Wave propagation in a straight elastic rod subjected to initial finite extension and twist

G. EASON (GLASGOW)

THE PROPAGATION of waves in a straight elastic rod subjected to an initial finite extension and twist is considered. The basic equations due to Green and Laws are assumed. It is found that effects arising from the initial twist may be important; in particular, they give a linking between certain of the modes. Some numerical results are presented in graphical form.

Rozważono propagację fal w prostym pręcie sprężystym poddanym działaniu skończonych odkształceń rozciągających i skręcających. Stwierdzono, że skręcenie wstępne prowadzić może do istotnych efektów; w szczególności może ono powodować sprzężenie pewnych postaci drgań. Podstawowe równania problemu przyjęto w postaci zaproponowanej przez Greena i Lawsa. W postaci graficznej przedstawiono pewne przykłady liczbowe.

Рассмотрено распространение волн в прямолинейном упругом стержне, подвергнутом действию конечных растягивающих и скручивающих деформаций. Констатировано, что вступительное скручивание может привести к существенным эффектам; в частности может оно вызывать сопряжение некоторых типов колебаний. Основные уравнения проблемы приняты в виде предложенном Грином и Лоусом. В графическом виде представлены некоторые числовые примеры.

1. Introduction

WAVE propagation in elastic rods has been studied for many years. Initially the basic equations were derived using assumptions that were not always clearly defined. More recently a number of approaches have been proposed which derive the basic equations from the full three-dimensional theory of elasticity or, alternatively, make use of the director theory. In general these two approaches result in the same basic equations. The basic equations adopted here are based on equations derived by GREEN and LAWS [1] using the director theory and adapted by GREEN, KNOPS and LAWS [2] to the case of a rod which is given an initial finite deformation followed by a superposed small deformation.

The particular problem considered here is that of a straight elastic rod which is subjected to an initial finite extension and twist. The propagation of infinitesimal waves in the rod is then investigated. It was shown by GREEN, KNOPS and LAWS [2] that wave propagation in a straight elastic rod subjected to a finite extension only is very similar to wave propagation in an unstressed rod. The extensional, torsional and two flexural modes remain mutually independent. The effect of an initial twist in addition to an extension is to cause a linking between some of the modes. The extensional and torsional modes interact, as do the two flexural modes. Comparison is made with results obtained from a classical theory such as that given by LOVE [3] and it is found that the present theory predicts the existence of three waves not predicted by the classical theory. It appears

likely that these new waves are high frequency effects, except when there is considerable compression and twist. Some numerical results are presented in graphical form. Related work on the propagation of waves in elastic rods is that due to EASON [4], COHEN and WHITMAN [5] and ANTMAN and LIM [6].

The basic equations, in the absence of body forces, for an elastic, isothermal rod subjected to an initial finite deformation followed by a superposed small deformation, as derived by GREEN, KNOPS and LAWS [2], are summarised is Sect. 2. These equations apply to a rod of any shape with any initial deformation. In Sect. 3 the static solution of these equations when specialised to a straight rod subjected to finite extension and twist is written down. The corresponding equations for the superposed small deformation are established in Sect. 4 and the grouping of the equations into two sets is noted. The extension-twist motion is considered in detail in Sect. 5 and the main features of the five waves that may propagate are established. A modified extension twist motion corresponding to the classical theory is discussed in Sect. 6. In this theory two waves only may propagate. The connection between the two theories is noted. The equations governing the flexural motion of the rod are analysed in Sect. 7 for the completely general case. It is found that four waves may propagate, this number remaining unchanged from the classical theory. Finally, in Sect. 8, the flexural motion of a symmetric rod is examined. For such a rod it is found that the equations may be factorised and the analysis is simplified.

In all cases the results obtained here reduce to those obtained by GREEN, KNOPS and LAWS [2] for the rod with finite extension only when the applied torsion is set equal to zero. They also reduce to those given by GREEN, LAWS and NAGHDI [7] when there is no initial finite stress.

2. The basic equations

The basic equations governing the deformation of an elastic rod that is subjected to a large deformation followed by a small superposed deformation have been given by GREEN, KNOPS and LAWS [2]. General results were obtained by making use of a theory involving directors. Their results will be summarised here and the equations of motion for an isothermal elastic rod subjected to an initial finite extension and twist will then be deduced.

A rod is defined to be a curve embedded in Euclidean 3-space at each point of which there are two assigned directors. Three configurations of the rod are considered; the initial configuration with the position of the curve $\overline{\mathscr{C}}$ and directors $\overline{\mathbf{A}}_\alpha$, the first deformed configuration with curve $\mathscr{C}$ and directors $\mathbf{A}_\alpha$ and the final configuration c with directors $\mathbf{a}_\alpha$. The convention adopted throughout is that Greek indices take the values 1, 2 and Latin indices the values 1, 2, 3. The equation of $\overline{\mathscr{C}}$ is taken to be

$$(2.1) \qquad\qquad \overline{\mathbf{R}} = \overline{\mathbf{R}}(\theta),$$

the equation of $\mathscr{C}$ is

$$(2.2) \qquad\qquad \mathbf{R} = \mathbf{R}(\theta),$$

and the equation of c at time t is given by

$$(2.3) \qquad \mathbf{r} = \mathbf{r}(\theta, t) = \mathbf{R}(\theta) + \varepsilon \mathbf{u}(\theta, t),$$

where ε is a small real parameter and θ denotes a convected coordinate. In the following analysis powers of ε above the first are neglected. In addition to the directors a third vector $\overline{\mathbf{A}}_3$ in the initial configuration, $\mathbf{A}_3$ in the first deformed configuration and $\mathbf{a}_3$ in the final configuration is introduced with

$$(2.4) \qquad \overline{\mathbf{A}}_3 = \overline{\mathbf{A}}_3(\theta) = \partial \overline{\mathbf{R}}/\partial \theta, \quad \mathbf{A}_3 = \mathbf{A}_3(\theta) = \partial \mathbf{R}/\partial \theta, \quad \mathbf{a}_3 = \mathbf{a}_3(\theta, t) = \partial \mathbf{r}/\partial \theta.$$

It is assumed that $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] > 0$.

It is convenient to introduce the reciprocal base vectors $\overline{\mathbf{A}}^i$, $\mathbf{A}^i$, $\mathbf{a}^i$ where

$$(2.5) \qquad \overline{\mathbf{A}}^i \cdot \overline{\mathbf{A}}_j = \mathbf{A}^i \cdot \mathbf{A}_j = \mathbf{a}^i \cdot \mathbf{a}_j = \delta^i_j,$$

and δ^i_j is the Kronecker delta. In the final configuration

$$(2.6) \qquad \mathbf{a}_i = \mathbf{a}_i(\theta, t) = \mathbf{A}_i(\theta) + \varepsilon \mathbf{b}_i(\theta, t),$$

$$(2.7) \qquad \mathbf{a}^i = \mathbf{a}^i(\theta, t) = \mathbf{A}^i(\theta) - \varepsilon \mathbf{b}^i(\theta, t),$$

with

$$(2.8) \qquad \mathbf{b}_3 = \partial \mathbf{u}/\partial \theta.$$

In Eqs. (2.6) and (2.7)

$$(2.9) \qquad \mathbf{b}_i = b_{ij} \mathbf{A}^j = b_i^{\cdot j} \mathbf{A}_j,$$

$$(2.10) \qquad \mathbf{b}^i = b_j^{\cdot i} \mathbf{A}^j.$$

It is convenient to introduce symmetrical quantities related to the strains in the rod defined by

$$(2.11) \qquad \overline{A}_{ij} = \overline{\mathbf{A}}_i \cdot \overline{\mathbf{A}}_j, \quad A_{ij} = \mathbf{A}_i \cdot \mathbf{A}_j, \quad a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j,$$

$$(2.12) \qquad \overline{A}^{ij} = \overline{\mathbf{A}}^i \cdot \overline{\mathbf{A}}^j, \quad A^{ij} = \mathbf{A}^i \cdot \mathbf{A}^j, \quad a^{ij} = \mathbf{a}^i \cdot \mathbf{a}^j.$$

In Eqs. (2.11) and (2.12)

$$(2.13) \qquad a_{ij} = A_{ij} + \varepsilon(b_{ij} + b_{ji}),$$

$$(2.14) \qquad a^{ij} = A^{ij} - \varepsilon A^{ik}(b_k^{\cdot j} + b_k^{\cdot i}).$$

In addition, define

$$(2.15) \qquad \Gamma_{ij} = A_{ij} - \overline{A}_{ij}, \quad \gamma_{ij} = a_{ij} - \overline{A}_{ij} = \Gamma_{ij} + \varepsilon(b_{ij} + b_{ji}).$$

Quantities related to the curvatures in the rod are defined by

$$(2.16) \qquad \overline{K}_{ij} = \overline{\mathbf{A}}_j \cdot \partial \overline{\mathbf{A}}_i/\partial \theta, \quad K_{ij} = \mathbf{A}_j \cdot \partial \mathbf{A}_i/\partial \theta, \quad \varkappa_{ij} = \mathbf{a}_j \cdot \partial \mathbf{a}_i/\partial \theta,$$

$$(2.17) \qquad \overline{K}_i^{\cdot j} = \overline{A}^{rj} \overline{K}_{ir}, \quad K_i^{\cdot j} = A^{rj} K_{ir}, \quad \varkappa_i^{\cdot j} = a^{rj} \varkappa_{ir},$$

with

$$(2.18) \qquad \varkappa_{ij} = K_{ij} + \varepsilon \lambda_{ij}, \quad \varkappa_i^{\cdot j} = K_i^{\cdot j} + \varepsilon \mu_i^{\cdot j}.$$

In Eq. (2.18)

(2.19) $$\lambda_{ij} = A_{js}\,\partial b_i^{\cdot s}/\partial\theta + b_i^{\cdot s}K_{sj} + b_j^{\cdot s}K_{is},$$

(2.20) $$\mu_i^{\cdot j} = \partial b_i^{\cdot j}/\partial\theta + b_i^{\cdot k}K_k^{\cdot j} - b_k^{\cdot j}K_i^{\cdot k}.$$

Also defined are

(2.21) $$\Sigma_{ij} = K_{ij} - \bar{K}_{ij}, \qquad \sigma_{ij} = \varkappa_{ij} - \bar{K}_{ij} = \Sigma_{ij} + \varepsilon\lambda_{ij}.$$

Note that $b_i^{\cdot j}$ and b_{ij} are related by

(2.22) $$b_i^{\cdot j} = A^{kj}b_{ik}.$$

The local equation of mass conservation is

(2.23) $$\varrho\,\sqrt{a_{33}} = \beta(\theta),$$

where β is a function of θ and ϱ is the mass per unit length of c.

