Arch. Mech., 43, 6, pp. 715–741, Warszawa 1991

Perfectly plastic plates loaded by boundary bending moments: relaxation and homogenization

J. J. TELEGA (WARSZAWA)

EFFECTIVE MODELS of perfectly plastic plates with periodic structure are obtained by the method of Γ-convergence. It is shown that loading of the boundary by bending moments essentially influences the process of homogenization and the final results. Relevant relaxation problems are discussed in detail.

1. Introduction

IN OUR PREVIOUS PAPERS [50, 51] the problem of homogenization of perfectly plastic plates clamped at a part Γ_0 of boundary and subjected to transverse forces at the complementary part Γ_1 was solved. Distributed loading, e.g. body forces, was also taken into account. New aspects (and difficulties) appear when the plate is additionally subject to boundary bending moments. The aim of the present paper is to solve such more complicated problem of homogenization provided that the plate reveals fine periodic structure.

The space $HB(\Omega)$, defined by (3.2), plays an essential role in the mathematical study of perfectly plastic plates [19, 20, 22, 54]. In our case Ω will represent the mid-plane of the plate. The trace operator γ_0 is strongly continuous even if the underlying topology of $HB(\Omega)$ is the weak one. Unfortunately, it is not so with the trace operator γ_1 (the trace operators are introduced and discussed in the paper [21]). Hence the need for relaxation of the relevant functionals. If boundary moments are absent then only one boundary trace condition on Γ_0 has to be relaxed [50, 51]. Physically this is obvious since on Γ_0 a plastic hinge can appear. An additional relaxation functional has to be introduced for dealing with boundary moments. Moreover, the space of virtual fields on Γ_1 is treated as independent from virtual fields on Ω.

In Sect. 5 the study of relaxation is performed. It seems that the results obtained can also be used for solving some existence problems. Demengel's existence results for perfectly plastic plates are restricted to the case $M_n^0 \equiv 0$ [19, 22, 54], where M_n^0 is the boundary moment. We observe also that little is known about existence theorems in the case of limit analysis of plates, cf. [16].

Homogenization problems are studied in Sect. 6. Assuming that the plate exhibits fine periodic structure, the effective model is constructed. To this end the method of Γ-convergence is applied. The study of homogenization in the case of limit analysis reveals the influence of the boundary moments on the limit load multiplier of the macroscopic plate. The latter is found by solving *two* limit analysis problems: one, more conventional, on Ω while the other on Γ_1. Similar result was earlier reported by BOUCHITTÉ and SUQUET [11] in the three-dimensional case, cf. also Ref. [46]. Knowing the limit load multiplier for the homogenized plate one can next perform homogenization of an elasto-perfectly plastic plate made of a Hencky material. The homogenization problems solved essentially exploit the relaxation technique developed in Sect. 5.

The effective model of a plastic plate with periodic structure depends upon the problem studied (limit analysis problem or elasto-plastic Hencky plate) and boundary conditions. Rigorously, it is inferred from the epi-limit of appropriate sequence of functionals. However, the homogenized or effective plate properties in the open domain Ω are given by the function j^h, specified by the formula (6.4), for the Hencky plate. Similarly, for the limit analysis problem such role is played by the dissipation function j_d^h, cf. (6.6).

For more information on homogenization problems in plasticity the reader should refer to [7, 23, 24, 38, 41, 44–48, 50, 51, 53].

2. Elements of the theory of epi-convergence. Epi-convergence and duality

2.1. Epi-convergence

Detailed presentation of the theory of epi-convergence, which is a particular case of the so called Γ-convergence, is available in the book by ATTOUCH [1], see also Refs [2, 3, 9, 55].

DEFINITION 2.1. *Let (X, τ) be a metrisable topological space and $\{F_\varepsilon\}_{\varepsilon>0}$ a sequence of functionals from X into $\overline{\mathbb{R}}$, the extended reals.*

(a) *The τ-epi-limit inferior $\tau\text{-}li_e F_\varepsilon$, denoted also by F^i, is the functional on X defined by*

$$F^i(u) = \tau - li_e F_\varepsilon(u) = \min_{\{u_\varepsilon \xrightarrow{\tau} u\}} \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)\,.$$

(b) *The τ-epi-limit superior $\tau - ls_e F_\varepsilon$, denoted also by F^s, is the functional on X defined by*

$$F^s(u) = \tau - ls_e F_\varepsilon(u) = \min_{\{u_\varepsilon \to u\}} \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)\,.$$

(c) *The sequence $\{F_\varepsilon\}_{\varepsilon>0}$ is said to be τ-epi-convergent if $F^i = F^s$. Then we write*

$$F = \tau - \lim{}_e F_\varepsilon\,.$$

PROPERTIES

Let $F_\varepsilon : (X, \tau) \to \overline{\mathbb{R}}$ be a sequence of functionals which is τ-epi-convergent, $F = \tau - \lim_e F_\varepsilon$. Then the following properties hold:

(i) The functionals F^i and F^s are τ-lower semicontinuous (τ — l.s.c.).

(ii) If the functionals F_ε are convex then $F^s = \tau - ls_e F_\varepsilon$ is also convex. Hence the epi-limit $F = \tau - \lim_e F_\varepsilon$ is a τ-closed (τ — l.s.c.) convex functional.

(iii) If $\Phi : X \to \mathbb{R}$ is a τ-continuous functional, called perturbation functional, then

$$\tau - \lim{}_e(F_\varepsilon + \Phi) = \tau - \lim{}_e F_\varepsilon + \Phi = F + \Phi\,.$$

(iv)

$$F(u) = \tau - \lim{}_e F_\varepsilon(u) \Leftrightarrow \begin{cases} \forall u_\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} u, \quad F(u) \le \liminf\limits_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon), \quad u \in X; \\ \forall u \in X\ \exists u_\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} u \quad \text{such that} \quad F(u) \ge \limsup\limits_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon). \end{cases}$$

In practical situations the last property plays an important role. Very useful is also the following theorem.

The effective model of a plastic plate with periodic structure depends upon the problem studied (limit analysis problem or elasto-plastic Hencky plate) and boundary conditions. Rigorously, it is inferred from the epi-limit of appropriate sequence of functionals. However, the homogenized or effective plate properties in the open domain Ω are given by the function j^h, specified by the formula (6.4), for the Hencky plate. Similarly, for the limit analysis problem such role is played by the dissipation function j_d^h, cf. (6.6).

For more information on homogenization problems in plasticity the reader should refer to [7, 23, 24, 38, 41, 44–48, 50, 51, 53].

2. Elements of the theory of epi-convergence. Epi-convergence and duality

2.1. Epi-convergence

Detailed presentation of the theory of epi-convergence, which is a particular case of the so called Γ-convergence, is available in the book by ATTOUCH [1], see also Refs [2, 3, 9, 55].

DEFINITION 2.1. *Let (X,τ) be a metrisable topological space and $\{F_\varepsilon\}_{\varepsilon>0}$ a sequence of functionals from X into $\overline{\mathbb{R}}$, the extended reals.*

(a) *The τ-epi-limit inferior $\tau\text{-}li_eF_\varepsilon$, denoted also by F^i, is the functional on X defined by*

$$F^i(u) = \tau - li_eF_\varepsilon(u) = \min_{\{u_\varepsilon \xrightarrow{\tau} u\}} \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)\,.$$

(b) *The τ-epi-limit superior $\tau - ls_eF_\varepsilon$, denoted also by F^s, is the functional on X defined by*

$$F^s(u) = \tau - ls_eF_\varepsilon(u) = \min_{\{u_\varepsilon\to u\}} \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)\,.$$

(c) *The sequence $\{F_\varepsilon\}_{\varepsilon>0}$ is said to be τ-epi-convergent if $F^i = F^s$. Then we write*

$$F = \tau - \lim{}_e F_\varepsilon\,.$$

PROPERTIES

Let $F_\varepsilon : (X,\tau) \to \overline{\mathbb{R}}$ be a sequence of functionals which is τ-epi-convergent, $F = \tau - \lim_e F_\varepsilon$. Then the following properties hold:

(i) The functionals F^i and F^s are τ-lower semicontinuous (τ — l.s.c.).

(ii) If the functionals F_ε are convex then $F^s = \tau - ls_eF_\varepsilon$ is also convex. Hence the epi-limit $F = \tau - \lim_e F_\varepsilon$ is a τ-closed (τ — l.s.c.) convex functional.

(iii) If $\Phi : X \to \mathbb{R}$ is a τ-continuous functional, called perturbation functional, then

$$\tau - \lim{}_e(F_\varepsilon + \Phi) = \tau - \lim{}_e F_\varepsilon + \Phi = F + \Phi\,.$$

(iv)

$$F(u) = \tau - \lim{}_e F_\varepsilon(u) \Leftrightarrow \begin{cases} \forall u_\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} u, \quad F(u) \le \liminf\limits_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon), \quad u \in X; \\ \forall u \in X\ \exists u_\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} u \quad \text{such that} \quad F(u) \ge \limsup\limits_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon). \end{cases}$$

In practical situations the last property plays an important role. Very useful is also the following theorem.

respectively. If $F^i(x) = F^s(x)$, *for each* $x \in X$, *then we write* (cf. Def. 2.1)

$$F = \tau - \lim_{\varepsilon \to 0} {}_e F_\varepsilon .$$

In metrisable topological space the Definitions 2.1 and 2.3 coincide. For instance, in a general topological space we have

$$\inf_{\{x_\varepsilon \xrightarrow{\tau} x\}} \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \geq \sup_{V \in N_\tau(x)} \liminf_{\varepsilon \to 0} \inf_{u \in V} F_\varepsilon(u) . \tag{2.2}$$

When (X, τ) is metrisable space, in (2.2) equality holds.

2.2. Epi-convergence and duality

Having a sequence of functionals $\{F_\varepsilon\}_{\varepsilon>0}$ one can construct the sequence of conjugate functionals F_ε^* by using the Fenchel transformation

$$F_\varepsilon^*(u^*) = \sup\{\langle u^*, u\rangle - F_\varepsilon(u) | u \in X\}, \qquad u^* \in X^* . \tag{2.3}$$

As usual $(X^*, X, \langle .,. \rangle)$ is a dual pair, cf. Refs [28, 36].

Now a natural question arises: what is a relation between the epi-convergence of the sequences $\{F_\varepsilon\}_{\varepsilon>0}$ and $\{F_\varepsilon^*\}_{\varepsilon>0}$, respectively? Existing results are confined to convex problems, see Refs. [1–3, 5, 6, 55]. ATTOUCH [2, 3] investigated such an interrelation provided that X is a reflexive separable Banach space. More general results were obtained by AZÉ [6] who assumes that X is a separable Banach space.

By $\Gamma_0(X)$ we denote the space of convex lower semicontinuous and proper functions on X, cf. Refs [28, 36].

THEOREM 2.3 [6]. *Let* X *be a separable Banach space and* $\{F_\varepsilon\}_{\varepsilon>0} \subset \Gamma_0(X)$. *Assume that*

(i) $F = s - \lim_e F_\varepsilon$,

(ii) $\lim_\varepsilon \sup F_\varepsilon^*(u_\varepsilon^*) < +\infty \Rightarrow \sup_\varepsilon \|u_\varepsilon^*\|_{X^*} < \infty$.

Then

$$F^* = w^* - \lim {}_e F_\varepsilon^* . \tag{2.4}$$

■

In the assumption (i) s denotes the strong topology of X whereas in (2.4) w^* is the weak-$*$ topology of the dual space X^*. The last theorem is important for studying homogenization of perfectly plastic solids and structures like plates loaded at their boundary, see Sect. 6. This theorem is also important for the formulation of the duality theory proposed by AZÉ [5]. Azé's theory is a convenient tool for performing the dual homogenization of solids and structures under the assumption of convexity, cf. Ref. [52]. However Azé's theory will not be exploited in the present paper.

3. Space $HB(\Omega)$. Convex functions and functionals of a measure. Representation of convex functionals of a measure

Before passing to the study of perfectly plastic plates it seems appropriate to introduce relatively less known, but essential, notions and results.

$\Omega \subset \mathbb{R}^2$ will always denote a bounded domain of the plane. E_s^2 is the space of symmetric 2×2 matrices. By $\mathsf{M}^1(\Omega, \mathsf{E}_s^2)$ we denote the space of bounded measures with values in $\mathbf{E}_s^2$ [9, 10, 21, 32, 54]. The necessity of introducing of such a space follows from the fact that for perfectly plastic plates the gradient of displacement or of velocity can be discontinuous. Hence the second derivatives are no longer functions but measures.

Let $C_0(\Omega, \mathbf{E}_s^2)$ be the space of continuous functions on Ω with values in $\mathbf{E}_s^2$ and vanishing on the boundary, that is for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that $|\varphi(x)| < \varepsilon$ for all $x \in \Omega \setminus K_\varepsilon$. The norm is given by

$$\forall \varphi \in C_0(\Omega, \mathbf{E}_s^2) \quad \|\varphi\|_\infty = \sup\{|\varphi(x)|, x \in \Omega\}\,. \tag{3.1}$$

The triple $(\mathbf{M}^1(\Omega, \mathbf{E}_s^2), C_0(\Omega, \mathbf{E}_s^2), \langle .,. \rangle)$ constitutes the dual pair, where in this case $\langle \cdot, \cdot \rangle$ is the duality bracket between $\mathbf{M}^1(\Omega, \mathbf{E}_s^2)$ and $C_0(\Omega, \mathbf{E}_s^2)$.

