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# Nonequilibrium semi-empirical temperature in materials with thermal relaxation

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A NEW SEMI-EMPIRICAL temperature scale is introduced in terms of which the classical Fourier law is formulated. Some results concerning the relations between the physical models describing the heat conduction in solids and the mathematical theory of the quasi-linear hyperbolic systems of conservation laws are presented. The hyperbolicity of the heat conduction equation is discussed together with thermo-mechanical coupling. In the case of small strains a symmetric system of equations is obtained that guarantees the well-posedness of the Cauchy problem.

## 1. Introduction

THE FIRST THEORY of heat conduction with finite wave speed has been proposed by CATTANEO [1,2] and VERNOTTE [3]. However, it seems that it was MAXWELL [4] who was the first to modify Fourier's law. Hence we will refer to

$$T\dot{\mathbf{q}} + \mathbf{q} = -k\nabla\vartheta \tag{1.1}$$

as to M.C.V. (Maxwell–Cattaneo–Vernotte) equation, where $T > 0$ is a suitable relaxation time, $\vartheta$ denotes the absolute temperature, $k$ is the conductivity and $\mathbf{q}$ the heat flux vector.

There is a number of experimental results which confirm the validity of Eq. (1.1). For example, [5, 6], for dielectric solids at low temperatures the following relation

$$k/Tc_v \cong \nu^2$$

holds, where $\nu$ is the average of the phonon velocity, $c_v$ is the specific heat. However, a proper thermodynamic setup for Eq. (1.1) is difficult to form, because of the second law of thermodynamics [7–9]. On the other hand, a more complex model can be obtained by supposing $T$ and $k$ to be functions of the absolute temperature, [7–12]. As far as anisotropic bodies are concerned, $T$ and $k$ should be represented by suitable tensor functions [13, 14].

It is said that Cattaneo proposed Fourier's constitutive law to be replaced by the constitutive relation (1.1) (see, for instance, [15]). It seems to us that this is impossible in the framework of Continuum Mechanics because Fourier's law is a constitutive equation whereas M.C.V. is an evolution equation. In his original paper Cattaneo derived both Fourier's law and Eq. (1.1) in the framework of Statistical Mechanics, so one can say that both relations play the role of balance equations, with different approximation of the mean kinetic energy of the molecules ([16, 17]).

Consider, for instance, an isotropic rigid conductor and suppose that the following constitutive equation holds

$$\mathbf{q} \equiv \mathbf{q}^*(\vartheta, \nabla\vartheta);$$

then the representation theorem of isotropic vector functions states that there exists a scalar-valued function $k(\vartheta, |\nabla\vartheta|)$ such that

$$\mathbf{q} = -k\nabla\vartheta. \tag{1.2}$$

In such a case the validity of Fourier's law is required by Mathematics. This means that in Continuum Mechanics (1.1) has to be regarded as a suitable evolution equation for $\mathbf{q}$ which holds together with the constitutive equation.

For example, for a rigid material with memory, the functional version of the Fourier's law

$$\mathbf{q}(t) = -\int_0^\infty kT^{-1}\exp(-T^{-1}s)\mathbf{g}(t-s)\,ds, \qquad \mathbf{g} = \operatorname{grad}\vartheta \tag{1.3}$$

leads to Eq. (1.1) after differentiation with respect to time [15].

Heat conduction with finite wave speed can be analyzed in the context of the theory of materials with memory [15, 18, 19]. When the system is very close to an equilibrium state and linearization of the constitutive functionals is made, we get the Fourier's law in the integral form (1.3) and , after time differentiation, the M.C.V. equation. Such an approach is very well grounded from the logical point of view and leads to a completely satisfactory wave analysis. However, the evolution equations so obtained are integro-differential equations for which proving the well-posedness theorems is very difficult.

A different approach consists in employing internal state variables in modelling the behavior of the body. This model is obtained by enriching the set of the independent variables appearing in the constitutive equations by additional quantities called internal state variables or hidden variables. Suitable kinetic equations for the evolution of the internal state variable are then postulated. Those additional equations are evolutionary first-order differential equations [20–25]. In such a context the well-posing of the Cauchy problem can be proved in the framework of the theory of the symmetric hyperbolic systems and conservation laws [7–9, 12]. However, the wave analysis leads to four characteristic wave velocities [9], except for [12], where a particular form of internal energy was assumed. At first glance, the physical meaning of two of them is not evident. The same criticism can arise by studying heat conduction in the scheme of Relativity and Extended Thermodynamics [9, 16, 17, 26–28]. The generalized thermoelasticity with two relaxation times and second order time derivative in the heat equation [29–31] cause another type of criticism.

## 2. A new semi-empirical temperature scale

In the papers [33, 34] we introduced a new temperature scale which, in our opinion, seems to be very useful in studying heat propagation problems. In order to discuss the meaning of the new temperature we present here some physical considerations, starting from the results proved by CATTANEO in [1].