Following GREEN, KNOPS and LAWS [2] it is found that in the final configuration the force components n^i, director force components $p^{\alpha i}$ and quantities $\pi^{\alpha i}$ are given by

(2.24) $$n^i = N^i + \varepsilon(\nu^i - b_k^{\cdot i}N^k),$$

(2.25) $$p^{\alpha i} = P^{\alpha i} + \varepsilon(\xi^{\alpha i} - b_k^{\cdot i}P^{\alpha k}),$$

(2.26) $$\pi^{\alpha i} = \Pi^{\alpha i} + \varepsilon(\omega^{qi} - b_k^{\cdot i}\Pi^{\alpha k}),$$

where majuscules such as N^i refer to values in the first deformed configuration. In the absence of assigned force and assigned director force the equations of equilibrium in the first deformed configuration take the forms

(2.27) $$\frac{\partial N^i}{\partial\theta} + K_r^{\cdot i}N^r = 0,$$

(2.28) $$\Pi^{12} - \Pi^{21} + P^{\gamma 2}K_\gamma^{\cdot 1} - P^{\gamma 1}K_\gamma^{\cdot 2} = 0,$$

(2.29) $$\Pi^{\beta 3} + P^{\alpha 3}K_\alpha^{\cdot\beta} - P^{\alpha\beta}K_\alpha^{\cdot 3} - N^\beta = 0,$$

(2.30) $$\Pi^{\alpha i} = \frac{\partial P^{\alpha i}}{\partial\theta} + K_r^{\cdot i}P^{\alpha r}.$$

In the final configuration the equations of motion are found to be

(2.31) $$\frac{\partial\nu^i}{\partial\theta} + K_r^{\cdot i}\nu^r = \beta\frac{\partial^2 u^i}{\partial t^2},$$

(2.32) $$\omega^{12} - \omega^{21} - b_r^{\cdot 2}\Pi^{1r} + b_r^{\cdot 1}\Pi^{2r} + P^{\gamma 2}\mu_\gamma^{\cdot 1} - P^{\gamma 1}\mu_\gamma^{\cdot 2}$$
$$+ K_\gamma^{\cdot 1}(\xi^{\gamma 2} - b_r^{\cdot 2}P^{\gamma r}) - K_\gamma^{\cdot 2}(\xi^{\gamma 1} - b_r^{\cdot 1}P^{\gamma r}) = 0,$$

(2.33) $$\omega^{\beta 3} - b_r^{\cdot 3}\Pi^{\beta r} - \nu^\beta + b_r^{\cdot\beta}N^r + K_\alpha^{\cdot\beta}(\xi^{\alpha 3} - b_r^{\cdot 3}P^{\alpha r})$$
$$+ \mu_\alpha^{\cdot\beta}P^{\alpha 3} - K_\alpha^{\cdot 3}(\xi^{\alpha\beta} - b_r^{\cdot\beta}P^{\alpha r}) - \mu_\alpha^{\cdot 3}P^{\alpha\beta} = 0,$$

where

(2.34) $$\mathbf{u} = \mathbf{A}_i u^i = \mathbf{A}^i u_i,$$

(2.35) $$\omega^{\alpha i} = \frac{\partial\xi^{\alpha i}}{\partial\theta} + K_r^{\cdot i}\xi^{\alpha r} - \beta y^{\alpha\beta}\frac{\partial^2 b_\beta^{\cdot i}}{\partial t^2}.$$

The final term in Eq. (2.35) is the director inertia term.

The Helmholtz free energy per unit mass, A, for an isothermal elastic rod is given by

$$(2.36) \qquad A = A(\gamma_{ij}, \sigma_{\alpha i}, \overline{A}_{ij}, \overline{K}_{\alpha i}),$$

where γ_{ij}, $\sigma_{\alpha i}$, $\overline{A}_{ij}$, $\overline{K}_{\alpha i}$ are defined by Eqs. (2.15), (2.21), (2.12) and, (2.16) respectively. It is found that (see GREEN, KNOPS and LAWS [2])

$$(2.37) \qquad N^3 - P^{\alpha 3} K_\alpha^{\cdot 3} = 2\beta \frac{\partial A}{\partial \Gamma_{33}},$$

$$(2.38) \qquad N^\beta - P^{\alpha 3} K_\alpha^{\cdot \beta} = \beta \frac{\partial A}{\partial \Gamma_{\beta 3}},$$

$$(2.39) \qquad \Pi^{\alpha\beta} + \Pi^{\beta\alpha} - P^{\gamma\beta} K_\gamma^{\cdot \alpha} - P^{\gamma\alpha} K_\gamma^{\cdot \beta} = 4\beta \frac{\partial A}{\partial \Gamma_{\alpha\beta}},$$

$$(2.40) \qquad P^{\alpha i} = \beta \frac{\partial A}{\partial \Sigma_{\alpha i}},$$

where A is evaluated in the first deformed configuration. The constitutive equations in the final configuration are

$$(2.41) \quad \nu^3 - b_r^{\cdot 3} N^r - K_\alpha^{\cdot 3}(\xi^{\alpha 3} - b_r^{\cdot 3} P^{\alpha r}) - P^{\alpha 3} \mu_\alpha^{\cdot 3} = 4\beta \frac{\partial^2 A}{\partial \Gamma_{33}^2} b_{33}$$

$$+ 2\beta \frac{\partial^2 A}{\partial \Gamma_{\beta 3} \partial \Gamma_{33}}(b_{\beta 3} + b_{3\beta}) + 2\beta \frac{\partial^2 A}{\partial \Gamma_{\alpha\beta} \partial \Gamma_{33}}(b_{\alpha\beta} + b_{\beta\alpha}) + 2\beta \frac{\partial^2 A}{\partial \Sigma_{\alpha i} \partial \Gamma_{33}} \lambda_{\alpha i},$$

$$(2.42) \quad \nu^\beta - b_r^{\cdot \beta} N^r - K_\alpha^{\cdot \beta}(\xi^{\alpha 3} - b_r^{\cdot 3} P^{\alpha r}) - P^{\alpha 3} \mu_\alpha^{\cdot \beta} = 2\beta \frac{\partial^2 A}{\partial \Gamma_{33} \partial \Gamma_{\beta 3}} b_{33}$$

$$+ \beta \frac{\partial^2 A}{\partial \Gamma_{\alpha 3} \partial \Gamma_{\beta 3}}(b_{\alpha 3} + b_{3\alpha}) + \beta \frac{\partial^2 A}{\partial \Gamma_{\lambda\mu} \partial \Gamma_{\beta 3}}(b_{\lambda\mu} + b_{\mu\lambda}) + \beta \frac{\partial^2 A}{\partial \Sigma_{\alpha i} \partial \Gamma_{\beta 3}} \lambda_{\alpha i},$$

$$(2.43) \quad \omega^{\alpha\beta} + \omega^{\beta\alpha} - b_r^{\cdot \beta} \Pi^{\alpha r} - b_r^{\cdot \alpha} \Pi^{\beta r} - P^{\gamma\beta} \mu_\gamma^{\cdot \alpha} - P^{\gamma\alpha} \mu_\gamma^{\cdot \beta} - K_\gamma^{\cdot \alpha}(\xi^{\gamma\beta} - b_r^{\cdot \beta} P^{\gamma r})$$

$$- K_\gamma^{\cdot \beta}(\xi^{\gamma\alpha} - b_r^{\cdot \alpha} P^{\beta r}) = 8\beta \frac{\partial^2 A}{\partial \Gamma_{33} \partial \Gamma_{\alpha\beta}} b_{33} + 4\beta \frac{\partial^2 A}{\partial \Gamma_{\lambda 3} \partial \Gamma_{\alpha\beta}}(b_{\lambda 3} + b_{3\lambda})$$

$$+ 4\beta \frac{\partial^2 A}{\partial \Gamma_{\lambda\mu} \partial \Gamma_{\alpha\beta}}(b_{\lambda\mu} + b_{\mu\lambda}) + 4\beta \frac{\partial^2 A}{\partial \Sigma_{\gamma i} \partial \Gamma_{\alpha\beta}} \lambda_{\gamma i},$$

$$(2.44) \quad \xi^{\alpha i} - b_r^{\cdot i} P^{\alpha r} = 2\beta \frac{\partial^2 A}{\partial \Gamma_{33} \partial \Sigma_{\alpha i}} b_{33} + \beta \frac{\partial^2 A}{\partial \Gamma_{\beta 3} \partial \Sigma_{\alpha i}}(b_{\beta 3} + b_{3\beta})$$

$$+ \beta \frac{\partial^2 A}{\partial \Gamma_{\lambda\mu} \partial \Sigma_{\alpha i}}(b_{\lambda\mu} + b_{\mu\lambda}) + \beta \frac{\partial^2 A}{\partial \Sigma_{\beta r} \partial \Sigma_{\alpha i}} \lambda_{\beta r},$$

where, again, A is evaluated in the first deformed configuration and

$$(2.45) \qquad \lambda_{ij} = A_{js} \frac{\partial b_i^{\cdot s}}{\partial \theta} + b_i^{\cdot s} K_{sj} + b_j^{\cdot s} K_{is}.$$

The general theory developed thus far is very complicated and in order to introduce some simplification it is assumed that A is independent of $\overline{A}_{ij}$ and $\overline{K}_{\alpha i}$ so that, from Eq. (2.36),

$$(2.46) \qquad A = A(\gamma_{ij}, \sigma_{\alpha i}).$$

In addition it is assumed that A is invariant under the transformations

(2.47)
$$\mathbf{a}_1 \to \pm\mathbf{a}_1, \quad \mathbf{a}_2 \to \pm\mathbf{a}_2, \quad \mathbf{a}_3 \to \pm\mathbf{a}_3,$$
$$\overline{\mathbf{A}}_1 \to \pm\overline{\mathbf{A}}_1, \quad \overline{\mathbf{A}}_2 \to \pm\overline{\mathbf{A}}_2, \quad \overline{\mathbf{A}}_3 \to \pm\overline{\mathbf{A}}_3,$$

so that, if A is a polynomial, it is a function of (see GREEN, KNOPS and LAWS [2])

