Arch. Mech., 33, 5, pp. 753-761, Warszawa 1981

Nonlinear micropolar continuum model of a composite reinforced by elements of finite rigidity
Part I. Equations of motion and constitutive relations

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A NONLINEAR orthotropic continuum model of an elastic composite reinforced by a single family of rods is considered. Finite rigidity of fibers and local nonhomogeneity of the stress field in the neighbourhood of the elements subject to bending are accounted for by introducing the assumption on the dependence of the elastic energy of the model continuum upon the curvature of the fibers. In the energy balance the body couples, rotary inertia and torsional rigidity of the rods are disregarded. The reinforcement elements are also assumed to be kinematically associated with the matrix. Considerations concerning the energy balance yield the equations of motion and the constitutive relations which represent a particular case of a micropolar continuum with constrained rotations, with the boundary conditions expressed in terms of moments and rotations which possess a clearly determined physical interpretation.

Rozpatruje się nieliniowy ortotropowy model ciągły sprężystego kompozytu zbrojonego jedną rodziną prętów. Skończoną sztywność elementów zbrojenia na zginanie oraz lokalną niejednorodność pola naprężenia w otoczeniu elementu uwzględnia się poprzez wprowadzenie założenia o zależności energii sprężystej modelowego ośrodka ciągłego od krzywizny prętów. W bilansie energetycznym pomija się momenty masowe, bezwładność momentową oraz sztywność prętów na skręcanie. Zakłada się także, że elementy zbrojenia są kinematycznie związane z matrycą. W wyniku rozważań nad bilansem energetycznym otrzymuje się równania ruchu i równania konstytutywne stanowiące szczególną realizację modelu anizotropowego ośrodka momentowego ze związanymi obrotami o jasno sprecyzowanym sensie fizycznym warunków brzegowych w momentach lub obrotach.

Рассматривается нелинейная ортотропная сплошная модель упругого композиционного материала, армированного одним семейством стержней. Конечная жесткость армировки при изгибе и локальные концентрации напряжений, возникающие в окрестности изгибаемого элемента учитываются посредством предположения о зависимости упругой энергии модельной среды от кривизны стержней. Не учитываются массовые моменты, моментная инерция и конечная жесткость элементов армировки при кручении. Предполагается, что армировка кинематически связана с матрицей. Из уравнений энергетического баланса выводятся уравнения движения и определяющие уравнения. Полученная модель является таким частным случаем анизотропной моментной модели с кинематически связанными оборотами, для которого краевые условия на моменты и вращение имеют четкое физическое истолкование.

Introduction

M. ARCISZ [1] in her paper considered a model of deterioration due to the loss of internal stability of a fiber-reinforced material. Physical interpretation of such type of fracture lies beyond the frames of continuum models although it appears rather obvious: at the stability loss of a fiber, its bending produces — under sufficiently small radii of curvature — a strong nonhomogeneity of the stress field and this leads to detaching of the fibers from the matrix. The material considered in that paper was simple and characterized by a symmetric stress tensor, i.e. the reinforcement was taken into account only by introducing anisotropic material properties of the medium. It seems to be of interest, from both the scientific and technological points of view, to consider the case of finite rigidity of the reinforcement combined with a considerable flexibility of the matrix. For instance, one could think of an insulating material of the foam plastic type reinforced by steel wires to increase its strength in compression. One could not exclude the possibility of applying such materials to structural purposes since their insulating properties in the direction perpendicular to the fibers would remain practically unchanged, while their strength measured along the fibers would considerably increase.

Let us now consider the elastic material reinforced by a single family of fibers or, more precisely, by elastic rods. It will be assumed that a certain vector **A** characterizing the direction of reinforcement can be ascribed to each material particle of the composite. The cases of reinforcement of finite bending stiffness will be considered. Such material will be referred to as transversely isotropic elastic micropolar material. The elastic energy of the material will be assumed to be a function not only of the stress tensor invariants and the vector **A**, but also of the vector of curvature of the reinforcement elements; our considerations will be confined to the cases in which these elements in the unstrained state will be parallel and uniformly distributed. Torsional stiffnes of the reinforcement will not be taken into account, and this assumption seems to be justified by the fact that the cases in which this stiffness might play an important role are rarely encountered in practice.

