural and easier to use in curvilinear system of coordinates. For this reason the formulation of the equations of fluid mechanics in the language of differential forms may be useful for the theoretical and numerical purposes. The paper contains such a formulation.

THE LANGUAGE of differential forms (or more exactly "exterior differential forms") is more convenient than that of vector fields for the following reasons:

1. An exterior derivative, which is a natural differentiation, does not require any additional structure, whereas in order to differentiate vectors one needs a connection, i.e. the covariant derivative.
2. It is easier to work with forms when coordinates are changed. This may be important for numerical calculations.
3. The geometrical aspects are more pronounced, hence it is often easier to find conservation laws.

Unfortunately the vector notations seems to be more intuitive. At least we are more used to it.

Differential forms can be identified with antisymmetric covariant tensor fields [1]. Let us consider some examples. The simplest example, the Pffaffian form, which gave the name for other exterior forms

$$\overset{1}{\omega} = a_i(x)\, dx^i$$

is a 1-form, i.e. an entity which can be integrated along contours in the space $\mathbf{E}^n$ of $x^1, \ldots, x^n$. Let a contour C be given by $x^i = x^i(s)$ then

$$\int\limits_C a_i(x)\, dx^i = \int a_i(x(s)) x^i(s)\, ds\,,$$

$a_i(x)$ are the components of this 1-form and $d\overset{i}{x}, \ldots, d\overset{n}{x}$ are the basic forms. Similarly

$$\overset{2}{\omega} = a_{ij}(x)\, dx^i \wedge dx^j \tag{1}$$

is a 2-form; it can be integrated over two-dimensional surfaces, e.g. if $x^i = x^i(\alpha, \beta)$, $(\alpha, \beta) \in \Omega \subset \mathbf{R}^2$, is such a surface then

$$\int\limits_{\Sigma^2} a_{ij}\, dx^i \wedge dx^j = \int a_{ij}(x)(x^i_{,\alpha} x^j_{,\beta} - x^i_{,\beta} x^j_{,\alpha})\, d\alpha\, d\beta\,. \tag{2}$$

As it follows from Eq. (2) the product $\wedge$, which is called the exterior product is antisymmetric, i.e.

$$dx^i \wedge dx^j = -\, dx^j \wedge dx^i\,.$$

Therefore only the antisymmetric part of the coefficients matrix a_{ij} is meaningful. In general,

$$\overset{k}{\omega} = a(x)_{i_1 \ldots i_k} \, dx^1 \wedge \ldots \wedge dx^k, \qquad a(x)_{i_1 \ldots i_n},$$

where $a(x)_{i_1 \ldots i_n}$ are scalar coefficients of $\overset{k}{\omega}$. It is a k-form and consequently can be integrated over k-dimensional surfaces in $\mathbf{E}^n$. Let Σ^k be given in the following parametric form: $x^i = x^i(\alpha^1, \ldots, \alpha^k)$, $(\alpha^1, \ldots, \alpha^k) \in \Omega^k \subset \mathbf{R}^k$

$$\int_{\Sigma^k} \overset{k}{\omega} = \int a_{i_1, \ldots, i_k} D^{i_1, \ldots, i_k} \, d\alpha^1, \ldots, d\alpha^k,$$

where

$$D^{i_1 \ldots i_k} = \det \begin{vmatrix} x^{i_1}_{,\alpha^1}, & x^{i_1}_{,\alpha^2}, & \ldots, & x^{i_1}_{,\alpha^k} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ x^{i_k}_{,\alpha^1}, & x^{i_k}_{,\alpha^2}, & \ldots, & x^{i_k}_{,\alpha^k} \end{vmatrix}.$$

Again the only antisymmetric part of $\{a_{i_1 \ldots i_k}\}$ is meaningful which permits us to identify forms with antisymmetric covariant tensor fields.