(2.48)
$$\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{12}^2, \gamma_{23}^2, \gamma_{13}^2, \gamma_{12}\gamma_{13}\gamma_{23},$$
$$\sigma_{11}^2, \sigma_{11}\sigma_{22}, \sigma_{22}^2, \sigma_{12}^2, \sigma_{12}\sigma_{21}, \sigma_{21}^2, \sigma_{13}^2, \sigma_{23}^2,$$
$$\sigma_{11}\sigma_{12}\sigma_{13}\sigma_{23}, \sigma_{11}\sigma_{21}\sigma_{13}\sigma_{23}, \sigma_{12}\sigma_{22}\sigma_{13}\sigma_{23}, \sigma_{21}\sigma_{22}\sigma_{13}\sigma_{23},$$
$$\gamma_{12}\sigma_{11}\sigma_{12}, \gamma_{12}\sigma_{11}\sigma_{21}, \gamma_{12}\sigma_{12}\sigma_{22}, \gamma_{12}\sigma_{21}\sigma_{22}, \gamma_{12}\sigma_{13}\sigma_{23},$$
$$\gamma_{13}\sigma_{11}\sigma_{13}, \gamma_{13}\sigma_{22}\sigma_{13}, \gamma_{13}\sigma_{12}\sigma_{23}, \gamma_{13}\sigma_{21}\sigma_{23},$$
$$\gamma_{23}\sigma_{11}\sigma_{23}, \gamma_{23}\sigma_{22}\sigma_{23}, \gamma_{23}\sigma_{12}\sigma_{13}, \gamma_{23}\sigma_{21}\sigma_{13},$$
$$\gamma_{12}\gamma_{13}\sigma_{11}\sigma_{23}, \gamma_{12}\gamma_{13}\sigma_{22}\sigma_{23}, \gamma_{12}\gamma_{13}\sigma_{12}\sigma_{13}, \gamma_{12}\gamma_{13}\sigma_{21}\sigma_{13},$$
$$\gamma_{12}\gamma_{23}\sigma_{11}\sigma_{13}, \gamma_{12}\gamma_{23}\sigma_{22}\sigma_{13}, \gamma_{12}\gamma_{23}\sigma_{12}\sigma_{23}, \gamma_{12}\gamma_{23}\sigma_{21}\sigma_{23},$$
$$\gamma_{13}\gamma_{23}\sigma_{11}\sigma_{12}, \gamma_{13}\gamma_{23}\sigma_{11}\sigma_{21}, \gamma_{13}\gamma_{23}\sigma_{12}\sigma_{22}, \gamma_{13}\gamma_{23}\sigma_{31}\sigma_{22}, \gamma_{13}\gamma_{23}\sigma_{13}\sigma_{23}.$$

Furthermore, the quantity $y^{\alpha\beta}$ in Eq. (2.35) is such that

(2.49)
$$y^{12} = y^{21} = 0, \quad y^{11} = \alpha_1, \quad y^{22} = \alpha_2.$$

This completes the summary of the basic equations. The particular problem of wave propagation in an initially straight rod which is subjected to finite extension and twist will now be examined.

3. The initial deformation

GREEN, KNOPS and LAWS [2] considered the propagation of waves in a straight elastic rod subjected to an initial finite simple extension. The basic equations for that problem are very similar to those for a similar rod that is not subjected to finite extension since the extensional, torsional and flexural modes remain independent of one another. The problem to be considered here is that of a straight elastic rod subjected to small deformation superposed on finite extension and twist. In this case the basic equations differ considerably from those for an undeformed rod since linking between the modes now occurs.

The problem of a straight elastic rod subjected to static finite extension and torsion in the absence of body forces has been discussed by GREEN, KNOPS and LAWS [2]. It is found that the initial state of the rod and the first deformed configuration are defined by (using Eqs. (2.1), (2.2), (2.4), (2.11), (2.15), (2.16), and (2.17))

(3.1)
$$\overline{\mathbf{R}} = \theta\overline{\mathbf{A}}_3, \quad \overline{A}_{IJ} = \overline{\mathbf{A}}_I \cdot \overline{\mathbf{A}}_J = \delta_{IJ}, \quad \overline{K}_{IJ} = 0,$$

(3.2)
$$\mathbf{A}_1 = \lambda_1 \{\overline{\mathbf{A}}_1 \cos(\psi\theta) + \overline{\mathbf{A}}_2 \sin(\psi\theta)\},$$
$$\mathbf{A}_2 = \lambda_2 \{-\overline{\mathbf{A}}_1 \sin(\psi\theta) + \overline{\mathbf{A}}_2 \cos(\psi\theta)\},$$

(3.3)
$$\mathbf{R} = \lambda_3 \overline{\mathbf{R}} = \lambda_3 \theta\overline{\mathbf{A}}_3,$$

(3.4)
$$A_{11} = \lambda_1^2, \quad A_{22} = \lambda_2^2, \quad A_{33} = \lambda_3^2, \quad A_{ij} = 0, \quad i \neq j,$$
$$A^{11} = 1/\lambda_1^2, \quad A^{22} = 1/\lambda_2^2, \quad A^{33} = 1/\lambda_3^2, \quad A^{ij} = 0, \quad i \neq j,$$

(3.5) $\quad \Gamma_{11} = \lambda_1^2 - 1, \quad \Gamma_{22} = \lambda_2^2 - 1, \quad \Gamma_{33} = \lambda_3^2 - 1, \quad \Gamma_{ij} = 0, \quad i \neq j,$

(3.6) $\quad K_{12} = -K_{21} = \lambda_1 \lambda_2 \psi = K,$

(3.7) $\quad K_2^{\cdot 1} = -K/\lambda_1^2, \quad K_1^{\cdot 2} = K/\lambda_2^2,$

(3.8) $\quad \Sigma_{12}^{\cdot} = -\Sigma_{21} = K,$

with K_{ij}, $K_j^{\cdot i}$ and Σ_{ij} zero otherwise. From Eq. (2.23)

(3.9)
$$\beta = \bar{\varrho} = \varrho \lambda_3,$$

and Eqs. (2.27)–(2.30) and Eqs. (2.37)–(2.40) are satisfied by

(3.10)
$$N^3 = N = 2\beta \frac{\partial A}{\partial \Gamma_{33}}, \quad N^\beta = 0;$$

(3.11)
$$P^{12} = \beta \frac{\partial A}{\partial \Sigma_{12}}, \quad P^{21} = \beta \frac{\partial A}{\partial \Sigma_{21}},$$
$$P^{11} = P^{22} = P^{\beta 3} = 0;$$

$$\Pi^{12} = \Pi^{21} = \Pi^{\beta 3} = 0,$$

(3.12)
$$\Pi^{11} = P^{21} K_2^{\cdot 1} + 2\beta \frac{\partial A}{\partial \Gamma_{11}},$$

$$\Pi^{22} = P^{12} K_1^{\cdot 2} + 2\beta \frac{\partial A}{\partial \Gamma_{22}};$$

(3.13)
$$K_2^{\cdot 1}(P^{12} - P^{21}) = 2\beta \frac{\partial A}{\partial \Gamma_{11}},$$

$$K_1^{\cdot 2}(P^{21} - P^{12}) = 2\beta \frac{\partial A}{\partial \Gamma_{22}}.$$

If λ_3, ψ and A are given, then Eqs. (3.10)–(3.13) determine N^3, P^{12}, P^{21}, Π^{11}, Π^{22}, λ_1 and λ_2. No restriction is placed on the form of A in the present discussion.

The values given by Eqs. (3.1)–(3.13) are now to be substituted into the equations of Sect. 2 that determine the infinitesimal motion. The resulting equations are discussed in the next section.

4. The superposed small deformation

The particular equations determining the infinitesimal motion of the rod in the absence of body forces will now be derived from the general equations of Section 2 using Eqs. (3.1)–(3.13).

Equation (2.31) results in the three equations

(4.1)
$$\frac{\partial v^1}{\partial \theta} - K A^{11} v^2 \doteq \beta A^{11} \frac{\partial^2 u_1}{\partial t^2},$$

(4.2)
$$\frac{\partial v^2}{\partial \theta} + KA^{22}v^1 = \beta A^{22}\frac{\partial^2 u_2}{\partial t^2},$$

(4.3)
$$\frac{\partial v^3}{\partial \theta} = \beta A^{33}\frac{\partial^2 u_3}{\partial t^2},$$

where the result

(4.4)
$$u^i = A^{ii}u_i, \quad (i \text{ not summed})$$

has been used.

Equations (2.32) and (2.33), with the results of Sect. 3, give

(4.5) $\omega^{12} - \omega^{21} + P^{12}\mu_1^{\cdot 1} - P^{21}\mu_2^{\cdot 2} - KA^{11}\xi^{22} - KA^{22}\xi^{11}$
$$+ KA^{22}b_2^{\cdot 1}(P^{12} + P^{21}) + KA^{11}b_1^{\cdot 2}(P^{12} + P^{21}) = 0,$$

(4.6) $\omega^{13} - v^1 - KA^{11}\xi^{23} - \mu_2^{\cdot 3}P^{21} + KA^{11}b_1^{\cdot 3}(P^{12} + P^{21}) + b_3^{\cdot 1}N = 0,$

(4.7) $\omega^{23} - v^2 + KA^{22}\xi^{13} - \mu_1^{\cdot 3}P^{12} - KA^{22}b_2^{\cdot 3}(P^{12} + P^{21}) + b_3^{\cdot 2}N = 0.$

From Eq. (2.35) it is found that

(4.8)
$$\omega^{11} = \frac{\partial \xi^{11}}{\partial \theta} - KA^{11}\xi^{12} - \beta\alpha_1 \frac{\partial^2 b_1^{\cdot 1}}{\partial t^2},$$

(4.9)
$$\omega^{22} = \frac{\partial \xi^{22}}{\partial \theta} + KA^{22}\xi^{21} - \beta\alpha_2 \frac{\partial^2 b_2^{\cdot 2}}{\partial t^2},$$

(4.10)
$$\omega^{12} = \frac{\partial \xi^{12}}{\partial \theta} + KA^{22}\xi^{11} - \beta\alpha_1 \frac{\partial^2 b_1^{\cdot 2}}{\partial t^2},$$

(4.11)
$$\omega^{21} = \frac{\partial \xi^{21}}{\partial \theta} - KA^{11}\xi^{22} - \beta\alpha_2 \frac{\partial^2 b_2^{\cdot 1}}{\partial t^2},$$

(4.12)
$$\omega^{13} = \frac{\partial \xi^{13}}{\partial \theta} - \beta\alpha_1 \frac{\partial^2 b_1^{\cdot 3}}{\partial t^2},$$

(4.13)
$$\omega^{23} = \frac{\partial \xi^{23}}{\partial \theta} - \beta\alpha_2 \frac{\partial^2 b_2^{\cdot 3}}{\partial t^2}.$$

When Eqs. (4.10) and (4.11) are substituted into Eq. (4.5) and the relations (2.20) and (2.22) are used, there results

(4.14) $\dfrac{\partial}{\partial \theta}(\xi^{12^{\cdot}} - \xi^{21}) + P^{12}A^{11}\dfrac{\partial b_{11}}{\partial \theta} - P^{21}A^{22}\dfrac{\partial b_{22}}{\partial \theta} = \beta\alpha_1 A^{22}\dfrac{\partial^2 b_{12}}{\partial t^2} - \beta\alpha_2 A^{11}\dfrac{\partial^2 b_{21}}{\partial t^2}.$

Similarly, Eq. (4.6) and (4.7) with Eqs. (4.12) and (4.13) give

(4.15) $\dfrac{\partial \xi^{13}}{\partial \theta} - A^{11}K\xi^{23} - v^1 - P^{21}A^{33}\dfrac{\partial b_{23}}{\partial \theta} + A^{11}Nb_{31} + A^{11}A^{33}KP^{12}b_{13} = \beta\alpha_1 A^{33}\dfrac{\partial^2 b_{13}}{\partial t^2},$