For mathematical study of the aforementioned discontinuities it is natural to introduce the space $HB(\Omega)$ [19–22, 25, 54].

$$HB(\Omega) = \{u \in W^{1,1}(\Omega) \mid \kappa(u) \in \mathbf{M}^1(\Omega, \mathbf{E}_s^2)\}\,, \tag{3.2}$$

where $\kappa(u) = (u_{,\alpha\beta}) = \left(\dfrac{\partial^2 u}{\partial x_\alpha \partial x_\beta}\right) = \nabla\nabla u = D^2 u$ (in the sense of distributions). By (x_α) $(\alpha = 1, 2)$ the Cartesian coordinates of the plane have been denoted. $W^{1,1}(\Omega)$ is the usual Sobolev space

$$W^{1,1}(\Omega) = \{u \in L^1(\Omega) \mid u_{,\alpha} \in L^1(\Omega)\}\,. \tag{3.3}$$

We shall also extensively use the following Sobolev space

$$W^{2,1}(\Omega) = \{u \in W^{1,1}(\Omega) \mid \kappa_{\alpha\beta}(u) \in L^1(\Omega)\}\,. \tag{3.4}$$

As usual, $L^1(\Omega)$ is the space of Lebesgue integrable functions. All the introduced spaces are nonreflexive Banach spaces (unfortunately !).

The natural norm on the space $HB(\Omega)$ is defined by

$$\|u\|_{HB(\Omega)} = \|u\|_{W^{1,1}(\Omega)} + \|\kappa(u)\|_{\mathbf{M}^1(\Omega, \mathbf{E}_s^2)}\,, \tag{3.5}$$

where

$$\|\kappa(u)\|_{\mathbf{M}^1(\Omega, \mathbf{E}_s^2)} = \sup\{\langle \kappa(u), \varphi \rangle \mid \varphi \in C_0(\Omega, \mathbf{E}_s^2), |\varphi(x)| \leq 1\}\,. \tag{3.6}$$

In (3.6) $|\cdot|$ denotes the Euclidean norm.

Let $\Pi_1(\Omega)$ be the space of distributions with vanishing second derivatives. Then (3.6) defines the norm on the quotient space $HB(\Omega)/\Pi_1(\Omega)$, equivalent to the norm induced by the norm of the space $HB(\Omega)$, provided that the domain Ω has the cone property [21]. DEMENGEL [21] proved that the injection

$$HB(\Omega) \subset W^{1,1}(\Omega) \tag{3.7}$$

is compact.

As we have already noted, in the case of perfectly plastic plates the components of the tensor of changes of curvature (or their rates) are usually bounded measures and not L^1 functions. Hence such notions as the total dissipation and the density of plastic dissipation, depending upon those measures, have to be precisely defined. To this end very useful are convex functions and functionals of a measure. Let us briefly discuss these notions. Firstly, however, some assumptions have to be introduced.

(H_1) | Let $f : (x, \mathbf{e}) \in \Omega \times \mathbf{E}_s^2 \to \mathbf{R} \cup \{+\infty\}$ be a normal convex integrand.

We recall that then f is convex and l.s.c. with respect to $\mathbf{e} \in \mathbb{E}^2_s$ and measurable with respect to $(x, \mathbf{e}) \in \Omega \times \mathbb{E}^2_s$ [28]. If $\mu \in \mathbf{M}^1(\Omega, \mathbb{E}^2_s)$ then $\mu \ll dx$ denotes that μ is absolutely continuous with respect to the Lebesgue measure $dx = dx_1 dx_2$ and $\mu = \mathbf{h}(x)dx$. We set

$$(3.8) \qquad \widetilde{F}(\mu) = \begin{cases} \int_\Omega f(x, \mathbf{h}(x))dx, & \text{if } \mu \ll dx \quad \text{and} \quad \mathbf{h} \in L^1(\Omega, \mathbb{E}^2_s), \\ +\infty, & \text{otherwise.} \end{cases}$$

Now we make further assumptions:

(H_2) | $\exists \varphi_0 \in C_0(\Omega, \mathbb{E}^2_s)$, $\exists b \in L^1(\Omega)$ such that $f(x, \mathbf{e}) \geq \varphi_{0\alpha\beta}(x) e_{\alpha\beta} - b(x)$, $\forall (x, \mathbf{e}) \in \Omega \times \mathbb{E}^2_s$,

(H_3) | $\exists \mathbf{v}_0 \in L^1(\Omega, \mathbb{E}^2_s)$ such that $\int_\Omega f(x, \mathbf{v}_0(x))\, dx < +\infty$.

BOUCHITTÉ [9] (see also [12]) proved that under the assumptions (H_1)–(H_3) the functional

$$(3.9) \quad F(\mu) = \sup\{\langle \mu, \varphi \rangle - \int_\Omega f^*(x, \varphi(x))\, dx \mid \varphi \in C_0(\Omega, \mathbb{E}^2_s)\}, \ \mu \in \mathbf{M}^1(\Omega, \mathbb{E}^2_s),$$

represents the lower semicontinuous regularization of the functional $\widetilde{F}$ in the weak-$*$ topology $\sigma(\mathbf{M}^1(\Omega, \mathbb{E}^2_s)), C_0(\Omega, \mathbb{E}^2_s))$. Here f^* denotes the polar (conjugate) function of $f(x, \cdot)$, that is

$$(3.10) \qquad f^*(x, \mathbf{e}) = \sup\{\mathbf{e}^* : \mathbf{e} - f(x, \mathbf{e}) \mid \mathbf{e} \in \mathbb{E}^2_s\}, \quad \mathbf{e}^* \in \mathbb{E}^2_s,$$

where $\mathbf{e}^* : \mathbf{e} = e^*_{\alpha\beta} e_{\alpha\beta}$. The functional $F(\mu)$, denoted also by $\int_\Omega \overline{f}(\mu)$, is a convex functional of a measure $\mu \in \mathbf{M}^1(\Omega, \mathbb{E}^2_s)$ and $\overline{f}(\mu)$ is the convex function of that measure, being itself a measure.

The approach used by BOUCHITTÉ and VALADIER [12] is an alternative one to that primarily proposed by DEMENGEL and TEMAM [25], see also [54]. HADHRI [33, Chapter VII] refined the results due to DEMENGEL and TEMAM and demonstrated that the Lemma 1.1 given in [25] is not valid, when the supremum is taken over $\mathcal{D}_f(L^1_\mu)$. The same regards the formula given as Eq. (II.4.22) in the book by TEMAM [54].

For better comprehension of our subsequent considerations it seems instructive to give the definition due to HADHRI [33], adapted to our case. The definition of Demengel and Temam follows if $f(x, \mathbf{e}) = g(\mathbf{e})$. Let f be a function which satisfies the following assumptions:

(H'_1) | $f : \Omega \times \mathbb{E}^2_s \to \mathbb{R}$ is a convex normal integrand, hence the function $f(x, \cdot)$ is continuous.

(H_4) | $\exists k_1 > 0$, $\exists k_0 \in L^1(\Omega)$; $\forall (x, \mathbf{e}) \in \Omega \times \mathbb{E}^2_s$, $f(x, \mathbf{e}) \leq k_1 |\mathbf{e}| + k_0(x)$.

(H_5) | $\exists k_2 \in L^1(\Omega)$, $\forall (x, \mathbf{e}) \in \Omega \times \mathbb{E}^2_s$, $f_d(x, \mathbf{e}) \leq f(x, \mathbf{e}) + k_2(x)$.

Here f_d is the principal part (the recession function) of f defined on $\Omega \times \mathbb{E}^2_s$ by

$$(3.11) \quad f_d(x, \mathbf{e}) = \lim_{t \to \infty} \frac{1}{t} f(x, t\mathbf{e}) = \lim_{\rho \to 0} \rho f\left(x, \frac{\mathbf{e}}{\rho}\right) = \sup\{\mathbf{e}^* : \mathbf{e} \mid \mathbf{e}^* \in \operatorname{dom} f^*(x, \cdot)\}.$$

As usual, $\operatorname{dom} f^*(x, \cdot)$ is the effective domain of the function $f^*(x, \cdot)$ [40]. In applications to limit analysis problems the recession function is nothing else as the dissipation density.

In a first instant Hadhri assumes that

$$\forall(x,\mathbf{e}) \in \Omega \times \mathbb{E}_s^2, \quad f(x,\mathbf{e}) \geq f(x,)) = 0\,. \tag{3.12}$$

Further we define

$$V_f = \{\mathbf{v} \in C_c(\Omega,\mathbb{E}_s^2) \mid f^*(\cdot,\mathbf{v}(\cdot)) \in L^1(\Omega)\}\,, \tag{3.13}$$

where $C_c(\Omega,\mathbb{E}_s^2)$ is the space of continuous functions with compact supports in Ω and with values in $\mathbb{E}_s^2$. The norm is given by (3.1).

For $\mu \in \mathbf{M}^1(\Omega,\mathbb{E}_s^2)$ and $\varphi \in C_c^+(\Omega) = \{\varphi \in C_c(\Omega) \mid \forall x \in \Omega, \varphi(x) \geq 0\}$ we set

$$\langle \overline{f}(\mu),\varphi\rangle = \sup\{\langle \mu, \mathbf{v}\varphi\rangle - \int\limits_\Omega f^*(x,\mathbf{v}(x))\varphi(x)\,dx \mid \mathbf{v} \in V_f\}\,. \tag{3.14}$$

If φ does not belong to $C_c^+(\Omega)$, by means of the decomposition

$$\forall\varphi \in C_c(\Omega), \quad \exists(\varphi^+,\varphi^-) \in [C_c^+(\Omega)]^2, \quad \varphi = \varphi^+ - \varphi^-\,, \tag{3.15}$$

we can set

$$\langle \overline{f}(\mu),\varphi\rangle = \langle \overline{f}(\mu),\varphi^+\rangle - \langle \overline{f}(\mu),\varphi^-\rangle\,. \tag{3.16}$$

Suppose now that f is defined by

$$f(x,\mathbf{e}) = f_1(x,\mathbf{e}) + \mathbf{a}(x) : \mathbf{e} + b(x)\,, \tag{3.17}$$

where $b \in L^1(\Omega)$, $\mathbf{a} : \Omega \to \mathbb{E}_s^2$ is a bounded Borel function and f_1 satisfies (3.12). Then for $\varphi \in C_c(\Omega)$ we set

$$\langle \overline{f}(\mu),\varphi\rangle = \langle \overline{f}_1(\mu),\varphi\rangle + \langle \mathbf{a} : \mu,\varphi\rangle + \int\limits_\Omega b\varphi\,dx\,. \tag{3.18}$$

In the sequel, for the sake of simplicity, the bar over a function of a measure will be omitted.

Having defined a convex function of a measure, one can construct a convex functional of the same measure [25, 33]. On the other hand, the approach used in [9, 12] is a reverse one; for interrelations the reader should refer to Ref. [12].

The following lemma is important for mathematical investigations of perfectly plastic plates, including homogenization problems [19, 21, 22, 25, 54].

LEMMA 3.1. Let f be a convex function on $\mathbb{E}_s^2$ satisfying the following property

$$\exists\Lambda_0 \geq \lambda_0 > 0, \quad \exists k_0 > 0, \quad \forall\mathbf{e} \in \mathbb{E}_s^2, \quad \lambda_0|\mathbf{e}| - k_0 \leq f(\mathbf{e}) \leq \Lambda_0(1 + |\mathbf{e}|)\,. \tag{3.19}$$

Let F be a functional on $W^{1,1}(\Omega)$ defined by

$$F(u) = \begin{cases} \int\limits_\Omega f(\kappa(u)), & \text{if} \quad u \in HB(\Omega), \\ +\infty, & \text{if} \quad u \in W^{1,1}(\Omega) \setminus HB(\Omega). \end{cases} \tag{3.20}$$

Then

(i) F is the lower semicontinuous regularization of the functional

$$\widetilde{F}(u) = \begin{cases} F(u), & \text{if } u \in C^\infty(\Omega) \cap W^{2,1}(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.21}$$

in the strong topology of the space $W^{1,1}(\Omega)$. The functional F is convex and continuous on $HB(\Omega)$.

(ii) For each $u \in HB(\Omega)$ there exists a sequence $\{u_n\}_{n=0}^{\infty} \subset C^\infty(\Omega) \cap W^{2,1}(\Omega)$ such that $\gamma_0 u_n = \gamma_0 u$, $\gamma_1 u_n = \gamma_1 u$, $\forall n \in \mathbb{N}$,

$$u_n \xrightarrow[n\to\infty]{W^{1,1}(\Omega)} u, \qquad \int_\Omega |\kappa(u_n)| \to \int_\Omega |\kappa(u)|, \qquad F(u_n) \to F(u),$$

where γ_0 and γ_1 are the trace operators; $\gamma_0 u \in \gamma_0(W^{2,1}(\Omega))$, $\gamma_1 u \in L^1(\Gamma)$, $\Gamma = \partial\Omega$. ■

Obviously $\mathbb{N}$ is the set of natural numbers. Further we observe that the functional given by (3.20) is to be understood as a convex functional of a measure, since for $u \in HB(\Omega) \setminus W^{2,1}(\Omega)$ one has $\kappa(u) \in \mathbf{M}^1(\Omega, \mathbb{E}_s^2)$. We also recall that $|\mu|$ is the total variation measure associated to μ; further by $(d\mu)/(d|\mu|)$ the density of μ with respect to $|\mu|$ will be denoted.

