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Stability problems for inelastic solids with defects and imperfections

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THE OBJECTIVE of the present paper is to investigate the influence of internal defects and imperfections in the dissipative material of a body on the instability of plastic flow. The importance of constitutive modelling for the instability and fracture phenomena is discussed. The modified material structure with internal state variables is formulated. Conditions for stability and asymptotic stability of the equilibrium intrinsic state are investigated. A model of an elastic-viscoplastic material with internal defects and imperfections is proposed. Moreover, a physical motivation for this model is given. A description of the model is presented within the framework of the modified material structure with internal state variables. It is postulated that the evolution equations for some of the internal state variables have the form of the partial differential equations, namely the diffusion equations. The initial-boundary-value problem is considered. Particular attention is given to the investigation of the influence of the diffusion cooperative effects on the onset of instability by necking.

Celem pracy jest zbadanie wpływu wewnętrznych defektów i imperfekcji w dysypatywnym materiale ciała na niestateczność plastycznego płynięcia. Przedyskutowano znaczenie modelowania konstytutywnego dla zjawisk niestateczności i zniszczenia. Sformułowano zmodyfikowaną strukturę materialną z parametrami wewnętrznymi. Zbadano warunki stateczności i asymptotycznej stateczności dla stanu równowagi wewnętrznej. Zaproponowano model materiału sprężysto-lepkoplastycznego z wewnętrznymi defektami i imperfekcjami. Opis tego modelu został przedstawiony w ramach zmodyfikowanej struktury materialnej z parametrami wewnętrznymi. Przyjęto, że równania ewolucji dla niektórych parametrów wewnętrznych mają postać równań różniczkowych cząstkowych, mianowicie równań dyfuzji. Rozważono problem początkowo-brzegowy. Szczególną uwagę zwrócono na zbadanie wpływu współdziałających efektów dyfuzyjnych na powstawanie niestateczności postaci szyjkowania.

Целью работы является исследование влияния внутренних дефектов и имперфекций в диссипативном материале тела на неустойчивость пластического течения. Обсуждено значение определяющего моделирования для явлений неустойчивости и разрушения. Сформулирована модифицированная материальная структура с внутренними параметрами. Исследованы условия устойчивости и асимптотической устойчивости для состояния внутреннего равновесия. Предложена модель упруго-вязкопластического материала с внутренними дефектами и имперфекциями. Описание этой модели представлено в рамках модифицированной материальной структуры с внутренними параметрами. Принято что уравнения эволюции для некоторых внутренних параметров имеют вид дифференциальных уравнений в частных производных, именно уравнений диффузии. Рассмотрена начально-краевая задача. Особенное внимание обращено на исследование влияния взаимодействующих диффузных эффектов на возникновение неустойчивости типа шейки.

1. Introduction

NOWADAYS stability phenomena have become major problems of investigations in mechanics of continuous media. Many recent theoretical and experimental investigations have been focussed on some aspects of instability of plastic flow. The instability phenomena may be treated as a prelude to fracture initiation and therefore is a matter of great interest.

In plastic flow problems the onset of instability is usually connected with the localization of plastic deformations (cf. J. R. RICE [34]).

It is very well recognized and confirmed experimentally that there are two main modes of localization of plastic deformations, namely necking and localization in the direction of pure shear (cf. A. K. CHAKRABARTI and J. W. SPRETNAK [4]).

Experimental investigations have shown that the intrinsic failure (necking or instability in the direction of pure shear) is strongly dependent upon such effects as temperature, strain rate, diffusion, defects and imperfections in a material and coupling between different dissipative mechanisms (cf. P. J. WRAY [35], T. B. COX and J. R. LOW [8] and W. PAVINICH and R. RAJ [26]).

A. NEEDLEMAN and J. R. RICE [23] have proved that the onset of localization does depend critically on the assumed constitutive law.

The influence of strain rate effects on the necking phenomena has been the subject of consideration in the recent papers by J. W. HUTCHINSON and K. W. NEELE [16] and by G. K. GHOSH [11, 12] for the simplified one-dimensional case of loading and by the author [30] for straining cylindrical specimens in axisymmetric state of stress.

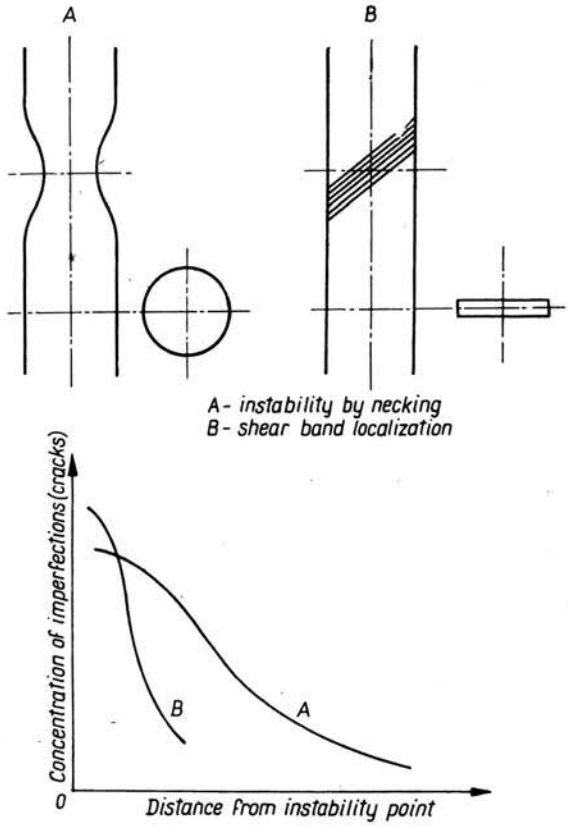

FIG. 1. Distribution of intergranular cracks along the length of a specimen suggested by experimental results.

All these investigations have shown the great importance of the constitutive modelling for instability and fracture phenomena.

Basing on experimental observations (cf. A. K. CHAKRABARTI and J. W. SPRETNAK [4], W. PAVINICH and R. RAJ [26] and P. J. WRAY [35]), we can come to the conclusion that the concentration of the internal imperfections (cracks at grain interface, voids nucleating on grain boundaries) in a material during straining process of a specimen does depend on the boundary conditions. Thus the evolution of the concentration of imperfections has to be described by means of partial differential equations.

The transport phenomenon of imperfections (kinetics of crack growth) during the deformation process plays an important role and frequently has a predominant influence on the onset of instability and, consequently, on the fracture mechanism, cf. Fig. 1.