Such simplifying assumptions make it possible to construct a relatively simple variant of the couple stress theory with constrained rotations confined to a single specified vector.

1. Geometric relations

Let the positions of the material particles X of the body, identified by the prescribed values of the material coordinates u^α, $\alpha = 1, 2, 3$, be given in the form

$$\mathring{\mathbf{R}}(X) = \mathring{\mathbf{R}}(u^\alpha). \tag{1.1}$$

The positions of the particles X in the body at instant t are denoted by

$$\mathbf{R}(X, t) = \mathbf{R}(u^\alpha, t). \tag{1.2}$$

Let in the undeformed body a certain specified family ξ of material fibres be given such that each particle X belongs to one and only one material fiber. Thus, under the given parametrization of fibers belonging to the family ξ, a vector **A** tangent to the material fiber may be ascribed to each material particle X. If the parametric equation of the fiber containing a particle X is written as

$$\mathbf{r} = \mathbf{r}(s) = \mathbf{r}(u^\alpha(S)) \quad (\alpha = 1, 2, 3), \tag{1.3}$$

then the vector **A** assumes the form

$$\mathbf{A} = \frac{d\mathbf{r}}{dS}. \tag{1.4}$$

Let S denote a material parameter, i.e. such a paramater which retains its value in the process of deformation; then the contravariant components of the vector $\mathring{\mathbf{A}}$ in the underformed state written in base $\mathring{\mathbf{g}}_\alpha \equiv \partial\mathring{\mathbf{R}}/\partial u^\alpha$:

$$\mathring{A}^\alpha = \mathring{\mathbf{A}} \cdot \mathring{\mathbf{g}}^\alpha,$$

and the components of $\mathbf{A}$ written in base $\mathbf{g}_\alpha \equiv \partial\mathbf{R}/\partial u^\alpha$:

$$A^\alpha = \mathbf{A} \cdot \mathbf{g}^\alpha$$

will be the same.

Denoting by $g_{\mu\nu}$ the metric tensor in deformed state, we may determine in the deformed body the field of unit vectors tangent to the family ξ,

$$a^\alpha \overset{\mathrm{df}}{\equiv} \frac{A^\alpha}{\sqrt{g_{\mu\nu} A^\mu A^\nu}}. \tag{1.5}$$

Curvature of the material fibers of the family ξ is expressed in terms of the modulus of the vector $\boldsymbol{\varkappa}$ determined by the formula

$$\boldsymbol{\varkappa} = \frac{d\mathbf{a}}{ds}, \tag{1.6}$$

where s denotes the natural parametrization in the deformed state. Equation (1.6) may be rewritten in the form

$$\boldsymbol{\varkappa} = \frac{d\mathbf{a}}{dS}\frac{dS}{ds}. \tag{1.7}$$

Since s is a natural parameter, then

$$\left|\frac{d\mathbf{R}}{ds}\right| = 1, \tag{1.8}$$

and hence

$$\left|\frac{d\mathbf{R}}{dS}\frac{dS}{ds}\right| = \left|\frac{d\mathbf{R}}{dS}\right|\left|\frac{dS}{ds}\right| = 1. \tag{1.9}$$

If the parametrization s is selected so that the sense of vector $\mathbf{a}$ complies with that of vector $\mathbf{A}$, Eq. (1.9) yields

$$\frac{dS}{ds} = \frac{1}{\left|\frac{d\mathbf{R}}{dS}\right|} = \frac{1}{\sqrt{A^\mu A^\nu g_{\mu\nu}}}. \tag{1.10}$$

On using Eqs. (1.10), the formula (1.7) may be written as

$$\boldsymbol{\varkappa} = \frac{1}{\sqrt{A^\mu A^\nu g_{\mu\nu}}}\frac{d\mathbf{a}}{dS} = \frac{1}{\sqrt{A^\mu A^\nu g_{\mu\nu}}}\frac{d}{dS}\left(\frac{\mathbf{A}}{\sqrt{A^\mu A^\nu g_{\mu\nu}}}\right), \tag{1.11}$$

while the differentiation with respect to S is written in the form

$$\frac{d}{dS}(\cdot) = \frac{\partial(\cdot)}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial S} \tag{1.12}$$

whence, for instance,

$$\frac{d\mathbf{R}}{dS} = \frac{\partial \mathbf{R}}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial S} = \mathbf{g}_\alpha \frac{\partial u^\alpha}{\partial S}. \tag{1.13}$$