Let

$$\mu = \mu_a(x)dx + \mu_s\,, \tag{3.22}$$

be the Lebesgue decomposition of μ into absolutely continuous and singular parts with respect to dx. For instance, if $\mu = \kappa(v)$ ($v \in HB(\Omega)$ denotes a velocity field) then μ_s represents the singular part of the tensor of changes of curvature on discontinuity lines.

Now we can formulate a fundamental result related to the integral representation of the functional (3.9).

LEMMA 3.2. Under the assumptions (H_1)–(H_3) and

$$\begin{gathered}\forall\varepsilon > 0 \quad \exists\delta > 0 \quad \forall(x, z) \in \Omega \times \Omega, \quad \forall \mathbf{e} \in \mathbb{E}_s^2, \\ |x - z| < \delta \Rightarrow |f(x, \mathbf{e}) - f(z, \mathbf{e})| < \varepsilon(1 + |\mathbf{e}|)\,,\end{gathered} \tag{3.23}$$

we have

$$F(\mu) = \int_\Omega f(x, \mu_a(x))\,dx + \int_\Omega f_d\left(x, \frac{d\mu_s}{d|\mu_s|}(x)\right)d|\mu_s|, \qquad \mu \in \mathbf{M}^1(\Omega, \mathbb{E}_s^2)\,. \tag{3.24}$$

For more details on integral representations of functionals of measures the reader should refer to [9, 10, 12, 14].

4. Plate models and extremum principles

In the sequel by $\Omega \subset \mathbb{R}^2$ we shall always denote a sufficiently regular and bounded domain. $\overline{\Omega}$ is the middle plane of plate. By $\mathbf{M} = (M_{\alpha\beta})$, $\kappa = (\kappa_{\alpha\beta})$ and $\mathbf{D} = (D_{\alpha\beta\lambda\mu})$ we denote the moment tensor, the curvature tensor (or its rate in the case of limit analysis) and the tensor of elastic moduli, respectively. The last one is not necessarily isotropic and $D_{\alpha\beta\lambda\mu} = D_{\beta\alpha\lambda\mu} = D_{\lambda\mu\alpha\beta}$. We assume that

$$\exists c > 0 \quad \forall \mathbf{e} \in \mathbb{E}_s^2, \quad D_{\alpha\beta\lambda\mu}(x)e_{\alpha\beta}e_{\lambda\mu} \geq c e_{\alpha\beta}e_{\alpha\beta}\,, \tag{4.1}$$

for almost every (a.e.) $x \in \Omega$. Our next assumption is $D_{\alpha\beta\lambda\mu} \in L^\infty(\Omega)$, hence

$$\exists c_1 \geq c > 0 \quad \forall \mathbf{e} \in \mathbb{E}_s^2, \quad D_{\alpha\beta\lambda\mu}(x)e_{\alpha\beta}e_{\lambda\mu} \leq c_1 e_{\alpha\beta}e_{\alpha\beta}, \quad \text{a.e. } x \in \Omega\,. \tag{4.2}$$

Let $C = C \subset \mathbb{E}_s^2$ be a bounded convex set of plastically admissible moments such that $\mathbf{0} \in \operatorname{int} C$. For instance, in the case of the Huber–Mises yield criterion the elasticity convex set C has the following form [34, 42, 43]:

(4.3) $$C = \{\mathbf{M} = (M_{\alpha\beta}) \mid M_{11}^2 - M_{11}M_{22} + M_{22}^2 + 3M_{12}^2 - M_0^2 \leq 0\},$$

where M_0 stands for the limit bending moment.

Let us set

(4.4) $$j^*(\mathbf{M}) = \begin{cases} \frac{1}{2}A_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu}, & \text{if} \quad \mathbf{M} \in C, \\ +\infty, & \text{if} \quad \mathbf{M} \notin C, \end{cases}$$

where $\mathbf{A} = \mathbf{D}^{-1}$. Equivalently one can write

(4.5) $$j^*(\mathbf{M}) = \frac{1}{2}A_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu} + I_C(\mathbf{M}).$$

Here I_C denotes the indicator function of the elasticity convex C [40]. It is worth noting that C represents also the effective domain of the function j^*, or $C = \operatorname{dom} j^*$. Moreover, we have

(4.6) $$j(\mathbf{e}) = (j^*)^*(\mathbf{e}), \qquad \mathbf{e} \in \mathbb{E}_s^2,$$

where

(4.7) $$j(\mathbf{e}) = \sup\{\mathbf{M} : \mathbf{e} - j^*(\mathbf{M}) \mid \mathbf{M} \in \mathbb{E}_s^2\} = \sup\{\mathbf{M} : \mathbf{e} - j^*(\mathbf{M}) \mid \mathbf{M} \in C\}, \ \mathbf{e} \in \mathbb{E}_s^2.$$

The function j is convex, lower semicontinuous and has the following property [19, 54]

(4.8) $$\exists \Lambda_0 \geq \lambda_0 > 0, \qquad \lambda_0(|\mathbf{e}| - 1) \leq j(\mathbf{e}) \leq \Lambda_0(1 + |\mathbf{e}|), \qquad \mathbf{e} \in \mathbb{E}_s^2.$$

Moreover, $j(\mathbf{0}) = 0$ and $j(\mathbf{e}) \geq 0$ for each $\mathbf{e} \in \mathbb{E}_s^2$.

The constitutive equation describing the elasto-plastic plate, within Hencky–Nadai–Ilushin model, has the following subdifferential form:

(4.9) $$\boldsymbol{\kappa} \in \partial j^*(\mathbf{M}) = (A_{\alpha\beta\lambda\mu}M_{\lambda\mu}) + \partial I_C(\mathbf{M}).$$

Thus we have

(4.10) $$\boldsymbol{\kappa} = \boldsymbol{\kappa}^e + \boldsymbol{\kappa}^p,$$

where

(4.11) $$\kappa_{\alpha\beta}^e = A_{\alpha\beta\lambda\mu}M_{\lambda\mu}, \qquad \boldsymbol{\kappa}^p \in \partial I_C(\mathbf{M}).$$

The recession function j_d is now the support function of the elasticity convex C and is calculated according to

$$j_d(\mathbf{e}) = \lim_{\rho \to 0^+} \rho j\left(\frac{\mathbf{e}}{\rho}\right) \doteq \sup\{\mathbf{M} : \mathbf{e} \mid \mathbf{M} \in C\}, \qquad \mathbf{e} \in \mathbb{E}_s^2.$$

The function j_d is convex, lower semicontinuous and positively homogeneous on $\mathbb{E}_s^2$; moreover it satisfies the following condition

(4.13) $$\exists \Lambda_0' \geq \lambda_0' > 0, \qquad \lambda_0'|\mathbf{e}| \leq j_d(\mathbf{e}) \leq \Lambda_0'(1 + |\mathbf{e}|), \qquad \mathbf{e} \in \mathbb{E}_s^2.$$

In the case of limit analysis j_d is the density of plastic dissipation. For nonhomogeneous materials the functions j, j^* and j_d depend explicitly on $x \in \Omega$. However, nonhomogeneities are not essential for considerations of the present and next sections. Nevertheless it should be noted that developments of these two sections remain valid in such a more general context. Both the function j, given by (4.6), and the function j_d are finite and convex, hence they are continuous [40]. Their growth is only linear, that is they are sublinear functions. The last property leads naturally to assuming the space $HB(\Omega)$ as a space of kinematical fields.

EXAMPLE 4.1. Let the elasticity convex of plastically admissible moments be given by

$$C = \{\mathbf{M} = (M_{\alpha\beta}) \mid a(\mathbf{M}) \le k\}. \tag{4.14}$$

Here the function a is not necessarily isotropic. If k depends on $x \in \Omega$ then obviously also C does. In such a rather general case it seems impossible to find explicit forms of the functions j and j_d. Especially complicated is the problem of finding j. Some particular examples are given by Demengel for isotropic plates [19, thèse].

If the function a has the following form

$$a(\mathbf{M}) = \frac{1}{2} F_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu}, \tag{4.15}$$

then

$$j_d(\mathbf{e}) = (2G_{\alpha\beta\lambda\mu} e_{\alpha\beta} e_{\lambda\mu} k)^{1/2}, \qquad \mathbf{G} = \mathbf{F}^{-1}, \tag{4.16}$$

provided that $\mathbf{F}$ is positive definite.

For more information on mechanical aspects of plastic plates, see Refs. [34, 42, 43].

Let $\Gamma = \partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and assume that the measure (length) of Γ_0 is strictly positive. The bar over a set denotes its closure and $\emptyset$ is the empty set. Along the part Γ_0 of the boundary kinematical conditions will be imposed. By B the transverse loading of the plate is denoted. We assume that $B \in \mathbf{M}(\Omega) = [C(\overline{\Omega})]^*$. It means that concentrated forces are admissible. One observes that the embedding $HB(\Omega) \subset C(\overline{\Omega})$ is continuous, provided that Ω has piecewise uniform C^2-regularity property [21, Remarque 3.2]. For instance, such an imbedding holds for rectangular plates. Hence we conclude that the linear form

$$L_1(u) = \langle B, u \rangle_{\mathbf{M}(\Omega)\times C(\overline{\Omega})}, \tag{4.17}$$

is continuous on $HB(\Omega)$, thus it is also weakly continuous. The functional L of the total loading of the plate is assumed to be given by

$$L(u) = L_1(u) + \int_{\Gamma_1} Q^0 u \, d\Gamma - \int_{\Gamma_1} M_n^0 \frac{\partial u}{\partial \mathbf{n}} d\Gamma = L_2(u) - \int_{\Gamma_1} M_n^0 \frac{\partial u}{\partial \mathbf{n}} d\Gamma. \tag{4.18}$$

To conform to notations used in structural mechanics we prefer to employ the notation $\partial u/\partial\mathbf{n}$ instead of $\gamma_1 u$ in the loading functional L. The largest domain $D(L)$ of L is the space $HB(\Omega)$; particularly, L is well defined for $u \in W^{2,1}(\Omega \subset HB(\Omega)$. Here $Q^0 \in L^\infty(\Gamma_1)$, $M_n^0 \in L^\infty(\Gamma_1)$ and Q_0, M_n^0 are prescribed shearing forces and bending moments, respectively. Unfortunately, the functional L is no longer weakly continuous on $HB(\Omega)$, except the case when $M_n^0 \equiv 0$.

Let us pass to the formulation of dual extremum principles. To this end we introduce the set K_0 of kinematically admissible fields

$$K_0 = \{u \in W^{2,1}(\Omega) \mid \gamma_0 u = 0, \gamma_1 u = 0, \text{ on } \Gamma_0\}. \tag{4.19}$$

For a given load multiplier λ the kinematical principle is formulated as follows.

PROBLEM P_λ

Find

$$\inf\left\{ \int_\Omega j[\boldsymbol{\kappa}(u)]\,dx - \lambda L(u) \mid u \in K_0 \right\}.$$

The dual problem can be derived by using Rockafellar's theory of duality [19, 54].

PROBLEM P_λ^*

Find

$$\sup\left\{-\int_\Omega j^*(\mathbf{M})\,dx \mid \mathbf{M}\in S_\lambda\right\},$$

where

$$S_\lambda = \{\mathbf{M}\in L^\infty(\Omega,\mathbf{E}_s^2) \mid M_{\alpha\beta,\beta\alpha}+\lambda B=0, \quad \text{in } \Omega; \\ Q=\lambda Q^0, \quad M_n=\lambda M_n^0, \quad \text{on } \Gamma_1; \quad \mathbf{M}(x)\in C, \quad \text{a.e. } x\in\Omega\}. \tag{4.20}$$

Here Q is the effective shear force. Strictly speaking, instead of Q and M_n we should use trace operators, say b_0 and b_1. In general we have $b_0(\mathbf{M})\in[\gamma_0(W^{2,1}(\Omega))]^*$, $b_1(\mathbf{M})\in L^\infty(\Gamma_1)$, provided that $\mathbf{M}\in S(\Omega)=\{\mathbf{M}\in L^\infty(\Omega,\mathbf{E}_s^2), M_{\alpha\beta,\beta\alpha}\in L^1(\Omega)\}$, see [19, 54].

For $\mathbf{M}\in C^2(\Omega,\mathbf{E}_s^2)$ we have

$$Q = M_{\alpha\beta,\beta}n_\alpha + \frac{\partial}{\partial s}(M_{\alpha\beta}n_\beta\tau_\alpha), \tag{4.21}$$

$$M_n = M_{\alpha\beta}n_\alpha n_\beta. \tag{4.22}$$

In this formulae $\mathbf{n}$ is the outward normal unit vector to $\Gamma=\partial\Omega$, $\boldsymbol{\tau}$ is the tangent unit vector and s denotes a curvi-linear abscissa on Γ measured positively in the direction of $\boldsymbol{\tau}$. Obviously, the load parameter λ cannot be arbitrary. It is limited by the limit load multiplier $\overline{\lambda}=\inf P_{LA}$.