Consider a gas in macroscopic mechanical equilibrium and let $Q$ be a given physical quantity related to a molecule of the gas. We will denote by $G$ an average of $Q$ at a given point of the gas. For the sake of simplicity we will suppose $G$ to be constant on each plane belonging to a family of parallel planes, orthogonal to a given direction $r$. However, the value of $G$ can change from one to another plane so that, denoting by $x$ a point of $r$ and supposing $G$ to be independent of the time, we get

$$G = G(x).$$

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$$G = G(x).$$

Moreover,

$$\alpha\beta = G - T\ \partial G/\partial t, \tag{2.11}$$

so that $\beta$ is related to the average of the kinetic energy as well as to its time derivative. The second term on the right-hand side of (2.11), containing the mean acceleration of the molecules, can be regarded as a contribution of internal forces among the molecules, which usually are neglected in the statistical definition of temperature. By Eq. (2.9) it follows

$$\partial\beta/\partial t = \partial\vartheta/\partial t - T\ \partial^2\vartheta/\partial t^2. \tag{2.12}$$

By using (2.9) we can eliminate $\partial\vartheta/\partial t$ in Eq. (2.12) so that

$$\partial(\alpha\beta)/\partial t = (\alpha/T)(\vartheta - \beta) - T\ \partial(\alpha\vartheta)/\partial t. \tag{2.13}$$

In our approach the last term in (2.13) will be neglected because $T$ is very small (cf. [29, 32]) as compared to $1/T$. We finally obtain

$$\partial(\alpha\beta)/\partial t = (\alpha/T)(\vartheta - \beta). \tag{2.14}$$

We call (2.14) *canonical kinetic equation* and the function $\beta$ defined by (2.14) *semi-empirical temperature*. We would like to stress that the new temperature has an empirical basis: we need the absolute temperature in order to define that. Moreover, it is related to the mean kinetic energy of the molecules of the gas under consideration, so it manifests material properties of the medium. However, $\beta$ is related to $\vartheta$ by the first order evolutionary equation so its definition is analytical, in a sense. From the physical point of view $\beta$ is the absolute temperature minus a frictional term due to the internal Van der Waals type forces. Since the velocities of the molecules are reduced by such forces, the final results is a drop in the temperature. The most important effect which is obtained by assuming $\beta$ as a new temperature scale, is the transformation of the type of the heat equation from parabolic to hyperbolic. This is easily proved by substituting in the heat equation the expression of $\vartheta$ given by Eq. (2.14).

In Eq. (2.14) $\partial\beta/\partial t$ is equal to a linear function of $\vartheta$ and $\beta$. It is worth noticing that we can change the scale of $\beta$ without affecting its physical meaning. In order to take into account this possibility we generalize Eq. (2.14) as follows([1]):

$$\dot\beta = \mathfrak{f}_{(\alpha,k,T)}(\vartheta,\beta), \tag{2.15}$$

where $\mathfrak{f}_{(\alpha,k,T)}$ is a suitable functional of the material functions (or constants) $\alpha, k$ and $T$, depending on the semi-empirical temperature as well as on the absolute one. The subscript $(\alpha, k, T)$ means that, in constructing $\mathfrak{f}$, we need to take into account those functions; the last equation will be called a *normed kinetic equation*.

REMARK 1. A comparison of the present approach with the Maxwell–Cattaneo–Vernotte equation can be made after the spatial differentation which leads to the so-called prolongated kinetic equation

$$\nabla\dot\beta = (\partial\mathfrak{f}_{(\alpha,k,T)}/\partial\vartheta)\nabla\vartheta + (\partial\mathfrak{f}_{(\alpha,k,T)}/\partial\beta)\nabla\beta, \tag{2.16}$$

where $\partial/\partial x$ was substituted by the gradient operator $\nabla$. Here the variable $q$ becomes a $3D$-vector $\mathbf{q}$.

([1]) In what follows the superposed dot denotes the time derivative.

In the case the canonical kinetic equation (2.14) we get

$$\nabla\dot{\beta} = (1/T)\nabla\vartheta - (1/T)\nabla\beta, \tag{2.17}$$

if $\alpha$ does not depend on $x$. Owing to Eqs. (2.10) and (2.17), we finally obtain

$$\dot{\overline{T\mathbf{q}}} + \mathbf{q} = -k\nabla\vartheta, \tag{2.18}$$

i.e. an equation of the M.C.V.-type is recovered.

REMARK 2. The semi-empirical temperature $\beta$ is uniquely defined by (2.15) if a suitable initial condition

$$\beta(x, t_0) = \beta_0 \tag{2.19}$$

at an initial time $t_0$ is given and $\mathfrak{f}$ is Lipschitz continuous.

We are admitting two supplementary assumptions on $\mathfrak{f}$:

$$\text{a)}\quad (\partial\mathfrak{f}/\partial\beta) \leq 0; \qquad \text{b)}\quad (\partial\mathfrak{f}/\partial\vartheta) > 0.$$

The requirement a) assures the stability of the solution of Eq. (2.15), while the other is needed in order to make $\beta$ an increasing function of $\vartheta$ and to preserve the natural order relation '$\leq$' between different temperatures when the relaxation time is zero.

REMARK 3. Since $\alpha$ plays the role of specific heat, the product $\alpha\beta$ represents the thermal energy of the body, that is due to the kinetic energy of the molecules as well as to the frictional internal forces. This energy is calculated with respect to the reference thermal level represented by $\vartheta^0$. Hence we can conclude that the kinetic equation (2.14) (or (2.15)) is nothing else but a balance law for the thermal energy with vanishing flux.