By Λ^k we denote the set (it is a module over the ring of C^∞ functions) of k-forms on $\mathbf{E}^n$. Forms can be multiplied

$$a \wedge b = a_{i_1, \ldots, i_k} b_{j_1, \ldots, j_r} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_r}.$$

The multiplication denoted by $\wedge$ is called the exterior multiplication and it inherits the following properties:

1) $a \in \Lambda^k$, $b \in \Lambda^r \Rightarrow a \wedge b \in \Lambda^{k+r}$;

2) $a, b \in \Lambda^1 \Rightarrow a \wedge b = -b \wedge a$, thus $a \wedge a = 0$ for $a \in \Lambda^1$. This implies that $a^k \wedge b^r = (-1)^{k \cdot r} b^r \wedge a^k$ for $a \in \Lambda^k$, $b \in \Lambda^r$. Thus $a \wedge a = 0$ for forms of odd ranks.

3) $\Lambda^k = \{0\}$ for $k > \dim \mathbf{E}$, Λ^0 — zero forms are scalar functions.

If $a \in \Lambda^0$ and $\overset{k}{\omega} \in \Lambda^k$ then we identify $a \wedge \overset{k}{\omega} = a \overset{k}{\omega}$, where the coefficients of a k-form $a \overset{k}{\omega}$ are obtained by multiplying the coefficients of $\overset{k}{\omega}$ by the scalar function a.

Exterior derivate

There exists a natural differentiation

$$d : \Lambda^k \longrightarrow \Lambda^{k+1}.$$

For zero forms, i.e. for scalars it is defined by

$$a \in \Lambda^0 \Rightarrow da = \frac{\partial a}{\partial x_i} dx^i \in \Lambda^1.$$

Similarly for 1-forms we have

$$a = a_i \, dx^i \in \Lambda^1 \Rightarrow da = \frac{\partial a_i}{\partial x^j} dx^j \wedge dx^i \in \Lambda^2,$$

and, in general, for k-forms

$$a \in \Lambda^k \Rightarrow da = a_{i_1, \ldots, i_k, j} \, dx^j \wedge dx^1 \wedge \ldots \wedge dx^k \in \Lambda^{k+1},$$

where

$$a_{i_1,\ldots,i_k,j} = \frac{\partial}{\partial x^j} a_{i_1,i_2,\ldots,i_k} .$$

Stokes theorem

Let $\Sigma^k \subset \mathbf{E}^n$ be an oriented piece of k-dimensional surface and let $\partial\Sigma$ denote the boundary of Σ (Fig.1); suppose it is piecewise smooth.

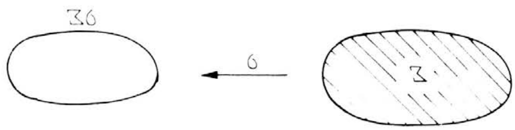

FIG. 1.

For k-dimensional $\Sigma^k \subset \mathbf{E}$ and $\omega^{k-1} \in \Lambda^{k-1}$ we have

$$\int\limits_{\Sigma^k} d\omega^{k-1} = \int\limits_{\partial\Sigma^k} \omega^{k-1} , \tag{3}$$

which is sometimes written as

$$\langle d\omega^{k-1}, \Sigma^k \rangle = \langle \omega^{k-1}, \partial\Sigma^k \rangle .$$

Thus ∂ and d are dual with respect to formula (3).