(4.16) $\dfrac{\partial \xi^{23}}{\partial \theta} + A^{22}K\xi^{13} - v^2 - P^{12}A^{33}\dfrac{\partial b_{13}}{\partial \theta} + A^{22}Nb_{32} - A^{22}A^{33}KP^{21}b_{23} = \beta\alpha_2 A^{33}\dfrac{\partial^2 b_{23}}{\partial t^2}.$

No further restriction is placed on the Helmholtz free energy beyond that implied by Eqs. (2.46) and (2.47) so that the constitutive relations are quite general. In order to write them down, it is convenient to write

$$
\begin{array}{lll}
k_1 = \beta \dfrac{\partial^2 A}{\partial \Gamma_{11}^2}, & k_2 = \beta \dfrac{\partial^2 A}{\partial \Gamma_{22}^2}, & k_3 = \beta \dfrac{\partial^2 A}{\partial \Gamma_{33}^2}, \\[2mm]
k_4 = 4\beta \dfrac{\partial^2 A}{\partial \Gamma_{12}^2}, & k_5 = \beta \dfrac{\partial^2 A}{\partial \Gamma_{23}^2}, & k_6 = \beta \dfrac{\partial^2 A}{\partial \Gamma_{13}^2}, \\[2mm]
k_7 = 2\beta \dfrac{\partial^2 A}{\partial \Gamma_{11} \partial \Gamma_{22}}, & k_8 = 2\beta \dfrac{\partial^2 A}{\partial \Gamma_{11} \partial \Gamma_{33}}, & k_9 = 2\beta \dfrac{\partial^2 A}{\partial \Gamma_{22} \partial \Gamma_{33}}, \\[2mm]
k_{10} = \beta \dfrac{\partial^2 A}{\partial \Sigma_{11}^2}, & k_{11} = \beta \dfrac{\partial^2 A}{\partial \Sigma_{22}^2}, & k_{12} = \beta \dfrac{\partial^2 A}{\partial \Sigma_{12}^2}, \\[2mm]
k_{13} = \beta \dfrac{\partial^2 A}{\partial \Sigma_{21}^2}, & k_{14} = 2\beta \dfrac{\partial^2 A}{\partial \Sigma_{12} \partial \Sigma_{21}}, & k_{15} = \beta \dfrac{\partial^2 A}{\partial \Sigma_{23}^2}, \\[2mm]
k_{16} = \beta \dfrac{\partial^2 A}{\partial \Sigma_{13}^2}, & k_{17} = 2\beta \dfrac{\partial^2 A}{\partial \Sigma_{11} \partial \Sigma_{22}}, & k_{18} = \beta \dfrac{\partial^2 A}{\partial \Gamma_{11} \partial \Sigma_{12}}, \\[2mm]
k_{19} = \beta \dfrac{\partial^2 A}{\partial \Gamma_{11} \partial \Sigma_{21}}, & k_{20} = \beta \dfrac{\partial^2 A}{\partial \Gamma_{22} \partial \Sigma_{12}}, & k_{21} = \beta \dfrac{\partial^2 A}{\partial \Gamma_{22} \partial \Sigma_{21}}, \\[2mm]
k_{22} = 2\beta \dfrac{\partial^2 A}{\partial \Gamma_{33} \partial \Sigma_{12}}, & k_{23} = \beta \dfrac{\partial^2 A}{\partial \Gamma_{33} \partial \Sigma_{21}}, & k_{24} = \beta \dfrac{\partial^2 A}{\partial \Gamma_{12} \partial \Sigma_{11}}, \\[2mm]
k_{25} = \beta \dfrac{\partial^2 A}{\partial \Gamma_{12} \partial \Sigma_{22}}, & k_{26} = \beta \dfrac{\partial^2 A}{\partial \Gamma_{23} \partial \Sigma_{13}}, & k_{27} = \beta \dfrac{\partial^2 A}{\partial \Gamma_{13} \partial \Sigma_{23}}.
\end{array}
$$

(4.17)

All other second derivatives of A vanish in the deformed configuration discussed in Sect. 3. Equation (2.41) with Eqs. (2.20), (2.22), (2.45) and the results of Sect. 3 gives

$$
\begin{aligned}
(4.18) \quad \nu^3 = {} & \{2k_8 + K A^{11}(k_{22} - k_{23})\} b_{11} + \{2k_9 + K A^{22}(k_{22} - k_{23})\} b_{22} \\
& + (4k_3 + A^{33}N) \frac{\partial u_3}{\partial \theta} + k_{22} \frac{\partial b_{12}}{\partial \theta} + k_{23} \frac{\partial b_{21}}{\partial \theta}.
\end{aligned}
$$

By a similar process Eqs. (2.42), (2.43) and (2.44) give rise to

$$
\begin{aligned}
(4.19) \quad \nu^1 = {} & (k_6 - A^{11}K k_{27}) b_{13} + (k_6 + A^{11}N - 2k_{27}A^{11}K + A^{11}A^{11}K^2 k_{15}) b_{31} \\
& + (k_{27} - A^{11}K k_{15}) \frac{\partial b_{23}}{\partial \theta},
\end{aligned}
$$

$$
\begin{aligned}
(4.20) \quad \nu^2 = {} & (k_5 + A^{22}K k_{26}) b_{23} + (k_5 + A^{22}N + 2A^{22}K k_{26} + A^{22}A^{22}K^2 k_{16}) b_{32} \\
& + (k_{26} + A^{22}K k_{16}) \frac{\partial b_{13}}{\partial \theta},
\end{aligned}
$$

$$
\begin{aligned}
(4.21) \quad & \frac{\partial \xi^{11}}{\partial \theta} - A^{11}K(\xi^{12} - \xi^{21}) - (P^{21}A^{11} + 2k_{19}) \frac{\partial b_{21}}{\partial \theta} - 2k_{18} \frac{\partial b_{12}}{\partial \theta} \\
& - 2k_8 \frac{\partial u_3}{\partial \theta} - \{4k_1 + 2A^{11}K(k_{18} - k_{19}) - K A^{11}A^{11}(P^{12} - 2P^{21})\} b_{11} \\
& - \{2k_7 + 2A^{22}K(k_{18} - k_{19}) - K A^{11}A^{22}P^{21}\} b_{22} = \beta \alpha_1 A^{11} \frac{\partial^2 b_{11}}{\partial t^2},
\end{aligned}
$$

(4.22) $\quad \dfrac{\partial \xi^{22}}{\partial \theta} - A^{22}K(\xi^{12} - \xi^{21}) - (P^{12}A^{22} + 2k_{20})\dfrac{\partial b_{12}}{\partial \theta} - 2k_{21}\dfrac{\partial b_{21}}{\partial \theta}$

$$-2k_9\dfrac{\partial u_3}{\partial \theta} - \{2k_7 + A^{11}A^{22}KP^{12} + 2A^{11}K(k_{20} - k_{21})\}b_{11}$$

$$-\{4k_2 + 2A^{22}K(k_{20} - k_{21}) + A^{22}A^{22}K(P^{21} - 2P^{12})\}b_{22} = \beta\alpha_2 A^{22}\dfrac{\partial^2 b_{22}}{\partial t^2},$$

(4.23) $\quad \dfrac{\partial}{\partial \theta}(\xi^{12} + \xi^{21}) - (4k_{24} + P^{12}A^{11})\dfrac{\partial b_{11}}{\partial \theta} - (4k_{25} + A^{22}P^{21})\dfrac{\partial b_{22}}{\partial \theta}$

$$-2\{k_4 - A^{11}A^{22}K(P^{12} - P^{21})\}(b_{12} + b_{21}) = \beta\alpha_1 A^{22}\dfrac{\partial^2 b_{12}}{\partial t^2} + \beta\alpha_2 A^{11}\dfrac{\partial^2 b_{21}}{\partial t^2},$$

(4.24) $\quad \xi^{11} = 2k_{24}b_{12} + (2k_{24} + A^{11}P^{12})b_{21} + k_{10}\dfrac{\partial b_{11}}{\partial \theta} + \dfrac{1}{2}k_{17}\dfrac{\partial b_{22}}{\partial \theta},$

(4.25) $\quad \xi^{12} = \left\{2k_{18} + A^{11}K\left(k_{12} - \dfrac{1}{2}k_{14}\right)\right\}b_{11} + \left\{2k_{20} + A^{22}P^{12}\right.$

$$\left. + A^{22}K\left(k_{12} - \dfrac{1}{2}k_{14}\right)\right\}b_{22} + k_{22}\dfrac{\partial u_3}{\partial \theta} + k_{12}\dfrac{\partial b_{12}}{\partial \theta} + \dfrac{1}{2}k_{14}\dfrac{\partial b_{21}}{\partial \theta},$$

(4.26) $\quad \xi^{21} = \left\{2k_{19} + A^{11}P^{21} - A^{11}K\left(k_{13} - \dfrac{1}{2}k_{14}\right)\right\}b_{11}$

$$+ \left\{2k_{21} - A^{22}K\left(k_{13} - \dfrac{1}{2}k_{14}\right)\right\}b_{22} + k_{23}\dfrac{\partial u_3}{\partial \theta} + \dfrac{1}{2}k_{14}\dfrac{\partial b_{12}}{\partial \theta} + k_{13}\dfrac{\partial b_{21}}{\partial \theta},$$

(4.27) $\quad \xi^{22} = \dfrac{1}{2}k_{17}\dfrac{\partial b_{11}}{\partial \theta} + k_{11}\dfrac{\partial b_{22}}{\partial \theta} + (2k_{25} + A^{22}P^{21})b_{12} + 2k_{25}b_{21},$

(4.28) $\quad \xi^{13} = (k_{26} + A^{33}P^{12})b_{23} + (k_{26} + A^{22}Kk_{16})b_{32} + k_{16}\dfrac{\partial b_{13}}{\partial \theta},$

(4.29) $\quad \xi^{23} = (k_{27} + A^{33}P^{21})b_{13} + (k_{27} - A^{11}Kk_{15})b_{31} + k_{15}\dfrac{\partial b_{23}}{\partial \theta},$

where it is found that

(4.30) $\qquad b_{31} = \dfrac{\partial u_1}{\partial \theta} - A^{22}Ku_2, \quad b_{32} = \dfrac{\partial u_2}{\partial \theta} + A^{11}Ku_1.$

The equations (4.1), (4.2), (4.3), (4.14), (4.15), (4.16) and (4.18)–(4.30) form the basic equations of the problem. These equations fall naturally into two sets of interlinked equations which may be analysed separately. The set of equations governing extensional-torsional motion consists of Eqs. (4.3), (4.14), (4.18) and (4.21)–(4.27). The remaining equations, namely Eqs. (4.1), (4.2), (4.15), (4.16), (4.19), (4.20), (4.28) and (4.29) with Eq. (4.30) describe motion in which two flexural modes are linked. When the rod is initially unstressed or is subjected to simple extension only, then the equations governing extensional, torsional and the two flexural modes are mutually independent. It is this interlinking of modes that is the principal new feature in the present discussion.