## 3. Rigid conductor

We can now speak about a material with thermal relaxation that is characterized by the existence of a time-valued material function $T$ which takes very small positive values; the function is responsible for a wave structure of heat pulses. For processes the durations of which are greater than $T$, a diffusive type of heat propagation is practically observed. If the function $T$ is constant (and equal to $\tau$ in some range of absolute temperature $\vartheta$) and very small but not vanishing, then $T$ is just the "relaxation time". We will deal in this section with an isotropic rigid conductor with constant relaxation time.

In order to construct a constitutive theory we recall two first thermodynamic laws

$$\rho\dot{\varepsilon} = -\operatorname{Div}\mathbf{q} + \rho r, \tag{3.1}$$

$$\rho\dot{\eta} \geq -\operatorname{Div}(\mathbf{q}/\vartheta) + \rho(r/\vartheta). \tag{3.2}$$

In Eq. (3.1) $\rho$ is the mass density, $\varepsilon$ is the specific internal energy, $r$ the body heat supply, $\eta$ the specific entropy. Now we are dealing with a rigid conductor, hence we have to assign three constitutive functions, it is assumed that

$$\begin{aligned} \varepsilon &\equiv \varepsilon^*(\vartheta, \nabla\vartheta,) \\ \eta &\equiv \eta^*(\vartheta, \nabla\vartheta), \\ \mathbf{q} &\equiv \mathbf{q}^*(\vartheta, \nabla\vartheta). \end{aligned} \tag{3.3}$$

However, we will assume that

$$\partial\vartheta/\partial\eta \neq 0, \tag{3.4}$$

so the absolute temperature can be expressed as a function of $\eta$ and the remaining thermodynamic variables. Moreover, instead of Eq. (3.3) we assume the following constitutive equations:

$$\begin{aligned} \varepsilon &\equiv \widehat{\varepsilon}(\eta, \nabla\beta), \\ \eta &\equiv \widehat{\eta}(\eta, \nabla\beta), \\ \mathbf{q} &\equiv \widehat{\mathbf{q}}(\eta, \nabla\beta). \end{aligned} \tag{3.5}$$

Owing to Eqs. (3.2) and (3.5), we get

$$\rho(\partial\varepsilon/\partial\eta - \vartheta)\dot{\eta} + (\mathbf{q}\vartheta^{-1}(\partial\mathfrak{f}/\partial\vartheta)^{-1} + \rho\partial\varepsilon/\partial\nabla\beta)\cdot\nabla\dot{\beta} \\ -\mathbf{q}\cdot\nabla\beta\vartheta^{-1}(\partial\mathfrak{f}/\partial\vartheta)^{-1}\partial\mathfrak{f}/\partial\beta \leq 0. \tag{3.6}$$

In order to satisfy Eq. (3.6) the following relations must hold

$$\begin{aligned} \vartheta &= \partial\varepsilon/\partial\eta, \\ \mathbf{q} &= -\rho\vartheta(\partial\mathfrak{f}/\partial\vartheta)\partial\varepsilon/\partial\nabla\beta, \end{aligned} \tag{3.7}$$

together with the reduced entropy inequality

$$-\mathbf{q}\cdot\nabla\beta(\partial\mathfrak{f}/\partial\vartheta)^{-1}\partial\mathfrak{f}/\partial\beta \leq 0. \tag{3.8}$$

As far as (3.8) is concerned, Eq. $(3.7)_1$ together with the requirement a) yields

$$(\partial\varepsilon/\partial\nabla\beta)\cdot\nabla\beta \geq 0. \tag{3.9}$$

In the case of an isotropic conductor we can conclude the existence of a scalar function $K(\eta, |\nabla\beta|)$ such that

$$\mathbf{q} = K(\eta, |\nabla\beta|)\nabla\beta. \tag{3.10}$$

We suppose $K$ to be constant so that Eqs. (2.16) and (2.18) yield

$$T = -(\partial\mathfrak{f}/\partial\beta)^{-1}, \tag{3.11}$$

i.e. $\mathfrak{f}$ is a linear function with respect to $\beta$. Let us write $T = \tau_0$ where $\tau_0$ is the relaxation time measured at a given reference temperature $\vartheta^0$. Let us notice that a suitable form of the specific internal energy, satisfying (3.9), is

$$\varepsilon(\eta, \nabla\beta) = \varepsilon_1(\eta) + 1/2\,\varepsilon_2\nabla\beta^2, \tag{3.12}$$

where $\varepsilon_2$ is a positive constant and $\varepsilon_1(\eta)$ describes the equilibrium part of internal energy. Dimensional analysis, together with the requirement $\varepsilon_2 = 0$ when $\tau_0 = 0$, lead to

$$\varepsilon_2 = \pm K\tau_0/\rho\vartheta^0, \tag{3.13}$$

where we choose plus or minus in order $\varepsilon_2$ to be positive. In order to conclude whether $K$ is non-positive, we substitute (3.10) into (3.8), to obtain