Due to the fact that $\mathbf{A} = A^\alpha \mathbf{g}_\alpha$,

$$A^\alpha = \frac{\partial u^\alpha}{\partial S}, \tag{1.14}$$

and Eq. (1.11) assumes the form

$$\begin{aligned} \boldsymbol{\varkappa} &= \frac{1}{\sqrt{A^\mu A^\nu g_{\mu\nu}}}\frac{\partial}{\partial u^\alpha}\left(\frac{\partial \mathbf{R}}{\partial u^\beta}\frac{A^\beta}{\sqrt{A^\mu A^\nu g_{\mu\nu}}}\right)A^\alpha \\ &= a^\alpha \frac{\partial}{\partial u^\alpha}(a^\beta \mathbf{g}_\beta) = a^\alpha\left(\frac{\partial a^\beta}{\partial u^\alpha}\mathbf{g}_\beta + a^\mu \frac{\partial \mathbf{g}_\mu}{\partial u^\alpha}\right) = a^\alpha\left(\frac{\partial a^\beta}{\partial u^\alpha}\mathbf{g}_\beta + a_\mu \Gamma^\beta_{\mu\alpha}\mathbf{g}_\beta\right), \end{aligned} \tag{1.15}$$

in which $\Gamma^\beta_{\mu\alpha} \equiv \dfrac{\partial \mathbf{g}_\mu}{\partial u^\alpha}\cdot \mathbf{g}^\beta$ is the Christoffel symbol and, finally, multiplication of both sides by $\mathbf{g}^\nu$ yields

$$\varkappa^\nu \equiv \boldsymbol{\varkappa}\cdot \mathbf{g}^\nu = a^\alpha\left(\frac{\partial a^\nu}{\partial u^\alpha} + a^\mu \Gamma^\nu_{\mu\alpha}\right) = a^\alpha a^\nu_{,\alpha}, \tag{1.16}$$

the comma denoting the covariant derivative.

It is known that the contravariant components of the metric tensor $\mathring{g}^{\alpha\beta} = \mathring{\mathbf{g}}^\alpha \cdot \mathring{\mathbf{g}}^\beta$ may be treated as components of the tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ [3, 4] in the base $\mathbf{g}_\alpha$, and hence instead of $\mathring{g}^{\alpha\beta}$ we shall write $B^{\alpha\beta}$ (obviously, in general, $\mathring{g}_{\alpha\beta} \neq B_{\alpha\beta}$ and also $\mathring{A}_\alpha \neq A_\alpha$).

2. Elastic energy density and its material derivative

As it was mentioned in the Introduction, the elastic energy will be assumed to be a function of the strain tensor $\mathbf{B}$ and the curvature vector $\boldsymbol{\varkappa}$. Using an approach similar to that used in [1], the elastic energy of a transversely isotropic body may be treated as an isotropic function of the tensor $\mathbf{B}$, vector $\boldsymbol{\varkappa}$ and vector $\mathbf{A}$; obviously, it cannot be dependent on the method of parametrization and hence it may depend on the direction of $\mathbf{A}$ and not on its modulus in the undeformed state. According to SPENCER [2], in the case of transversal isotropy for μ symmetric tensors and ν vectors, there exist $6\mu+3\nu-1$ independent invariants. This means that in the case of a single tensor $\mathbf{B}$ and single vector $\boldsymbol{\varkappa}$, the number of independent invariants should be equal to 8; however, in the case considered here we have $a^\alpha g_{\alpha\beta} a^\beta = 1$ and $a^\alpha_{,\gamma} g_{\alpha\beta} a^\beta = a^\alpha_{,\gamma} a_\alpha = 0$, and

$$\varkappa^\nu a_\nu = a^\alpha a^\nu_{,\alpha} a_\nu = 0, \tag{2.1}$$

and also

$$\varkappa^\nu A_\nu = 0. \tag{2.2}$$

Vector $\varkappa$ is hence *ex definitione* perpendicular to $\mathbf{A}$. This relation reduces the number of independent invariants to 7([1]).