Another important property of d: similary as $\partial^2 = \partial \circ \partial = \emptyset$ (i.e. for $\Omega \subset \mathbf{R}^n$, $\partial(\partial\Omega) = \emptyset$ is an empty set) we have

$$d^2 := d \circ d = 0.$$

EXAMPLE

$$\phi \in \Lambda^0, \qquad d\phi = \phi_{,i} dx^i .$$

By the definition of d and by Property 2 we have

$$d^2\phi = \phi_{,ij}\, dx^i \wedge dx^j = \phi_{,ij}\, dx^j \wedge dx^i = -\phi_{,ij}\, dx^j \wedge dx^i .$$

Thus $d^2\phi = 0$. In any system of Cartesian coordinates in $\mathbf{R}^3$ we define the pseudovector field $\boldsymbol{\omega}$ associated with any $\omega \in \Lambda^2(\mathbf{R}^2)$ as

$$\boldsymbol{\omega} = (\omega^i = \frac{1}{2}\varepsilon^{ijk}\omega_{jk}), \qquad i,j,k = 1,2,3 .$$

Then the following isomorphism can be proved:

Differential forms notation		Vector notation
ω	$\longleftrightarrow$	$\boldsymbol{\omega}$,
$d\omega$	$\longleftrightarrow$	$\operatorname{div}\boldsymbol{\omega}$,
$d(\mathbf{v}_L \rfloor \omega)$	$\longleftrightarrow$	$\operatorname{rot}(\boldsymbol{\omega} \times \mathbf{v}_L)$,
$\mathbf{v}_L \rfloor \omega$	$\longleftrightarrow$	$\boldsymbol{\omega} \times \mathbf{v}_L$,

where $\mathbf{v}_L \rfloor \omega = \omega_{\alpha\beta}(\mathbf{v}_L^\alpha\, dx^\beta - \mathbf{v}_L^\beta\, dx^\alpha)$. This permits us to express our equation in the vector notation.

Pull-back of differential forms

Let us consider two manifolds M and N of dimensions m and n, respectively, and let $f : M \to N$ be a smooth mapping, then f induces a mapping of tangent bundles

$$TM \xrightarrow{f} TN\,,$$

i.e. the Jacobi matrix of f transforms the vectors tangent to M into vectors tangent to N (Fig.2). It turns out, however, that forms are transported in the opposite direction than vectors

$$\Lambda^k M \longleftarrow \Lambda^k N\,.$$

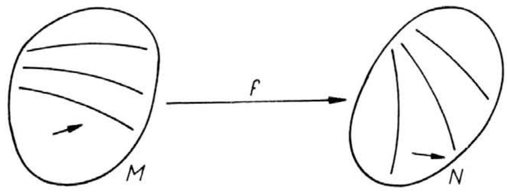

FIG. 2.

EXAMPLE

If $\omega = a_1\, dx^1 + a_2\, dx^2$ is a 1-form on $\mathbf{R}^2$ and $f : \mathbf{R}^1 \to \mathbf{R}^2$ given by $s \xmapsto{f} x^i(s)$ (Fig.3) then the differentials dx^i can be expressed as $dx^i = \dfrac{\partial x^i}{\partial s}\, ds$ on $\mathbf{R}^1$. Inserting this into expression for ω we obtain

$$\left(a_1 \frac{\partial x^1}{\partial s} + a_2 \frac{\partial x^2}{\partial s} \right) ds\,,$$

which indeed is a 1-form on M (= $\mathbf{R}^1$ in our example).

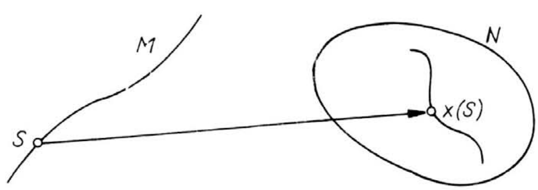

FIG. 3.

Restriction of a form to submanifold

This is a special case of pull-back of a differential form. The identity mapping i generates the pull-back of forms from N to M (Fig.4). If M can be described as $x^p = x^{p+1} = \ldots = x^n = 0$ then the restriction can be obtained by substituting: $dx^p = 0, \ldots, dx^n = 0$.

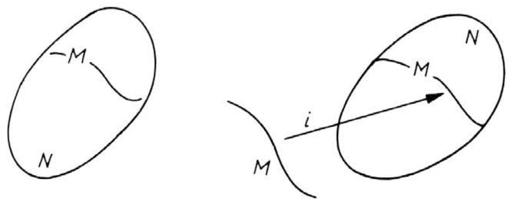

FIG. 4.

