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Investigation of shock wave structure in elasto-visco-plastic bars using the asymptotic method

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THE PROPAGATION of a shock wave in a bar of elasto-visco-plastic material is investigated by an asymptotic method. The influence of the transverse motion of the bar, the static diagram of the material, as well as the strain rate influence on the shock wave profile are investigated. It is shown, that depending on the ratio of geometrical and physical parameters characterizing the bar, the shock wave is of a qualitatively different type, either monotone or oscillating. The problem of the influence of approximative effects on the structure of the shock wave in the numerical integration is investigated using the same method of solution.

Zastosowano metodę asymptotyczną do badania procesu rozchodzenia się fali uderzeniowej w prętach sprężysto-lepkoplastycznych. Zbadano wpływ ruchu pręta w kierunku poprzecznym, statycznej charakterystyki materiału jak również prędkości odkształcenia na profil fali uderzeniowej. Wykazano, że — w zależności od układu geometrycznych i fizycznych parametrów charakteryzujących pręt — typy powstałych fal uderzeniowych mogą być jakościowo różne, monotoniczne lub oscylujące. Tą samą metodą przeanalizowano zagadnienie wpływu przybliżeń na strukturę fali uderzeniowej w procesie całkowania numerycznego.

В работе асимптотическим методом исследуется распространение ударной волны в стержне из упруго вязко-пластического материала. Исследуется влияние поперечного движения, статической диаграммы материала, а также скорости деформации на профиль ударного перехода. Показано, что в зависимости от соотношения между геометрическими и физическими параметрами стержня, профиль ударной волны будет либо монотонным, либо осцилирующим. Тем же методом исследуется вопрос о влиянии на структуру сдарной волны аппроксимационных эффектов при численном решении конечно-разноутных уравнений распространения волн.

1. Introduction

IT IS WELL known that the elementary theory of longitudinal waves in an elastic bar based on the hypothesis of the plain sections gives satisfactory results, provided $r_0 \ll \Lambda$, where r_0 is the characteristic transverse dimension and Λ is the wave length. On the other hand, the theory based on the exact equations of the elasticity theory is rather complicated. This is why a number of authors tried to create an approximate one-dimensional theory which would take into account the influence of the transverse motion and would permit the correct description of the short impulse propagation.

The first work in this field was the work of LOVE [1]. He considered that the transverse displacement $w = -\nu r\partial u/\partial x$, where ν is the Poisson coefficient and u is the longitudinal displacement component. Having additionally introduced the term corresponding to the transverse motion in the expression for the kinetic energy, Love by the variational method obtained the equation

$$(1.1) \qquad \frac{\partial\sigma}{\partial x} - \varrho\frac{\partial v}{\partial t} = \varrho r_0^2\nu^2\frac{\partial^3 v}{\partial x^2\partial t}, \qquad v = \frac{\partial u}{\partial t}$$

which differs from the elementary one by the right hand term corresponding to the kinetic energy of the transverse motion.

MINDLIN and HERMANN [2] „weakened" Love's assumptions. They asumed that

$$u_x = u(x,t), \qquad u_y = \frac{r}{r_0} w(x,t)$$

and from the precise equations of the elasticity theory by integrating over r they obtained a fourth-order equations system with respect to the derivatives over x for the functions u and w. Still more precise equations were obtained in [3], where in the expansions u_x and u_y over the radius r two terms in each equation were left:

$$u_x = u_0(x,t)+r^2u_1(x,t)+\ \dots,$$
$$u_y = rw_0(x,t)+r^3w_1(x,t)+\ \dots.$$

This led to a sixth-order system of differential Eqs. in x for the unknown functions u_i and w_i.

In [4] it was shown that at $t \gg r/c$, where $c = E^{1/2}\varrho^{-1/2}$, the solutions on the basis of all the three theories are equivalent to that obtained on the basis of Love's equation (1.1). Moreover, for the problem of a constant velocity impact the solutions coincide with the asymptotic solution obtained in [5] on the basis of precise equations of the elasticity theory.

Since Love's hypothesis is of a purely kinematic character, the results of [4] give every reason to proceed just as well from the equation of motion (1.1) in the investigation of nonelastic waves in the bars, when $t_0 \gg r/c$.

In [6] an asymptotic method was proposed for the solution of problems of wave propagation in elasto-visco-plastic bars, and on the basis of an elementary theory, the shock wave structure was investigated. Let us examine this problem proceeding from Eq. (1.1).

2. Investigation of the shock wave structure in an elasto-visco-plastic bar

For the description of elasto-plastic properties of a material, depending on the strain rate, let us proceed from the Sokolovsky–Malvern model

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{E}\frac{\partial \sigma}{\partial t} + \frac{\operatorname{sgn}\sigma}{\tau}\hat{\Phi}[|\sigma| - \sigma_S(\varepsilon)],$$
$$\hat{\Phi}(z) = \begin{cases} \Phi(z), & z \geqslant 0, \\ 0, & z < 0, \end{cases} \tag{2.1}$$

where the type of the function $\Phi(z)$ and the value of the constant τ are defined on the basis of dynamic experimental data given, for example, in [7]; $\sigma = \sigma_S(\varepsilon)$ is the static tension-compression diagram of the material.