$$K(\partial\mathfrak{f}/\partial\vartheta)^{-1} \leq 0. \tag{3.14}$$

Due to assumption b) we conclude that $K \leq 0$, then we put

$$K = -k_0.$$

Under such hypothesis

$$\varepsilon_2 = (k_0\tau_0)/(\rho\vartheta^0) \tag{3.15}$$

and

$$\mathbf{q} = -k_0 \nabla \beta. \tag{3.16}$$

Taking into account (3.16), (3.15), (3.11) and $(3.7)_2$ and integrating with respect to $\vartheta$, we finally get

$$\mathfrak{f}(\vartheta, \beta) = \vartheta^0 \tau_0^{-1} \log(\vartheta/\vartheta^0) + \mathfrak{f}_1(\beta^0, \beta),$$

where $\mathfrak{f}_1$ is a linear function of $\beta$ and $\beta^0$, ($\beta^0$ is the semi-empirical temperature corresponding to $\vartheta = \vartheta^0$). Dimensional analysis together with requirement a) lead to

$$\mathfrak{f}(\vartheta, \beta) = \vartheta^0 \tau_0^{-1} \log(\vartheta/\vartheta^0) - (\beta - \beta^0)/\tau_0. \tag{3.17}$$

By Eq. (3.17), the following kinetic equation results:

$$\dot{\beta} = \vartheta^0 \tau_0^{-1} \log(\vartheta/\vartheta^0) - (\beta - \beta^0)/\tau_0, \tag{3.18}$$

which can be inverted to the form

$$\vartheta = \vartheta^0 \exp[(\tau_0/\vartheta^0)\dot{\beta} + \vartheta^{0-1}(\beta - \beta^0)]. \tag{3.19}$$

Using Eq. (3.19) together with (3.12), (3.16) and (3.2) we finally obtain the following equation, governing the heat propagation in the isotropic rigid conductor:

$$\rho \tau_0 c_v \vartheta \ddot{\beta} + \rho c_v \vartheta \dot{\beta} + k_0 \tau_0 \nabla \beta \nabla \dot{\beta} - k_0 \vartheta^0 \Delta \beta - \rho r \vartheta^0 = 0. \tag{3.20}$$

It is a quasi-linear hyperbolic second order evolution equation. Well-posedness of a Cauchy problem for (3.20) was proved in [35]. Its characteristic equation is

$$\rho \tau_0 c_v \vartheta \Lambda^2 + k_0 \tau_0 \nabla \beta \cdot \mathbf{n} \Lambda - k_0 \vartheta^0 = 0, \tag{3.21}$$

where $\Lambda$ is the wave speed and the unit vectors $\mathbf{n}$ indicate the direction of propagation. We have for $\Lambda$ two solutions

$$\Lambda_{\pm} = (-k_0 \tau_0 \nabla \beta \cdot \mathbf{n} \pm (k_0^2 \tau_0^2 (\nabla \beta \cdot \mathbf{n})^2 + 4 \rho \vartheta^0 \vartheta c_v k_0 \tau_0)^{1/2})(2 \rho c_v \vartheta \tau_0)^{-1}. \tag{3.22}$$

The hyperbolic condition is

$$c_v k_0 \tau_0 > 0, \tag{3.23}$$

which is satisfied when $k_0$, $\tau_0$, and $c_v$ are positive. Since for heat conductors $k_0$ does not vanish, specific heat $c_v$ is normally assumed positive, the only constraint implied by (3.23) is

$$\tau_0 > 0,$$

which could be considered as an assumption on the coefficients of Eq. (3.20). When the gradient of $\beta$ is perpendicular to the direction of propagation, the speed $\Lambda_+$ is equal to $-\Lambda_-$ and we are faced with the symmetric propagation. On the other hand, when

$$k_0^2 \tau_0^2 (\nabla \beta \cdot \mathbf{n})^2 + 4 \rho \vartheta^0 \vartheta c_v(\vartheta) k_0 \tau_0 = 0 \tag{3.24}$$

Eq. (3.20) becomes parabolic. However, it may happen only when $\tau_0 = 0$. In such a case (3.20) reduces to the parabolic heat equation and the wave speed becomes infinite. Let us rewrite (3.18) in the more proper form

$$\tau_0 \dot{\beta} = \vartheta^0 \log(\vartheta/\vartheta^0) - (\beta - \beta^0).$$

Hence the limit case with $\tau_0 = 0$ leads to the relation

$$\beta = \beta^0 + \vartheta^0 \log(\vartheta/\vartheta^0),$$

so that the new temperature scale is logarithmic.

REMARK 4. When $\tau_0 = 0$ and $c_v(\vartheta) = c_{v0}/\vartheta$, then Eq. (3.20) reduces to the linear equation with constant coefficients for the variable $\beta$.

REMARK 5. When $\tau_0 = 0$, then

$$\mathbf{q} = K(\vartheta)\nabla\vartheta = -(k_0\vartheta^0/\vartheta)\nabla\vartheta$$

follows from the previous relation between $\vartheta$ and $\beta$.

Finally, we want to list some interesting properties of field equations of the model under consideration in the case of shock and acceleration wave propagation.