EXAMPLE

Let $\omega = a_1\, dx^1 + a_2\, dx^2$ be a 1-form on $\mathbf{R}^2$ and let the submanifold be given by $x^2 =$ const than the restriction of ω to the submanifold $x^2 =$ const is a 1-form $a_1(x^1, \text{const})\, dx^1$.

Commutation property

The exterior derivative has a very nice property: given a mapping $f: M \to N$, then we have

$$f \circ d = d \circ f\,,$$

where d on the left-hand side acts on N whereas d on the right acts on M.

EXAMPLE

Let $f: y^i = f^i(x^1, \ldots, x^p), \quad i = 1, \ldots, n,$

$$dy^i = f^i_{,x^s}(x^1, \ldots, x^p)\, dx^s\,,$$

and

$$\xi_{i_1 \ldots i_s}\, dy^1 \wedge \ldots \wedge dy^s$$

is a r-form on N, then

$$\xi_{i_1 \ldots i_s} \frac{\partial f_1}{\partial x^{i_1}} \cdots \frac{\partial f_s}{\partial x^{i_r}} dx^{i_1} \wedge \ldots \wedge dx^{i_r}$$

is a r-form on M induced by f from N.

Ideal barotropic fluid in the language of differential forms

As the starting point we take the action form for an "average particle" of the fluid (ev. test particle)

$$A \stackrel{\text{def}}{=} -H\, dt + \mathbf{p} \cdot d\mathbf{x}\,.$$

In the barotropic flows the entropy is constant, therefore the variation of the energy when a particle is moved from one position, say x_1, to another, say x_2, is equal to the variation of chemical potential $\delta E = \mu(x_1) - \mu(x_2)$. Thus in our case (after taking the mass of the particle as unity), we have

$$A = -\left(\frac{1}{2}v^2 + \int \frac{dp}{\rho(p)}\right) dt + v_i\, dx^i\,.$$

This is a 1-form in the Galilean space-time. Let us take a closed contour in the space-time. From the definition of A, $\oint_C A$ is "number of vortex lines" within the the contour of integration (Fig.5). This number can be, however, expressed as an integral of the "vortex line density" over the two-dimentional surface S spanned on C

$$\int_S J \Rightarrow \int_S J = \int_{\partial S} A\,.$$

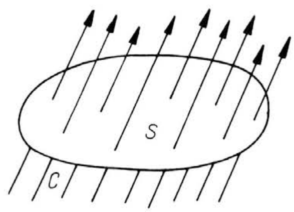

FIG. 5.

Applying Stokes theorem to $\int_{\delta S} A$

$$\int_{\partial S} A = \int_S dA\,,$$

we obtain differential equations

$$dA = J\,.$$

Clearly, J must be a 2-form as it is integrated over two-dimensional surfaces!

Let $\mathbf{v}_L$ — the velocity field of those vortex lines (we know: in a Cartesian system of coordinates $\mathbf{v}_L = \mathbf{v}$ for ordinary perfect fluid. There are fluids, however, for which $\mathbf{v}_L \neq \mathbf{v}$ [2]). In the system of coordinates which is co-moving with the vortex lines we have

$$J = \omega_{ij}\, dx^i \wedge dx^j, \qquad i,j = 1,2,3, \qquad \text{(there is no } dt).$$

$\omega = \omega_{ij}\, dx^i \wedge dx^j$ can be treated as a 2-form defined on three-dimensional space ($t =$ const). In the laboratory system of coordinates

$$dx^i \longrightarrow dx^i - v_L^i\, dt\,,$$

$$J = \omega + (\mathbf{v}_L \rfloor \omega) \wedge dt\,.$$

Taking exterior derivative of $dA = J$ we obtain

$$dJ = 0 \qquad \text{conservation of vorticity}.$$

The Lie derivative (introduced by Ślebodziński)