Therefore it is of great importance to formulate a model of inelastic material in which not only properties essential for the plastic flow process but also fundamental for critical situations like instability and fracture phenomena are taken into considerations.

It is obvious that the constitutive model of an inelastic material appropriate for the adequate description of instability and fracture has to contain additional information specific for these phenomena.

Although the instability phenomenon is determined by many factors it seems that the synergic nature of coupling effects is most essential. During the evolution of the deformation process of a body these cooperative effects usually lead to the onset of the particular instability mode.

A description of such a complex problem is the subject of the present paper. It can be achieved within the framework of a modified structure with internal state variables. The modification concerns the evolution equations for the internal state variables. It is postulated that some evolution equations can have the form of the partial differential equations. The equilibrium intrinsic state for a body is defined and the conditions for its stability and asymptotic stability are investigated. In this study the results of the analytical theory of semi-groups are used.

A model of an elastic-viscoplastic material with internal defects and imperfections is proposed. This model is justified by the physical mechanisms of a polycrystalline matter flow in some regions of temperature and strain rate changes. A thermo-activated mechanism is assumed to cause elastic-viscoplastic flow, a mechanism of grain-boundary sliding with accommodated diffusion introduces an evolution of defects and a mechanism of void growth plus its diffusion field insert the evolution of imperfections in the material considered.

To describe this model a set of the internal state variables is introduced. In this set one scalar internal variable is interpreted as the scalar measure of the concentration of defects and the other as the scalar measure of the concentration of imperfections. It is postulated that the evolution equations for both of these internal state variables have the form of the diffusion equations.

Some simplifications and particular cases of the constitutive equations proposed are discussed.

As an example of application, the straining process of a cylindrical bar specimen under the condition of uniform elevated temperature is considered. The tensile velocity

is assumed to be constant. The problem is treated as axisymmetric and quasi-static. The material of the specimen is assumed to be elastic-viscoplastic with defects and imperfections. Particular forms of the material functions involved in the theory are assumed on the basis of available axperimental data for mild steel. Numerical calculations are obtained and discussed. It was found that the onset of the instability by the necking mode is very much sensitive to the diffusion cooperative effects.

2. Modified material structure with internal state variables

In the previous papers [27] the author developed a general thermodynamic theory of dissipative materials. In this theory the notion of the intrinsic state in a particle X of a body $\mathscr{B}$ plays an important role. The intrinsic state σ of a particle X is defined as a pair — the local deformation-temperature configuration $\mathbf{P}(t) = (\mathbf{C}(t), \vartheta(t), \nabla\vartheta(t))$ of a particle X and its method of preparation $\mathbf{A}(t)$, i.e.

$$\sigma = (\mathbf{P}(t), \mathbf{A}(t)), \quad \mathbf{P}(t) \in \mathscr{G}, \quad \mathbf{A}(t) \in \mathscr{K}. \tag{2.1}$$

A method of preparation $\mathbf{A}(t)$ of the deformation-temperature configuration of a particle X represents the way of inserting the additional information required to define uniquely the intrinsic state of a particle X and is needed to describe the internal dissipation of a material.

Through the particular realization of a method of preparation an internal state variable material structure can be constructed. It is postulated that $\mathbf{A}(t) = \boldsymbol{\alpha}(t)$ is an element of a finite dimensional vector space $\mathscr{V}_n$.

The principle of determinism for the material structure with internal state variables is expressed by the constitutive equations

$$\mathbf{Z}(t) = \hat{\mathbf{S}}(\mathbf{P}(t), \boldsymbol{\alpha}(t)), \quad \mathbf{Z}(t) \in \mathscr{S} \tag{2.2}$$

and can be stated as follows: A unique value of the response $\mathbf{Z}(t)$ of a material at X, i.e. unique values of the free energy $\psi(t)$, the entropy $\eta(t)$, the Piola–Kirchhoff stress tensor $\mathbf{T}(t)$, and the heat flux vector $\mathbf{q}(t)$, corresponds to every intrinsic state $\sigma \in \Sigma \subset \mathscr{G} \times \mathscr{V}_n$.

The mapping $\hat{\mathbf{S}}: \Sigma \to \mathscr{S}$ is called the response function and represents the free energy response function $\hat{\Psi}$, the entropy response function $\hat{\mathrm{N}}$, the stress response function $\hat{\mathbf{T}}$ and the heat response function $\hat{\mathbf{Q}}$, i.e.

$$\hat{\mathbf{S}} \equiv \{\hat{\Psi}, \hat{\mathrm{N}}, \hat{\mathbf{T}}, \hat{\mathbf{Q}}\}. \tag{2.3}$$

The evolution for the material structure with internal state variables is postulated by the initial-value problem for the operator differential equation of the form

$$\partial_t \boldsymbol{\alpha}(t) = \mathscr{L}\boldsymbol{\alpha}(t) + \hat{\mathbf{f}}(\sigma), \quad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0 \in \mathscr{D}(\mathscr{L}), \tag{2.4}$$

where $\mathscr{L}$ is a linear, in general unbounded, operator with domain $\mathscr{D}(\mathscr{L})$ and range $\mathscr{R}(\mathscr{L})$ both contained in a real Hilbert space $\mathscr{H}$ and $\hat{\mathbf{f}}$ is a (nonlinear) function defined on $\mathscr{H}$ into $\mathscr{H}$.

We need to endow the method of preparation space $\mathscr{K} \equiv \mathscr{V}_n$ with the properties of the real Hilbert space $\mathscr{H}$.

Let a body $\mathscr{B}$ be an open domain in the n-dimensional Euclidean space E with boundary $\partial\mathscr{B}$ which consists of a bounded part $\partial\mathscr{B}_1$ and an unbounded part $\partial\mathscr{B}_2$. Then the domain $\mathscr{D}(\mathscr{L})$ is defined as follows:

$$\text{2.5)} \quad \mathscr{D}(\mathscr{L}) = \{\boldsymbol{\alpha}(t) \in \mathscr{H},\ \mathbf{a}(t, \mathbf{X})\partial_n\boldsymbol{\alpha}(t) + \mathbf{b}(t, \mathbf{X})\boldsymbol{\alpha}(t) = 0,\ t \in [0, d_P],\ \mathbf{X} \in \partial\mathscr{B}_1,\ \lim_{\mathbf{X}\to\partial\mathscr{B}_2} \boldsymbol{\alpha}(t, \mathbf{X}) = 0\},$$

where $\mathbf{n}$ is the unit outward normal vector on $\partial\mathscr{B}_1$, $\mathbf{a}$ and $\mathbf{b}$ are bounded functions on $[0, d_P] \times \partial\mathscr{B}_1$. $\mathscr{B}$ is a bounded domain if $\partial\mathscr{B}_2$ is empty and $\mathscr{B}$ is whole space E if $\partial\mathscr{B}_1$ is empty. In general, $\partial\mathscr{B}$ consists of a bounded part and an unbounded part such as the exterior of a sphere or a half-space, etc.