Let us introduce the dimensionless variables, where v_0 is the initial velocity and t_0 is the characteristic time of the problem

$$\bar{x} = x/ct_0, \quad \bar{t} = t/t_0, \quad \bar{v} = v/v_0, \quad \bar{\sigma} = \sigma/\sigma_0, \quad \bar{\varepsilon} = \varepsilon/\varepsilon_0,$$
$$\sigma_0 = \varrho c v_0, \quad E\varepsilon_0 = \sigma_0. \tag{2.2}$$

By adding to Eqs. (1.1) and (2.1) the equations of compatibility of velocity and deformaion fields, we obtain a complete system of equations:

(2.3) $$\frac{\partial \sigma}{\partial x} = \frac{\partial v}{\partial t} + \gamma \frac{\partial^3 v}{\partial x^2 \partial t}, \quad \frac{\partial v}{\partial x} = \frac{\partial \varepsilon}{\partial t}, \quad \omega\left(\frac{\partial \varepsilon}{\partial t} - \frac{\partial \sigma}{\partial t}\right) = \hat{\Phi}[|\sigma| - \sigma_S(\varepsilon)].$$

Here and further on the dash over the dimensionless variables is omitted.

We shall consider the case when both parameters γ and ω are simultaneously small, but generally speaking, have different orders of smallness. It is convenient to note this parametrically as follows:

(2.4) $$\gamma = k_1 \delta^m, \quad \omega = k_2 \delta^n,$$

where δ is a small value, $k_1 \sim O(1)$, $k_2 \sim O(1)$ and m and n are given integers. We shall seek the solution of the equation as a power expansion over a small parameter δ:

(2.5) $$U = U^{(0)} + \delta U^{(1)} + \delta^2 U^{(2)} + \ldots,$$

where U is the solution vector of the system (2.3).

By substituting Eq. (2.5) into Eq. (2.3) and equating the terms with the same powers of δ for a zero approximation, we obtain

(2.6) $$c^2(\varepsilon_0) \frac{\partial \varepsilon_0}{\partial x} = \frac{\partial v_0}{\partial t}, \quad \frac{\partial \varepsilon_0}{\partial t} = \frac{\partial v_0}{\partial x}, \quad c^2(\varepsilon) = \frac{d\sigma_S(\varepsilon)}{d\varepsilon}.$$

It is a well-known system of equations that describes the wave propagation in a bar of a nonlinear-elastic material with the dependence $\sigma = \sigma_S(\varepsilon)$.

For the following approximations we obtain the same equations as for a nonhomogeneous linear plastic m aterial, the right hand parts depending on the previous approximations. For instance, the first approximation is described by the system

(2.7) $$\frac{\partial \sigma^{(1)}}{\partial x} - \frac{\partial v^{(1)}}{\partial t} = k_1 \frac{\partial^3 v^0}{\partial x^2 \partial t}, \quad \frac{\partial \varepsilon^{(1)}}{\partial t} = \frac{\partial v^{(1)}}{\partial x},$$
$$\sigma^{(1)} - \left.\frac{d\sigma_S}{d\varepsilon}\right|_{\varepsilon = \varepsilon^{(0)}} \varepsilon^{(1)} = \frac{k_2}{\Phi'_{(0)}} \left(\frac{\partial \varepsilon^{(0)}}{\partial t} - \frac{\partial \sigma^{(0)}}{\partial t}\right),$$

etc.

The right hand parts with the coefficients k_1 and k_2 may be either present or absent depending on the relationship between m and n. But the zero approximation (2.6) does not change its form. Here we shall restrict ourselves to obtaining the solution of a zero approximation uniformly precise over the whole region of motion. It is at once clear that the solution of the system (2.6) does not give such a solution. Really, the system (2.6) is a quasi-linear system of equations which admits the availability of discontinuous solutions at continuous initial and boundary conditions if the function $\sigma_S(\varepsilon)$ is concave ($\sigma_S''(\varepsilon) > 0$), and at the discontinuous boundary or initial conditions if $\sigma_S(\varepsilon)$ is a piecewise function. (The last case is very important, as such an approximation is often used in calculations for curves having $\sigma_S''(\varepsilon) < 0$) .

At the same time it is easy to prove that the original system of equations (2.3) is parabolic and odes not admit discontinuous solutions. That is why the solutions of the systems (2.3) and (2.6) in the vicinity of the discontinuity lines $x = x(t)$ will differ by $O(1)$.