We pass now to limit analysis problems.

PROBLEM P_{LA}

Find

$$\inf\left\{\int_\Omega j_d[\boldsymbol{\kappa}(v)]\,dx \mid v\in K_0, L(v)=1\right\}.$$

PROBLEM P_{LA}^*

Find

$$\sup\{\lambda(\mathbf{M}) \mid \mathbf{M}\in S_\lambda\}.$$

We recall that a field u occuring in the problem P_λ is the displacement field, while for the problem $P_{LA}\,v$ is a velocity field.

The above problems are essential for developments which follows. The next lemma provides deeper insight into some interrelations between them, cf. Ref. [19–22, 54].

LEMMA 4.1. The following conditions are equivalent:

(i) $\inf P_\lambda = \sup P_\lambda^* > -\infty$.

(ii) $S_\lambda \neq \emptyset$ ($\emptyset$—empty set).

(iii) $\overline{\lambda}=\inf P_{LA}\geq\lambda$.

5. Relaxation

In perfectly plastic solids and structures like plates discontinuities are often present [34, 42, 43]. The presence of kinematical discontinuities implies that the space $W^{2,1}(\Omega)$ is too small to incorporate them. Hence the need for the space $HB(\Omega)$ to which a minimizer of the kinematical problem belongs, cf. [19–22, 54]. Further, the weak convergence of a sequence $\{u_n\} \subset HB(\Omega)$ to $u \in HB(\Omega)$ does not necessarily imply the strong convergence of $\gamma_1 u_n$ to $\gamma_1 u$ in $L^1(\Gamma)$. An additional condition is required to ensure such convergence [22].

A relaxation of some boundary conditions turns out to be a convenient method in the mathematical treatement of boundary value problems of perfect plasticity, cf. Refs. [19–22, 44, 46, 54].

Let us set

$$\widehat{L}\left(u, \left(\frac{\partial u}{\partial \mathbf{n}}\right)^{+}\right) = L_1(u) + \int_{\Gamma_1} Q^0 u \, d\Gamma - \int_{\Gamma_1} \left(\frac{\partial u}{\partial \mathbf{n}}\right)^{+} M_n^0 \, d\Gamma , \tag{5.1}$$

where the functions u and $\left(\frac{\partial u}{\partial \mathbf{n}}\right)^{+}$ are *two independent arguments* of the loading functional $\widehat{L}$. The function $\left(\frac{\partial u}{\partial \mathbf{n}}\right)^{+}$ may be thought of as the external trace $(\gamma_1 u)^+$ on Γ_1, provided that $u \in HB(\Omega)$. Physically, the kinematic field on which the moment M_n^0 acts is treated as an independent field. The domain of the functional $\widehat{L}$ is the space $W^{2,1}(\Omega) \times L^1(\Gamma_1)$. In the next section we shall also have $D(\widehat{L}) = HB(\Omega) \times \mathbf{M}^1(\Gamma_1)$, provided that $M_n^0 \in C_0(\Gamma_1)$. The symbol $(\cdot)^+$ should not be confused with the positive part of a function.

Now we pass to the formulation of the relaxed problems.

PROBLEM RP_λ

Find

$$\inf\left\{ \int_\Omega j[\kappa(u)]dx + \int_{\Gamma_0} j_d[\underline{\mathcal{F}}(-\gamma_1 u)]d\Gamma + \int_{\Gamma_1} j_d\left[\underline{\mathcal{F}}\left(\left(\frac{\partial u}{\partial \mathbf{n}}\right)^{+} - \gamma_1 u\right)\right]d\Gamma \right.$$
$$\left. -\lambda \widehat{L}\left(u, \left(\frac{\partial u}{\partial \mathbf{n}}\right)^{+}\right) \;\middle|\; u \in W^{2,1}(\Omega), \left(\frac{\partial u}{\partial \mathbf{n}}\right)^{+} \in L^1(\Gamma_1), \gamma_0 u = 0, \quad \text{on } \Gamma_0 \right\},$$

where

$$\mathcal{F}_{\alpha\beta}(p) = p n_\alpha n_\beta . \tag{5.2}$$

PROBLEM RP_{LA}

Find

$$\inf\left\{ \int_\Omega j_d[\kappa(v)]dx + \int_{\Gamma_0} j_d[\underline{\mathcal{F}}(-\gamma_1 v)d\Gamma \right.$$
$$+ \int_{\Gamma_1} j_d\left[\underline{\mathcal{F}}\left(\left(\frac{\partial v}{\partial \mathbf{n}}\right)^{+} - \gamma_1 v\right)\right]d\Gamma \;\middle|\; v \in W^{2,1}(\Omega),$$
$$\left. \left(\frac{\partial v}{\partial \mathbf{n}}\right)^{+} \in L^1(\Gamma_1), \gamma_0 v = 0 \quad \text{on } \Gamma_0, \widehat{L}\left[v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^{+}\right] = 1 \right\},$$

We shall prove that

$$P_{LA}^* = RP_{LA}^* . \tag{5.3}$$

To this end the theory of duality presented in [28] will be employed. Let

$$\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) : \left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^{+}\right) \to \left[\kappa(v), \gamma_1 v_{|\Gamma_0}, \left(\left(\frac{\partial v}{\partial \mathbf{n}}\right)^{+} - \gamma_1 v\right)_{|\Gamma_1}\right],$$

$$V_1 = \{v \in W^{2,1}(\Omega) \mid v = 0, \quad \text{on } \Gamma_0\}, \qquad V = V_1 \times L^1(\Gamma_1),$$
$$\Lambda : V \to Y = L^1(\Omega, \mathsf{E}_s^2) \times L^1(\Gamma_0) \times L^1(\Gamma_1), \qquad \mathbf{p} = (\mathbf{p}_1, p_2, p_3) \in Y,$$
$$\mathbf{p}^* \in Y^* = L^\infty(\Omega, \mathsf{E}_s^2) \times L^\infty(\Gamma_0) \times L^\infty(\Gamma_1), \qquad G(\mathbf{p}) = G_1(\mathbf{p}_1) + G_2(p_2) + G_3(p_3),$$

where

$$G_1(\mathbf{p}_1) = \int_\Omega j_d(\mathbf{p}_1)dx, \qquad G_2(p_2) = \int_{\Gamma_0} j_d[\underline{\mathcal{F}}(-p_2)]d\Gamma,$$
$$G_3(p_3) = \int_{\Gamma_1} j_d[\underline{\mathcal{F}}(p_3)]d\Gamma.$$

We set

$$F\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) = \begin{cases} 0, & \text{if } \quad \widehat{L}\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) = 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Under the above notations the problem RP_{LA} means finding

$$\inf\left\{G\left[\Lambda\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right)\right] + F\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) \;\middle|\; \left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) \in V\right\}.$$

The dual problem has the following form [28]:

$$(RP_{LA}^*) \qquad \sup\{-G^*(\mathbf{p}^*) - F^*(-\Lambda^*(\mathbf{p}^*) \mid \mathbf{p}^* \in Y^*\}.$$

Because the dissipation density j_d is a support function, hence

$$(5.4) \qquad G_1^*(\mathbf{p}_1^*) = \begin{cases} 0, & \text{if} \quad \mathbf{p}_1^*(x) \in C \quad \text{for a.e.} \quad x \in \Omega, \\ +\infty, & \text{otherwise.} \end{cases}$$

By using DEMENGEL'S results [19], we have

$$(5.5) \qquad G_2^*(p_2^*) = \begin{cases} 0, & \text{if} \quad \mathbf{p}_1^* \in S_\lambda,\ b_1(\mathbf{p}_1^*) = -p_2^*, \text{ on } \Gamma_0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Similarly one obtains

$$(5.6) \qquad G_3^*(p_3^*) = \begin{cases} 0, & \text{if} \quad \mathbf{p}_1^* \in S_\lambda,\ b_1(\mathbf{p}_1^*) = p_3^*, \text{ on } \Gamma_1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Setting $\mathbf{M} = \mathbf{p}_1^*$ one has $b_1(\mathbf{M}) = M_n = M_{\alpha\beta}n_\alpha n_\beta$ (on the boundary Γ), provided that $\mathbf{M}$ is sufficiently regular, for instance $\mathbf{M} \in C^2(\Omega, \mathsf{E}_s^2)$. It remains to determine $F^*(-\Lambda^*\mathbf{p}^*)$. We have

$$(5.7) \quad F^*(-\Lambda^*\mathbf{p}^*) = \sup\left\{\left\langle -\Lambda^*\mathbf{p}^*, \left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right)\right\rangle_{V^*\times V} - F\left[v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right] \;\middle|\; \left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) \in V\right\}$$
$$= \sup\left\{\left\langle -\mathbf{p}^*, \Lambda\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right)\right\rangle_{Y^*\times Y} \;\middle|\; \widehat{L}\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) = 1, \left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) \in V\right\}$$

$$(5.7)\ _{[\text{cont.}]} \quad = \sup\left\{ - \int_{\Omega} p^*_{1\alpha\beta}\kappa_{\alpha\beta}(v)\,dx - \int_{\Gamma_0} p^*_2\gamma_1 v\,d\Gamma - \int_{\Gamma_1} p^*_3\left[\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ - \gamma_1 v\right] d\Gamma \;\middle|\; \widehat{L}\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) = 1, \left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) \in V \right\}.$$

Let $\left(v_0, \left(\frac{\partial v_0}{\partial \mathbf{n}}\right)^+\right) \in V$ be such that $\widehat{L}\left[v_0, \left(\frac{\partial v_0}{\partial \mathbf{n}}\right)^+\right] = 1$. For an arbitrary $\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) \in V$ we set

$$(5.8) \qquad \left(\tilde{v}, \left(\frac{\partial \tilde{v}}{\partial \mathbf{n}}\right)^+\right) = \left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) + \left[1 - \widehat{L}\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right)\right]\left(v_0, \left(\frac{\partial v_0}{\partial \mathbf{n}}\right)^+\right).$$

Hence it follows that $\left(\tilde{v}, \left(\frac{\partial \tilde{v}}{\partial \mathbf{n}}\right)^+\right) \in V$ and $\widehat{L}\left(\tilde{v}, \left(\frac{\partial \tilde{v}}{\partial \mathbf{n}}\right)^+\right) = 1$. Taking a function $\left(\tilde{v}, \left(\frac{\partial \tilde{v}}{\partial \mathbf{n}}\right)^+\right)$ in (5.7) instead of $\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right)$ we get

$$(5.9) \quad F^*(-\Lambda^*\mathbf{p}^*) = \sup\left\{ - \int_{\Omega} p^*_{1\alpha\beta}\kappa_{\alpha\beta}(v)\,dx - \int_{\Gamma_0} p^*_2\gamma_1 v\,d\Gamma - \int_{\Gamma_1} p^*_3\left[\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ - \gamma_1 v\right] d\Gamma + \lambda\left[1 - \widehat{L}\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right)\right] \;\middle|\; \left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) \in V \right\},$$

where

$$\lambda = \int_{\Omega} p^*_{1\alpha\beta}\kappa_{\alpha\beta}(v_0)\,dx + \int_{\Gamma_0} p^*_2\gamma_1 v_0\,d\Gamma + \int_{\Gamma_1} \left(\frac{\partial v_0}{\partial \mathbf{n}}\right)^+ p^*_3\,d\Gamma.$$

To calculate the supremum in (5.7) we first take $\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ = 0$ and $v \in \mathcal{D}(\Omega)$. Then we readily obtain

$$(5.10) \quad F^*(-\Lambda^*\mathbf{p}^*) \geq \lambda + \sup\left\{ \int_{\Omega} [-p^*_{1\alpha\beta}\kappa_{\alpha\beta}(v) - \lambda B v]\,dx \mid v \in \mathcal{D}(\Omega)\right\}$$
$$= \begin{cases} \lambda, & \text{if} \quad p^*_{1\alpha\beta,\beta\alpha} + \lambda B = 0, \text{ in } \Omega, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us return to (5.9). Integrating by parts one obtains

$$F^*(-\Lambda^*p^*) = \lambda + \sup\left\{ - \int_{\Omega} (p^*_{1\alpha\beta,\beta\alpha}v + \lambda B v)\,dx + \int_{\Gamma_0} [-p^*_2 - b_1(p^*_1)]\gamma_1 v\,d\Gamma + \int_{\Gamma_1} [b_0(\mathbf{p}^*_1) - \lambda Q^0]\gamma_0 v\,d\Gamma + \int_{\Gamma_1} [-b_1(p^*_1) + p^*_3]\gamma_1 v\,d\Gamma + \int_{\Gamma_1} \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ (-p^*_3 + \lambda M^0_n)\,d\Gamma \;\middle|\; \left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) \in V \right\}.$$

Taking into account of (5.10) and knowing that v and $\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+$ are independent functions one finally arrives at the following relation:

(5.11) $F^*(-\Lambda^*\mathbf{p}^*)$

$$= \begin{cases} \lambda, & \text{if } p^*_{1\alpha\beta,\beta\alpha} + \lambda B = 0, \quad \text{in } \Omega; \quad b_1(\mathbf{p}^*_1) = -p^*_2, \quad \text{on } \Gamma_0; \\ & \text{and} \quad b_0(\mathbf{p}^*_1) = \lambda Q^0, \quad p^*_3 = b_1(p^*_1) = \lambda M^0_n, \quad \text{on } \Gamma_1; \\ +\infty, & \text{otherwise.} \end{cases}$$

By taking now account of (5.4)–(5.6) and (5.11) in the general form of the problem RP^*_{LA} and setting $\mathbf{p}^*_1 = \mathbf{M}$, we deduce that $P^*_{LA} = RP^*_{LA}$. Thus by employing the duality theorem [28] we write

$$\inf P_{LA} = \sup P^*_{LA} = \sup RP^*_{LA} = \inf RP_{LA}\,. \tag{5.12}$$

Similar considerations can be carried out for the problem RP_λ. Thus one can show that $RP^*_\lambda = P^*_\lambda$ and $\inf P_\lambda = \sup P^*_\lambda = \sup RP^*_\lambda = \inf RP^*_\lambda$.