A modified material structure with internal state variables is defined by the constitutive equations (2.2) and the initial-boundary-value problem (2.4) and (2.5).

The main feature of the material structure proposed is a more general evolution of the internal state variables in a body $\mathscr{B}$. The evolution is governed by the partial differential equations and, as a result, depends on the boundary conditions assumed.

Of course, we can obtain a classical formulation (cf. B. D. COLEMAN and M. E. GURTIN [6]) under the condition that the linear operator $\mathscr{L}$ vanishes.

3. Thermodynamic restrictions

DEFINITION 1. *Every pair* $(\mathbf{P}, \mathbf{Z})$ *such that*

(i) $\mathrm{Dom}\,\mathbf{P} = \mathrm{Dom}\,\mathbf{Z}$;

(ii) *The constitutive equations* (2.2) *and the evolution equations* (2.4) *are satisfied*;

(iii) *For every instant of time* $t \in [0, d_P]$ *the dissipation principle in the form of the Clausius-Duhem inequality*

$$\text{(3.1)} \qquad -\dot{\psi}(t) - \dot{\vartheta}(t)\eta(t) + \frac{1}{2\varrho}\,\mathrm{tr}[\mathbf{T}(t)\dot{\mathbf{C}}(t)] - \frac{1}{\varrho\vartheta(t)}\,\mathbf{q}(t)\cdot\nabla\vartheta(t) \geqq 0$$

is satisfied; *will be called an admissible local thermodynamic process.*

We postulate such assumptions for the response functions $\hat{\mathbf{S}} \equiv \{\hat{\Psi}, \hat{\mathbf{N}}, \hat{\mathbf{T}}, \hat{\mathbf{Q}}\}$ and for the differential operator $\mathscr{L}$ and the nonlinear function $\hat{\mathbf{f}}$ that the dissipation principle leads to the results as follows

$$\text{(3.2)} \qquad \begin{gathered} \partial_{\nabla\vartheta(t)}\hat{\Psi}(\cdot) = 0, \quad \mathbf{T}(t) = 2\varrho\partial_{\mathbf{C}(t)}\hat{\Psi}(\cdot), \quad \eta(t) = -\partial_{\vartheta(t)}\hat{\Psi}(\cdot), \\ \hat{\mathrm{d}}(\sigma) = -\partial_{\boldsymbol{\alpha}(t)}\hat{\Psi}(\cdot)\cdot\partial_t\boldsymbol{\alpha}(t) - \frac{1}{\varrho\vartheta(t)}\,\hat{\mathbf{Q}}(\sigma)\cdot\nabla\vartheta(t) \geqq 0 \end{gathered}$$

for every $t \in [0, d_P]$.

The internal dissipation function is defined by the expression

$$\text{(3.3)} \qquad \hat{\mathrm{i}}(\sigma) = -\frac{1}{\vartheta(t)}\,\partial_{\boldsymbol{\alpha}(t)}\hat{\Psi}(\cdot)\cdot[\mathscr{L}\boldsymbol{\alpha}(t) + \hat{\mathbf{f}}(\sigma)].$$

Our considerations were based on the Clausius–Duhem inequality as a fundamental dissipation principle. Different results can be obtained when the I. MÜLLER [20, 21] con-

ception is used as a basic thermodynamic postulate. A discussion of constitutive restrictions according to the I. MÜLLER idea for the modified internal state variable structure will be given in a separate paper.

It is easy to prove that in both cases the thermodynamic condition is weaker than that needed for stability requirements, (cf. results obtained by R. L. FOSDICK and K. R. RAJAGOPAL [10] for fluids of third grade and by M. E. GURTIN [13] for global stability).

4. Stability of equilibrium intrinsic states

The following definitions specify what we mean by a solution, an equilibrium solution and the stability of an equilibrium intrinsic state.

DEFINITION 2. *By a solution* $\boldsymbol{\alpha}(t)$ *of* Eq. (2.4) *we mean the following*: (i) $\boldsymbol{\alpha}(t)$ *is uniformly Lipschitz continuous in* t *for each* $t \geqq 0$ *with* $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0 \in \mathcal{D}(\mathcal{L})$ *in a Hilbert space* $\mathcal{H}$; (ii) $\boldsymbol{\alpha}(t) \in \mathcal{D}(\mathcal{L})$ *for each* $t \geqq 0$ *and the strong derivative* $\partial_t \boldsymbol{\alpha}(t) = \mathcal{L}\boldsymbol{\alpha}(t) + \hat{\mathbf{f}}(\sigma)$ *exists and is strongly continuous except at a countable number of values* t.

DEFINITION 3. *An element* $\boldsymbol{\alpha}^*$ *of* $\mathcal{D}(\mathcal{L})$ *is an equilibrium solution of* Eq. (2.4) *if and only if*

$$\mathcal{L}\boldsymbol{\alpha}^* + \hat{\mathbf{f}}(\mathbf{P}^*, \boldsymbol{\alpha}^*) = 0 \tag{4.1}$$

for all $t \geqq 0$ *and for* $\mathbf{P}^* = (\mathbf{C}^*(\mathbf{X}), \vartheta^*(\mathbf{X}), 0)$. *Then the state*

$$\sigma^* = \left(\mathbf{C}^*(\mathbf{X}), \vartheta^*(\mathbf{X}), 0, \boldsymbol{\alpha}^*(\mathbf{X})\right) \tag{4.2}$$

is called an equilibrium intrinsic state for the material at $\mathbf{X}$ *of a body* $\mathcal{B}$.