For further considerations the following set of invariants J_k will be assumed:

$$
\begin{aligned}
J_1 &= B^{\alpha\beta} g_{\alpha\beta},\\
J_2 &= \frac{1}{2}(I^2 - B^{\alpha\delta}B^{\beta\gamma}g_{\gamma\delta}g_{\alpha\beta}),\\
J_3 &= \frac{\det g_{\alpha\beta}}{\det \mathring{g}_{\alpha\beta}} = \det g_{\alpha\beta}\det \mathring{g}^{\alpha\beta},\\
J_4 &= A^\mu A^\nu g_{\mu\nu},\\
J_5 &= A^\alpha B^{\gamma\delta} A^\beta g_{\alpha\gamma} g_{\delta\beta},\\
J_6 &= \varkappa^\mu \varkappa^\nu g_{\mu\nu},\\
J_7 &= \varkappa^\mu \varkappa^\nu B^{\alpha\beta} g_{\mu\alpha} g_{\nu\beta}.
\end{aligned}
\tag{2.3}
$$

The material derivatives of the invariants J_k will also be needed. To this end the material derivatives of the corresponding tensor components must be determined. First of all let us point out the obvious identities:

$$\overset{\cdot}{\overline{A^\alpha}} = \overset{\cdot}{\overline{\mathring{A}^\alpha}} = 0, \tag{2.4}$$

$$\overset{\cdot}{\overline{B^{\alpha\beta}}} = \overset{\cdot}{\overline{\mathring{g}^{\alpha\beta}}} = 0. \tag{2.5}$$

The symbols of material derivatives are written above the bars to stress the fact that derivatives of the components are considered here, and not the components of the derivatives of the tensors and vectors([2]).

Furthermore we obtain

$$
\begin{aligned}
\overset{\cdot}{\overline{g_{\alpha\beta}}} &= \dot{\mathbf{g}}_\alpha \cdot \mathbf{g}_\beta + \mathbf{g}_\alpha \cdot \dot{\mathbf{g}}_\beta = \left(\overset{\cdot}{\overline{\frac{\partial \mathbf{R}}{\partial u^\alpha}}}\right) \cdot \mathbf{g}_\beta + \mathbf{g}_\alpha \left(\overset{\cdot}{\overline{\frac{\partial \mathbf{R}}{\partial u^\alpha}}}\right)\\
&= \frac{\partial \dot{\mathbf{R}}}{\partial u^\alpha} \cdot \mathbf{g}_\beta + \mathbf{g}_\alpha \cdot \frac{\partial \dot{\mathbf{R}}}{\partial u^\beta} = \frac{\partial \mathbf{v}}{\partial u^\alpha} \cdot \mathbf{g}_\beta + \mathbf{g}_\alpha \cdot \frac{\partial \mathbf{v}}{\partial u^\beta} = v_{\beta,\alpha} + v_{\alpha,\beta}.
\end{aligned}
\tag{2.6}
$$

From the equality

$$\overset{\cdot}{\overline{g_{\alpha\beta} g^{\beta\gamma}}} = \overset{\cdot}{\overline{\delta^\gamma_\alpha}} = 0 \tag{2.7}$$

it follows directly that

$$\overset{\cdot}{\overline{g^{\alpha\beta}}} = -g^{\alpha\delta} g^{\beta\gamma}(v_{\delta,\gamma} + v_{\gamma,\delta}) \tag{2.8}$$

([1]) To put it more clearly: in order to orient the vectors $\mathbf{A}$ and $\varkappa$ in the base of eigenvectors of tensor $\mathbf{B}$, six numbers must be prescribed what, together with the three invariants of $\mathbf{B}$, amounts to nine invariants; however, taking into account the fact that we are not interested in the modulus of $\mathbf{A}$ and that the vectors $\mathbf{A}$ and $\varkappa$ are orthogonal, we have seven invariants left.