A further feature of the equations obtained here is that, compared with the classical theory such as that given in Love [3], additional equations occur in the extensional-twist

motion. These extra equations which are Eqs. (4.21), (4.22), (4.23), (4.24), (4.27) and a combination of Eqs. (4.25) and (4.26) give rise to the new waves which are found in the subsequent analysis. The flexural modes obtained here are essentially the same as those given by the classical theory.

Some of the quantities appearing in Eqs. (4.1)–(4.30) have been introduced in a rather abstract way. Compared with the classical theory v^i represent force resultants, ξ^{13}, ξ^{23} and $(\xi^{12} - \xi^{21})$ represent the classical bending moments and the remaining ξ^{ij} are additional bending moments relating to bending of the cross-section. The quantities μ_i are the displacement components and b_{ij} relate to the twist of the cross-section. A comparison of the equations obtained here and the traditional ones is not difficult.

The two sets of equations governing the two different motions will now be discussed separately.

5. Extension-twist motion

The extension twist motion is governed by Eqs. (4.3), (4.14), (4.18) and Eqs. (4.21)–(4.27). These equations may be combined and rearranged to give

$$(5.1) \quad \left(G_1 \frac{\partial^2}{\partial \theta^2} - \beta_3 \frac{\partial^2}{\partial t^2}\right) u_3 + G_2 \frac{\partial b_{11}}{\partial \theta} + G_3 \frac{\partial b_{22}}{\partial \theta} + k_{22} \frac{\partial^2 b_{12}}{\partial \theta^2} + k_{23} \frac{\partial^2 b_{21}}{\partial \theta^2} = 0,$$

$$(5.2) \quad -G_2 \frac{\partial u_3}{\partial \theta} + \left(k_{10} \frac{\partial^2}{\partial \theta^2} - G_4 - \Gamma_1 \frac{\partial^2}{\partial t^2}\right) b_{11}$$

$$+ \left(\frac{1}{2} k_{17} \frac{\partial^2}{\partial \theta^2} - G_5\right) b_{22} + \frac{1}{2}(G_6 - G_7) \frac{\partial b_{12}}{\partial \theta} + \frac{1}{2}(G_6 + G_7) \frac{\partial b_{21}}{\partial \theta} = 0,$$

$$(5.3) \quad -G_3 \frac{\partial u_3}{\partial \theta} + \left(\frac{1}{2} k_{17} \frac{\partial^2}{\partial \theta^2} - G_5\right) b_{11} + \left(k_{11} \frac{\partial^2}{\partial \theta^2} - G_8 - \Gamma_2 \frac{\partial^2}{\partial t^2}\right) b_{22}$$

$$+ \frac{1}{2}(G_9 - G_{10}) \frac{\partial b_{12}}{\partial \theta} + \frac{1}{2}(G_9 + G_{10}) \frac{\partial b_{21}}{\partial \theta} = 0,$$

$$(5.4) \quad (k_{22} - k_{23}) \frac{\partial^2 u_3}{\partial \theta^2} + G_7 \frac{\partial b_{11}}{\partial \theta} + G_{10} \frac{\partial b_{22}}{\partial \theta}$$

$$+ \left\{\left(k_{12} - \frac{1}{2} k_{14}\right) \frac{\partial^2}{\partial \theta^2} - \frac{1}{2}(\Gamma_3 - \Gamma_4) \frac{\partial^2}{\partial t^2}\right\} b_{12}$$

$$- \left\{\left(k_{13} - \frac{1}{2} k_{14}\right) \frac{\partial^2}{\partial \theta^2} - \frac{1}{2}(\Gamma_3 + \Gamma_4) \frac{\partial^2}{\partial t^2}\right\} b_{21} = 0,$$

$$(5.5) \quad (k_{22} + k_{23}) \frac{\partial^2 u_3}{\partial \theta^2} - G_6 \frac{\partial b_{11}}{\partial \theta} - G_9 \frac{\partial b_{22}}{\partial \theta}$$

$$+ \left\{\left(k_{12} + \frac{1}{2} k_{14}\right) \frac{\partial^2}{\partial \theta^2} - 2G_{11} - \frac{1}{2}(\Gamma_3 - \Gamma_4) \frac{\partial^2}{\partial t^2}\right\} b_{12}$$

$$+ \left\{\left(k_{13} + \frac{1}{2} k_{14}\right) \frac{\partial^2}{\partial \theta^2} - 2G_{11} - \frac{1}{2}(\Gamma_3 + \Gamma_4) \frac{\partial^2}{\partial t^2}\right\} b_{21} = 0,$$

where

(5.6)
$$\beta_3 = \beta A^{33},$$

(5.7)
$$\Gamma_1 = \beta\alpha_1 A^{11}, \qquad\qquad \Gamma_2 = \beta\alpha_2 A^{22},$$
$$\Gamma_3 = \beta(\alpha_2 A^{11} + \alpha_1 A^{22}), \qquad \Gamma_4 = \beta(\alpha_2 A^{11} - \alpha_1 A^{22}),$$

$$G_1 = 4k_3 + NA^{33},$$
$$G_2 = 2k_8 + A^{11}K(k_{22} - k_{23}),$$
$$G_3 = 2k_9 + A^{22}K(k_{22} - k_{23}),$$
$$G_4 = 4k_1 + 4A^{11}K(k_{18} - k_{19}) - A^{11}A^{11}K(P^{12} - P^{21}) + A^{11}A^{11}K^2(k_{12} + k_{13} - k_{14}),$$
$$G_5 = 2k_7 + 2A^{22}K(k_{18} - k_{19}) + 2A^{11}K(k_{20} - k_{21}) + A^{11}A^{22}K(P^{12} - P^{21})$$
$$+ A^{11}A^{22}K^2(k_{12} + k_{13} - k_{14}),$$

(5.8)
$$G_6 = 2(2k_{24} - k_{18} - k_{19}) - A^{11}K(k_{12} - k_{13}) + A^{11}(P^{12} - P^{21}),$$
$$G_7 = 2(k_{18} - k_{19}) + A^{11}K(k_{12} + k_{13} - k_{14}) + A^{11}(P^{12} - P^{21}),$$
$$G_8 = 4k_2 + 4A^{22}K(k_{20} - k_{21}) - A^{22}A^{22}K(P^{12} - P^{21}) + A^{22}A^{22}K(k_{12} + k_{13} - k_{14}),$$
$$G_9 = 2(2k_{25} - k_{20} - k_{21}) - A^{22}K(k_{12} - k_{13}) - A^{22}(P^{12} - P^{21}),$$
$$G_{10} = 2(k_{20} - k_{21}) + A^{22}K(k_{12} + k_{13} - k_{14}) + A^{22}(P^{12} - P^{21}),$$
$$G_{11} = k_4 - A^{11}A^{22}K(P^{12} - P^{21}).$$

Solutions of Eqs. (5.1)–(5.5) are sought of the form

(5.9)
$$(u_3, b_{11}, b_{22}, b_{12}, b_{21}) = (\hat{u}_3, \hat{b}_{11}, \hat{b}_{22}, \hat{b}_{12}, \hat{b}_{21}) e^{i(\xi\theta - \omega t)},$$

where $\hat{u}_3$ etc., ξ and ω are constants. It is found, after some manipulations, that non-trivial solutions exist provided that

(5.10)
$$\begin{vmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} \end{vmatrix} = 0.$$

Here

$$D_{11} = (G_1\xi^2 - \beta_3\omega^2), \qquad D_{22} = (k_{10}\xi^2 + G_4 - \Gamma_1\omega^2),$$
$$D_{33} = (k_{11}\xi^2 + G_8 - \Gamma_2\omega^2), \qquad D_{44} = \{(k_{12} + k_{13} + k_{14})\xi^2 + 4G_{11} - \Gamma_3\omega^2\},$$
$$D_{55} = \{(k_{12} + k_{13} - k_{14})\xi^2 - \Gamma_3\omega^2\},$$
$$D_{12} = D_{21} = G_2\xi, \qquad D_{13} = D_{31} = G_3\xi,$$
$$D_{14} = D_{41} = -(k_{22} + k_{23})\xi^2, \qquad D_{15} = D_{51} = (k_{22} - k_{23})\xi^2,$$
$$D_{23} = D_{32} = \left(\frac{1}{2}k_{17}\xi^2 + G_5\right), \qquad D_{24} = D_{42} = G_6\xi,$$
$$D_{25} = D_{52} = G_7\xi, \qquad D_{34} = D_{43} = G_9\xi,$$
$$D_{35} = D_{53} = G_{10}\xi,$$
$$D_{45} = D_{54} = \{(k_{13} - k_{12})\xi^2 - \Gamma_4\omega^2\}.$$

Equation (5.10) may be regarded as a quintic equation for ω^2 once ξ is prescribed, so that there are five values for ω^2 for each value of ξ. In the classical theory of wave propagation in rods such as that in LOVE [3] only two of these roots occur. The remaining three roots arise due to the presence of the terms in b_{11}, b_{22} and $(b_{12}+b_{21})$ in the basic equations (or, alternatively, the presence of ξ^{11}, ξ^{22} and $(\xi^{12}+\xi^{21})$). It is not possible to solve Eq. (5.10) in the completely general case but it is possible to obtain some information concerning the roots.

When $\xi = 0$, Eq. (5.10) reduces to

$$(5.11) \qquad \begin{vmatrix} -\beta_3\omega^2 & 0 & 0 & 0 & 0 \\ 0 & (G_4-\Gamma_1\omega^2) & G_5 & 0 & 0 \\ 0 & G_5 & (G_8-\Gamma_2\omega^2) & 0 & 0 \\ 0 & 0 & 0 & (4G_{11}-\Gamma_3\omega^2) & -\Gamma_4\omega^2 \\ 0 & 0 & 0 & -\Gamma_4\omega^2 & -\Gamma_3\omega^2 \end{vmatrix} = 0,$$

so that

$$(5.12) \qquad \omega^2 = 0 \text{ twice,}$$

$$(5.13) \qquad \omega^2 = 4\Gamma_3 G_{11}/(\Gamma_3^2 - \Gamma_4^2),$$

$$(5.14) \qquad 2\omega^2\Gamma_1\Gamma_2 = (\Gamma_1 G_8 + \Gamma_2 G_4) \pm \{(\Gamma_1 G_8 + \Gamma_2 G_4)^2 - 4\Gamma_1\Gamma_2(G_4 G_8 - G_5^2)\}^{1/2} =$$
$$= \Gamma_1 G_8 + \Gamma_2 G_4 \pm \{(\Gamma_1 G_8 - \Gamma_2 G_4)^2 + 4\Gamma_1\Gamma_2 G_5^2\}^{1/2}.$$

In the $\xi - \omega$ plane two of the five curves pass through the origin, from Eq. (5.12), and there are three cut-off frequencies given by Eqs. (5.13) and (5.14). The cut-off frequency (5.13) is real provided that

$$(5.15) \qquad G_{11} > 0,$$

that is

$$(5.16) \qquad A^{11}A^{22}K(P^{12}-P^{21}) < k_4.$$

It appears likely that, for sufficiently large values of K, G_{11} may become negative and the cut-off frequency (5.13) may become purely imaginary. The second form of Eq. (5.14) indicates that ω^2 is always real; it is also positive if

$$(5.17) \qquad G_4 G_8 > G_5^2.$$

This inequality is difficult to analyse due to the complicated nature of G_4, G_5 and G_8. However, if the undeformed rod and the applied deformation are symmetrical with respect to the 1 and 2 directions so that

$$(5.18) \qquad G_4 = G_8,$$

then Eq. (5.17) is replaced by

$$(5.19) \qquad G_4 + G_5 > 0,$$

and

$$(5.20) \qquad G_4 - G_5 > 0.$$

The inequality (5.20) is satisfied provided that

$$(5.21) \qquad 4k_1 - 2k_7 > 2A^{11}A^{11}K(P^{12} - P^{21}),$$

and it is possible for this condition to be violated for suitable values of K. Thus in the symmetric rod it is possible to have an imaginary cut-off frequency. Presumably this is also possible in the non-symmetric case also.