Therefore, in the vicinity of $x = x(t)$ the formal expansion (2.5) is unfit. The reason is that we do not take into account the rapid change of the solution in the direction orthogonal to the discontinuity lines $x = x(t)$. To construct the expansion in the vicinity of the discontinuity line, let us use new variables in Eqs. (2.3)

$$\beta = \frac{x(t)-x}{\lambda}, \quad \alpha = t. \tag{2.8}$$

Here λ is a small parameter which characterizes the rate of the solution change in the vicinity of the shock transition. The stretching over the coordinate β by the value of λ^{-1} is introduced for the derivatives over x to be the values of the order of $O(1)$.

With new variables the system of equations (2.3) will be noted as

$$\begin{gathered}\frac{1}{\lambda}\frac{\partial\sigma}{\partial\beta} = \frac{\gamma}{\lambda^2}\left(\frac{D}{\lambda}\frac{\partial}{\partial\beta}+\frac{\partial}{\partial\alpha}\right)\frac{\partial^2 v}{\partial\beta^2} - \frac{D}{\lambda}\frac{\partial v}{\partial\beta} - \frac{\partial v}{\partial\alpha}, \quad D = x'(t),\\ \frac{1}{\lambda}\frac{\partial v}{\partial\beta} = \frac{\partial\varepsilon}{\partial\alpha} + \frac{D}{\lambda}\frac{\partial\varepsilon}{\partial\beta}, \quad \lambda\left(\frac{D}{\lambda}\frac{\partial}{\partial\beta}+\frac{\partial}{\partial\alpha}\right)(\varepsilon-\sigma) = \hat{\Phi}[\sigma-\sigma_S(\varepsilon)].\end{gathered} \tag{2.9}$$

Since two small parameters of different orders are present in our system, three different cases depending on their order should be considered separately for the case of a concave static diagram $\sigma_S(\varepsilon)$ and for the case of a segmental linear diagram, where $\sigma'(\varepsilon_{k+1}) < \sigma'(\varepsilon_k)$ at $\varepsilon_{k+1} > \varepsilon_k$.

The concave diagram

The small parameter λ characterizing the rate of the solution change is defined — on the basis of the comparison of the orders of senior terms in each of the equations of the system (2.9).

1. If $2n > m$, then in the first equation the senior terms corresponding to the inertia of the transverse and longitudinal motion are of the same order and $\lambda = \gamma^{1/2}$.

2. If $2n < m$, then in the third equation the terms corresponding to elastic and visco-plastic forces are of the same order and $\lambda = \omega$.

3. If $2n = m$, then in Eqs. (2.9) all the above listed terms are of the same order and $\lambda = \gamma^{1/2} = \omega$.

We shall look for the solution in the vicinity of the discontinuity line in the form of a power series over λ

$$U = U_0 + \lambda U_1 + \lambda^2 U_2 + \ldots + \lambda^n U_n + \ldots.$$

For the zero approximation U_0 we get the following system of equations:

$$\begin{gathered}\frac{\partial\sigma_0}{\partial\beta} = D\left(\frac{\partial v_0}{\partial\beta} + b_1\frac{\partial^3 v_0}{\partial\beta^3}\right), \quad D\frac{\partial\varepsilon_0}{\partial\beta} = -\frac{\partial v_0}{\partial\beta},\\ \Phi[\sigma-\sigma_S(\varepsilon_0)] = b_2 D\frac{\partial}{\partial\beta}(\varepsilon_0-\sigma_0).\end{gathered} \tag{2.10}$$

The coefficients b_i are introduced for the convenience of notation $b_1 = b_2 = 1$, when $2n = m$, at $2n > m$, $b_1 = 1$, $b_2 = 0$ and $2n < m$, $b_1 = 0$, $b_2 = 1$. We shall investigate the general case when both coefficients differ from zero. The system (2.10) is a system of common differential equations. It is interesting to note that at $S = \text{const}$ it coincides with the system of equations describing stationary solutions of precise equations (2.3).

Boundary conditions for the system (2.10) are found from the coalescence conditions of the slowly changing solution of the system (2.6) and the quickly changing solution (2.10). If we denote the solution (2.6) before the front U^- and behind the front U^+, then β the boundary conditions may be noted as follows

$$U_0|_{\beta=-\infty} = U^-, \quad U_0|_{\beta=\infty} = U^+. \tag{2.11}$$

As a rule, in most metal bars $D^2 \ll 1$; then, in the constitutive equation (2.10) for the sake of simplicity we may neglect the term corresponding to elastic deformation in comparison with visco-plastic deformation.

Integrating the first equation (2.10) and excluding σ and v, we obtain

$$b_1 \frac{d^2\varepsilon}{d\beta^2} + F\left(b_2 D \frac{d\varepsilon}{d\beta}\right) - D^2\varepsilon + \sigma_S(\varepsilon) - (\sigma^- - D^2\varepsilon^-) = 0. \tag{2.12}$$

Here $F(z)$ is the inverse function of $\Phi(z)$.