([2]) Let us note that Eqs. (2.4) and (2.5) do not imply the vanishing of the material derivatives of A_α or $B_{\alpha\beta}$.

and

$$\dot{\mathbf{g}}^{\beta} = -\mathbf{g}^{\mu} g^{\beta\nu}_{\cdot} v_{\mu,\nu}. \tag{2.9}$$

Equation (1.15) is now used to calculate $\dot{\overline{\varkappa^{\gamma}}}$,

$$\dot{\overline{\varkappa^{\gamma}}} = \overline{\dot{\boldsymbol{\varkappa}} \cdot \mathbf{g}^{\gamma}} = \overline{\frac{A^{\alpha}\mathbf{g}^{\gamma}}{\sqrt{g_{\mu\nu}A^{\mu}A^{\nu}}} \cdot \frac{\partial}{\partial u^{\alpha}}\left(\frac{\partial \mathbf{R}}{\partial u^{\beta}} \frac{A^{\beta}}{\sqrt{g_{\mu\nu}A^{\mu}A^{\nu}}}\right)}^{\bullet}. \tag{2.10}$$

After transformations and using Eqs. (2.4)–(2.9), (1.5) and (1.16) we obtain

$$\dot{\overline{\varkappa^{\gamma}}} = -2\varkappa^{\gamma}a^{\varphi}a^{\theta}v_{\varphi,\theta} - a^{\gamma}(\varkappa^{\varphi}a^{\theta} + \varkappa^{\theta}a^{\varphi})v_{\varphi,\theta} - a^{\gamma}a^{\alpha}a^{\varphi}a^{\theta}v_{\varphi,\theta\alpha} + a^{\alpha}a^{\beta}v^{\gamma}_{,\beta\alpha}. \tag{2.11}$$

Using Eqs. (2.4)–(2.11), the following relations may easily be found:

$$\dot{J}_1 = B^{\mu\nu}(v_{\mu,\nu} + v_{\nu,\mu}), \tag{2.12}$$

$$\dot{J}_2 = (IB^{\mu\nu} - B^{\mu\alpha}B^{\nu\beta}g_{\alpha\beta})(v_{\mu,\nu} + v_{\nu,\mu}), \tag{2.13}$$

$$\dot{J}_4 = A^{\mu}A^{\nu}(v_{\mu,\nu} + v_{\nu,\mu}), \tag{2.14}$$

$$\dot{J}_5 = (A^{\mu}B^{\nu\beta}A_{\beta} + A^{\nu}B^{\mu\beta}A_{\beta})(v_{\mu,\nu} + v_{\nu,\mu}), \tag{2.15}$$

$$\dot{J}_6 = (\varkappa^{\mu}\varkappa^{\nu} - 2\varkappa^2 a^{\mu}a^{\nu})(v_{\mu,\nu} + v_{\nu,\mu}) + 2\varkappa^{\gamma}a^{\mu}a^{\nu}v_{\gamma,\mu\nu}, \tag{2.16}$$

$$\dot{J}_7 = -2(\varkappa_{\alpha}\varkappa_{\beta}B^{\alpha\beta}a^{\mu}a^{\nu} + a_{\alpha}\varkappa_{\beta}B^{\alpha\beta}\varkappa^{\mu}a^{\nu} - \varkappa_{\beta}B^{\nu\beta}\varkappa^{\mu})(v_{\mu,\nu} + v_{\nu,\mu}) - 2(\varkappa_{\alpha}a_{\beta}B^{\alpha\beta}a^{\gamma} - \varkappa_{\beta}B^{\beta\gamma})a^{\mu}a^{\nu}v_{\gamma,\mu\nu}. \tag{2.17}$$

In order to determine $\dot{J}_3$, use must be made of the relation [3]:

$$J_3 = \left(\frac{\varrho_0}{\varrho}\right)^2, \tag{2.18}$$

where ϱ_0, ϱ denote the mass densities in the undeformed and actual states, respectively.