Let us find now the infimum over $\left(\dfrac{\partial v}{\partial \mathbf{n}}\right)^+$ in the problem RP_{LA}. To this end for a fixed v one has to investigate the following minimization problem

$$(P_v) \quad \inf\left\{ \int_{\Gamma_1} j_d\left[\underline{\mathcal{F}}\left(\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ - \gamma_1 v\right)\right] d\Gamma \;\middle|\; \hat{L}\left[v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right] = 1, \quad \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ \in L^1(\Gamma_1)\right\}.$$

By applying the theory of duality used previously we shall formulate the dual problem P^*_v. We take $\Lambda\left(\dfrac{\partial v}{\partial \mathbf{n}}\right)^+ = \left(\dfrac{\partial v}{\partial \mathbf{n}}\right)^+$ and set

$$G(p) = \int_{\Gamma_1} j_d[\underline{\mathcal{F}}(p - \gamma_1 v)]\,d\Gamma\,,$$

$$F\left[\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right] = \begin{cases} 0, & \text{if } \hat{L}\left[v, \left(\dfrac{\partial v}{\partial \mathbf{n}}\right)^+\right] = 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us find the conjugate functions. We have

$$G^*(p^*) = \sup\left\{ \langle p^*, p\rangle_{L^\infty(\Gamma_1)\times L^1(\Gamma_1)} - \int_{\Gamma_1} j_d[\underline{\mathcal{F}}(p - \gamma_1 v)]\,d\Gamma \;\middle|\; p \in L^1(\Gamma_1)\right\} \tag{5.13}$$

$$= \begin{cases} \langle p^*, \gamma_1 v\rangle_{L^\infty(\Gamma_1)\times L^1(\Gamma_1)}, & \text{if } p^*(s) \in C_b(s) \quad \text{for a.e.} \quad s \in \Gamma_1, \\ +\infty, & \text{otherwise,} \end{cases}$$

where

$$C_b(s) = C \cdot \mathbf{n}(s) \cdot \mathbf{n}(s) \overset{\text{df}}{=} \{z \mid \exists \mathbf{M} \in C, M_n(s) = z\}\,. \tag{5.14}$$

It should be noted that the set C_b of plastically admissible bending moments along Γ_1 depends, in general, on $s \in \Gamma_1$, because the normal unit vector $\mathbf{n}$ does.

Further one has

$$F^*(-\Lambda^* p^*) = \sup\left\{\left\langle -p^*, \Lambda\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right\rangle_{L^\infty(\Gamma_1)\times L^1(\Gamma_1)} \;\middle|\; \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ \in L^1(\Gamma_1),\right. \tag{5.15}$$

$$\left. \hat{L}\left[v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right] = 1\right\}$$

$$= \sup\left\{\left\langle -p^*, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right\rangle_{L^\infty(\Gamma_1)\times L^1(\Gamma_1)} + \varrho\left[1 - L_2(v) + \int_{\Gamma_1}\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ M_n^0 d\Gamma\right] \;\middle|\; \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ \in L^1(\Gamma_1)\right\} = \begin{cases} \varrho(1 - L_2(v)), & \text{if } p^* = \varrho M_n^0, \\ +\infty, & \text{otherwise.}\end{cases} \tag{5.15 [cont.]}$$

By taking account of (5.13) and (5.15) we obtain

$$(P_v^*) \quad \sup\{\langle -p^*, \gamma_1 v\rangle_{L^\infty(\Gamma_1)\times L^1(\Gamma_1)} - \varrho(1 - L_2(v)) \mid p^* \in L^\infty(\Gamma_1), p^*(s) \in C_b(s), p^* = \varrho M_n^0\} = \lambda_b|1 - L(v)|,$$

where

$$\lambda_b = \sup\{\varrho \in \mathbf{R} \mid \varrho M_n^0(s) \in C_b(s), \quad \text{a.e.} \quad s \in \Gamma_1\}. \tag{5.16}$$

6. Homogenization

6.1 Periodic structure of the plate and some general results

In the paper [51] results concerning homogenization of perfectly plastic plates were reported provided that $M_n^0 \equiv 0$. Prior to dealing with a plate loaded by boundary bending moments, the general theorem concerning the epi-limit on the space $HB(\Omega)$ will be given. The complementary homogenized potential will also be derived.

Let a perfectly plastic plate be periodically heterogeneous. Heterogeneities enter by means of a periodicity of the function j, or j_d in the case of limit analysis. In other words the function $j(y, \kappa)$ is Y-periodic with respect to y, cf. Refs [1–3, 8, 9, 15, 23, 24, 35, 37, 44–48, 50–53, 55]. Here, as usually in the homogenization theory, Y is the so-called basic cell. Now it is two-dimensional, for instance $Y = (0, y_1^0) \times (0, y_2^0)$. It means that the moduli $A_{\alpha\beta\lambda\mu}$ and the elasticity convex C are Y-periodic. The plate itself has εY-periodic structure defined by $A^\varepsilon_{\alpha\beta\lambda\mu}(x) = A_{\alpha\beta\lambda\mu}(x/\varepsilon)$ and $C^\varepsilon(x) = C(x/\varepsilon)$, where $\varepsilon > 0$ is a small parameter and $x \in \Omega$. For instance, the periodic elasticity convex corresponding to the relation (4.14) is given by

$$C^\varepsilon(x) = \{\mathbf{M} \in \mathbf{E}_s^2 \mid a(\mathbf{M}) \le k^\varepsilon(x), x \in \Omega\}. \tag{6.1}$$

Now the yield limit $k^\varepsilon(x) = k(x/\varepsilon)$ varies εY-periodically over Ω.

Before passing in the next subsection to the homogenization of perfectly plastic plates we shall first formulate a general homogenization theorem for a sequence of functionals with linear growth and the epi-limit on the space $HB(\Omega)$.

THEOREM 6.1. *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain possessing the uniform C^2-regularity property, except possibly at a finite number of points of $\Gamma = \partial\Omega$. Let $j : (y, \mathbf{e}) \in \mathbf{R}^2 \times \mathbf{E}_s^2 \to j(y, \mathbf{e}) \in \mathbf{R}$ be a measurable function, convex in $\mathbf{e}$, Y-periodic in y and such that*

$$\exists \Lambda_0 \ge \lambda_0 > 0, \quad \exists k_0 > 0, \quad \lambda_0|\mathbf{e}| - k_0 \le j(y, \mathbf{e}) \le \Lambda_0(1 + |\mathbf{e}|),$$

for each $\mathbf{e} \in \mathbf{E}_s^2$. For each $\varepsilon > 0$ we define the functional F^ε on $W^{1,1}(\Omega)$ by

$$F^\varepsilon(u) = \begin{cases} \int_\Omega j\left(\frac{x}{\varepsilon}\right), \kappa(u(x))dx, & \textit{if} \quad u \in W^{2,1}(\Omega), \\ +\infty, & \textit{if} \quad u \in W^{1,1}(\Omega) \setminus W^{2,1}(\Omega). \end{cases} \tag{6.2}$$

Then $W^{1,1}(\Omega) - \lim_e F^\varepsilon = F^h$, *where*

$$F^h(u) = \begin{cases} \int\limits_\Omega j^h(\kappa(u)), & \text{if} \quad u \in HB(\Omega), \\ +\infty, & \text{if} \quad u \in W^{1,1}(\Omega) \setminus HB(\Omega), \end{cases} \tag{6.3}$$

and

$$j^h(\mathbf{e}) = \inf\left\{ \frac{1}{|Y|} \int\limits_Y j(y, \kappa_y(w) + \mathbf{e}) dy \mid w \in W_{\text{per}} \right\}, \quad \mathbf{e} \in \mathbf{E}_s^2. \tag{6.4}$$

Here

$$W_{\text{per}} = \left\{ w \in W^{2,1}(Y) \;\middle|\; w \text{ and } \frac{\partial w}{\partial y_\alpha} \text{ have equal traces at the opposite sides of } Y \right\}, \tag{6.5}$$

and

$$\kappa_{y\alpha\beta}(w) = \frac{\partial^2 w}{\partial y_\beta \partial y_\alpha}.$$

■

The proof of the above theorem is lengthy and is given in the paper [53]. In principle it parallels that invented by BOUCHITTÉ [8, 9] for the epi-limit on the space BV of functions with bounded variation [32]. It is worth noting that Bouchitté's proof exploits the density of piecewise affine continuous functions in the space $W^{1,1}$. In the paper [53] the following results due to DESCLOUX [26] is used.

LEMMA 6.1. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with Lipschitzian boundary, $1 \le p < +\infty$, and $f \in W^{2,p}(\Omega)$. There exists a sequence of functions $f_n \in C^1(\Omega)$ with piecewise constant second derivatives and such that

$$\lim_{n \to 0} \|f - f_n\|_{W^{2,p}(\Omega)} = 0.$$

■

Surprisingly, but this lemma seems to have been unknown till now. It is of interest not only for homogenization problem of plates. This result can find wider applications in numerical analysis of some two-dimensional problems, for instance in finite element methods. Its generalization to dimensions higher than two is not known.

In order to find the function $(j_d)^h$ we simply replace the potential j by its recession function j_d in Th. 6.1. In fact, deeper result holds true

$$j_d^h := (j^h)_d = (j_d)^h. \tag{6.6}$$

Prior to proving that $(j^h)_d = (j_d)^h$, we shall determine the polar function or the homogenized complementary potential $(j^h)^*$. To this end by $\boldsymbol{\nu} = (\nu_\alpha)$ and $\mathbf{t} = (t_\alpha)$ we denote the outer unit normal and tangent vectors to ∂Y, respectively. The following formula for the complementary homogenized potential will be derived

$$(j^h)^*(\mathbf{e}^*) = \inf\left\{ \frac{1}{|Y|} \int\limits_Y j^*(y, \mathbf{m}(y) + \mathbf{e}^*) \, dy \mid \mathbf{m} \in \mathfrak{M}_{\text{per}} \right\}, \quad \mathbf{e}^* \in \mathbf{E}_s^2, \tag{6.7}$$

where

(6.8) $$\mathfrak{M}_{\text{per}} = \{\mathbf{m} \in L^\infty(Y, \mathsf{E}^2_s) \mid \operatorname{div}_y \operatorname{div}_y \mathbf{m} = 0, \quad \text{in } Y : m_\nu \text{ takes equal and } q \text{ opposite values at opposite sides of } Y; \sum_{i=1}^{4} \mathbf{m}_{12}(O_i) = 0, \quad \int_Y \mathbf{m}(y)dy = 0\},$$

and

(6.9) $$m_\nu = m_{\alpha\beta}\nu_\alpha\nu_\beta, \quad q = \nu_\alpha \frac{\partial m_{\alpha\beta}}{\partial y_\beta} + \frac{\partial m_t}{\partial \xi}, \quad m_t = m_{\alpha\beta}\nu_\alpha t_\beta,\ O_i \quad (i = 1,2,3,4)$$

are vertices of the cell Y while ξ parametrizes ∂Y.