Let us conduct qualitative research of the integral curves of Eq. (2.12). For this purpose let us examine the phase plane (e, ε) and transform Eq. (2.12) to the form of

$$e = d\varepsilon/d\beta, \quad de/d\varepsilon = [D^2\varepsilon - \sigma_S(\varepsilon) + (\sigma^- - D^2\varepsilon^-) - F(b_2De)]/b_1 e = M(e, \varepsilon)/L(e, \varepsilon). \tag{2.13}$$

Equation (2.13) has two singular points which correspond to the values before the front $\varepsilon = \varepsilon^-$, $e = 0$ and behind it $\varepsilon = \varepsilon^+$, $e = 0$. Thus the integral curve corresponding to the unknown solution of the shock transition must pass through both singular points. Let us prove that the solution of this problem exists and is unique if $M(e, \varepsilon)$ is a differentiable function. The curves $M(e, \varepsilon) = 0$, $L(e, \varepsilon) = 0$ shown in Fig. 1 and 2 divide

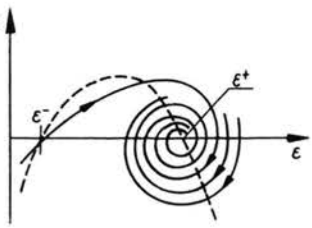

FIG. 1.

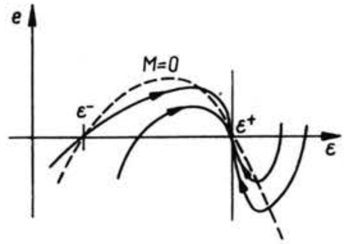

FIG. 2.

the semiplane in the vicinity of singular points into four domains, where the components M and L change their sign. The field of directions of the integral curves is shown in Figs. 1 and 2. The type of singular points is defined on the basis of the type of the roots of the equation

$$\begin{vmatrix} \frac{\partial M}{\partial e} - \chi & \frac{\partial L}{\partial e} \\ \frac{\partial M}{\partial \varepsilon} & \frac{\partial L}{\partial \varepsilon} - \chi \end{vmatrix} = 0.$$

Substituting the values $M(e, \varepsilon)$ and $L(e, \varepsilon)$, we find

$$\chi_{1,2} = -\frac{b_2DF'(0)}{2} \pm \sqrt{\left[\frac{b_2DF'(0)}{2}\right] + b_1[D^2 - \sigma_S'(\varepsilon^{\mp})]}.$$

Taking into account the fact that $D^2 - \sigma_S(\varepsilon^-) > 0$ and $D^2 - \sigma_S(\varepsilon^+) < 0$, we conclude that in the point $\varepsilon = \varepsilon^-$, $e = 0$ is always the saddle point and in the point $\varepsilon = \varepsilon^+$, $e = 0$ is the focus if $b_1|D^2 - \sigma_S'(\varepsilon^+)| > \left[\frac{b_2 DF'(0)}{2}\right]$ and is a nod of the reverse inequality bodies.

The type of the integral curves passing through both singular points corresponding to these two possibilities is shown in Fig. 1 and Fig. 2, respectively.

The character of change ε in the shock transition zone is shown in Figs. 3 and 4. From those figures one can see that when viscous forces prevail over the forces of transverse

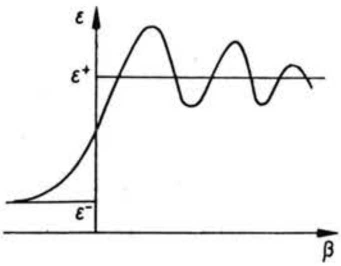

FIG. 3.

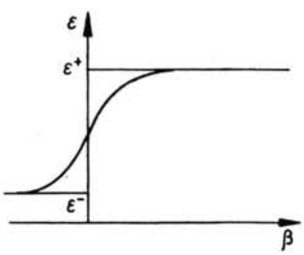

FIG. 4.

inertia, the wave profile is smooth and when the inverse inequality holds, oscillations are observed behind the front.

Note that at $F'(0) = 0$ the singular point $\varepsilon = \varepsilon^+$, $e = 0$ will be the centre, i.e. behind the wave front the investigation on the basis of the linear approximation gives nonattenuating oscillations. However, the oscillations will be attenuating if we take into account the nonlinear terms of Eq. (2.13).

When $b = 0$, from Eq. (2.13) we see that the integral curve tends to the line segment $M = 0$, connecting the singular points; the solution in this case is of the type shown in Fig. 4.