The known relation

$$\dot{\varrho} = -\varrho g^{\alpha\beta}v_{\alpha,\beta}, \tag{2.19}$$

yields

$$\dot{J}_3 = J_3 g^{\alpha\beta}(v_{\alpha,\beta} + v_{\beta,\alpha}). \tag{2.20}$$

Let $w = w(J_k)$ denote the elastic energy density. Denoting

$$\tau^{\mu\nu} \equiv 2\left\{\frac{\partial w}{\partial J_1}B^{\mu\nu} + \frac{\partial w}{\partial J_2}(J_1 B^{\mu\nu} - B^{\mu\alpha}B^{\nu\beta}g_{\alpha\beta}) + \frac{\partial w}{\partial J_3}J_3 g^{\mu\nu} + \frac{\partial w}{\partial J_4}A^{\mu}A^{\nu} + \frac{\partial w}{\partial J_5}2A^{\mu}B^{\nu\beta}A_{\beta} + \frac{\partial w}{\partial J_6}(\varkappa^{\mu}\varkappa^{\nu} - 2\varkappa^2 a^{\mu}a^{\nu}) + \frac{\partial w}{\partial J_7}2(\varkappa_{\alpha}\varkappa_{\beta}B^{\alpha\beta}a^{\mu}a^{\nu} + a_{\alpha}\varkappa_{\beta}B^{\alpha\beta}\varkappa^{\mu}a^{\nu} - \varkappa_{\beta}B^{\nu\beta}\varkappa^{\mu})\right\}, \tag{2.21}$$

we obtain

$$\varrho\dot{w} = \varrho\tau^{(\mu\nu)}v_{\mu,\nu} + 2\varrho\left[\frac{\partial w}{\partial J_6}\varkappa^{\mu} - \frac{\partial w}{\partial J_7}(\varkappa_{\alpha}a_{\beta}B^{\alpha\beta}a^{\mu} - \varkappa_{\beta}B^{\beta\mu})\right]a^{\gamma}a^{\nu}v_{\mu,\nu\gamma}. \tag{2.22}$$

3. Energy balance, constitutive relations, equations of motion

Let us postulate the energy balance which takes into account the work done by the moment on the corresponding rotation of vector **a**; it will also be assumed that the fibers do not exhibit any torsional rigidity, and postulate the existence of such tensor of moments $M^{\alpha\beta}$ that the moment vector M^{β} at a surface with the unit normal **n** is expressed by the formula

$$M^{\beta} = M^{\gamma\beta} n_{\gamma}. \tag{3.1}$$

Assumption of zero torsional rigidity yields

$$M^{\beta} a_{\beta} = 0. \tag{3.2}$$

Under such assumptions the energy balance for each material region D and for each velocity field assumes the form

$$\overline{\int_{D} \varrho w dV}^{\,\cdot} = \int_{\partial D} n_{\nu} T^{\nu\mu} v_{\mu} dS + \int_{D} \varrho f^{\mu} v_{\mu} dV + \int_{\partial D} n_{\gamma} M^{\gamma\beta} \varepsilon_{\beta\varphi\varrho} a^{\varphi} (\dot{\mathbf{a}})^{\varrho} dS - \overline{\int_{D} \frac{\varrho v^2}{2} dV}^{\,\cdot}. \tag{3.3}$$

Here

$$(\dot{\mathbf{a}})^{\varrho} \equiv \dot{\mathbf{a}} \cdot \mathbf{g}^{\varrho} = (\dot{\overline{a^{\alpha}}} \mathbf{g}_{\alpha} + a^{\alpha} \dot{\mathbf{g}}_{\alpha}) \cdot \mathbf{g}^{\varrho}, \tag{3.4}$$

and **f** is the density of body forces.

Due to the fact that

$$\dot{\overline{a^{\alpha}}} = \frac{\dot{\overline{A^{\alpha}}}}{\sqrt{g_{\varphi\theta} A^{\varphi} A^{\theta}}} = -\frac{1}{2} \frac{A^{\alpha} A^{\mu} A^{\nu}}{\sqrt{g_{\varphi\theta} A^{\varphi} A^{\theta}}^{\,3}} (v_{\mu,\nu} + v_{\nu,\mu}) = -\frac{1}{2} a^{\alpha} a^{\mu} a^{\nu} (v_{\mu,\nu} + v_{\nu,\mu}), \tag{3.5}$$

and

$$\dot{\mathbf{g}}_{\alpha} \cdot \mathbf{g}^{\varrho} = \frac{\partial \mathbf{v}}{\partial u^{\alpha}} \cdot \mathbf{g}^{\varrho} = v^{\varrho}_{.\alpha}, \tag{3.6}$$

