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Stress tensor in sedimenting dilute suspensions of heavy particles

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THE FLOW of dilute suspension of identical, spherical, heavy particles, sedimenting in the presence of a constant body or inertia force, is considered and the stress tensor for such motion is determined. Comparing the "classical" viscous shear stress tensor with the momentum flux of sedimenting particles, some quantitative criteria are formulated, determining the magnitudes of particle sizes, body forces and shear velocities, for which the momentum flux of particles dominates over "classical" viscous stress.

CONSIDERING the momentum transport in continuous media, the stress tensor is introduced, which is due to the double influence of molecular flux and of forces interacting between molecules. The stress tensor τ_{ij} of suspensions may also be divided into two parts [1]

$$\tau_{ij} = \tilde{\tau}_{ij} + \bar{\tau}_{ij}, \tag{1}$$

where

$$\tilde{\tau}_{ij} = -\varphi\varrho_p w_{pi} w_{pj} - (1-\varphi)\varrho_l w_{li} w_{lj} \tag{2}$$

is the "ordered" momentum flux of both phases moving relatively to the suspension with the mean velocities w_{pi} and w_{li} which satisfy the relation

$$\varphi\varrho_p w_{pi} + (1-\varphi)\varrho_l w_{li} = 0. \tag{3}$$

The subscripts p or l refer to the particles or the liquid phase, φ denotes the volume fraction of the particles, and ϱ_p and ϱ_l denote the densities of both phases. A Cartesian frame of reference x_i is introduced here. The second part $\bar{\tau}_{ij}$ is due to other factors, as internal forces interaction, and to chaotic molecular and molar momentum flux.

For the dilute suspension of heavy particles

$$\varphi \ll 1, \quad \frac{\varrho_l}{\varrho_p} \ll 1, \tag{4}$$

an approximation of the formula (2) may be presented:

$$\tilde{\tau}_{ij} = -\varphi \cdot \left(1 + \varphi\frac{\varrho_p}{\varrho_l}\right) \varrho_p w_{pi} w_{pj}. \tag{5}$$

The second part $\bar{\tau}_{ij}$ of the stress tensor τ_{ij} may also be approximately evaluated:

$$\bar{\tau}_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij}, \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{6}$$

where p is the pressure, δ_{ij} — the unit tensor, μ — the viscosity coefficient of the liquid phase and ε_{ij} — the strain tensor of suspension moving with the mean velocity u_i.

In sedimenting suspension the relative velocity w_{pi} is generated mainly by the external body force g_i or by the inertia force, due to the acceleration $a_i = du_i/dt$ of the suspension.

The evaluation of the influence of these body and inertia forces on the stress tensor $\bar{\tau}_{ij}$ is the main aim of this work. Additionally, on the basis of quantitative evaluations, some criteria will be formulated and conditions will be determined, for which the sedimentation exerts a dominant influence on the shear stress.

In the presence of the body force g_i, we will locally consider a laminar flow of a dilute suspension containing identical heavy, spherical particles of the radius r. In developed sedimentation, the particles are assumed to move uniformly with an almost constant mean relative velocity w_{pi}. We will assume that the relative motion is sufficiently slow to make use of Stokes' formula for the drag.

Although the relative velocity w_{pi} of the particles is assumed to be constant, their absolute velocity u_i+w_{pi} varies. By applying the differentation rules

$$\frac{d(u_i+w_{pi})}{dt} = \frac{\partial(u_i+w_{pi})}{dt} + (u_k+w_{pk})\frac{\partial(u_i+w_{pi})}{\partial x_k}$$

$$= \frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} + w_{pk}\frac{\partial u_i}{\partial x_k} = \frac{du_i}{dt} + w_{pk}\frac{\partial u_i}{\partial x_k},$$

we obtain the absolute acceleration a_{pi} of the particle

$$a_{pi} = a_i + w_{pk}\frac{\partial u_i}{\partial x_k}, \tag{7}$$

where a_i is the acceleration in the mean motion of the suspension. The balance equation of forces acting on a particle

$$\frac{4}{3}\pi r^3(\varrho_p-\varrho_l)g_i - 6\pi\mu r(w_{pi}-w_{li}) = \frac{4}{3}\pi r^3\varrho_p\left(a_i + w_{pk}\frac{\partial u_i}{\partial x_k}\right), \tag{8}$$

taking into account the relations (3) and (4), may be reduced to the form

$$\left(\nu\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)w_{pk} = f_i, \quad f_i = g_i - a_i, \tag{9}$$

where f_i is the effective body force and

$$\nu = \frac{9\left(1+\varphi\dfrac{\varrho_p}{\varrho_l}\right)\mu}{2r^2\varrho_p} \tag{10}$$

is a dimensional auxiliary constant.