Piecewise diagram

The given solution in the shock transition field is true if only the square of the shock wave velocity is $D^2 \not\equiv \sigma_S'(\varepsilon)$. This condition is severed when the diagram $\sigma = \sigma_S(\varepsilon)$ is a piecewise convex one, i.e. $\sigma_S'(\varepsilon_{k+1}) < \sigma_S'(\varepsilon_k)$ at $\varepsilon_{k+1} > \varepsilon_k$. In the case of such a diagram, shock waves appear in the solution of the system (2.6) at the discontinuity boundary and initial conditions, and they coincide with the characteristics of the system. Provided $D^2 \equiv \sigma_S'(\varepsilon)$, some of the main terms of Eqs. (2.9) disappear and this makes it necessary to take into account the terms omitted earlier. To make it obvious, let us substitute $z = \sigma - \sigma_S(\varepsilon)$ in Eq. (2.9)

$$(2.14) \qquad \frac{1}{\lambda}\left[\frac{\partial z}{\partial \beta} + \sigma_S'(\varepsilon)\frac{\partial \varepsilon}{\partial \beta}\right] = \frac{D}{\lambda}\frac{\partial v}{\partial \beta} + \frac{\partial v}{\partial \alpha} - \frac{\gamma}{\lambda^2}\left(\frac{D}{\lambda}\frac{\partial}{\partial \beta} + \frac{\partial}{\partial \alpha}\right)\frac{\partial^2 v}{\partial \beta^2}$$

$$-\frac{\partial v}{\partial \beta} = \lambda\frac{\partial \varepsilon}{\partial \alpha} + D\frac{\partial \varepsilon}{\partial \beta}\,\omega\left(\frac{D}{\lambda}\frac{\partial}{\partial \beta} + \frac{\partial}{\partial \alpha}\right)\big(\varepsilon - z - \sigma_S(\varepsilon)\big) = \Phi(z).$$

In the first equation the underlined terms were reduced; the force of inertia in the longitudinal direction corresponds to the term $\partial v/\partial \alpha$; this is why the conditions $\lambda = \gamma^{1/3}$ and $z \sim O(\lambda)$ should be observed. Since we consider the general case, when the terms corresponding to the inertia of the transverse motion and viscous resistance are of the same order, from the third equation we get $\lambda^{k+1} = \omega$. The value of k is defined by the behaviour of the function $\Phi(z) \sim Cz^k$ at $z \to 0$. Hence we find that $\omega \sim O\left(\gamma\frac{k+1}{3}\right)$. From the above analysis it follows that the solution of Eqs. (2.14) is to be sought in the form of the following expansion:

$$v = v_0+\lambda v_1+\lambda^2 v_2+ \ldots, \qquad \varepsilon = \varepsilon_0+\lambda\varepsilon_1+ \ldots +\lambda^n\varepsilon_n+ \ldots,$$
$$z = \lambda z_0+\lambda^2 z_1+ \ldots +\lambda^n z_n+ \ldots. \tag{2.15}$$

A comparison of the power expansions for concave and λ convex diagrams makes it clear that the differences in the solutions for two types of diagrams are connected with the deviation of the actual dependence $\sigma = \sigma(\varepsilon)$ in the transition zone from the static law $\sigma = \sigma_S(\varepsilon)$. For a concave diagram this deviation is of the order $O(1)$, and for a convex one — $O(\lambda)$. Substituting the expansions (2.15) into Eqs. (2.14), we find the following equations for the zero approximation

$$\frac{\partial z_0}{\partial \beta} = 2\frac{\partial v_0}{\partial \alpha} - b_1 D\frac{\partial^3 v_0}{\partial \beta^3}, \qquad \frac{\partial v_0}{\partial \beta} = -D\frac{\partial \varepsilon_0}{\partial \beta},$$

$$z_0^k = -D(1-D^2)\cdot b_2\frac{\partial \varepsilon_0}{\partial \beta} = (1-D^2)\cdot b_2\frac{\partial v_0}{\partial \beta}.$$

Excluding z_0 and ε_0 we get the equation

$$A\frac{\partial}{\partial \beta}\left(\frac{\partial v_0}{\partial \beta}\right)^p = 2\frac{\partial v_0}{\partial \alpha} - b_1 D\frac{\partial^3 v_0}{\partial \beta^3}, \qquad A = b_2(1-D^2)^p, \qquad p = k^{-1}. \tag{2.16}$$

The solution of this partial differential equation should satisfy the condition (2.11). In the problem considered the initial conditions are of no importance; they may be of any type not contradicting the conditions (2.11). In the general case, the solution of Eq. (2.16) in the closed form is impossible to obtain. Nevertheless, in a particular case when the conditions before and behind the front do not vary, we can get the solution of the posed problem in the class of automodel solutions. Supposing that $v_0 = v(\xi)$, where $\xi = \beta\alpha^q$, from Eq. (2.16) we find

$$pAv''(v')^{p-1}\alpha^{q(p+1)} = 2v'q\alpha^{q-1}\beta - b_1 Dv'''. \tag{2.17}$$

We can see from Eq. (2.17) that there always exists an automodel solution when both effects act simultaneously at $p = 2$, $q = -1/3$. From Eq. (2.17) we find that it should satisfy the equation

$$b_1 Du''+2Au'u+\frac{2}{3}\xi u = 0, \qquad u = v' \tag{2.18}$$

and the boundary conditions

$$v|_{\xi=-\infty} = v^-, \qquad v|_{\xi=\infty} = v^+, \qquad u < M, \qquad \xi \to \pm\infty.$$

The qualitative research of this problem shows the solution gives a smooth profile at $b_1 \ll 1$ and finite b_2, and behind the wave front there will be oscillations about the value $v = v_0$ when b_1 is the finite value and b_2 is small.