DEFINITION 4. *An equilibrium intrinsic state* σ^* *is said to be stable (with respect to initial perturbations of* $\boldsymbol{\alpha}$*) if given any* $\varepsilon > 0$, *there exists a* δ *such that*

$$||\boldsymbol{\alpha}(0) - \boldsymbol{\alpha}^*|| < \delta \Rightarrow ||\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}^*|| < \varepsilon \tag{4.3}$$

for all $t \geqq 0$, *where* $\boldsymbol{\alpha}(t)$ *is any solution of*

$$\partial_t \boldsymbol{\alpha}(t) = \mathcal{L}\boldsymbol{\alpha}(t) + \hat{\mathbf{f}}\left(\mathbf{P}^*, \boldsymbol{\alpha}(t)\right), \quad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0 \in \mathcal{D}(\mathcal{L}). \tag{4.4}$$

Then a subset $\mathcal{D}(\mathcal{L})$ *of* $\mathcal{H}$ *is said to be a stability region of the equilibrium intrinsic state* σ^*.

DEFINITION 5. *An equilibrium intrinsic state* σ^* *is said to be asymptotically stable if* (i) *it is stable, and* (ii)

$$\lim_{t \to \infty} ||\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}^*|| = 0, \tag{4.5}$$

where $\boldsymbol{\alpha}(t)$ *is any solution of* Eq. (4.4).

From the analytical theory of semi-groups (cf. E. HILLE and R. S. PHILLIPS [14], K. YOSIDA [36], T. KATO [17], F. E. BROWDER [2] G. LUMER and R. S. PHILLIPS [18], G. J. MINTY [19], C. V. PAO [24] and C. V. PAO and G. VOGT [25]) we know that if the operator $\mathcal{L}_1 \equiv \mathcal{L} + \hat{\mathbf{f}}(\mathbf{P}^*, \cdot)$ generates a nonlinear semi-group $\{\mathrm{T}_t; t \geqq 0\}$, then a solution

$$\partial_t \boldsymbol{\alpha}(t) = \mathcal{L}\boldsymbol{\alpha}(t) + \hat{\mathbf{f}}(\mathbf{P}^*, \boldsymbol{\alpha}(t)), \quad t \geqq 0 \tag{4.6}$$

starting at $t = 0$ from any element $\boldsymbol{\alpha}_0 \in \mathcal{D}(\mathcal{L})$ is given by

$$\boldsymbol{\alpha}(t; \boldsymbol{\alpha}_0) = \mathrm{T}_t \boldsymbol{\alpha}_0 \quad \text{for all} \quad t \geqq 0 \quad \text{with} \quad \boldsymbol{\alpha}(0; \boldsymbol{\alpha}_0) = \boldsymbol{\alpha}_0. \tag{4.7}$$

Thus the existence of a solution to Eq. (4.4) is ensured and the stability property can be determined from the family of nonlinear operators $\{\mathrm{T}_t; t \geqq 0\}$.

The problem is to impose conditions on the operators $\mathcal{L}$ and $\hat{\mathbf{f}}$ such that the operator $\mathcal{L}_1 \equiv \mathcal{L} + \hat{\mathbf{f}}(\mathbf{P}^*, \cdot)$ generates a nonlinear contraction or negative contraction semi-group in $\mathcal{H}$ from which the existence, uniqueness and stability or asymptotic stability of the solution (4.4) are insured.

Basing on the results presented by C. V. PAO and W. G. VOGT [25], we can prove the theorems as follows:

THEOREM 1. *If a nonlinear operator* $\mathcal{L}_1 \equiv \mathcal{L} + \mathbf{f}(\mathbf{P}^*, \cdot)$ is dissipative, i.e.

$$(\mathcal{L}_1 \boldsymbol{\alpha} - \mathcal{L}_1 \boldsymbol{\omega}, \boldsymbol{\alpha} - \boldsymbol{\omega}) \leqq 0 \quad \text{for} \quad \boldsymbol{\alpha}, \boldsymbol{\omega} \in \mathcal{D}(\mathcal{L}) \tag{4.8}$$

then (i) *there exists a unique solution of the initial-boundary value problem* (4.4) *in the sense of Definition* 1;

(ii) *any equilibrium intrinsic state* σ^*, *if it exists, is stable*;

(iii) *a stability region of* σ^* *is* $\mathcal{D}(\mathcal{L})$.

THEOREM 2. *If a nonlinear operator* $\mathcal{L}_1 \equiv \mathcal{L} + \hat{\mathbf{f}}(\mathbf{P}^*, \cdot)$ *is strictly dissipative, i.e.*

$$(\mathcal{L}_1 \boldsymbol{\alpha} - \mathcal{L}_1 \boldsymbol{\omega}, \boldsymbol{\alpha} - \boldsymbol{\omega}) \leqq -\beta ||\boldsymbol{\alpha} - \boldsymbol{\omega}||^2, \boldsymbol{\alpha}, \boldsymbol{\omega} \in \mathcal{D}(\mathcal{L}) \tag{4.9}$$

then all the results in Theorem 1 *hold, and moreover, if an equilibrium intrinsic state* σ^* *exists it is asymptotically stable.*

In Eqs. (4.8) and (4.9) the symbol $(\cdot, \cdot)$ denotes the inner product in a Hilbert space $\mathcal{H}$. The supremum of all numbers β satisfying Eq. (4.9) is called the dissipative constant of $\mathcal{L}_1$.

It is noteworthy that to satisfy for instance the condition (4.9) in Theorem 2 we have to assume:

1. The linear operator $\mathcal{L}$ is the infinitesimal generator of a negative contraction semi-group of class C_0 (for definition see K. YOSIDA [36]) with the contraction constant β^*.

2. The nonlinear function $\hat{\mathbf{f}}$ is Lipschitz continuous on $\mathcal{H}$ with Lipschitz constant $k < \beta^*$, i.e.

$$||\hat{\mathbf{f}}(\mathbf{P}^*, \boldsymbol{\alpha}) - \hat{\mathbf{f}}(\mathbf{P}^*, \boldsymbol{\omega})|| \leqq k ||\boldsymbol{\alpha} - \boldsymbol{\omega}|| \quad \text{for all} \quad \boldsymbol{\alpha}, \boldsymbol{\omega} \in \mathcal{H}. \tag{4.10}$$

The Lipschitz continuity of $\hat{\mathbf{f}}$ in the asymptotic stability theorem can be weakened by using the condition

$$\left(\hat{\mathbf{f}}(\mathbf{P}^*, \boldsymbol{\alpha}) - \hat{\mathbf{f}}(\mathbf{P}^*, \boldsymbol{\omega}), \boldsymbol{\alpha} - \boldsymbol{\omega}\right) \leqq k ||\boldsymbol{\alpha} - \boldsymbol{\omega}||^2 \tag{4.11}$$

with $k < \beta^*$ for all $\boldsymbol{\alpha}, \boldsymbol{\omega} \in \mathcal{H}$.