Let us take a vector field V defined on the manifold N. The field V gives rise to a flow, i.e. to the one-parameter group of transformations

$$\phi_\tau : N \longrightarrow N, \qquad \phi_0 = id\,.$$

This induces the transport of all tensor fields on N. Let it be given by

$$x = \phi(x_0, \tau), \qquad \phi = (x_0, 0) = x_0\,.$$

Given a tensor field T, then the tensor defined at $x = \phi(x_0, \tau)$ can be transported back to x_0, $\phi_\tau^{-1}T$, then having the family of tensors depending on τ and defined at the same point

x_0 we can differentiate it with respetc to τ, thus obtaining what is called the Lie derivative. In particular, this is true for vector fields (Fig.6.). Comparing vectors at different τ at the same point one can compute

$$\mathcal{L}_V X = \frac{d}{d\tau}\phi_{-\tau} \circ X = \lim_{\tau=0} \frac{X_\tau - X_0}{\tau} .$$

Similarly for other tensors (forms).

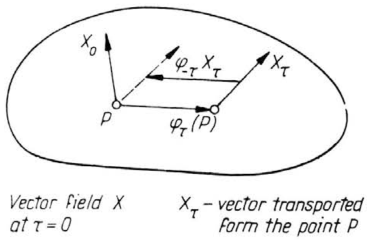

FIG. 6.

The following formula interrelates the Lie derivative and exterior derivative

$$\mathcal{L}_V J = V \rfloor dJ + d(V \rfloor J)$$

for any form J. We define $V = (1, \mathbf{v}_L)$ — vector field on space-time. If J is our vorticity current form then

$$V \rfloor J = 0 ,$$

which is easy to check in the co-moving system of coordinates: $V = (1, 0)$, J has no dt in it; dJ is also vanishing thus

$$\mathcal{L}_V J = 0 .$$

In addition $J \wedge J = 0$ (co-moving system of coordinates).

Summarizing:

$$A \stackrel{\text{def}}{=} -\left(\frac{1}{2}v^2 + \int \frac{dp}{\rho(p)}\right) dt + v_i\, dx^i, \qquad J \stackrel{\text{def}}{=} \omega + (\mathbf{v}_L \rfloor \omega) \wedge dt ,$$

$$dA = J \Rightarrow dJ = 0 \Rightarrow \mathcal{L}_V J = 0 ,$$

are the equations of an ideal fluid in the differential form notation. There are equivalent, respectively, to

$$\begin{cases} \dfrac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla v + \nabla_p = \boldsymbol{\omega} \times (\mathbf{v} - \mathbf{v}_L) , \\ \operatorname{rot} \mathbf{v} = \boldsymbol{\omega} , \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \omega}{\partial t} + \operatorname{rot}(\boldsymbol{\omega} \times \mathbf{v}_L) = 0 , \\ \operatorname{div} \boldsymbol{\omega} = 0 . \end{cases}$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \mathbf{v}_L \cdot \nabla\right) \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}_L + \boldsymbol{\omega} \operatorname{div} \mathbf{v}_L + 0 ,$$

when expressed by using the vector notation in Cartesian coordinates.

For an ordinary fluid $\mathbf{v}_L = \mathbf{v}$, but there are other possibilities; for superfluid He^4, $\mathbf{v}_L \neq \mathbf{v}$, similarly in MHD.

The lacking continuity equation can also be written as

$$dq = 0\,,$$

where $q = \rho V \rfloor (dx \wedge dy \wedge dz \wedge dt)$. Thus

$$dA = J, \qquad dq = 0$$

are the basic equations of perfect barotropic fluid.

To illustrate how the differential forms machinery works consider, as an example, the problem of vorticity prescribed at the boundary (Fig.7).