we obtain

$$(\dot{\mathbf{a}})^{\varrho} = (a^{\nu} g^{\varrho\mu} - a^{\varrho} a^{\nu} a^{\mu}) v_{\mu,\nu}, \tag{3.7}$$

whence it follows

$$\varepsilon_{\beta\varphi\varrho} a^{\varphi} (\dot{\mathbf{a}})^{\varrho} = \varepsilon_{\beta\varphi\varrho} a^{\varphi} a^{\nu} g^{\varrho\mu} v_{\mu,\nu}. \tag{3.8}$$

Reducing Eq. (3.3) to the volume integral and taking into account the fact that Eq. (3.3) must hold for each material subdomain D, we obtain for each point and each velocity field

$$-\varrho \dot{w} + (T^{\nu\mu} v_{\mu})_{,\nu} + \varrho f^{\mu} v_{\mu} + (M^{\gamma\beta} \varepsilon_{\beta\varphi\varrho} a^{\varphi} a^{\nu} g^{\varrho\mu} v_{\mu,\nu})_{,\gamma} - \varrho w^{\mu} v_{\mu} = 0, \tag{3.9}$$

where $\mathbf{w} \equiv \dot{\mathbf{v}}$.

Calculating the covariant derivatives, substituting for w the expression (2.22) and rearranging the terms at the consecutive velocity gradients, we obtain

$$\begin{aligned}(T^{\nu\mu}_{\;\;,\nu} + \varrho f^{\mu} - \varrho w^{\mu}) v_{\mu} + (T^{\nu\mu} + S^{\gamma\nu\mu}_{\;\;\;\;,\gamma} - \varrho \tau^{(\mu\nu)}) v_{\mu,\nu} \\ + \left[M^{\gamma\beta} \varepsilon_{\beta\varphi\varrho} a^{\varphi} g^{\varrho\mu} - 2\varrho \frac{\partial w}{\partial J_6} \varkappa^{\mu} a^{\gamma} + 2\varrho \frac{\partial w}{\partial J_7} (\varkappa_{\alpha} a_{\beta} B^{\beta\mu} a^{\mu} - \varkappa_{\beta} B^{\beta\mu}) a^{\gamma} \right] a^{\nu} v_{\mu,\nu\gamma} = 0,\end{aligned} \tag{3.10}$$

with the notation

$$S^{\gamma\nu\mu} \equiv M^{\gamma\beta}\varepsilon_{\beta\varphi\varrho}a^{\varphi}a^{\nu}g^{\varrho\mu}. \tag{3.11}$$

The equality (3.10) must be fulfilled for each velocity field and at every point. Selection of the suitable velocity fields at a fixed point enables us to prescribe arbitrary values of v_μ and $v_{\mu,\nu}$, hence the following relations must also hold true:

$$T^{\nu\mu}_{\ \ ,\nu}+\varrho f^{\mu}-\varrho w^{\mu} = 0, \tag{3.12}$$

$$T^{\nu\mu}+S^{\gamma\nu\mu}_{\ \ \ ,\gamma}-\varrho\tau^{(\mu\nu)} = 0. \tag{3.13}$$

The values of $v_{\eta,\nu\gamma}$ cannot be chosen arbitrarily since they are restrained by the condition of the space to be Euclidean,

$$v_{\mu,\nu\gamma} = v_{\mu,\gamma\nu}. \tag{3.14}$$

Let us note that, on the ohter hand, the values $a^{\nu}v_{\mu,\nu\gamma}$ may assume arbitrary values equal to the components of an arbitrary tensor of rank 2([3]), so that we can write on the basis of Eq. (3.10)

$$M^{\gamma\beta}\varepsilon_{\beta\varphi\varrho}a^{\varphi}g^{\varrho\mu} = 2\varrho\frac{\partial w}{\partial J_6}\varkappa^{\mu}a^{\gamma}-2\varrho\frac{\partial w}{\partial J_7}(\varkappa_\alpha a_\beta B^{\alpha\beta}a^{\mu}-\varkappa_\beta B^{\beta\mu})a^{\gamma}, \tag{3.15}$$

whence

$$S^{\gamma\nu\mu} = 2\varrho\left[\frac{\partial w}{\partial J_6}\varkappa^{\mu}-\frac{\partial w}{\partial J_7}(\varkappa_\alpha a_\beta B^{\alpha\beta}a^{\mu}-\varkappa_\beta B^{\mu\beta})\right]a^{\nu}a^{\gamma}. \tag{3.16}$$