It can be easily proved that stability and asymptotic stability are invariant if the inner product $(\cdot, \cdot)$ of $\mathcal{H}$ is replaced by an equivalent inner product $(\cdot, \cdot)_1$ with respect to which a nonlinear operator $\mathcal{L}_1 \equiv \mathcal{L} + \hat{\mathbf{f}}(\mathbf{P}^*, \cdot)$ is dissipative or strictly dissipative (cf. C. V. PAO and W. G. VOGT [25]).

These results are of great importance for practical application to the study of stability conditions for dissipative bodies with internal defects and imperfections.

5. Physical motivation

Recent physical investigations have proved the importance of interface kinetics in the explanation of a complicated nature of polycrystalline matter flow in some regions of temperature and strain rate changes. When strain rate is small ($10^{-8}-10^{-4}$ s^{-1}) and the temperature is high enough to permit diffusion and when strains are large (as in superplastic flow), the flow process can be modelled by a grain-boundary sliding mechanism with diffusional accommodation. For higher strain rates ($10^{-4}-10^{-2}$ s^{-1}) dislocation creep flow is more dominant.

The main dissipative mechanisms for mild steel at room temperature for a strain rate range from 10^{-12} to 10^{6} s^{-1} are shown in Fig. 2.

M. F. ASHBY and R. A. VERRALL [1] suggested a model for a polycrystalline material deformed at temperature above 0.3 ϑ_M (where ϑ_M is the melting temperature). This model

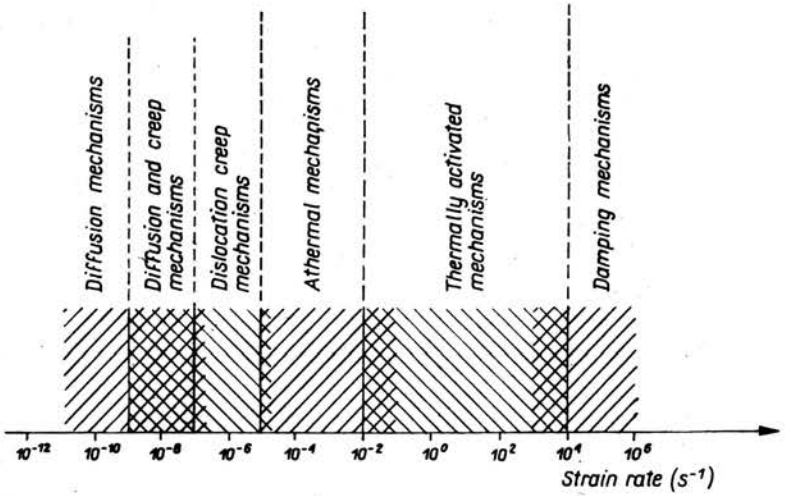

FIG. 2. Dissipative mechanisms for mild steel at room temperature for strain rate range from 10^{-12} to 10^{6} s^{-1}.

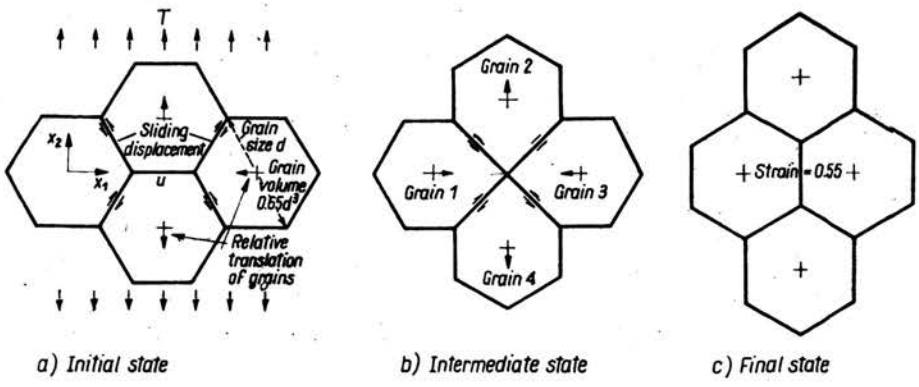

FIG. 3. The isothermal deformation process of a group of four grains, at constant pressure and stress (After) M. F. ASHBY and R. A. VERRALL [1]).

is called non-uniform diffusion-accommodated flow and is generalization of the quasi-uniform diffusional flow model proposed by C. HERRING [15] and R. L. COBLE [5]. At large strains the model of M. F. ASHBY and R. A. VERRALL [1] proceeds faster than Herring-Coble creep.

The characteristic assumptions of the Ashby–Verrall model are as follows (cf. Fig. 3):

(i) The grains do not suffer the same change as the specimen.

(ii) The grain switches its neighbours.

(iii) The number of grains across the cross-section of the specimen does not remain constant.

(iv) The translation of the grains involve also their rotation. In this model grains behave as if they consisted of a rigid core surrounded by a plastic mantle.

As it is shown in Fig. 3 the grains themselves suffer accommodation strains which permit them to remain stuks together. In this model these accommodation strains are accomplished by diffusion.

There are two independent diffusion paths in a polycrystal, namely bulk diffusion through the grains and diffusion via the high-conductivity paths which the grain boundaries provide.

It has been assumed by M. F. ASHBY and R. A. VERRALL [1] that superplastic flow is almost always boundary-diffusion controlled.

Basing on the dissipative process for the model, Ashby and Verral assumed that the work drives four irreversible mechanisms:

(i) The diffusion process by which the grains temporarily change shape, suffering accommodation strains.

(ii) The interface reaction. Grain or phase boundaries may be imperfect sinks or sources for point defects.

(iii) Grain boundary sliding. Work is done against the boundary viscosity when sliding occurs.