In (5.10) write

$$(5.22) \qquad \omega = \zeta\xi,$$

and let $\xi \to \infty$ so that ω also becomes large. The resulting equation is

(5.23)

$$\begin{vmatrix} (G_1 - \beta_3\zeta^2) & 0 & 0 & -(k_{22}+k_{23}) & (k_{22}-k_{23}) \\ 0 & (k_{10}-\Gamma_1\zeta^2) & \frac{1}{2}k_{17} & 0 & 0 \\ 0 & \frac{1}{2}k_{17} & (k_{11}-\Gamma_2\zeta^2) & 0 & 0 \\ -(k_{22}+k_{23}) & 0 & 0 & \{(k_{12}+k_{13}+k_{14})-\Gamma_3\zeta^2\} & \{(k_{13}-k_{12})-\Gamma_4\zeta^2\} \\ (k_{22}-k_{23}) & 0 & 0 & \{(k_{13}-k_{12})-\Gamma_4\zeta^2\} & \{(k_{12}+k_{13}-k_{14})-\Gamma_3\zeta^2\} \end{vmatrix} = 0,$$

which may be replaced by a quadratic and a cubic equation in ζ^2. The quadratic equation has roots

$$(5.24) \quad 2\Gamma_1\Gamma_2\zeta^2 = (\Gamma_1 k_{11} + \Gamma_2 k_{10}) \pm \left\{(\Gamma_1 k_{11} + \Gamma_2 k_{10})^2 - 4\Gamma_1\Gamma_2\left(k_{10}k_{11} - \frac{1}{4}k_{17}^2\right)\right\}^{1/2}$$

$$= (\Gamma_1 k_{11} + \Gamma_2 k_{10}) \pm \{(\Gamma_1 k_{11} - \Gamma_2 k_{10})^2 + \Gamma_1\Gamma_2 k_{17}^2\}^{1/2},$$

so that these values of ζ^2 are real and positive provided that

$$(5.25) \qquad 4k_{10}k_{11} > k_{17}^2,$$

a condition that will always hold. The cubic equation is more difficult to analyse in detail. The product of the roots of the cubic equation is given by

$$(5.26) \qquad \Delta_1 = \begin{vmatrix} G_1 & k_{22} & k_{23} \\ k_{22} & k_{12} & \frac{1}{2}k_{14} \\ k_{23} & \frac{1}{2}k_{14} & k_{13} \end{vmatrix}$$

and $\Delta_1 > 0$ provided that

$$(5.27) \qquad G_1\left(k_{12}k_{13} - \frac{1}{4}k_{14}^2\right) > (k_{12}k_{23}^2 + k_{13}k_{22}^2) - k_{14}k_{22}k_{23}.$$

For suitable negative values of N this condition may be violated and the system of equations then ceases to be fully hyperbolic.

The discussion presented here suggests that typical curves in the $\xi-\omega$ plane are of the form shown in Fig. 1 when the cut-off frequencies are real and $\Delta_1 > 0$. It appears unlikely that these curves intersect the ξ-axis. It is of interest to note that the change in character
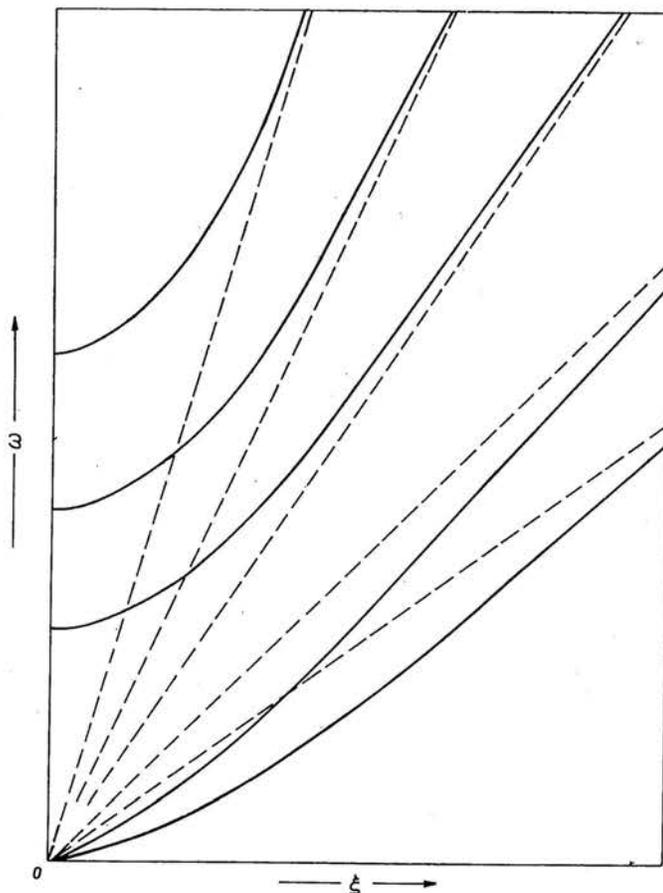
FIG. 1. Variation of ω with ξ; extensional-twist motion.

of the cut-off frequencies determined by Eqs. (5.16) and (5.17) is determined by the value of the twist K and is independent of the axial load N whereas the change in the quantity Δ_1 depends on N only and is independent of K.

6. Extension-twist motion, classical theory

The classical theory of rods such as that to be found in LOVE [3] gives rise to a set of equations which is different from that derived in Sect 5. The terms in Eqs. (5.1)–(5.5) which involve b_{11}, b_{22} and $(b_{12}+b_{21})$ are neglected, as are Eqs. (5.2), (5.3) and (5.5). The modified forms of Eqs. (5.1) and (5.4) are

$$(6.1) \qquad \left(G_1 \frac{\partial^2}{\partial \theta^2} - \beta_3 \frac{\partial^2}{\partial t^2} \right) u_3 + G_{12} \frac{\partial^2 b}{\partial \theta^2} = 0,$$

$$(6.2) \qquad G_{12} \frac{\partial^2 u_3}{\partial \theta^2} + \left(G_{13} \frac{\partial^2}{\partial \theta^2} - \Gamma_3 \frac{\partial^2}{\partial t^2} \right) b = 0,$$

where

(6.3)
$$b = \frac{1}{2}(b_{12} - b_{21}),$$

(6.4)
$$G_{12} = k_{22} - k_{23}, \quad G_{13} = k_{12} + k_{13} - k_{14}.$$

It should be noted that Eqs. (5.2), (5.3) and (5.5) are not satisfied exactly unless G_2, G_3, G_7, G_{10} and Γ_4 are zero. In addition it is required that $k_{23} = -k_{22}$ and $k_{13} = k_{12}$, which will not normally be the case.

Substituting from Eq. (5.9) it is found that non-trivial solutions exist provided that

(6.5)
$$\begin{vmatrix} (G_1\xi^2 - \beta_3\omega^2) & G_{12}\xi^2 \\ G_{12}\xi^2 & (G_{13}\xi^2 - \Gamma_3\omega^2) \end{vmatrix} = 0.$$

Equation (6.5) gives a quadratic equation for ω^2 in terms of ξ^2. This has solutions

(6.6)
$$2\beta_3\Gamma_3\omega^2/\xi^2 = (\beta_3 G_{13} + \Gamma_3 G_1) \pm \{(\beta_3 G_{13} + \Gamma_3 G_1)^2 - 4\beta_3\Gamma_3(G_1 G_{13} - G_{12}^2)\}^{1/2}$$
$$= (\beta_3 G_{13} + \Gamma_3 G_1) \pm \{(\beta_3 G_{13} - \Gamma_3 G_1)^2 + 4\beta_3\Gamma_3 G_{12}^2\}^{1/2}.$$

The second form of Eq. (6.6) indicates that ω^2/ξ^2 is real. This quantity is also positive (i.e. ω/ξ is real) provided that $G_1 G_{13} > G_{12}^2$ from the first form of Eq. (6.6). This condition reduces to

(6.7)
$$4k_3 + NA^{33} > \frac{(k_{22} - k_{23})^2}{k_{12} + k_{13} - k_{14}},$$

and this inequality is violated for sufficiently large negative values of N. The basic equations then cease to be hyperbolic.

Assuming that Eq. (6.7) is satisfied, then Eq. (6.6) gives a pair of straight lines in the $\xi - \omega$ plane. These both pass through the origin and correspond to the bottom two curves in Fig. 1. This suggests that the three additional curves in Fig. 1 are probably only of importance at high frequencies except when K is sufficiently large for one of the cut-off frequencies to become relatively small.