The expression for $M^{\gamma\theta}$ which does not appear in the equations of motion is, nevertheless, necessary in formulating the boundary conditions; it is determined in the following manner: Eq. (3.15) is multiplied by $g_{\mu\xi}\varepsilon^{\xi\varkappa\theta}a_{\varkappa}$:

$$M^{\gamma\beta}(\delta^{\varkappa}_{\beta}\delta^{\theta}_{\varphi}-\delta^{\theta}_{\beta}\delta^{\varkappa}_{\varphi})a^{\varphi}a_{\varkappa} = 2\varrho\left[\frac{\partial w}{\partial J_6}\varkappa_{\xi}\varepsilon^{\xi\varkappa\theta}a_{\varkappa}-\frac{\partial w}{\partial J_7}\varkappa_\beta B^{\beta\mu}g_{\mu\xi}\varepsilon^{\xi\varkappa\theta}a_{\varkappa}\right]a^{\gamma}. \tag{3.17}$$

Furthermore, from the assumption, for each $\mathbf{n}$ we have

$$n_\gamma M^{\gamma\beta}a_\beta = 0, \tag{3.18}$$

whence

$$M^{\gamma\beta}a_\beta = 0, \tag{3.19}$$

and Eq. (3.17) may be written in the form

$$M^{\gamma\theta} = 2\varrho a^{\gamma}\left[\frac{\partial w}{\partial J_6}b^{\theta}-\frac{\partial w}{\partial J_7}\varkappa_\beta B^{\beta\mu}\varepsilon_{\mu\nu\varphi}a^{\nu}g^{\varphi\theta}\right]. \tag{3.20}$$

Here $b^{\theta} = a_{\varkappa}\varepsilon^{\alpha\varphi\theta}\varkappa_{\varphi}$ are components of the unit binormal vector multiplied by the modulus of vector $\boldsymbol{\varkappa}$. The set of equations (3.12), (3.13) and (3.16) is written in the simple form

$$T^{\nu\mu}_{\ \ ,\nu}+\varrho f^{\mu} = \varrho w^{\mu}, \tag{3.21}$$

([3]) Let a tensor have the components $A_{\mu\gamma}$; then from the condition $a^{\nu}v_{\mu,\nu\gamma} = A_{\mu\gamma}$ follow 9 equations for 18 independent components of the tensor $v_{\mu,\nu\gamma}$.

where

$$T^{\nu\mu} = \varrho\tau^{(\nu\mu)} - 2\left[\varrho\left(\frac{\partial w}{\partial J_6}\varkappa^\mu - \frac{\partial w}{\partial J_7}(\varkappa_\alpha a_\beta B^{\alpha\beta}a^\mu - \varkappa_\beta B^{\beta\mu})a^\nu a^\gamma\right)\right]_{,\gamma}.$$

The boundary conditions expressed in terms of stresses or displacements are written in the usual manner, while the additional conditions in terms of moments or rotations are formulated as follows:

a) Moments:

$$M^\beta|_{\partial D} = n_\gamma M^{\gamma\beta}|_{\partial D}, \tag{3.22}$$

$M^{\gamma\beta}$ being taken from Eq. (3.20). It is easily observed that for the boundary regions of the bodies parallel to the reinforcement, the identity $M^\beta|_{\partial D} = 0$ must hold.

b) Rotations:

$$a^\gamma|_{\partial D} = N^\gamma, \tag{3.23}$$

N^γ being the field of unit vectors prescribed at the surface ∂D; if ∂D is a material surface parallel to the fibers, then the identity $\mathbf{N}\cdot\mathbf{n} = 0$ follows; the density ϱ in Eq. (3.21) is found from Eq. (2.18).

The second part of this paper will be devoted to the application of the model introduced here to the simplest cases of material instability.

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INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received January 29, 1981.