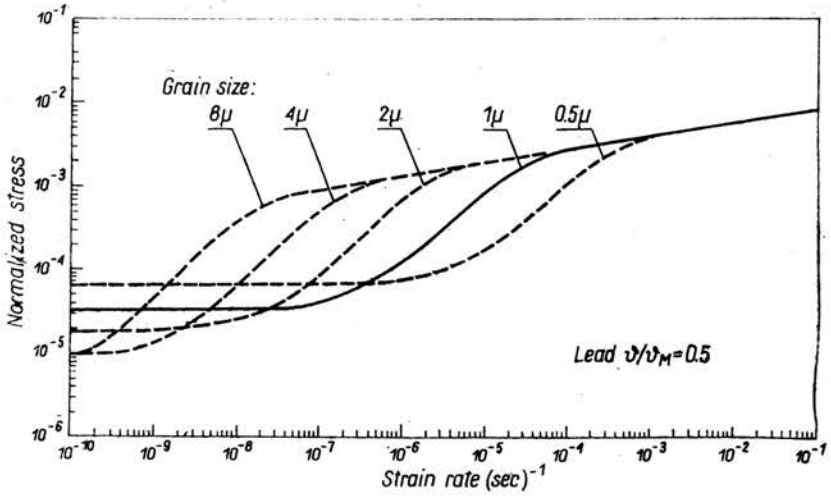

FIG. 4. The effect of changing the grain size on the stress-strain rate plot (After M F. ASHBY and R. A. VERRALL [1]).

(iv) Fluctuations of boundary area. The grain boundary area increases as the cluster of four grains moves from the initial to the intermediate state, storing free energy in the system. As the cluster moves from the intermediate state to the final position the boundary area decreases, releasing the energy (cf. Fig. 3).

Any polycrystalline material can deform by the mechanism described above. But its characteristics will be apparent only in the presence of competing mechanisms like that of dislocation creep or a thermally-activated process.

Many aspects of superplastic flow can be explained as a superposition of the non-uniform diffusional mechanism.

In Fig. 4 the effect of grain size for this model is shown and in Fig. 5 a comparison of the model predictions with the experimental data of R. C. COOK and N. R. RISEBROUGH [7] for a dilute zinc alloy is presented.

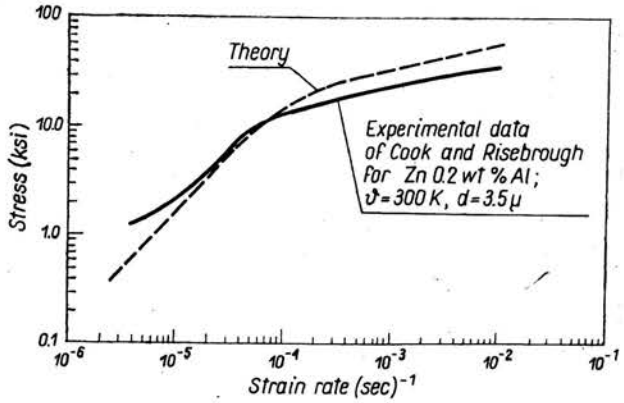

FIG. 5. The comparison of the measurements of R. C. COOK and N. R. RISEBROUGU [7] with theoretical results of the diffusional-accommodated model (After M. E. ASHBY and R. A. VERRALL [1]).

These results have lead to the conclusion that the threshold stress predicted by the model is generally too small but in the range of strain rate from 10^{-5} to 10^{-2} s^{-1} agreement of the theoretical results with experimental data is sufficiently good.

The main conclusion from the considerations of Ashby and Verrall concerns the importance of the diffusion effects. We can say that in some ranges of strain rate and temperature changes the diffusion effect is crucial for the appropriate description of the complex behaviour of polycrystalline materials.

Recently G. H. EDWARD and M. F. ASHBY [9] have analysed the growth of an array of grain boundary voids during creep deformation in order to describe intergranular fracture. They developed the coupled diffusion and power-law creep model. In this model each void grows by diffusion, but the void plus its diffusion field is contained within a cell of power-law creeping material. The coupled model predicts times and strain-to-fracture which appear to be consistent with observation.

On the other hand investigations on the growth of an array of grain boundary voids called here imperfections during a deformation process showed the importance of a coupled diffusion and thermo-activated mechanism to explain complex fracture phenomena in metals.

We propose that each void grows by diffusion, but the void plus its diffusion field is contained within a cell of a thermo-activated flowing material, cf. Fig. 6.

We assume that thermo-activated mechanism leads to elastic-viscoplastic flow of a material. The mechanism of grain-boundary sliding with accommodated diffusion intro-

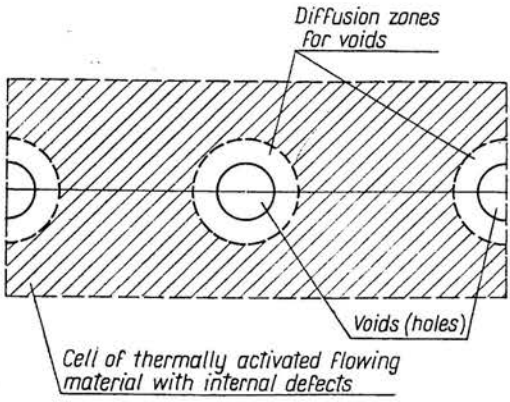

FIG. 6. A model of a thermally-activated flowing material with internal defects and imperfections (voids).

duces the evolution of defects and the mechanism of void growth plus its diffusion field inserts the evolution of imperfections in the material considered.

Basing on these physical considerations, we propose a model of an elastic-viscoplastic material with internal defects and imperfections.

Let E_p denote the inelastic strain, T the stress, ϑ temperature, ξ the scalar measure of the concentration of defects and ζ the scalar measure of the concentration of imperfections (voids) in a material.

The fundamental evolution equation for the inelastic strain E_p has the form (cf. [29])

$$\dot{E}_p = \gamma \exp\{-U(T^*)/k\vartheta\}, \tag{5.1}$$

where γ is the viscosity coefficient, $U(T^*)$ denotes the activation energy and k is the Boltzmann constant. The stress T^* is determined by the relation

$$T^* = T - T_A, \tag{5.2}$$

where T_A denotes the athermal component of stress.

Experimental results and physics of solids for the thermo-activated flow mechanism suggest the relations (cf. [29])

$$\begin{aligned} U(T^*) &= \varphi\left(\frac{T}{T_A} - 1\right), \\ \gamma &= \hat{\gamma}(\vartheta, E_p, \xi, \zeta), \\ T_A &= \hat{T}_A(\vartheta, E_p, \xi, \zeta). \end{aligned} \tag{5.3}$$

Then the evolution equation (5.1) takes the form

$$\dot{E}_p = \hat{\gamma}\Phi\left(\frac{T}{\hat{T}_A} - 1\right). \tag{5.4}$$

If the range of strain rate changes is such that the main dissipative mechanism is the dislocation creep, then we can assume

$$U(T^*) = \varphi\left(\frac{T}{T_0}\right), \tag{5.5}$$

where

$$T_0 = \hat{T}_0(\vartheta, E_p, \xi, \zeta). \tag{5.6}$$

For this case the evolution equation (5.1) has the form

$$\dot{E}_p = \hat{\gamma}\Phi\left(\frac{T}{\hat{T}_0}\right). \tag{5.7}$$

It is noteworthy that the Norton law of the creep theory results from Eq. (5.7) as a particular case.