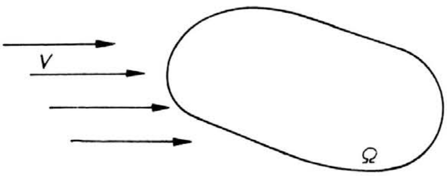

FIG. 7.

Let us now ask whether the vorticity $\boldsymbol{\omega}$ can be prescribed freely on $\partial\Omega$? In spite of the fact that $\dfrac{\partial \boldsymbol{\omega}}{\partial t} + \mathrm{rot}(\boldsymbol{\omega} \times \mathbf{v}_L) = 0$ is a nice evolution equation for ω, the answer is negative. The reason is that the system

$$dJ = 0 \Longleftrightarrow \begin{cases} \dfrac{\partial \boldsymbol{\omega}}{\partial t} + \mathrm{rot}(\boldsymbol{\omega} \times \mathbf{v}_L) = 0, \\ \mathrm{div}\boldsymbol{\omega} = 0 \end{cases}$$

is overdetermined, so $\boldsymbol{\omega}_{|\partial\Omega}$ must satisfy some constraints. One rather easy finds these constraints. Making the pull-back of dJ on the boundary $\partial\Omega$ one obtains $0 = i^*(dJ) = d(i^* J)$ and which in Cartesian system of coordinates is an equation for $\boldsymbol{\omega}^{\perp}$. Thus only $\boldsymbol{\omega}_{\|}$ can be prescribed freely.

Conservation of helicity

Let us define helicity current 3-form $\mathcal{H}$ in the space-time as

$$\mathcal{H} := A \wedge dA\,.$$

Applying the exterior derivative we have

$$d\mathcal{H} = dA \wedge dA = J \wedge J\,.$$

However, in the system of coordinates co-moving with the vortex lines we have $\mathbf{v}_L = 0$ and thus $J = \omega$, and therefore $J \wedge J = \omega \wedge \omega = 0$. Hence, $J \wedge J$ vanishes in any system of coordinates. Thus we have

$$d\mathcal{H} = 0\,.$$

In consequence, $\mathcal{H}$ represents a conserved current. The equations $d\mathcal{H} = 0$ in Cartesian coordinates can be written as [3]

$$\frac{\partial \mathcal{H}^0}{\partial t} + \mathrm{div}\,\mathcal{H} = 0\,,$$

where $\mathcal{H}^0 = \mathbf{v}\cdot\boldsymbol{\omega}$, $\mathcal{H} = \left(\frac{1}{2}v^2 + \int \frac{dp}{\varrho}\right)\boldsymbol{\omega} + \mathbf{v}\times(\mathbf{v}_L \times \boldsymbol{\omega})$. Helicity, $\int \mathcal{H}^0\, d^3x$, is an interesting quantity, because it expresses certain topological properties of the vorticity vector field [3,4,5].

To conclude, let us mention further possible generalizations. Let us consider the flow in which entropy of the fluid can change. Thus the density $\rho = \rho(p,T)$ becomes also a function of T. Then, starting from the some definition of A and J

$$A = -\left\{\frac{1}{2}v^2 + \mu\right\} dt + v_i\, dx_i\,, \qquad J = \omega + (\mathbf{v}_L \rfloor \omega)\wedge dt\,,$$

one can easy check that the flow equations can be written in this case as

$$dA = J - S\, dT \wedge dt\,,$$

where S — entropy, μ — chemical potential,

$$d\mu = \frac{1}{\rho}\, dp - S\, dT\,.$$

The equation for vorticity current which follows now is more complicated

$$dJ = dS \wedge dT \wedge dt\,,$$

and it implies

$$\mathcal{L}_V J = (\mathbf{v}\cdot\nabla)S\, dT \wedge dt - (\mathbf{v}\cdot\nabla T)\, dS \wedge dt + dS \wedge dT\,.$$

The same formalism can be used to describe superfluids, as for example superfluid He^4, and also in MHD.

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