Let us pass to the proof. Before, however, we observe that $\mathbf{e}^*$ is the macroscopic moment tensor. Thus we may write $\mathbf{e}^* = \mathbf{M}^h$. According to the definition of the conjugate function we have [40]

(6.10) $$\begin{aligned}(j^h)^*(\mathbf{e}^*) &= \sup\{\mathbf{e}^* : \mathbf{e} - j^h(\mathbf{e}) \mid \mathbf{e} \in \mathsf{E}^2_s\} \\ &= \sup_{\mathbf{e}\in\mathsf{E}^2_s} \frac{1}{|Y|}\left\{ \int_Y \mathbf{e}^* : \mathbf{e} dy - \inf_{w\in W_{\text{per}}} \int_Y j(y, \kappa_y(w) + \mathbf{e})dy \right\},\end{aligned}$$

since

$$\int_Y \mathbf{e}^* : \kappa_y(w)dy = 0\,.$$

Setting

$$X = \kappa_y(W_{\text{per}}) \oplus \mathsf{E}^2_s, \quad J(\boldsymbol{\theta}) = \int_Y j(y, \boldsymbol{\theta}(y))dy\,,$$

we can rewrite (6.10) in the following way

$$(j^h)^*(\mathbf{e}^*) = \frac{1}{|Y|}(J + I_X)^*(\mathbf{e}^*)\,.$$

Here $\oplus$ denotes the direct sum of the involved spaces and I_X is the indicator function of the space X. The polar functional $(J + I_X)^*$ is expressed in the form of the inf-convolution [36, 40]

(6.11) $$(J + I_X)^*(\mathbf{e}^*) = (J^* \,\square\, I_{\mathfrak{M}_{\text{per}}})(\mathbf{e}^*)\,,$$

provided that $\mathbf{e}^*$ is identified with a constant element of the space $L^\infty(Y, \mathsf{E}^2_s)$. We have [28]

$$J^*(\mathbf{m}) = \int_Y j^*(y, \mathbf{m}(y))dy\,.$$

The set $\mathfrak{M}_{\text{per}}$ of microscopically admissible moments is found from the following relation

$$\mathfrak{M}_{\text{per}} = X^\perp = [\kappa_y(W_{\text{per}})]^\perp \cap (\mathsf{E}^2_s)^\perp\,.$$

We have

$$(\mathsf{E}^2_s)^\perp = \{\mathbf{m} \in L^\infty(Y, \mathsf{E}^2_s) \mid \langle \mathbf{m}, \mathbf{e}\rangle_{\infty,1} = 0, \quad \forall \mathbf{e} \in \mathsf{E}^2_s\}\,.$$

For the sake of simplicity we use the following notation

$$\langle\cdot,\cdot\rangle_{\infty,1} = \langle\cdot,\cdot\rangle_{L^\infty(Y,\mathsf{E}^2_s)\times L^1(Y,\mathsf{E}^2_s)}\,.$$

Simple calculation gives

(6.12) $$(\mathbb{E}_s^2)^\perp = \{\mathbf{m} \in L^\infty(Y, \mathbb{E}_s^2) \mid \int_Y \mathbf{m}(y)dy = 0\}.$$

One has to find now $[\boldsymbol{\kappa}_y(W_{\text{per}})]^\perp$. The definition yields

$$0 = \langle \mathbf{m}, \boldsymbol{\kappa}_y(w)\rangle_{\infty,1} = \int_Y m_{\alpha\beta}(y)\kappa_{y\alpha\beta}(w(y))dy, \ \forall w \in W_{\text{per}}.$$

Integrating twice by parts, at least formally, we arrive at the relation

(6.13) $$0 = \langle \mathbf{m}, \boldsymbol{\kappa}_y(w)\rangle_{\infty,1} = \int_Y m_{\alpha\beta,\beta\alpha} w dy - \int_{\partial Y} q\gamma_0 w d\xi + \int_{\partial Y} m_\nu \gamma_1 w d\xi + \sum_{i=1}^4 m_{12}(O_i)w(O_i),$$

for each $w \in W_{\text{per}}$. Here q (the effective local shear force) and m_ν (the local bending moments) are given by (6.9). Hence (6.8) follows. Further, from Eq. (6.11) we get

$$(j^h)^*(\mathbf{e}^*) = \frac{1}{|Y|}\inf\{J^*(\mathbf{m}_1) + I_{\mathfrak{M}_{\text{per}}}(\mathbf{m}_2) \mid \mathbf{e}^* = \mathbf{m}_1 + \mathbf{m}_2, \mathbf{m}_\alpha \in \mathfrak{M}_{\text{per}}\}$$
$$= \frac{1}{|Y|}\inf\{J^*(\mathbf{e}^* - \mathbf{m}_2) \mid \mathbf{m}_2 \in \mathfrak{M}_{\text{per}}\}.$$

The set $\mathfrak{M}_{\text{per}}$ is a linear space, therefore we can write

(6.14) $$(j^h)^*(\mathbf{M}^h) = \inf\left\{\frac{1}{|Y|}\int_Y j^*(y, \mathbf{m}(y) + \mathbf{M}^h)dy \mid \mathbf{m} \in \mathfrak{M}_{\text{per}}\right\}, \mathbf{M}^h \in \mathbb{E}_s^2.$$

Thus the relation (6.7) has been proved.

Finding the function $[(j_d)^h]^*$ similarly as $(j^h)^*$ we conclude that $\text{dom}[(j_d)^h]^* = \text{dom}(j^h)^*$. Physically, those effective domains represent the (convex) set of plastically admissible macroscopic moments or the homogenized elasticity convex. This set is denoted by C^h, in other words

(6.15) $$C^h = \text{dom}[(j_d)^h]^* = \text{dom}(j^h)^*.$$

Hence we obtain

(6.16) $$(j_d)^h(\mathbf{e}) = \sup\{\mathbf{e}^* : \mathbf{e} \mid \mathbf{e}^* \in \text{dom}[(j_d)^h]^*\}$$
$$= \sup\{\mathbf{e}^* : \mathbf{e} \mid \mathbf{e}^* \in \text{dom}(j^h)^*\} = (j^h)_d(\mathbf{e}),$$

what proves Eq. (6.6).

6.2. Homogenization of plate loaded on the boundary

Now we pass to the homogenization of perfectly plastic plates provided that their periodic structure is the same as that described in the previous subsection. The loading functional is given by (4.18). However, now we assume once and for all that $M_n^0 \in C_0(\Gamma_1)$. Why such a stronger assumption is needed will become evident from our subsequent considerations. At this moment we note that a generalization to the case $M_n^0 \in L^\infty(\Gamma_1)$ remains an open problem.

We start by the formulation of the relaxed limit analysis problem for a fixed $\varepsilon > 0$.

PROBLEM RP_{LA}^ε

Find

$$\lambda^\varepsilon = \inf\Big\{ \int_\Omega j_d\Big[\frac{x}{\varepsilon}, \kappa(v)\Big]dx + \int_{\Gamma_0} j_d\Big[\frac{x}{\varepsilon}, \underline{\mathcal{F}}(-\gamma_1 v)\Big]d\Gamma + \int_{\Gamma_1} j_d\Big[\frac{x}{\varepsilon}, \underline{\mathcal{F}}\Big(\Big(\frac{\partial v}{\partial \mathbf{n}}\Big)^+ - \gamma_1 v\Big)\Big]d\Gamma \;\Big|\; v \in V_1, \Big(\frac{\partial v}{\partial \mathbf{n}}\Big)^+ \in L^1(\Gamma_1), \quad \widehat{L}\Big[v, \Big(\frac{\partial v}{\partial \mathbf{n}}\Big)^+\Big] = 1 \Big\}.$$

By using the results obtained in Sect. 5 we can write

$$\lambda^\varepsilon = \inf\Big\{ \int_\Omega j_d\Big[\frac{x}{\varepsilon}, \kappa(v)\Big]dx + \int_{\Gamma_0} j_d\Big[\frac{x}{\varepsilon}, \underline{\mathcal{F}}(-\gamma_1 v)\Big]d\Gamma + \lambda_b^\varepsilon|1 - L(v)| \;|\; v \in V_1 \Big\}, \tag{6.17}$$

where, cf. the formula (5.16)

$$\lambda_b^\varepsilon = \{\varrho \in \mathbf{R} \mid \varrho M_n^0 \in C_b^\varepsilon(s), s \in \Gamma_1\}, \tag{6.18}$$

and

$$C_b^\varepsilon(s) = C^\varepsilon \cdot \mathbf{n}(s) \cdot \mathbf{n}(s) = \{z \mid \exists \mathbf{M} \in C^\varepsilon(x), \quad x \in \Omega, \quad M_{\alpha\beta,\beta\alpha} \in L^2(\Omega),\ Q \in L^\infty(\Gamma_1),\ M_n(s) = z,\ s \in \Gamma_1\}. \tag{6.19}$$

After [19, 54] we could assume $M_{\alpha\beta,\beta\alpha} \in L^1(\Omega)$ (or even $M_{\alpha\beta,\beta\alpha} \in \mathbf{M}^1(\Omega)$).

Denoting by λ^h the limit load multiplier of the homogenized plate we have

$$\lambda^h = \lim_{\varepsilon \to 0} \lambda^\varepsilon = \min QP_{LA}^h, \tag{6.20}$$

where

$$\inf\Big\{ \int_\Omega j_d^h[\kappa(v)] + \int_{\Gamma_0} j_d^h[\underline{\mathcal{F}}(-\gamma_1 v)]d\Gamma + \lambda_b^h|1 - L(v)| \;|\; v \in HB(\Omega), \quad \gamma_0 v = 0, \text{ on } \Gamma_0 \Big\}, \tag{QP_{LA}^h}$$

and

$$\lambda_b^h = \sup\{\varrho \in \mathbf{R} \mid \varrho M_n^0(s) \in C_b^h(s) \cap \Delta(s),\ \forall s \in \Gamma_1\}. \tag{6.21}$$

Here $C_b^h(s) = C^h \cdot \mathbf{n}(s) \cdot \mathbf{n}(s)$, cf. (5.14) and (6.15). We recall that $M_n^0 \in C_0(\Gamma_1)$. More delicate and only partially solved is the problem of determination of the set $\Delta(s)$. The results of BOUCHITTÉ [9, Chapter III, Th. 4.8] permit to determine this set provided that $C^\varepsilon(x)$ takes only two values: C_1 and C_2. Physically it means that a plate exhibiting the periodic structure is made of two materials. For one of them the set of plastically admissible moments is given by C_1, while for the second by C_2.

Let us now recall the relevant results of BOUCHITTÉ [9] presented here in a form suitable for our purposes, cf. also [13]. By $\mathfrak{O}$ we denote a compact (or locally compact) metric space. In applications $\mathfrak{O}$ is usually a domain or its closure. We take:

μ — a positive measure ($\mu \in \mathbf{M}^1_+(\mathfrak{O})$), $\operatorname{supp}\mu = \mathfrak{O}$;

$\{A_\varepsilon\}_{\varepsilon>0}$ — a family of Borel subsets of the set $\mathfrak{O}$ such that $\mu(\partial A_\varepsilon) = 0$;

C_1, C_2 — continuous multivalued mappings with values in closed and convex subsets of $\mathbf{R}^d$ having 0 as an interior point;

$$C^\varepsilon(x) = \begin{cases} C_1(x), & \text{if } x \in A_\varepsilon, \\ C_2(x), & \text{if } x \in \mathfrak{O} \setminus A_\varepsilon, \end{cases} \tag{6.22}$$

$$K^\varepsilon = \{\varphi \in C_0(\mathfrak{O}, \mathbf{R}^d) \mid \varphi(x) \in C^\varepsilon(x), \mu \text{ a.e. on } \mathfrak{O}\}. \tag{6.23}$$

Of interest is obviously the passage to the limit with ε ($\varepsilon \to 0$).

LEMMA 6.2 [9]. If $\operatorname{int} A_\varepsilon \to A$ and $\operatorname{int}(\mathfrak{O} \setminus A_\varepsilon) \to B$ when $\varepsilon \to 0$ then A and B are closed sets such that $A \cup B = \mathfrak{O}$. Moreover, the sequence $\{K^\varepsilon\}_{\varepsilon>0}$ is strongly convergent in Kuratowski's sense to a convex set K_1 given by

$$K_1 = \{\varphi \in C_0(\mathfrak{O}, \mathbf{R}^d) \mid \varphi(x) \in \Delta(x), \ \forall x \in \mathfrak{O}\}, \tag{6.24}$$

where Δ is a l.s.c. multivalued mapping and

$$\Delta(x) = \begin{cases} C_1(x) \cap C(x), & \text{if} \quad x \in A \cap B, \\ C_1(x), & \text{if} \quad x \in A \setminus B, \\ C_2(x), & \text{if} \quad x \in B \setminus A. \end{cases} \tag{6.25}$$

■

REMARK 6.1. The convergence in Kuratowski's sense and its relation to homogenization problems are discussed by ATTOUCH [1]. More details on multivalued mappings the reader will find in the book by AUBIN and FRANKOWSKA [4].

The next lemma, practically very important, gives sufficient conditions ensuring that $A = B = \mathfrak{O}$ [9]. By $\chi^\varepsilon = \chi_{A_\varepsilon}$ we denote the characteristic function of A_ε.