It is sufficient to assume

$$\Phi\left(\frac{T}{T_0}\right) \equiv \left(\frac{T}{T_0}\right)^n \quad \text{and} \quad \dot{E}_0 \equiv \gamma. \tag{5.8}$$

Then, from Eq. (5.7) we have

$$\frac{\dot{E}_p}{\dot{E}_0} = \left(\frac{T}{T_0}\right)^n. \tag{5.9}$$

From the considerations above we reach the important conclusion that the parameters E_p, ξ and ζ can be assumed as fundamental internal state variables in the model proposed.

The viscosity coefficient γ and the athermal stress T_A (or T_0) can be treated as internal state variables or after the assumption of Eqs. (5.3) (or (5.6)) as material functions.

The evolution of the internal state variables ξ and ζ shall result from diffusion effects and shall be described by the second order partial differential equations (diffusion equations).

If the physical interpretations of the internal state variables ξ and ζ are assumed as the probabilistic densities respectively, for the distribution of defects and imperfections in a body, then after introducing additional conditions we can obtain for ξ and ζ the Fokker–Planck evolution equations.

6. Elastic-viscoplastic material with internal defects and imperfections

To describe this physical model of dissipative material within the framework of a modified structure with internal state variables let us introduce a set of the internal parameters as follows (cf. a more restrictive assumption of the author [30]):

$$\boldsymbol{\alpha}(t) = \{\mathbf{E}_p(t), \varkappa(t), \xi(t), \zeta(t)\}, \tag{6.1}$$

where $\mathbf{E}_p(t)$ is interpreted as the inelastic strain tensor, $\varkappa(t)$ as the isotropic work-hardening parameter, $\xi(t)$ as the scalar measure of the concentration of defects and $\zeta(t)$ as the scalar measure of the concentration of imperfections.

The evolution equations are postulated in the form

$$\partial_t \mathbf{E}_p(t) = \hat{\gamma}(\sigma)\left\langle \Phi\left(\frac{f(\cdot)}{\varkappa(t)} - 1\right)\right\rangle \partial_{\mathbf{T}(t)} f(\cdot),$$

$$\partial_t \varkappa(t) = \operatorname{tr}[\hat{\mathbf{K}}(\sigma)\,\partial_t \mathbf{E}_p(t)], \tag{6.2}$$

$$\partial_t \xi(t) = \nabla \mathbf{D}_1(\sigma)\nabla\xi(t) + \frac{1}{\tau_1(\sigma)}\left(\xi^* - \xi(t)\right) + \operatorname{tr}[\hat{\boldsymbol{\Xi}}_1(\sigma)\,\partial_t \mathbf{E}_p(t)],$$

$$\partial_t \zeta(t) = \nabla \mathbf{D}_2(\sigma)\nabla\zeta(t) + \frac{1}{\tau_2(\sigma)}\left(\zeta^* - \zeta(t)\right) + \operatorname{tr}[\hat{\boldsymbol{\Xi}}_2(\sigma)\,\partial_t \mathbf{E}_p(t)],$$

where

$$f(\cdot) = f\left(\mathbf{T}(t), \vartheta(t), \mathbf{E}_p(t), \xi(t), \zeta(t)\right) \tag{6.3}$$

denotes the quasi-static yield function (loading function), and the symbol $\langle\ [\ ]\ \rangle$ is understood according to the definition

$$\langle\ [\ ]\ \rangle = \begin{cases} 0 & \text{if} \quad f(\cdot) \leqq \varkappa(t), \\ [\ ] & \text{if} \quad f(\cdot) > \varkappa(t). \end{cases} \tag{6.4}$$

The material function Φ is interpreted as the excess of stress over the quasi-static yield condition and is motivated by the thermo-activated mechanism of viscoplastic flow, $\hat{\gamma}(\sigma)$ denotes the viscosity coefficient, $\hat{\mathbf{K}}(\sigma)$ is the material function which for practical application can be assumed as proportional to the stress tensor $\mathbf{T}(t)$, $\mathbf{D}_1(\sigma)$ and $\mathbf{D}_2(\sigma)$ denote the diffusion coefficients for defects and imperfections, respectively, $\frac{1}{\tau_1(\sigma)}$ and $\frac{1}{\tau_2(\sigma)}$ denote the velocities of generation of defects and imperfections, respectively, and $\hat{\boldsymbol{\Xi}}_1(\sigma)$ and $\hat{\boldsymbol{\Xi}}_2(\sigma)$ describe the generation of defects and imperfections, respectively, caused by plastic flow.

Practical applications need some simplifications. The most important of them is the assumption of one internal state variable ω instead of two ξ and ζ. Then ω is interpreted as the scalar measure of the concentration of defects and imperfections in the material. The evolution equation for ω is postulated in the form

$$\partial_t \omega(t) = \nabla \mathbf{D}(\sigma)\nabla\omega(t) + \frac{1}{\tau(\sigma)}\left(\omega^* - \omega(t)\right) + \operatorname{tr}[\hat{\boldsymbol{\Xi}}(\sigma)\,\partial_t \mathbf{E}_p(t)], \tag{6.5}$$

and has an obvious interpretation.

Under this simplification the internal dissipation is described by

$$\hat{\mathrm{i}}(\sigma) = -\frac{1}{\vartheta(t)}\left\{\hat{\gamma}(\sigma)\left\langle \Phi\left(\frac{f(\cdot)}{\varkappa(t)} - 1\right)\right\rangle \operatorname{tr}\left[\left(\partial_{\mathbf{E}_p(t)}\hat{\Psi} + \partial_{\varkappa}\hat{\Psi}\hat{\mathbf{K}}(\sigma) + \partial_{\omega(t)}\hat{\Psi}\hat{\boldsymbol{\Xi}}(\sigma)\right)\partial_{\mathbf{T}(t)} f(\cdot)\right] + \partial_{\omega(t)}\hat{\Psi}\left[\nabla \mathbf{D}(\sigma)\nabla\omega(t) + \frac{1}{\tau(\sigma)}\left(\omega^* - \omega(t)\right)\right]\right\}. \tag{6.7}$$

It is noteworthy that in the determination of the particular forms of the material functions $\mathbf{D}(\sigma)$, $\tau(\sigma)$ and $\hat{\boldsymbol{\Xi}}(\sigma)$, the conditions of stability (4.8) or asymptotic stability (4.9) should be taken into account.