By solving Eq. (9) and by introducing w_{pi} to (5) we obtain

$$\tilde{\tau}_{ij} = -\varphi\frac{\varrho_p}{\varrho_l}\left(1+\varphi\frac{\varrho_p}{\varrho_l}\right)\varrho_l \times \frac{\left[f_kW_{ki} + \nu\left(f_i\dfrac{\partial u_k}{\partial x_k} - f_k\dfrac{\partial u_i}{\partial x_k}\right) + f_i\nu^2\right]\left[f_kW_{kj} + \nu\left(f_j\dfrac{\partial u_k}{\partial x_k} - f_k\dfrac{\partial u_j}{\partial x_k}\right) + f_j\nu^2\right]}{\left(W + \nu U + \nu^2\dfrac{\partial u_k}{\partial x_k} + \nu^3\right)^2}, \tag{11}$$

where

(12)
$$W=\begin{vmatrix}\frac{\partial u_1}{\partial x_1}, & \frac{\partial u_1}{\partial x_2}, & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1}, & \frac{\partial u_2}{\partial x_2}, & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1}, & \frac{\partial u_3}{\partial x_2}, & \frac{\partial u_3}{\partial x_3}\end{vmatrix},\qquad W_{kl}=\begin{vmatrix}\frac{\partial u_{k+1}}{\partial x_{l+1}}, & \frac{\partial u_{k+1}}{\partial x_{l+2}}\\ \frac{\partial u_{k+2}}{\partial x_{l+1}}, & \frac{\partial u_{k+2}}{\partial x_{l+2}}\end{vmatrix},$$

$$U=\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3}+\frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}-\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}-\frac{\partial u_3}{\partial x_1}\frac{\partial u_1}{\partial x_3},$$

the subscripts $k+1$, $k+2$, $i+1$, $i+2$, go through cyclic order: 1, 2, 3, 1, 2, 3 ....

For heavy particles, the product $\varphi\varrho_p/\varrho_l$ should not obligatorily be small. We will assume it to be of the order of unity. For small particles ν is very large and $\tilde{\tau}_{ij}$, being in consequence of order ν^{-2}, may be disregarded. For heavy and large particles, when ν is sufficiently small, the tensor $\tilde{\tau}_{ij}$ depends strongly on the local motion of the suspension $\partial u_i/\partial x_k$. Analysing this dependence for arbitrary motion is very difficult. We will evaluate quantitatively the tensor $\tilde{\tau}_{ij}$ for a particular case of shear flow only.

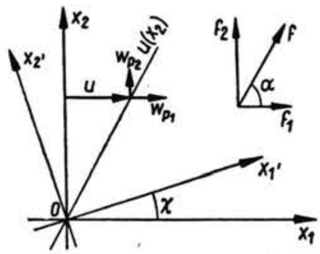

FIG. 1.

Let us consider a plane shear flow (Fig. 1) with $f_3=0$, $f_1=f\cos\alpha$, $f_2=f\sin\alpha$, the velocity u_i directed along the x_1 axis ($u_2=u_3=0$) and the gradient of velocity $\partial u_1/\partial x_2 = u'$ directed along the x_2 axis. For this case we have

(13)
$$\frac{\partial u_i}{\partial x_k}=\left\|\begin{matrix}0, & u', & 0\\ 0, & 0, & 0\\ 0, & 0, & 0\end{matrix}\right\|,\qquad W=U=W_{kl}=\frac{\partial u_k}{\partial x_k}=0.$$

Introducing Eq. (13) into Eq. (11) we obtain the tensor

(14)
$$\tilde{\tau}_{ij}=2G\left(\varphi\frac{\varrho_p}{\varrho_l}\right)\tau^*\cdot\left\|\begin{matrix}\left(\frac{u'}{\nu}\sin\alpha-\cos\alpha\right)^2, & -\left(\frac{u'}{\nu}\sin\alpha-\cos\alpha\right)\sin\alpha\\ -\left(\frac{u'}{\nu}\sin\alpha-\cos\alpha\right)\sin\alpha, & \sin^2\alpha\end{matrix}\right\|$$

which, in a new system of Cartesian coordinates $x_{1'}$, $x_{2'}$, inclined by angle

(15)
$$\chi=\frac{1}{2}\operatorname{arctg}\frac{1}{2}\left[\left(\frac{u'}{\nu}-\operatorname{ctg}\alpha\right)^{-1}-\left(\frac{u'}{\nu}-\operatorname{ctg}\alpha\right)\right]$$

in respect to x_1, x_2 (Fig. 1), may be presented in another form:

$$\tilde{\tau}_{i'j'} = \tau^* \cdot G\left(\varphi \frac{\varrho_p}{\varrho_l}\right) \cdot H\left(\frac{u'}{\nu}, \alpha\right)\left[-\left\|\begin{matrix}1, & 0\\ 0, & 1\end{matrix}\right\| + \left\|\begin{matrix}0, & 1\\ 1, & 0\end{matrix}\right\|\right], \tag{16}$$

where

$$\tau^* = \frac{2}{81}\frac{r^4\varrho_p^2\varrho_l f^2}{\mu^2}, \quad G(\Phi) = \frac{\Phi}{1+\Phi},$$

$$H\left(\frac{u'}{\nu}, \alpha\right) = \left(\frac{u'}{\nu}\sin\alpha - \cos\alpha\right)^2 + \sin^2\alpha = \left(\frac{u'}{\nu}\sin\alpha\right)^2 - \frac{u'}{\nu}\sin 2\alpha + 1. \tag{17}$$