As usual, we represent a shock front as a moving surface $\Sigma(t)$ separating two regions $\mathcal{R}^+(t)$ and $\mathcal{R}^-(t)$. We will assume that the propagation vector is directed from $\mathcal{R}^-(t)$ to $\mathcal{R}^+(t)$. Let $f(x,t)$ be a continuous function in the interior points of $\mathcal{R}^+$ and $\mathcal{R}^-$, and let $f^+(x,t)$ and $f^-(x,t)$ denote the limits of $f(x,t)$ on $\Sigma(t)$ when $x$ is approached from $\mathcal{R}^+$ and $\mathcal{R}^-$, respectively. The symbol $[\![f]\!] := f^+ - f^-$ will denote the jump of $f$ across $\Sigma(t)$. Supposing the body heat supply to be continuous across $\Sigma(t)$, the shock jump conditions (or the so-called generalized Rankine–Hugoniot condition) read

$$\begin{aligned} \rho V[\![\varepsilon]\!] &= [\![\mathbf{q}]\!]\cdot\mathbf{N}, \\ V[\![\beta]\!] &= 0, \\ V[\![\mathbf{q}]\!] &= -k_0[\![\mathfrak{f}]\!]\mathbf{N}, \end{aligned} \tag{3.25}$$

where $V$ and $\mathbf{N}$ are the normal velocity and the normal versor of the shock wave, respectively. The continuity of the new temperature, even in the case of a shock wave, is an essential property of the model proposed and is a consequence of the kinetic equation (3.18). Moreover, the relation $(3.25)_3$ takes the form

$$[\![\mathbf{q}]\!] = k_0(V\tau_0)^{-1}\vartheta^0\log\frac{\vartheta^-}{\vartheta^+}\mathbf{N}. \tag{3.26}$$

Finally, by $(3.25)_1$ and (3.26) we get

$$[\![\varepsilon]\!] = k_0(\rho V^2\tau_0)^{-1}\vartheta^0\log\frac{\vartheta^-}{\vartheta^+}. \tag{3.27}$$

Let us quote the results proved in [36] concerning acceleration wave propagation. In the aforementioned paper, the case of $1D$ rigid conductors was considered. There the following equation, describing the evolution of the amplitude of the acceleration waves

$$\frac{d}{dt}\alpha(t) - n\tau_0\alpha^2 + (m\tau_0)^{-1}\alpha = 0, \tag{3.28}$$

was obtained, where $\alpha := [\![\ddot\beta]\!]$, and $m$ and $n$ are suitable functions depending on $c_v$, as well as on $\vartheta, \vartheta^0, \beta, \beta^0$. Equation (3.28) is of Bernoulli type and, for $m > 0$, its solution $\alpha(t)$ blows up in a finite time, if the initial condition $a_0 = \alpha(0)$ satisfies the following inequality

$$-n\tau_0\alpha_0 + (m\tau_0)^{-1} < 0. \tag{3.29}$$

The finite blow-up time is given by

$$t_1 = m\tau_0\log(mn\alpha_0\tau_0^2/(mn\alpha_0\tau_0^2 - 1)).$$

At that time the formation of shock will be observed. Two limit cases of very small and very large $\tau_0$ can be considered. When $\tau_0 \to 0$ then $t_1 \to \infty$ and the solution of

(3.28) does not blow up in finite time. On the other hand, if $\tau_0 \to \infty$ then $t_1 \to 0$ and $\alpha(t)$ blows up immediately; however, this limit case is rather artificial.

If $\mathfrak{f}$ is given by (3.17) and $c_v$ is constant, the functions $m$ and $n$ reduce to $m = 1, n = \vartheta^{0-1}$, so that (3.29) is satisfied when

$$\alpha_0 > 2\vartheta^0/\tau_0^2.$$

Let us notice that the relaxatoion time is of order $10^{-9}$ s for most materials; assuming then $\vartheta^0 \cong 1K$, as in the case of helium II, the last inequality will be satisfied when $\alpha_0$ is of order $10^{18}$, which means that in such a situation of physical interest blow-up will not occur except, maybe, for laser pulses of duration of fractions of femtoseconds.

## 4. Thermoelastic solids

Let $\mathcal{S} \subset R$ be a thermoelastic solid with thermal relaxation in its reference configuration. Then the balance laws for the linear momentum and energy are

$$\rho\dot{\mathbf{v}} = \operatorname{Div}\mathbf{S} + \rho\mathbf{b}, \tag{4.1}$$

$$\overline{\rho(\varepsilon + 0.5\mathbf{vv})}^{\boldsymbol{\cdot}} - \operatorname{Div}(\mathbf{vS}) + \operatorname{Div}\mathbf{q} - \rho\mathbf{bv} - \rho r = 0,$$

where:

$\rho$ is the reference mass density;
$\mathbf{S}$ is the Piola–Kirchhoff stress tensor;
$\mathbf{v}$ and $\dot{\mathbf{v}}$ denote the particle velocity and acceleration, respectively;
$\mathbf{q}$ denotes the heat flux calculated in the reference configuration;
$\mathbf{b}$ is the body force;
$r$ is the body heat supply.