LEMMA 6.3. If the sequence $\{\chi^\varepsilon\}_{\varepsilon>0}$ converges in the topology $\sigma(L^\infty_\mu, L^1_\mu)$ to a function $\theta(x)$ such that $0 < \|\theta\|_{L^\infty_\mu} < 1$ then the sets $\operatorname{int} A_\varepsilon$ and $\operatorname{int}(\mathfrak{O} \setminus A_\varepsilon)$ converge to $\mathfrak{O}$. Preserving the notations of the previous lemma we then have $A = B = \mathfrak{O}$ and

$$\Delta(x) = C_1(x) \cap C_2(x), \ \forall x \in \mathfrak{O}.$$

■

EXAMPLE 6.1. Let us investigate a simple example illustrating the last lemma. Suppose that $\mathfrak{O} = \Gamma_1$ is an interval $[a, b]$, see Fig. 1. Such a plate has the form of a wafer, with layers perpendicular to the mid-plane. The basic cell Y has the following form: $Y = (0,1) = (0, a_1] \cup (a_1, a_2] \cup (a_2, 1)$, where $0 < a_1 < a_2 < 1$. The characteristic function of the set (interval) $A_2 = (a_1, a_2]$ is defined by

$$\chi_{A_2}(y) = \begin{cases} 1, & \text{if} \quad a_1 < y \le a_2, \\ 0, & \text{if} \quad y \in Y \setminus A_2. \end{cases} \tag{6.26}$$

The set A_ε is a sum of intervals $A_2^{\varepsilon,i} = (a_1^\varepsilon, a_2^\varepsilon]_i$, hence $A_\varepsilon = \bigcup_{i \in I(\varepsilon)} A_2^{\varepsilon,i}$. For a fixed $\varepsilon > 0$ the set A_ε (or rather its closure) represents that part of Γ_1 which is occupied by the material ②. The sequence of characteristic functions χ^ε converges to $(a_2 - a_1)$

in the weak-$*$ topology of the space $L^\infty(\Gamma_1)$, cf. Ref. [18, p. 21]. Thus in that case $\theta(x) = a_2 - a_1$, $0 < a_2 - a_1 < 1$. Now we have $\mu = ds$ and $A = B = \Gamma_1$; moreover

$$\Delta(s) = [C_1 \cdot \mathbf{n}(s) \cdot \mathbf{n}(s)] \cap [C_2 \cdot \mathbf{n}(s) \cdot \mathbf{n}(s)], \ \forall s \in A \cap B = \Gamma_1 ,$$

where $\mathbf{n}(s) = (1,0)$. Hence we conclude that the set Δ does not depend on $s \in \Gamma_1$. We recall that $C_1(C_2)$ denotes the set of plastically admissible states for the material ① (②). ■

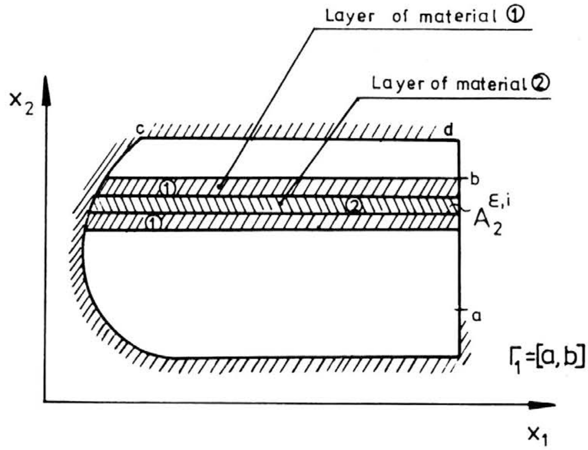

FIG. 1.

Let us return to the limit analysis problem RP^ε_{LA}. We set

$$(6.27) \qquad J^\varepsilon_{\Gamma_1}\left[\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ - \gamma_1 v\right] = \int\limits_{\Gamma_1} j_d\left[\frac{x}{\varepsilon}, \underline{\mathcal{F}}\left(\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ - \gamma_1 v\right)\right] d\Gamma .$$

The functional $J^\varepsilon_{\Gamma_1}$ is a conjugate one and we can write, cf. [13],

$$(6.28) \qquad J^\varepsilon_{\Gamma_1} = (I_{\widetilde{C}^\varepsilon_b})^* \quad (\text{in the duality } \langle\cdot,\cdot\rangle_{\mathbf{M}^1(\Gamma_1)\times C_0(\Gamma_1)}) ,$$

where

$$(6.29) \qquad \widetilde{C}^\varepsilon_b = \{\varphi \in C_0(\Gamma_1) \mid \varphi(s) \in C^\varepsilon_b(s), \ \text{a.e. } s \in \Gamma_1\} .$$

The sequence $\{I_{\widetilde{C}^\varepsilon_b}\}_{\varepsilon>0}$ epi-converges to $I_{\widetilde{C}_b}$ in the strong topology of the space $C_0(\Gamma_1)$, where, cf. [9, 13]

$$(6.30) \qquad \widetilde{C}_b = \{\varphi \in C_0(\Gamma_1) \mid \varphi(s) \in \Delta(s), \forall s \in \Gamma_1\} .$$

In the particular case covered by Lemma 6.2 the set $\Delta(s)$ can be characterized explicitly. Let us briefly discuss such a case. Suppose that $\overline{\Omega} = \overline{\Omega^\varepsilon_1} \cup \overline{\Omega^\varepsilon_2}$, where

$$\Omega^\varepsilon_1 = \{x \in \Omega \mid C^\varepsilon(x) = C_1\}, \quad \Omega^\varepsilon_1 = \{x \in \Omega \mid C^\varepsilon(x) = C_2\} .$$

Thus the plate is made of two materials. C_1 and C_2 are their closed and convex sets of plastically admissible moments. We set $A_\varepsilon = \overline{\Omega^\varepsilon_1} \cap \Gamma_1$. Thus we have

$$C^\varepsilon_b(s) = C^\varepsilon \cdot \mathbf{n}(s) \cdot \mathbf{n}(s) = \begin{cases} C_1 \cdot \mathbf{n}(s) \cdot \mathbf{n}(s), & \text{if} \quad s \in A_\varepsilon, \\ C_2 \cdot \mathbf{n}(s) \cdot \mathbf{n}(s), & \text{if} \quad s \in \Gamma_1 \setminus A_\varepsilon. \end{cases}$$

If $\operatorname{int} A_\varepsilon \to A$ and $\operatorname{int}(\Gamma \setminus A_\varepsilon) \to B$ when $\varepsilon \to 0$ then

$$\Delta(s) = \begin{cases} [C_1 \cdot \mathbf{n}(s) \cdot \mathbf{n}(s)] \cap [C_2 \cdot \mathbf{n}(s) \cdot \mathbf{n}(s)], & \text{if} \quad s \in A \cap B, \\ C_1 \cdot \mathbf{n}(s) \cdot \mathbf{n}(s), & \text{if} \quad s \in A \setminus B, \\ C_2 \cdot \mathbf{n}(s) \cdot \mathbf{n}(s), & \text{if} \quad s \in B \setminus A. \end{cases}$$

Now we continue the discussion of the general case. Let $d_{\Gamma_1}(s, \cdot)$ denote the support function of the set $C_b^h(s) \cap \Delta(s)$. The results of Sect. 3 permit to extend it to a convex function of a measure $\mu \in \mathbf{M}^1(\Gamma_1)$. Hence we can write

$$J_{\Gamma_1}(\mu) = \int_{\Gamma_1} d_{\Gamma_1}(s, \mu), \qquad \mu \in \mathbf{M}^1(\Gamma_1). \tag{6.31}$$

The space $\mathbf{M}^1(\Gamma_1)$ is dual to the separable space $C_0(\Gamma_1)$, so one can apply Th. 2.3. In our case we infer that the sequence $\{J^\varepsilon_{\Gamma_1}\}_{\varepsilon>0}$ is sequentially epi-convergent to the functional J_{Γ_1} in the topology $\sigma(\mathbf{M}^1(\Gamma_1), C_0(\Gamma_1))$. Summarizing we formulate

THEOREM 6.2. *Under the assumption $M_n^0 \in C_0(\Gamma_1)$ the sequence λ^ε of load multipliers converges to the multiplier λ^h, where*

$$(RP^h_{LA}) \qquad \lambda^h = \inf\Big\{ \int_\Omega j_d^h[\kappa(v)] + \int_{\Gamma_0} j_d^h[\underline{\mathcal{F}}(-\gamma_1 v)]d\Gamma + \int_{\Gamma_1} d_{\Gamma_1}[s, \mu - (\gamma_1 v)d\Gamma] \mid v \in HB(\Omega),$$
$$\gamma_0 v = 0 \quad on\ \Gamma_0; \qquad \mu \in \mathbf{M}^1(\Gamma_1), \widehat{L}(v,\mu) = 1\Big\}.$$

The sequence of functionals

$$\int_\Omega j_d\left[\frac{x}{\varepsilon}, \kappa(v)\right]dx + \int_{\Gamma_0} j_d\left[\frac{x}{\varepsilon}, \underline{\mathcal{F}}(-\gamma_1 v)\right]d\Gamma + \int_{\Gamma_1} j_d\left[\frac{x}{\varepsilon}, \underline{\mathcal{F}}\left(\left(\frac{\partial v}{\partial \mathbf{n}}\right)^+ - \gamma_1 v\right)\right]d\Gamma + I_{\widehat{K}}\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right),$$

where $v \in W^{2,1}(\Omega)$ and $\left(\dfrac{\partial v}{\partial \mathbf{n}}\right)^+ \in L^1(\Gamma_1)$, is sequentially epi-convergent in the topology $[w - HB(\Omega)] \times \sigma(\mathbf{M}^1(\Gamma_1), C_0(\Gamma_1))$ to the functional

$$\int_\Omega j_d^h[\kappa(v)] + \int_{\Gamma_0} j_d^h[\underline{\mathcal{F}}(-\gamma_1 v)]d\Gamma + \int_{\Gamma_1} d_{\Gamma_1}[s, \mu - (\gamma_1 v)d\Gamma] + I_{\widehat{\mathfrak{K}}}(v,\mu), \tag{6.32}$$

where

$$\widehat{K} = \left\{\left(v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right) \in W^{2,1}(\Omega) \times L^1(\Gamma_1) \mid \gamma_0 v = 0 \text{ on } \Gamma_0, \quad \widehat{L}\left[v, \left(\frac{\partial v}{\partial \mathbf{n}}\right)^+\right] = 1\right\},$$

$$\widehat{\mathfrak{K}} = \{(v,\mu) \in HB(\Omega) \times \mathbf{M}^1(\Gamma_1) \mid \gamma_0 v = 0 \text{ on } \Gamma_0, \quad \widehat{L}(v,\mu) = 1\}.$$

■

By employing an approach similar to that which lead to the final form of the problem P_v^*, we infer that the problem QP^h_{LA} results from RP^h_{LA} provided that in the latter one calculates the infimum over $\mu \in \mathbf{M}^1(\Gamma_1)$. In such a case on Γ_1 the duality is to be understood in the sense of $\langle \cdot, \cdot \rangle_{\mathbf{M}^1(\Gamma_1) \times C_0(\Gamma_1)}$.

From the preceding considerations we conclude that, see also [11],

$$\lambda^h = \min(\lambda^h_\Omega, \lambda^h_b)\,, \tag{6.33}$$

where λ^h_b is given by Eq. (6.21) and

$$\lambda^h_\Omega = \inf\Big\{ \int\limits_\Omega j^h_d[\boldsymbol{\kappa}(v)] + \int\limits_{\Gamma_0} j^h_d[\underline{\mathcal{F}}(-\gamma_1 v)]d\Gamma \mid v \in W^{2,1}(\Omega), \quad \gamma_0 v = 0 \text{ on } \Gamma_0, \quad L(v) = 1\Big\}. \tag{6.34}$$

We observe that in general $C^h_b(s) \cap \Delta(s) \subset C^h_b(s)$ and the inclusion can be strict. It means that the limit load of the homogenized plate explicitly depends on materials distribution along Γ_1. Thus the weaker material, if not appropriately distributed, can lower the limit load of the effective plate. For instance, considering Fig. 1 once again, if $\Gamma_1 = [c, d]$ then the set $\Delta(s)$ is determined by plastic properties of either weaker or stronger material, depending upon which of them is distributed along Γ_1.

Having performed homogenization in the case of limit analysis, we can pass to the homogenization of elastic perfectly plastic plates made of Hencky materials.

PROBLEM P^ε_λ ($\varepsilon > 0$ and fixed)

Find

$$\inf\Big\{ \int\limits_\Omega j\Big[\frac{x}{\varepsilon}, \boldsymbol{\kappa}(u)\Big]dx - \lambda L(u) \mid u \in K_0 \Big\},$$

where the loading functional is given by the relation (4.18) and $M^0_n \in C_0(\Gamma_1)$. Next we formulate the relaxed problem.

PROBLEM RP^ε_λ

Find

$$\inf\Big\{ \int\limits_\Omega j\Big[\frac{x}{\varepsilon}, \boldsymbol{\kappa}(u)\Big]dx + \int\limits_{\Gamma_0} j_d\Big[\frac{x}{\varepsilon}, \underline{\mathcal{F}}(-\gamma_1 u)\Big]d\Gamma + \int\limits_{\Gamma_1} j_d\Big[\frac{x}{\varepsilon}, \underline{\mathcal{F}}\Big(\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)^+ - \gamma_1 u\Big)\Big]d\Gamma - \lambda \widehat{L}\Big[u, \Big(\frac{\partial u}{\partial \mathbf{n}}\Big)^+\Big] \mid u \in W^{2,1}(\Omega), \gamma_0 u = 0, \text{ on } \Gamma_0; \Big(\frac{\partial u}{\partial \mathbf{n}}\Big)^+ \in L^1(\Gamma_1)\Big\}.$$

The following theorem solves the problem of the homogenization of the elastic-perfectly plastic plate made of the Hencky material and exhibiting the periodic structure. Obviously, the loading functional is given by (4.18) and $M^0_n \in C_0(\Gamma_1)$.