Besides, the automodel solution of Eq. (2.17) always exists if $b_1 = 0$ or $A = 0$.

In the considered case of the piecewise diagram of the material $b_1 = 0$, if $n < m(k+1)/3$ and $A = 0$ either in the case of a purely elastic material when $D^2 = 1$, or at $b_2 = 0$, when $n > m(k+1)/3$.

At $A = 0$ we find that $p = -1/3$ and Eq. (2.17) becomes

$$\frac{d^3 v}{d\xi_1^3} + \frac{2}{3}\xi_1 \frac{dv}{d\xi_1} = 0, \quad \xi_1 = \xi D^{-1/3}.$$

The solution satisfying the conditions (2.11) is of the type

$$v = v^- + (v^+ - v^-)/\pi\left[\pi/3 - \int_0^{\xi_1} \mathrm{Ai}(\tau)d\tau\right], \tag{2.19}$$

where $\mathrm{Ai}(x)$ is the Airy function. At $D = 1$ the solution (2.19) coincides with the asymptotic solution obtained for an elastic bar on the basis of the precise solution of a three-dimensional theory [5]. The character of the solution is qualitatively the same as in Fig. 3.

Let us consider more carefully the case when $b_1 = 0$

$$\left(\frac{dv}{d\xi}\right)^{-(p+2)} \frac{d^2 v}{d\xi^2} + \frac{Ap^2\xi}{p+1} = 0. \tag{2.20}$$

The solution is of different types depending on the value of k.

At $k = 1$ we find

$$v = v^- + \frac{v^+ - v^-}{2}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\xi^2}d\xi\right],$$
$$\xi = \frac{x - Dt}{2\sqrt{t}}\omega^{1/2}(1-D^2)^{-1/2}. \tag{2.21}$$

The effective shock wave width is defined according to the formula

$$\Delta x = \frac{|v^+ - v^-|}{\max\left(\frac{dv}{dx}\right)} = \left(\frac{2\pi t}{\omega}\right)^{1/2}(1-D^2)^{1/2},$$

wherefrom one can see that it does not depend on the shock wave intensity and increases in time proportionally to $t^{1/2}$.

If $k > 1$, the solution (2.20) is of the type

$$v = \frac{v^+ + v^-}{2} + \frac{v^+ - v^-}{2I_1}\int_0^{z} \frac{dz}{(1+z^2)^{|p|}}, \tag{2.22}$$

$$z = \xi\frac{|B|^{1/2}}{C}, \quad B = \frac{k(1-k)}{1+k}(1-D^2)^{-\frac{1}{k}}, \quad C = \left(\frac{v^- - v^+}{2I_1}|B|^{1/2}\right)^{\frac{1}{-2|p|+1}},$$

(22.2) [cont.]
$$\xi = \frac{x-Dt}{t^{k/(k+1)}}\omega^{1/(k+1)}, \quad I_1 = \int\limits_0^\infty \frac{dz}{(1+z^2)^{|p|}} = \frac{\Gamma(1/2)\Gamma\left(|p|-\frac{1}{2}\right)}{2\Gamma(|p|)}.$$

Here $\Gamma(z)$ is a gamma function. In this case the transition layer over the variable ξ has an infinite width, the tendency to the values v^+ and v^- over the infinity being nonexponential as in the case of $k = 1$, but a power one, so that the effective width Δx increases consider ably faster than in the linear case

$$\Delta x = \omega^{-1/k+1}t^{k/k+1}\Delta\xi,$$

where $\Delta\xi$ is the effective width of the transition layer over ξ.

At $k < 1$ the transition layer over the variable ξ has the finite thickness $2\xi_0$; instead of the conditions (2.11) one should take the boundary conditions over the interval $[0, \xi_0]$

(2.23)
$$v(0) = (v^+ - v^-)/2, \quad v(\xi_0) = v^+, \quad v'(\xi_0) = 0.$$

The solution of Eq. (2.20) satisfying these conditions is

$$v = \frac{v^+ + v^-}{2} + \frac{v^+ - v^-}{2I_0}\int\limits_0^{\xi/\xi_0}(1-z^2)^p dz,$$

(2.24)
$$\xi_0 = \left(\frac{v^+ - v^-}{2B^pI_0}\right)^{1/(2p+1)}, \quad p = \frac{k}{1-k},$$

$$I_0 = \int\limits_0^1 (1-z^2)^p, \quad dz = \frac{\Gamma(p+1)\Gamma(1/2)}{2\Gamma(p+3/2)}.$$

The shock wave width increases with time in this case but is slower than the effective width in the case of a linear function

$$\Delta x = 2\xi_0\omega^{-\frac{1}{k+1}}t^{\frac{k}{k+1}}.$$

Figure 5 shows the change of the shock wave profile with the change of k.