The symbols Div and $\nabla$ denote the gradient and divergence operators in the reference configuration, respectively. Together with (4.1) we are faced with the unilateral differential constraint

$$\rho\dot{\eta} \geq -\operatorname{Div}(\mathbf{q}/\vartheta) + \rho(r/\vartheta)$$

representing the second law of thermodynamics. In order to obtain the restrictions on the constitutive equations, we rewrite (4.1) in a more suitable form by using the free energy

$$\psi = \varepsilon - \vartheta\eta. \tag{4.3}$$

Then

$$\rho(\overline{\psi + \eta\vartheta}^{\boldsymbol{\cdot}}) - \mathbf{S}\cdot\nabla\mathbf{v} + \mathbf{q}\cdot(\nabla\vartheta)/\vartheta \leq 0 \ (^2). \tag{4.4}$$

We assume now the following constitutive equations:

$$\begin{aligned} \psi &= \psi^*(\vartheta, \mathbf{F}, \nabla\beta), & \eta &= \eta^*(\vartheta, \mathbf{F}, \nabla\beta), \\ \mathbf{q} &= \mathbf{q}^*(\vartheta, \mathbf{F}, \nabla\beta), & \mathbf{S} &= \mathbf{S}^*(\vartheta, \mathbf{F}, \nabla\beta), \end{aligned} \tag{4.5}$$

where $\mathbf{F}$ is the gradient of the deformation. Using (4.4) we can arrange (4.3) as follows:

$$\rho(\partial\psi/\partial\vartheta + \rho\eta)\dot{\vartheta} + (\rho\partial\psi/\partial\mathbf{F} - \mathbf{S})\cdot\dot{\mathbf{F}} + \rho(\partial\psi/\partial\nabla\beta)\cdot\nabla\dot{\beta} + \mathbf{q}\cdot(\nabla\vartheta)/\vartheta \leq 0.$$

($^2$) In this paper the symbol $\cdot$ means the full contraction (saturation) giving a scalar as the result.

Using the extended (prolongated) kinetic equation we obtain

$$(4.6)\quad (\rho\partial\psi/\partial\vartheta + \rho\eta)\dot{\vartheta} + (\rho\partial\psi/\partial\mathbf{F} - \mathbf{S})\cdot\dot{\mathbf{F}} \\ +(\rho\partial\psi/\partial\nabla\beta + \mathbf{q}/\vartheta(\partial\mathfrak{f}/\partial\vartheta)^{-1})\cdot\nabla\dot{\beta} + \mathbf{q}/\vartheta(\partial\mathfrak{f}/\partial\beta)(\partial\mathfrak{f}/\partial\vartheta)^{-1}\cdot\nabla\beta \leq 0.$$

Inequality (4.6) leads to three potential relations

$$(4.7)\qquad \partial\psi/\partial\vartheta = -\eta, \qquad \rho\partial\psi/\partial\mathbf{F} = \mathbf{S}, \qquad -\rho\vartheta\partial\psi/\partial\nabla\beta(\partial\mathfrak{f}/\partial\vartheta) = \mathbf{q}$$

and the reduced inequality

$$(4.8)\qquad \mathbf{q}(\partial\mathfrak{f}/\partial\beta)(\partial\mathfrak{f}/\partial\vartheta)^{-1}\cdot\nabla\beta \geq 0.$$

Two first relations are well known results in the theory of themoelastic solids, while the last one $(4.7)_3$ is a new potential relation for the heat flux. Finally the inequality (4.8) could be regarded as a compatibility condition for $\mathfrak{f}$. In particular, under the assumption a) $(4.7)_3$ and (4.8) together with the hypothesis of nonvanishing dependence of $\psi$ on $\nabla\beta$, we obtain

$$(4.9)\qquad (\partial\psi/\partial\nabla\beta)\cdot\nabla\beta \geq 0.$$

We make now the further constitutive assumptions

$$(4.10)\qquad \psi = a(\vartheta)(\operatorname{tr}\mathbf{F}\mathbf{F}^T - 3) + b(\vartheta)J + 0.5\varepsilon_2\nabla\beta^2, \qquad \mathbf{q} = -k_0\nabla\beta,$$

where $J = \det\mathbf{F}$, $\varepsilon_2 = (k_0\tau_0)/(\rho\vartheta^0)$, and $a(\vartheta)$ and $b(\vartheta)$ are temperature-dependent generalized elastic moduli. In order to satisfy both (4.12) and (4.7), it is necessary that

$$(4.11)\qquad (\vartheta^0/\tau_0)\vartheta^{-1} = \partial\mathfrak{f}/\partial\vartheta.$$