THEOREM 6.3. *For each load multiplier λ such that $0 \le \lambda < \lambda^h$* (cf. Th.6.2) *the sequence of problems RP^ε_λ is epi-convergent in the topology $[w - HB(\Omega)] \times \sigma[\mathbf{M}^1(\Gamma_1), C_0(\Gamma_1)]$ to*

$$(RP^h_\lambda) \quad \inf\Big\{ \int\limits_\Omega j^h[\boldsymbol{\kappa}(u)] + \int\limits_{\Gamma_0} j^h_d[\underline{\mathcal{F}}(-\gamma_1 u)]d\Gamma + \int\limits_{\Gamma_1} d_{\Gamma_1}[s, \mu - (\gamma_1 u)d\Gamma] - \lambda \widehat{L}(u,\mu) \mid u \in HB(\Omega), \gamma_0 u = 0 \text{ on } \Gamma_0; u \in \mathbf{M}^1(\Gamma_1)\Big\}.$$

■

We observe that the problem RP^h_λ represents the relaxed form of the following one

$$(P^h_\lambda) \qquad \inf\left\{ \int_\Omega j^h[\kappa(u)] - \lambda L(u) \mid u \in HB(\Omega), \gamma_0 u = 0, \gamma_1 u = 0, \text{ on } \Gamma_0 \right\}.$$

According to the considerations of Sect. 5 we have

$$\inf P^h_\lambda = \inf RP^h_\lambda . \tag{6.35}$$

In the last theorem the epi-convergence evidently concerns the relevant sequence of functionals.

7. Concluding remarks

Tracing back the idea of an independent treating of the external trace in the study of functionals with linear growth, one comes across the papers by FERRO [29–31]. This author investigated some problems related to minimization of functionals defined on the space BV of functions with bounded variation [32].

The homogenization problems solved are a particular case of the homogenization of a thin perfectly plastic solid with fine periodic structure. In such a case there are two small parameters characterizing it. One of them is the thickness, say h, while the other one is ε, the parameter of periodicity. In our study we have started from the two-dimensional model. Thus it has been tacitly assumed that the passage $h \to 0$ had been *a priori* performed. KOHN and VOGELIUS [35] proved that for linear elastic plate two-dimensional homogenization is justified provided that $\varepsilon \approx h^a$, $a < 1$, see also [15, 37]. Intuitively it seems that the same should hold for plastic plates. However, there is a delicate problem of boundary conditions. It is known that asymptotic methods of justification of elastic plate model cannot actually deal with mixed boundary conditions [17, 27]. Hence it follows that the problem of homogenization of a thin perfectly plastic body with periodic structure loaded on the boundary is a very complicated one. Obviously we mean the construction of two-dimensional models when $h \to 0$ and $\varepsilon \to 0$ for various combinations of these parameters. This challenging problem remains open.

Multi-layer plates are often used as structural elements. Suppose now that a thin multi-layered laminate made of perfectly plastic materials has a fine periodic structure in the transverse direction. A natural question arises how to construct the effective plate model. The answer is simple provided that the plate is clamped at the boundary. Firstly, one performs one-dimensional homogenization, thus obtaining a transversely homogeneous thin three-dimensional body. Towards this end one can use the results due to BOUCHITTÉ [8, 9]. Secondly, the passage $h \to 0$ yields the effective two-dimensional plate model. The last problem has recently been solved by PERCIVALE [39] and TANG QI [49]. It is worth noting that the homogenization process performed at step one smears out the layers or their number tends to infinity while h is kept constant.

References

1. H. ATTOUCH, *Variational convergence for functions and operators*, Pitman, Boston-London-Melbourne 1984.
2. H. ATTOUCH, *Variational properties of epi-convergence. Applications of limit analysis problems in mechanics and duality theory*, in: Multifunctions and Integrands, G. SALINETTI [Ed.], Lecture Notes in Mathematics 1091, Springer-Verlag, Berlin 1984, pp. 80–104.

3. H. ATTOUCH, *Epi-convergence and duality. Convergence of sequence of marginal and Lagrangian functions. Applications to homogenization problems in mechanics*, Publications AVAMAC, Université de Perpignan, No 84–11.
4. J.-P AUBIN and FRANKOWSKA, *Set-valued analysis*, Birkhäuser, Boston-Basel-Berlin 1990.
5. D. AZÉ, *Epi-convergence et dualité. Application à la convergence des variables primales et duales pour des suites de problèmes d'optimisation convexe*, Publications AVAMAC, Université de Perpignan, No 84–12.
6. D. AZÉ, *Convergence des variables duales dans des problèmes de transmission à travers des couches minces par des méthodes d'epi-convergence*, Ricerche di Mat., **35**, 125–159, 1986.
7. O. O. BARABANOV, *On limit surface loads in the theory of plasticity*, Prikl. Mat. Mekh., **53**, 824–829. 1989 [in Russian].
8. G. BOUCHITTÉ, *Convergence et relaxation de fonctionnelles du calcul des variations à croissance linéaire. Application à l'homogénéisation en plasticité*, Ann. Fac. Sci. Toulouse, **8**, 7–36, 1986–1987.
9. G. BOUCHITTÉ, *Calcul des variations en cadre non réflexif. Représentation et relaxation de fonctionnelles intégrales sur un espace de mesures. Application en plasticité et homogénéisation*, Thèse d'Etat, Université de Perpignan 1987.
10. G. BOUCHITTÉ and G. BUTTAZZO, *New lower semicontinuity results for non-convex functionals defined on measures*, Preprint SISSA, Trieste, 68 M, June 1989.
11. G. BOUCHITTÉ and P. SUQUET, *Charges limites, plasticité et homogénéisation : le cas d'un bord chargé*, C. R. Acad. Sci. Paris, Série I, **305**, 441–444, 1987.
12. G. BOUCHITTÉ and M. VALADIER, *Integral representation of convex functionals on a space of measures*, J. Funct. Anal., **80**, 398–420, 1988.
13. G. BOUCHITTÉ and M. VALADIER, *Multifonctions s.c.i. et régularisée s.c.i. essentielle. Fonctions de mesures dans le cas sous-linéaire*, in: Analyse Non Linéaire, H. ATTOUCH, J.-P. AUBIN, F.H. CLARKE and I. EKELAND [Eds.], Gauthier-Villars and C.R.M. Université de Montréal 1989, pp. 123–150.
14. G. BUTTAZZO, *Semicontinuity and integral representation in the calculus of variations*, Pitman, London 1989.
15. D. CAILLERIE, *Thin elastic and periodic plates*, Math. Meth. in the Appl. Sci., **6**, 159–191, 1984.
16. E. CHRISTIANSEN, *Limit analysis for plastic plates*, SIAM J. Math. Anal., **11**, 414–422.
17. P. G. CIARLET, *Plates and junctions in elastic multi-structures: an asymptotic analysis*, Massonet, Paris, Springer-Verlag, Berlin 1990.
18. B. DACOROGNA, *Direct methods in the calculus of variations*, Springer-Verlag, Berlin 1989.
19. F. DEMENGEL, *Problèmes variationnels en plasticité parfaite des plaques*, Thèse de 3-ème cycle, Université Paris-Sud, 1982 (abbreviated version: Numer. Funct. Anal. and Optimiz., **6**, 73–119, 1983).
20. F. DEMENGEL, *Fonctions convexes de mesure et problèmes variationnels. Aplication en mécanique non linéaire (élasticité, plasticité)*, Thèse de Doctorat d'Etat, Université Paris-Sud, 1986.
21. F. DEMENGEL, *Fonctions à hessien borné*, Ann. Inst. Fourier de L'Université Sci. et Med. de Grenoble, **34**, 155–190, 1984.
22. F. DEMENGEL, *Compactness theorems for spaces of functions with bounded derivatives and applications to limit analysis problems in plasticity*, Arch. Rat. Mech. Anal., **105**, 123–161, 1989.
23. F. DEMENGEL and TANG QI, *Homogénéisation en plasticité*, C. R. Acad. Sci. Paris, Série I, **303**, 339–341, 1986.
24. F. DEMENGEL and TANG QI, *Convex function of a measure obtained by homogenization*, SIAM J. Math. Anal., **21**, 409–435, 1990.
25. F. DEMENGEL and R. TEMAM, *Convex functions of a measure and applications*, Indiana Univ. Math. J., **33**, 673–709, 1984.
26. J. DESCLOUX, *Un problème d'approximation*, Ecole Polytechnique Fédérale de Lausanne 1990, unpublished manuscript.
27. P. DESTUYNDER, *Une théorie asymptotique des plaques mince en elasticité linéaire*, Masson, Paris 1986.
28. I. EKELAND and R. TEMAM, *Convex analysis and variational problems*, North-Holland, Amsterdam 1976.
29. F. FERRO, *Sul minimo di funzionali definiti sullo spazio delle funzioni di variazione limitata in n dimensioni*, Annali Mat. Pura Appl., **117**, 153–171.
30. F. FERRO, *Variational functionals defined on spaces of BV functions and their dependence on boundary data*, Annali Mat. Pura Appl., **122**, 269–287, 1979.
31. F. FERRO, *A minimal surface theorem with generalized boundary data*, Bolletino U.M.I. **16**-B, 962–971.
32. E. GIUSTI, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Boston-Basel-Stuttgart 1984.

33. T. HADHRI, *Présentation et analyse mathématique et numérique de quelques modèles pour des structures élastoplastiques homogènes ou hétérogènes*, Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris 1986.
34. M. JANAS, J.A. KÖNIG and A. SAWCZUK, *Plastic analysis of structures*, Wroclaw 1972 [in Polish].
35. R. V. KOHN and M. VOGELIUS, *A new model for thin plates with rapidly varying thickness. Part I*, Int. J. Solids Struct., **20**, 335–350, 1984; *Part II. A convergence proof*, Quart. Appl. Math., **43**, 1–12, 1985; *Part III. Comparison of different scalings*, Quart. Appl. Math., **44**, 35–48, 1986.
36. P. J. LAURENT, *Approximation et optimisation*, Herrmann, Paris 1972.
37. T. LEWIŃSKI, *Effective models of composite periodic plates, I. Asymptotic solution*, Int. J. Solids Structures, **27**, 1155–1172, 1991; *II. Simplifications due to symmetries*, ibid., pp. 1173–1184; *III. Two-dimensional approach*, ibid., 1185–1203.
38. J.-C. MICHEL, *Homogénéisation de matériaux élastoplastiques avec cavités*, Thèse de 3-ème cycle, Université Pierre et Marie Curie, Paris 1984.
39. D. PERCIVALE, *Perfectly plastic plates: a variational definition*, J. Reine Angew. Math., **411**, 39–50, 1990.
40. R. T. ROCKAFELLAR, *Convex analysis*, Princeton University Press, Princeton 1970.
41. E. SANCHEZ-PALENCIA, A. ZAOUI, [Eds.], *Homogenization techniques for composite media*, Lecture Notes in Physics 272, Springer-Verlag, Berlin 1987.
42. M. A. SAVE and C. E. MASSONET, *Plastic analysis and design of plastic plates, shells and disks*, North-Holland, Amsterdam 1972.
43. A. SAWCZUK, *Plasticité des plaques*, E.N.S.M. Nantes, mai-juin 1982, 106 pp.
44. P. M. SUQUET, *Plasticité et homogénéisation*, Thèse d'Etat, Université Pierre et Marie Curie, Paris 1982.
45. P. M. SUQUET, *Approach by homogenization of some linear and nonlinear problems in solid mechanics*, in: Plastic Behaviour of Anisotropic Solids, J.-P. BOEHLER [Ed.], Editions du CNRS, 1985, pp. 77–117.
46. P. M. SUQUET, *Discontinuities and plasticity*, in: Nonsmooth Mechanics and Applications, J. J. MOREAU and P. G. PANAGIOTOPOULOS [Eds.], Springer-Verlag, Wien-New York 1988, pp. 278–340.
47. TANG QI, *Homogénéisation des problèmes élastoplastiques en dimension trois*, Thèse, Université de Paris-Sud, 1986.
48. TANG QI, *A dual method in the calculation of homogenisation and applications*, Proc. Royal Soc. Edinb., **110A**, 321–334, 1988.
49. TANG QI, *Justification of elastoplastic plate models*, Asymptotic Analysis, **3**, 57–76, 1990.
50. J. J. TELEGA, *Variational methods and convex analysis in contact problems and homogenization*, IFTR Reports 38/90, 1990, 209 pp. [in Polish].
51. J. J. TELEGA, *Some results of homogenization in plasticity: plates and fissured solids*, in: Inelastic Solids and Structures, M. KLEIBER and J. A. KÖNIG, [Eds.], Pineridge Press, Swansea 1990, pp. 343–360.
52. J. J. TELEGA, *Homogenization and effective properties of plates weakened by partially penetrating fissures: convergence and duality*, Modél. Math. et Anal. Numér. [in print.].
53. J. J. TELEGA, *Epi-limit on IB and homogenization of heterogeneous perfectly plastic plates*, submitted.
54. R. TEMAM, *Mathematical problems in plasticity*, Gauthier-Villars, Paris, Trans-Inter-Scientia, Tonbridge 1985.
55. V. V. ZHIKOV, *Problems of convergence, duality and averaging for functionals of the calculus of variations*, Izv. AN SSSR, Ser. Mat., **47**, 961–998. [in Russian].

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received March 6, 1991.