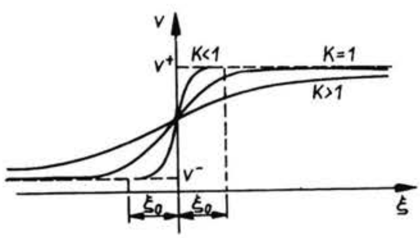

FIG. 5.

3. Shock wave structure at the difference approximation of equations

Now let us see how small additional terms arising in the process of numerical integration of equations as a result of a difference approximation influence the shock wave structure. It is convenient to investigate the simplest case of a discontinuity wave in plastic bar on the basis of an elementary theory. The system of equations

$$\partial\sigma/\partial x = \varrho\partial v/\partial t, \quad \partial\sigma/\partial t = E\partial v/\partial x \tag{3.1}$$

may be written in dimensionless variables

$$\bar{x} = x/l, \quad \bar{t} = tc/l, \quad \bar{\sigma} = \sigma/\varrho c v_0, \quad \bar{v} = v/v_0 \tag{3.2}$$

and in terms of Riemann invariants $q = \bar{\sigma}+\bar{v}$, $p = \bar{\sigma}-\bar{v}$

$$\partial p/\partial t+\partial p/\partial x = 0, \quad \partial q/\partial t-\partial q/\partial x = 0. \tag{3.3}$$

Let us consider the numerical predictor-corrector scheme for Eqs. (3.3)

$$\begin{aligned} p_{i+1/2}^{n+1/2} &= \frac{1}{2}(p_i^n+p_{i+1}^n)-\frac{\varkappa\tau}{h}(p_{i+1}^n-p_i^n),\\ p_{i-1/2}^{n+1/2} &= \frac{1}{2}(p_i^n+p_{i-1}^n)-\frac{\varkappa\tau}{h}(p_i^n-p_{i-1}^n),\\ p_i^{n+1} &= p_i^n-\frac{\tau}{h}(p_{i+1/2}-p_{i-1/2})^{n+1/2}. \end{aligned} \tag{3.4}$$

The notations are clear from Fig. 6 where the scheme pattern is shown.

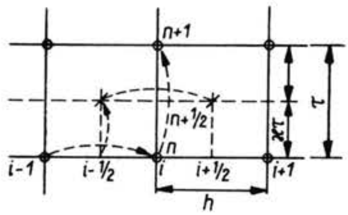

FIG. 6.

First, the solution may be defined at the intermediate step at the points with half-integral indices shown with crosses at the Lax scheme; and then at the points of the basic network shown with circles according to the "cross" scheme.

Excluding the values with half-integral indices, we get

$$p_i^{n+1} = p_i^n-\tau/2h(p_{i+1}^n-p_{i-1}^n)+\frac{\varkappa\tau^2}{2h^2}(p_{i+1}-2p_i+p_{i-1})^n, \tag{3.5}$$

where $1/2 < \varkappa \leqslant h/2\tau$ and $\tau/h = 1$. The scheme (2.5) at $\varkappa = 1/2$ turns into the Lax–Wendroff scheme of the second order of accuracy and at $\varkappa$ into the scheme of Courant–Isaacson–Rees [8]. Expanding all the terms of the finite-difference equation (3.5) at the point with the indices (n, i) over the powers of τ and h, we get an equivalent differential equation of an infinite order:

$$\begin{aligned} &\frac{\partial p}{\partial t}+\frac{\partial p}{\partial x}+\sum_{k=1}^{\infty}\frac{\tau^{2k}}{(2k+1)!}\left(\frac{\partial^{2k+1}p}{\partial t^{2k+1}}+\left(\frac{h}{\tau}\right)^{2k}\frac{\partial^{2k+1}p}{\partial x^{2k+1}}\right)\\ &\qquad+\sum_{k=1}^{\infty}\frac{\tau^{2k-1}}{(2k)!}\left[\frac{\partial^{2k}p}{\partial t^{2k}}-2\varkappa\left(\frac{h}{\tau}\right)^{2k-2}\frac{\partial^{2k}p}{\partial x^{2k}}\right] = 0. \end{aligned} \tag{3.6}$$

Since in any explicit scheme $\tau/h = K$, $K \leqslant 1$ Eq. (3.6) contains one small parameter $\tau \ll 1$ at senior differential terms.