A simple integration from $\vartheta^0$ to $\vartheta$ gives

$$(4.12)\qquad \mathfrak{f}(\vartheta,\beta) = \vartheta^0\tau_0^{-1}\log(\vartheta/\vartheta^0) + \mathfrak{f}(\vartheta^0,\beta),$$

where $\vartheta^0$ is a reference temperature. If, as in the case of the rigid conductor, we assume

$$(4.13)\qquad \mathfrak{f}(\vartheta,\beta) = \vartheta^0\tau_0^{-1}\log(\vartheta/\vartheta^0) - (\beta-\beta_0)/\tau_0,$$

then we can rewrite the balance of linear momentum (when the body forces vanish) as

$$(4.14)\qquad \ddot{\mathbf{u}} = \operatorname{Div}\partial\psi/\partial\mathbf{F},$$

where $\mathbf{u}$ means the displacement vector. Now, if we recall the well-known formula $(\partial J/\partial\mathbf{F}) = J\mathbf{F}^{-T}$, then we obtain

$$(4.15)\qquad \mathbf{S} = \partial\psi/\partial\mathbf{F} = 2a\mathbf{F} + bJ\mathbf{F}^{-T}.$$

Moreover, if we confine ourselves to the case of small strains and admit the following approximation

$$(4.16)\qquad \mathbf{F}^{-1} \cong \mathbf{I} - \nabla\mathbf{u},$$

then we get

$$(4.17)\qquad \ddot{\mathbf{u}} - A\Delta\mathbf{u} + \mathbf{B}\cdot\nabla\mathbf{u} + \mathbf{C} = 0,$$

where

$$A = 2a - b, \qquad \mathbf{B} = (b' - 2a')\nabla\vartheta, \qquad \mathbf{C} = -(2a' + b')\nabla\vartheta;$$

the symbol $'$ means the derivative with respect to $\vartheta$ and $\mathbf{I}$ is the identity tensor. We note that, owing to the extended kinetic equation, the functions $\mathbf{B}$ and $\mathbf{C}$ depend on $\vartheta$ and its derivatives: so a thermoelastic coupling is present. The heat equation will be obtained under some simplifications. First of all we assume the body is incompressible so that $\dot{J} = J \operatorname{Div} \mathbf{v} = 0$.

Moreover, by (4.1) it follows

$$\rho\dot{\varepsilon} - \mathbf{S} : \nabla \mathbf{v} + \operatorname{Div} \mathbf{q} - \rho r = 0. \tag{4.18}$$

Due to Eqs. (4.15) and (4.18) and taking into account the incompressibility, we get the required equation

$$\tau_0 D\ddot{\beta} + D\dot{\beta} + k_0\tau_0 \nabla\beta \cdot \nabla\dot{\beta} - k_0\vartheta^0 \Delta\beta - \rho r\vartheta^0 + \vartheta^0 A \Delta\mathbf{u} \cdot \dot{\mathbf{u}} = 0, \tag{4.19}$$

where

$$D = -\rho\vartheta^2[a''(\operatorname{tr}\mathbf{F}\mathbf{F}^T - 3) + b''J]. \tag{4.20}$$

It is evident that both the equations (4.17) and (4.19) are separately hyperbolic if the free energy satisfies the strong ellipticity condition; in the present case this equation is equivalent to a nonnegative $A$, i.e.

$$A \geq 0, \text{ i.e. } 2a(\vartheta) \geq b(\vartheta). \tag{4.21}$$

For Eq. (4.17) nonvanishing characteristics speeds are given by

$$\nu^{\pm} = \pm A^{1/2}, \tag{4.22}$$

what means that the propagation is symmetric. The characteristic equation of (4.19) is

$$\tau_0 D \Lambda^2 + k_0\tau_0 \nabla\beta \cdot \mathbf{n}\Lambda - k_0\vartheta^0 = 0, \tag{4.23}$$

where $\mathbf{n}$ is the direction of propagation. Equation (4.23) has two real solutions

$$\Lambda^{\pm} = [-k_0\tau_0\nabla\beta \cdot \mathbf{n} \pm [(k_0\tau_0\nabla\beta \cdot \mathbf{n})^2 + 4\tau_0 D k_0 \vartheta^0]^{0.5}](2D\tau_0)^{-1} \tag{4.24}$$

provided $D$ is positive. (Note that $D$ represents the product of $\vartheta$ and the specific heat). Symmetric propagation happens only if $\nabla\beta \cdot \mathbf{n} = 0$: the temperature gradient is orthogonal to the direction $\mathbf{n}$. This analysis correspond to the case in which thermomechanical coupling is neglected. Let us return to the full coupled system. In order to write (4.17) and (4.19) as a first order system let us put

$$\mathbf{H} = \nabla\mathbf{u}, \quad \mathbf{z} = \nabla\beta, \quad \gamma = \dot{\beta}.$$

We get, in a Cartesian material coordinate system $X_j$, $j = 1, 2, 3$,

$$\begin{gathered}
\dot{v}_i - A\partial H_{ij}/\partial X_j + B_j H_{ij} + C_i = 0, \\
\tau_0 D\dot{\gamma} + D\gamma + k_0\tau_0 z_i \dot{z}_i - k_0\vartheta^0 \partial z_i/\partial X_i - \rho r\vartheta^0 + \vartheta^0 A H_{ij}\dot{H}_{ij} = 0, \\
\dot{z}_i - \partial\gamma/\partial X_i = 0, \quad \dot{H}_{ij} - \partial v_i/\partial X_j = 0, \quad i, j = 1, 2, 3.
\end{gathered} \tag{4.25}$$