The tensor $\tilde{\tau}_{i'j'}$, proportional to τ^*, depends additionally through G and H on the nondimensional parameters: $\varphi\varrho_p/\varrho_l$, α and u'/ν. For very large u'/ν and $f_2 \neq 0$, it may grow proportionally to $(u'/\nu)^2$. To compare $\tilde{\tau}_{ij}$ with $\bar{\tau}_{ij}$ for $f_2 \neq 0$, we shell introduce the two following auxialiary dimensionless constants

$$A = \frac{\tau^*}{p} = \frac{2}{81}\frac{r^4\varrho_p^2\varrho_l f^2}{\mu^2 p}, \quad B = \frac{\tau^*\left(1+\varphi\dfrac{\varrho_p}{\varrho_l}\right)}{\mu\nu} = \frac{4}{729}\frac{r^6\varrho_p^3\varrho_l f^2}{\mu^4}. \tag{18}$$

For B and u'/ν sufficiently large: $B(u'/\nu) \gg 1$, the viscous term in $\bar{\tau}_{ij}$ may become negligibly small and the whole shear stress would be due to the momentum flux of particles. For A sufficiently large $A \gg 1$, the additional contribution of the sedimentiation term $\tilde{\tau}_{ij}$ to the pressure term in $\bar{\tau}_{ij}$ should be taken into account. For larger particles and a higher velocity gradient $u'/\nu \gg 1$, the factor $H(u'/\nu, \alpha) \approx (u'/\nu)^2\sin^2\alpha$ may additionally contribute to the sedimentation influence on the pressure.

As an example, we will consider a horizontal Couette flow of a sedimenting fog in the air at atmospheric pressure. For this case we have: the density of water $\varrho_p = 10^3$ kg/m^3, the density of air $\varrho_l = 1.2$ kg/m^3, the viscosity of air $\mu = 18 \cdot 10^{-6}$ kg/(ms), the gravity acceleration $f = g = 9.81$ m/s^2, the pressure $p = 10^5$ kg/(ms^2) and, in consequence, the constants

$$(1+\varphi\varrho_p/\varrho_l)/\nu = 1.2 \cdot 10^7 r^2 \text{ s/m}^2, \quad \tau^* = 8.8 \cdot 10^{15} r^4 \text{ kg/(m}^5\text{s}^2),$$
$$A = 8.8 \cdot 10^{10} r^4 \text{ 1/m}^4, \quad B = 6 \cdot 10^{27} r^6 \text{ 1/m}^6,$$

depend on r, as it is given in the Table 1.

We can see that for fogs with drops larger than 10 μm the sedimentation effects may dominate over the "classical" shear viscosity $\mu u'$. Yet larger drops of millimeter size could additionally influence the static pressure.

It seems that for suspensions with large and heavy particles the considered above influence of sedimentation performs an important role not only in the simple case of the Couette flow, but also in other motions containing shear flow regions. For such cases corresponding to large values of the characteristic numbers A, B, u'/ν and a moderate value of $\varphi\varrho_p/\varrho_l$, the "classical" viscosity may be disregarded and it should be replaced by sedimentation effects. These effects introduce into the flow a particular anisotropy. The motion

Table 1.

r	[mm]	0.001	0.01	0.1	1
$\dfrac{1+\varphi\dfrac{\varrho_p}{\varrho_l}}{\nu}$	[s]	$1.2\cdot 10^{-5}$	$1.2\cdot 10^{-3}$	0.12	12
τ^*	$\left[\dfrac{\mathrm{kg}}{\mathrm{ms}^2}\right]$	$8.8\cdot 10^{-9}$	$8.8\cdot 10^{-5}$	0.88	$8.8\cdot 10^{3}$
A	—	$8.8\cdot 10^{-14}$	$8.8\cdot 10^{-10}$	$8.8\cdot 10^{-6}$	0.088
B	—	$6\cdot 10^{-9}$	$6\cdot 10^{-3}$	$6\cdot 10^{3}$	$6\cdot 10^{9}$

along the resulting body forces encounters small resistance, but in the flow across the field of body forces the sedimentation provokes a considerable momentum flux of particles, which generates the resulting stress tensor.

References

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