7. Flexural motion

A motion which links together the two flexural modes is governed by Eqs. (4.1), (4.2), (4.15), (4.16), (4.19), (4.20), (4.28), (4.29) and (4.30). These equations combine to give

(7.1)
$$\left(H_1\frac{\partial^2}{\partial\theta^2} - H_2 - \beta_1\frac{\partial^2}{\partial t^2}\right)u_1 - H_3\frac{\partial u_2}{\partial\theta} + H_4\frac{\partial b_{13}}{\partial\theta} + \left(H_5\frac{\partial^2}{\partial\theta^2} - H_6\right)b_{23} = 0,$$

(7.2)
$$H_3\frac{\partial u_1}{\partial\theta} + \left(H_7\frac{\partial^2}{\partial\theta^2} - H_8 - \beta_2\frac{\partial^2}{\partial t^2}\right)u_2 + \left(H_9\frac{\partial^2}{\partial\theta^2} + H_{10}\right)b_{13} + H_{11}\frac{\partial b_{23}}{\partial\theta} = 0,$$

(7.3)
$$-H_4\frac{\partial u_1}{\partial\theta} + \left(H_9\frac{\partial^2}{\partial\theta^2} + H_{10}\right)u_2 + \left(k_{16}\frac{\partial^2}{\partial\theta^2} - H_{12} - \Gamma_5\frac{\partial^2}{\partial t^2}\right)b_{13} + H_{13}\frac{\partial b_{23}}{\partial\theta} = 0,$$

(7.4)
$$\left(H_5\frac{\partial^2}{\partial\theta^2} - H_6\right)u_1 - H_{11}\frac{\partial u_2}{\partial\theta} - H_{13}\frac{\partial b_{13}}{\partial\theta} + \left(k_{15}\frac{\partial^2}{\partial\theta^2} - H_{14} - \Gamma_6\frac{\partial^2}{\partial t^2}\right)b_{23} = 0,$$

where

(7.5) $\beta_1 = \beta A^{11}, \quad \beta_2 = \beta A^{22},$

(7.6) $\Gamma_5 = \beta \alpha_1 A^{33}, \quad \Gamma_6 = \beta \alpha_2 A^{33},$

$H_1 = k_6 + A^{11}N - 2A^{11}Kk_{27} + A^{11}A^{11}K^2k_{15},$

$H_2 = A^{11}A^{11}K^2H_7,$

$H_3 = (A^{22}H_1 + A^{11}H_7)K,$

$H_4 = k_6 - A^{11}K(k_{26}+k_{27}) - A^{11}A^{22}K^2k_{16},$

$H_5 = k_{27} - A^{11}Kk_{15},$

$H_6 = A^{11}K(k_5 + A^{22}Kk_{26}),$

(7.7) $H_7 = k_5 + A^{22}N + 2A^{22}Kk_{26} + A^{22}A^{22}K^2k_{16},$

$H_8 = A^{22}A^{22}K^2H_1,$

$H_9 = k_{26} + A^{22}Kk_{16},$

$H_{10} = A^{22}K(k_6 - A^{11}Kk_{27}),$

$H_{11} = k_5 + A^{22}K(k_{26}+k_{27}) - A^{11}A^{22}K^2k_{15},$

$H_{12} = k_6 - A^{11}A^{33}K(P^{12}-P^{21}),$

$H_{13} = (k_{26}-k_{27}) + A^{33}(P^{12}-P^{21}),$

$H_{14} = k_5 - A^{22}A^{33}K(P^{12}-P^{21}).$

Solutions of Eqs. (7.1)–(7.4) of the form

(7.8) $(u_1, u_2, b_{13}, b_{23}) = (\hat{u}_1, \hat{u}_2, \hat{b}_{13}, \hat{b}_{23})e^{-i(\xi\theta-\omega t)},$

where $\hat{u}_1$, $\hat{u}_2$, $\hat{b}_{13}$, $\hat{b}_{23}$, ξ and ω are constants are now considered. Non-trivial solutions exist provided that

(7.9) $$\begin{vmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{vmatrix} = 0,$$

(here

$D_{11} = (H_1\xi^2 + H_2 - \beta_1\omega^2), \quad D_{22} = (H_7\xi^2 + H_8 - \beta_2\omega^2),$

$D_{33} = (k_{16}\xi^2 + H_{12} - \Gamma_5\omega^2), \quad D_{44} = (k_{15}\xi^2 + H_{14} - \Gamma_6\omega^2),$

$D_{12} = D_{21} = H_3\xi, \quad D_{13} = D_{31} = H_4\xi, \quad D_{14} = D_{41} = (H_5\xi^2 + H_6),$

$D_{23} = D_{32} = -(H_9\xi^2 - H_{10}), \quad D_{24} = D_{42} = H_{11}\xi,$

$D_{34} = D_{43} = -H_{13}\xi),$

which may be regarded as a quartic equation for ω^2 in terms of ξ. As was the case with Eq. (5.10), it is not possible in general to write down analytical solutions of Eq. (7.9) but it is possible to obtain useful information concerning the four roots.

When $\xi = 0$, Eq. (7.9) factorises to give

(7.10) $$\begin{vmatrix} (H_2 - \beta_1\omega^2) & H_6 \\ H_6 & (H_{14} - \Gamma_6\omega^2) \end{vmatrix} = 0,$$

and

(7.11)
$$\begin{vmatrix} (H_8-\beta_2\omega^2) & H_{10} \\ H_{10} & (H_{12}-\varGamma_5\omega^2) \end{vmatrix} = 0,$$

Eq. (7.10) results in the values

$$(7.12) \quad 2\beta_1\varGamma_6\omega^2 = (\beta_1 H_{14}+\varGamma_6 H_2)\pm\{(\beta_1 H_{14}+\varGamma_6 H_2)^2-4\beta_1\varGamma_6(H_2 H_{14}-H_6^2)\}^{1/2}$$
$$= (\beta_1 H_{14}+\varGamma_6 H_2)\pm\{(\beta_1 H_{14}-\varGamma_6 H_2)^2+4\beta_1\varGamma_6 H_6^2\}^{1/2},$$

and (7.11) gives

$$(7.13) \quad 2\beta_2\varGamma_5\omega^2 = (\beta_2 H_{12}+\varGamma_5 H_8)\pm\{(\beta_2 H_{12}+\varGamma_5 H_8)^2-4\beta_2\varGamma_5(H_8 H_{12}-H_{10}^2)\}^{1/2}$$
$$= (\beta_2 H_{12}+\varGamma_5 H_8)\pm\{(\beta_2 H_{12}-\varGamma_5 H_8)^2+4\beta_2\varGamma_5 H_{10}^2\}^{1/2}.$$

All four values for ω^2 are real; they are also positive provided that

(7.14)
$$H_2 H_{14} > H_6^2,$$

from Eq. (7.12) and, from Eq. (7.13),

(7.15)
$$H_8 H_{12} > H_{10}^2.$$

The inequalities (7.14) and (7.15) depend on both N and K. It is possible that for appropriate values they may be violated so that the cut-off frequencies then become imaginary.

In order to examine the behaviour of the roots of Eq. (7.9) for large values of ω and ξ, substitute from Eq. (5.22) and let $\xi \to \infty$. The resulting equation again factorises to give

(7.16)
$$\begin{vmatrix} (H_1-\beta_1\zeta^2) & H_5 \\ H_5 & (k_{15}-\varGamma_6\zeta^2) \end{vmatrix} = 0,$$

and

(7.17)
$$\begin{vmatrix} (H_7-\beta_2\zeta^2) & H_9 \\ H_9 & (k_{16}-\varGamma_5\zeta^2) \end{vmatrix} = 0,$$

so that

$$(7.18) \quad 2\beta_1\varGamma_6\zeta^2 = (\beta_1 k_{15}+\varGamma_6 H_1)\pm\{(\beta_1 k_{15}+\varGamma_6 H_1)^2-4\beta_1\varGamma_6(H_1 k_{15}-H_5^2)\}^{1/2}$$
$$= (\beta_1 k_{15}+\varGamma_6 H_1)\pm\{(\beta_1 k_{15}-\varGamma_6 H_1)^2+4\beta_1\varGamma_6 H_5^2\}^{1/2},$$

and

$$(7.19) \quad 2\beta_2\varGamma_5\zeta^2 = (\beta_2 k_{16}+\varGamma_5 H_7)\pm\{(\beta_2 k_{16}+\varGamma_5 H_7)^2-4\beta_2\varGamma_5(H_7 k_{16}-H_9^2)\}^{1/2}$$
$$= (\beta_2 k_{16}+\varGamma_5 H_7)\pm\{(\beta_2 k_{16}-\varGamma_5 H_7)^2+4\beta_2\varGamma_5 H_9^2\}^{1/2}.$$

The expressions (7.18) and (7.19) give real values for ζ^2. The quantity ζ^2 is also positive provided that

(7.20)
$$H_1 k_{15} > H_5^2,$$

in Eq. (7.18) and

(7.21)
$$H_7 k_{16} > H_9^2,$$

in Eq. (7.19). The condition (7.20) reduces to

(7.22)
$$k_{15}(k_6+A^{11}N)-k_{27}^2 > 0,$$

and Eq. (7.21) reduces to

(7.23)
$$k_{16}(k_5 + A^{22}N) - k_{26}^2 > 0,$$

so that for suitable negative values of N these conditions may be violated. It is of interest to note that they depend on N but not on K in contrast with Eqs. (7.14) and (7.15).

Where ω is zero Eq. (7.9) may be written in the form

(7.24)
$$\begin{vmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{vmatrix} = 0,$$

here

$$D_{11} = (H_1 \xi^2 + A^{11} A^{11} K^2 H_7), \quad D_{22} = (H_7 \xi^2 + A^{22} A^{22} K^2 H_1),$$
$$D_{33} = (k_{16} \xi^2 + H_{12}), \quad D_{44} = (k_{15} \xi^2 + H_{14}),$$
$$D_{12} = D_{21} = \xi K (A^{22} H_1 + A^{11} H_7), \quad D_{13} = D_{31} = \xi (J_2 - A^{11} K H_9),$$
$$D_{14} = D_{41} = (H_5 \xi^2 + A^{11} K J_1), \quad D_{23} = D_{32} = -(H_9 \xi^2 - A^{22} K J_2),$$
$$D_{24} = D_{42} = \xi (J_1 + A^{22} K H_5), \quad D_{34} = D_{43} = -H_{13} \xi,$$

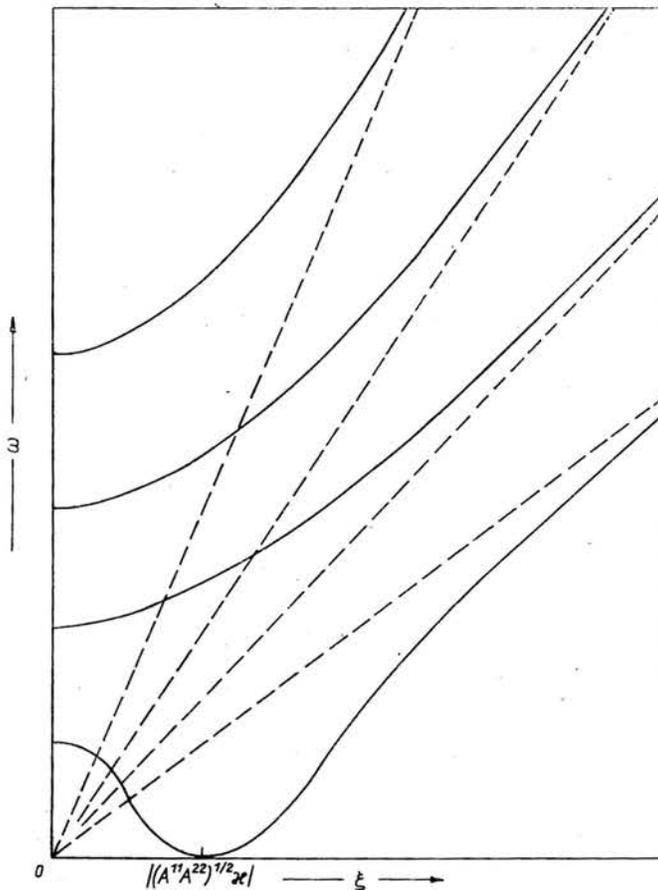FIG. 2. Variation of ω with ξ; flexure-flexure motion, general case.

where H_1, H_5, H_7 and H_9 are defined by Eq. (7.7) and

(7.25)
$$J_1 = k_5 + A^{22}Kk_{26},$$
$$J_2 = k_6 - A^{11}Kk_{27}.$$

By carrying out operations on the rows and columns of the determinant in Eq. (7.24) it is possible to express the equation in the form

(7.26) $(\xi^2 - A^{11}A^{22}K^2)^2 \begin{vmatrix} H_1 & 0 & J_2 & H_5\xi \\ 0 & H_7 & -H_9\xi & J_1 \\ J_2 & -H_9\xi & (k_{16}\xi^2 + H_{12}) & -H_{13}\xi \\ H_5\xi & J_1 & -H_{13}\xi & (k_{15}\xi^2 + H_{14}) \end{vmatrix} = 0,$

so that

(7.27)
$$\xi = \pm(A^{11}A^{22})^{1/2}K,$$

is a double root of the equation and

(7.28)
$$\xi = |(A^{11}A^{22})^{1/2}K|,$$

is always a positive root. The presence of this double root indicates that the lowest curve in the $\xi-\omega$ plane touches the ξ-axis at the point given by Eq. (7.28). It is possible, but unlikely, that other roots could also be obtained from Eq. (7.26).