Moreover, tedious but straightforward calculations show that (4.25) is a quasi-linear first order symmetric system of the type

$$\mathbf{A}^0(\mathbf{w})\partial\mathbf{w}/\partial t + \mathbf{A}^i(\mathbf{w})\partial\mathbf{w}/\partial X_i = \mathcal{G}(\mathbf{w})$$

for the unknown 16 row vector $\mathbf{w} = [\mathbf{v}, \gamma, \mathbf{z}, \mathbf{H}]^T$, where $\mathbf{A}^i$, $i = 1, 2, 3$ are suitable $16 \times 16$ matrices, the matrix $\mathbf{A}^0$ is positive defined for $A$ positive (cf. [9] for definition). This enables us to formulate

PROPOSITION 1. A local in time Cauchy problem for (4.25) is well-posed, i.e. solution to a Cauchy problem exists, is unique and continuously depends on the initial data, in the Sobolev space $\mathcal{H}^s$, for $s \geq 2$.

As far as acceleration waves are concerned, in thermoelastic materials with the Fourier law the blow-up occurs even if $\tau_0$ is equal to zero, [37]. In [36] an evolution equation for the amplitude was derived for thermoelastic solids with thermal relaxation. The equation is again of Bernoulli type and reduces to the equation obtained in [37] when $\tau_0$ is equal to zero.

At the end of the paper let us shortly discuss the important feature of the present approach related to the dependence of speeds of propagation of acceleration waves on the relaxation time. In a $1D$ case the coupled system of equations (4.1) and (2.15) with the $RHS$ given by (4.13) written on an acceleration wave-front, leads to a fourth-order algebraic characteristic equation (cf. the derivation performed in [36] for general constitutive equations and Eq. (4.13) there). Its coefficients depend on the state ahead of the front, as well as on the thermal relaxation time $\tau_0$. We are not going to repeat this equation and the long expressions for its coefficients. We are going, rather, to point out results of the limit analysis.

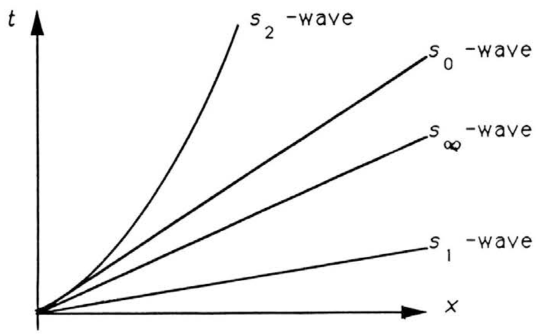


FIG. 1.

Let us denote by $s_1$ and $s_2$ the roots of the characteristic equation of the coupled system of equations derived for the $1D$ problem (cf. Fig. 1), where $s_1$ represents the fastest wave travelling into an unperturbed medium. (Note, that in the case of the wave propagating into an unperturbed region, the characteristic equation contains only even powers of the characteristic speed, except for a free term). When $\tau_0 \to 0$, the characteristic equation has two roots of opposite signs, $\pm s_0$. Let moreover $s_\infty$ denote the highest nonzero speed when $\tau_0 \to \infty$. The following propositions can be proved [36]:

PROPOSITION 2. If the fastest acceleration wave is moving into a body being in a state of rest, then its speed $s_1$ has the following properties:

$$\lim_{\tau_0 \to 0} s_1^2 = s_\infty^2, \qquad \lim_{\tau_0 \to \infty} s_1^2 = \infty,$$

$$s_1^2 > s_0^2, \quad s_1^2 > s_\infty^2$$

$s_1^2$ is a decreasing function of $\tau_0$. ■

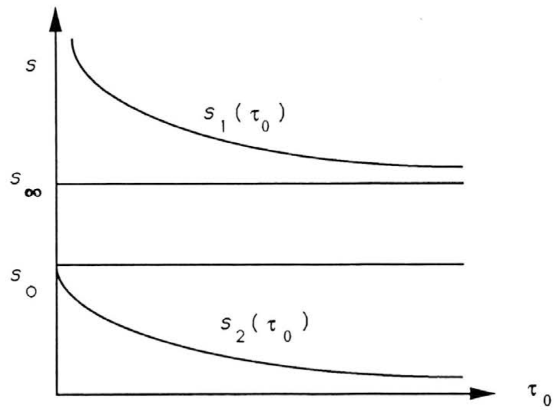


FIG. 2.

As far as the second (slower) wave is concerned, it propagates into a perturbed medium with speed $s_2$. However, we can assume that this state is not far from equilibrium. Under such a hypothesis we have (cf. Fig. 2)

PROPOSITION 3. If the second wave propagates into a state close to equilibrium, then

$$\lim_{\tau_0 \to 0} s_2^2 = s_0^2, \qquad \lim_{\tau_0 \to \infty} s_2 = 0;$$

moreover, the following inequalities hold

$$s_2^2 < s_0^2, \qquad s_2^2 < s_\infty^2,$$

and $s_2^2$ is a decreasing function of $\tau_0$. ∎

The above properties, together with the previously obtained result concerning the physical foundations of the new variable $\beta$ and its properties, form a strong background of the present approach in description of the second sound effect and thermal waves. The latter seems to be important at interfaces at very high temperatures as well [38].

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