To solve this equation, let us apply the same asymptotic methos we used in Sect. 2. As before, we seek the slowly changing solution of the type

$$p = p^{(0)} + \tau p^{(1)} + \ldots + \tau^n p^{(n)} + \ldots. \tag{3.7}$$

Substituting the series (3.7) into Eq. (2.5) for p we get the equation

$$\partial p^{(0)}/\partial t + \partial p^{(0)}/\partial x = 0 \tag{3.8}$$

coinciding with the approximated equation. Since our aim is to find a uniformly exact zero approximation, we shall not write out the rest of the approximations, but turn to the solution in the vicinity of the discontinuity $x = t$. As before, let us introduce the variables

$$\beta = \frac{x-t}{\lambda}, \qquad \alpha = t,$$

then Eq. (3.6) will be of the following type:

$$\frac{\partial p}{\partial \alpha} + \sum_{m=1}^{\infty} \frac{\tau^{2m}}{(\alpha m+1)!} \left[\left(\frac{\partial}{\partial \alpha} - \frac{1}{\lambda} \frac{\partial}{\partial \beta} \right)^{2m+1} + \left(\frac{h}{\tau} \right)^{2m} \frac{\partial^{2m+1} p}{\partial \beta^{2m+1}} \frac{1}{\lambda^{2m+1}} \right] + \sum_{k=1}^{\infty} \frac{\tau^{2k-1}}{2k!} \left[\left(\frac{\partial}{\partial \alpha} - \frac{1}{\lambda} \frac{\partial}{\partial \beta} \right)^{2k} - 2\varkappa \left(\frac{h}{\tau} \right)^{2k-2} \frac{\partial^{2k} p}{\partial \beta^{2k}} \frac{1}{\lambda^{2k}} \right]. \tag{3.9}$$

We shall seek the solution in the form

$$p = p_0 + \lambda p_1 + \ldots + \lambda^n p_n + \ldots. \tag{3.10}$$

It is easy to see that at $\varkappa > 1/2$ the main term of the expansion will be in the second sum, then $\lambda = \tau^{1/2}$ and from Eq. (3.9) for the zero approximation p_0 we find

$$\partial p_0/\partial \alpha = b^2 (\partial^2 p/\partial \beta^2), \tag{3.11}$$

where

$$b^2 = 1/2 < (2\varkappa - 1), \qquad \varkappa = > \frac{1}{2}.$$

This equation differs from Eq. (2.16) only by the coefficient; consequently, the effect of the approximating terms is qualitatively the same as that of viscosity. The solution of Eq. (3.11) will be of the same type as Eq. (2.21) and we can easily find the effective shock wave width with the help of the formula

$$\Delta\beta = \frac{p_0^+ - p_0^-}{\max\left(\frac{\partial p_0}{\partial \beta} \right)} = b\sqrt{\pi t}. \tag{3.12}$$

From Eq. (3.12) it is clear that the shock wave width is directly proportional to the value $\varkappa$ and is independent of the Courant number K, what is quite unexpected.

At $\varkappa = 1/2$ the scheme is of the second order of accuracy. In Eq. (3.9) now the first term of the first sum becomes the main one, then $\lambda = \tau^{2/3}$, and for a zero approximation we find

$$b^3 \frac{\partial^3 p_0}{\partial \beta^3} = \frac{\partial p_0}{\partial \alpha}, \quad b^3 = \frac{1}{6}(K^{-2}-1). \tag{3.13}$$

Equation (3.13) at $A = 0$ coincides with Eq. (3.16). Thus the effect of the approximation terms is the same as the transverse motion inertia one. We see that the approximation effects do not always look like viscous resistance, as it is often referred to identifying all the approximation effects at the discontinuity with the approximational viscosity influence.

For the effective shock wave width we find

$$\Delta\beta = \frac{1}{\mathrm{Ai}(0)} \pi b t^{1/3}, \tag{3.14}$$

wherefrom one can see that for the second order schemes the dependence of the shock wave width on time 1 is weaker than for the schemes of the first order of accuracy. It is interesting to note that the obtained results for the scheme (3.4) are common for any known schemes of the first and the second orders of accuracy. In Eqs. (3.11)–(3.14) only the coefficient b changes. For example, for the Lax scheme of the first order of accuracy [8] it can be shown that $b^2 = 1/4(K^{-2}-1)$ and for the "cross" scheme of the second order of accuracy $b^3 = 1/2(K^{-2}-1)$. That is why, by comparing the value of the coefficient b one can speak of a scheme of the minimal effective width. One can draw a conclusion about the stability of the scheme in the case of smooth solutions taking into account the first differential approximation in the parabolic form [9]. It was obtained from Eq. (3.6) leaving only the main term with the smallest power τ and using Eq. (3.3):

$$\frac{\partial p}{\partial \varkappa} + \frac{\partial p}{\partial t} = \frac{\tau}{2}(2\varkappa - 1)\frac{\partial^2 p}{\partial x^2}, \tag{3.15}$$

then $\varkappa = 1/2$, we have

$$\frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} = \frac{\tau^3}{6}(K^{-2}-1)\frac{\partial^2 p}{\partial x^2}. \tag{3.16}$$

The nonnegativity of the coefficients of the right hand part in Eqs. (3.15)–(2.16) is the necessity conditions of the stability of the scheme (3.4). These conditions coincide with the conditions of nonnegativity of the coefficients in Eqs. (3.11) and (3.13). Thus, for linear equations the conditions of stability for the continuous and discontinuous solutions are the same. For the nonlinear equations the situation is more complicated. One can show that such coincidence does not take place. The stable scheme for the smooth solutions can became unstable for the discontinuous one.

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