Figure 2 indicates schematically the type of curves to be expected in the $\xi-\omega$ plane when all cut-off frequencies are real and Eqs. (7.22) and (7.23) are not violated. In the flexural case the basic equations are those obtained from the classical theory and no simplification of the type encountered in the extension-twist case arises. A simplification does occur, however, when there is symmetry and this will be discussed in the next section.

8. Flexural motion with symmetry

The discussion of flexural motion which has been presented in Sect. 7 is completely general and assumes no symmetries other than those defined by Eqs. (2.47) and (2.49). When the undeformed rod and the initial, finite, deformations are symmetrical with respect to the 1 and 2 directions, some simplification of the basic equations and Eq. (7.9) occurs. It is assumed in this section that

(8.1) $\lambda_1 = \lambda_2 = \lambda, \quad A^{11} = A^{22} = A,$

(8.2) $\beta_1 = \beta_2 = \beta, \quad \Gamma_5 = \Gamma_6 = \Gamma,$

(8.3) $k_5 = k_6, \quad k_{15} = k_{16}, \quad k_{26} = -k_{27},$

so that Eq. (7.9) takes the form

(8.4) $\begin{vmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{vmatrix} = 0,$

here

$$D_{11} = D_{22} = \{H_1(\xi^2 + A^2K^2) - \beta\omega^2\}, \quad D_{33} = D_{44} = (k_{15}\xi^2 + H_{12} - \Gamma\omega^2),$$
$$D_{12} = D_{21} = 2AKH_1\xi, \quad D_{13} = D_{31} = D_{24} = D_{42} = H_4\xi,$$
$$D_{14} = D_{41} = D_{23} = D_{32} = (H_{10} - H_9\xi^2), \quad D_{34} = D_{43} = -H_{13}\xi.$$

After some manipulation it is found that this equation factorises, resulting in the pair of equations

(8.5) $\quad \{H_1(\xi - AK)^2 - \beta\omega^2\}\{k_{15}\xi^2 + H_{13}\xi + H_{12} - \Gamma\omega^2\} - (\xi - AK)^2(M_1 + M_2\xi)^2 = 0,$

(8.6) $\quad \{H_1(\xi + AK)^2 - \beta\omega^2\}\{k_{15}\xi^2 - H_{13}\xi + H_{12} - \Gamma\omega^2\} - (\xi + AK)^2(M_1 - M_2\xi)^2 = 0,$

where

(8.7) $$M_1 = k_5 + AKk_{26},$$

(8.8) $$M_2 = k_{26} + AKk_{15}.$$

Clearly Eq. (8.6) is obtained from Eq. (8.5) by changing the sign of ξ so that it is not necessary to pursue a detailed analysis of both equations.

Eq. (8.5) may be solved for ω^2 to give

(8.9) $\quad 2\beta\Gamma\omega^2 = \{H_1\Gamma(\xi - AK)^2 + \beta(k_{15}\xi^2 + H_{13}\xi + H_{12})\} \pm \{[H_1\Gamma(\xi - AK)^2 +$
$$+ \beta(k_{15}\xi^2 + H_{13}\xi + H_{12})]^2 - 4\beta\Gamma(\xi - AK)^2[(k_{15}\xi^2 + H_{13}\xi + H_{12})H_1$$
$$- (M_1 + M_2\xi)^2]\}^{1/2} = \{H_1\Gamma(\xi - AK)^2 + \beta(k_{15}\xi^2 + H_{13}\xi + H_{12})\}$$
$$\pm \{[H_1\Gamma(\xi - AK)^2 - \beta(k_{15}\xi^2 + H_{13}\xi + H_{12})]^2 + 4\beta\Gamma(\xi - AK)^2(M_1$$
$$+ M_2\xi)^2\}^{1/2},$$

and from the second of these it is clear that both values of ω^2 are real. These values are also positive provided that

(8.10) $$H_1(k_{15}\xi^2 + H_{13}\xi + H_{12}) > (M_1 + M_2\xi)^2.$$

The inequality (8.10) may be violated for certain combinations of N and K. Solutions of Eq. (8.6) may be obtained from Eq. (8.9) by replacing ξ by $-\xi$. The inequality corresponding to Eq. (8.10) is found to be

(8.11) $$H_1(k_{15}\xi^2 - H_{13}\xi + H_{12}) > (M_1 - M_2\xi)^2.$$

It is possible to plot curves in the $\omega - \xi$ plane by making use of Eq. (8.9) but due to the complicated nature of the equation it is helpful to have additional information. Cut-off frequencies occur when $\xi = 0$ and

(8.12) $$2\beta\Gamma\omega^2 = (A^2K^2H_1\Gamma + \beta H_{12}) \pm \{(A^2K^2H_1\Gamma - \beta H_{12})^2 + 4\beta\Gamma A^2K^2M_1^2\}^{1/2}.$$

These cut-off frequencies are real if

(8.13) $$H_1H_{12} > M_1^2.$$

When $\xi = 0$, Eqs. (8.5) and (8.6) are identical so that the cut-off frequencies (8.12) also arise from Eq. (8.6).

When ω is set equal to zero in Eq. (8.5), an equation for ξ is obtained of the form

(8.14) $$(\xi - AK)^2\{H_1(k_{15}\xi^2 + H_{13}\xi + H_{12}) - (M_1 + M_2\xi)^2\} = 0.$$

The value $\xi = AK$ is a double root of this equation and, provided that Eq. (8.10) is satisfied, it is the only root. Similarly, provided that Eq. (8.11) is satisfied, the only root of Eq. (8.6) when ω is zero is $\xi = -AK$. Consequently, there is always a double root of Eq. (8.4) when ω is zero and $\xi > 0$ given by

$$(8.15) \qquad\qquad\qquad\qquad \xi = |AK|.$$

The curves in the $\xi - \omega$ plane touch the ξ-axis at this point.

For large values of ξ, and with $\omega = \zeta\xi$, Eq. (8.9) gives

$$(8.16) \qquad\qquad 2\beta\Gamma\zeta^2 = (\Gamma H_1 + \beta k_{15}) \pm \{(\Gamma H_1 - \beta k_{15})^2 + 4\beta\gamma M_2^2\}^{1/2},$$

which is real; it is also positive if

$$(8.17) \qquad\qquad\qquad\qquad H_1 k_{15} > M_2^2.$$

which reduces to Eq. (7.22) or Eq. (7.23). Equation (8.6) also gives the values (8.16) for ζ^2 so that the four curves in the $\xi - \omega$ plane are asymptotic to two lines only.

Figure 3 gives a schematic representation of the four curves given by Eq. (8.4). The top and bottom curves are given by Eqs. (8.5) and (8.9) and the two intermediate curves
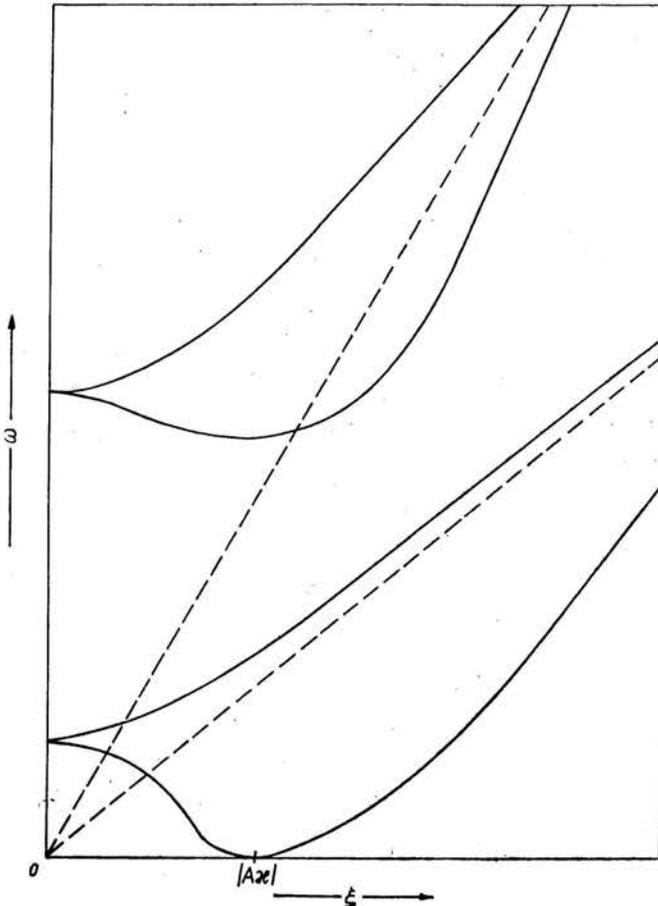

FIG. 3. Variation of ω with ξ; flexure-flexure motion, symmetric case.

by Eq. (8.6) when K is positive. When K is negative this situation is reversed. In drawing these curves it has been assumed that N and K are such that the frequency equation gives real roots for ω. As was the case in Sect. 7, no simplification occurs comparable with that observed for extension-twist motion.

9. Conclusion

Solutions have been obtained for the propagation of waves in a pre-strained straight elastic rod subjected to simple extension and twist. It is found that under these conditions there is a linking between the extensional and torsional waves and also between the two flexural modes. The propagation of the various waves has been investigated. It is found that in the extension-twist case three waves not predicted by the classical theory may propagate. No new waves exist in the flexural case. It is conjectured that in general these new waves are of interest at high frequencies; however, for suitable combinations of initial compression and twist they may be of importance at lower frequencies also.

Acknowledgement

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF STRATHCLYDE, GLASGOW, SCOTLAND, U.K.