For the formulation of the initial-boundary-value problem it will be useful to have the rate type material structure.

In the case of an isothermal process we have the constitutive equation for the stress tensor as follows:

$$\mathbf{T}(t) = \hat{\mathbf{T}}(\sigma_1) = \hat{\mathbf{T}}\big(\mathbf{E}(t), \mathbf{E}_p(t), \varkappa(t), \omega(t)\big). \tag{6.8}$$

Let us postulate that the intrinsic state σ_2 for the rate type material structure is determined by the expression

$$\sigma_2 = \big(\mathbf{E}(t), \mathbf{T}(t), \varkappa(t), \omega(t)\big). \tag{6.9}$$

The evolution equation for the stress tensor $\mathbf{T}(t)$ has the form (cf. [28])

$$\partial_t \mathbf{T}(t) = \hat{\boldsymbol{\beta}}_0 + \hat{\boldsymbol{\beta}}_1[\partial_t \mathbf{E}(t)], \tag{6.10}$$

where

$$\begin{aligned}\hat{\boldsymbol{\beta}}_0 = {} & \hat{\gamma}(\sigma_2)\left\langle \Phi\left(\frac{f(\cdot)}{\varkappa(t)} - 1\right)\right\rangle \{\partial_{\mathbf{E}_p(t)}\hat{\mathbf{T}}[\partial_{\mathbf{T}} f] + \partial_{\varkappa}\hat{\mathbf{T}}\,\mathrm{tr}[\hat{\mathbf{K}}\partial_{\mathbf{T}} f] \\ & + \partial_{\omega}\hat{\mathbf{T}}\,\mathrm{tr}[\hat{\boldsymbol{\Xi}}\partial_{\mathbf{T}} f]\} + \partial_{\omega}\hat{\mathbf{T}}\left[\nabla D(\sigma_2)\nabla\omega(t) + \frac{1}{\tau(\sigma_2)}\left(\omega^* - \omega(t)\right)\right], \\ \hat{\boldsymbol{\beta}}_1 = {} & \partial_{\mathbf{E}}\hat{\mathbf{T}}.\end{aligned} \tag{6.11}$$

The evolution equation for $\varkappa$ and ω have the form as postulated by Eqs. $(6.2)_2$ and (6.5)

$$\begin{aligned}\partial_t \varkappa(t) = {} & \hat{\gamma}(\sigma_2)\,\mathrm{tr}[\hat{\mathbf{K}}(\sigma_2)\,\partial_{\mathbf{T}} f]\left\langle \Phi\left(\frac{f(\cdot)}{\varkappa(t)} - 1\right)\right\rangle, \\ \partial_t \omega(t) = {} & \nabla D(\sigma_2)\nabla\omega + \frac{1}{\tau(\sigma_2)}\left(\omega^* - \omega(t)\right) + \hat{\gamma}(\sigma_2)\,\mathrm{tr}[\hat{\boldsymbol{\Xi}}(\sigma_2)\,\partial_{\mathbf{T}} f]\left\langle \Phi\left(\frac{f(\cdot)}{\varkappa(t)} - 1\right)\right\rangle.\end{aligned} \tag{6.12}$$

To keep our consideration as simple as possible we postulate

$$\begin{aligned}& \hat{\gamma}(\sigma_2) \equiv \gamma_0 = \text{const}, \\ & f(\cdot) \equiv (\mathrm{II}_{\mathbf{S}})^{1/2}, \quad \text{where} \quad \mathbf{S} = \mathbf{T} - \left(\frac{1}{3}\,\mathrm{T}_{ii}\right)\mathbf{I}, \\ & D(\sigma_2) \equiv D_0 = \text{const}, \\ & \tau(\sigma_2) \equiv \tau_0 = \text{const},\end{aligned} \tag{6.13}$$

then

$$\partial_{\mathbf{T}} f(\cdot) = \frac{\mathbf{S}}{(\mathrm{II}_{\mathbf{S}})^{1/2}}. \tag{6.14}$$

Basing on available experimental data the determination of the material constant γ_0 and the material function Φ for rate sensitive plastic materials was presented in the previous paper of the author [30]. Using the results of that paper we have

$$\Phi \equiv \left[\frac{(\mathrm{II}_{\mathbf{S}})^{1/2}}{\varkappa} - 1\right]^5 \quad \text{and} \quad \gamma_0 = 25.82\ \mathrm{s}^{-1}. \tag{6.15}$$

7. Applications to instability and fracture phenomena

In a separate paper of the author [32] the rate type constitutive equations for an elastic-viscoplastic material with defects and imperfections have been applied to the solution of the initial-boundary-value problem. It has been studied the tensile deformation of a circular cylinder bar of initial length $2L_0$ and initial radius R_0. The problem has been treated in the cylindrical coordinates r, θ, z. It has been assumed that the problem is axisymmetric and additionally that the deformations are symmetric about the mid plane $z = 0$.

Two initial-boundary-value problems have been formulated. In the first of them the ends of the specimen are assumed to be cemented to rigid grips, while in the second they remain shear free. In both cases the bar is strained parallel to its main axis with a constant velocity $\dot{U}$. The lateral surfaces of the bar are required to remain stress free in both cases. Similar problems for an elastic-plastic material were investigated by A. NEEDLEMAN [22] for a cylindrical bar and by M. A. BURKE and W. D. NIX [3] for a rectangular bar.

Particular attention has been given to investigate diffusion effects caused by the evolution of defects and imperfections on the onset of the instability of the straining process.

It has been found that the onset of the instability by the necking mode is strongly dependent upon the diffusion cooperative effects.

The diffusion process enhances necking and the strain rate effect changes the onset of the instability. The strain at the instability point decreases with an increase in strain rate while the load at the instability point increases with an increase in strain rate (cf. [30–32]).

These conclusions have proved the great importance and usefulness of the new theory developed.

References

1. M. F. ASHBY and R. A. VERRALL, *Diffusion-accommodated flow and superplasticity*, Acta Metallurgica, **21**, 149–163, 1973.
2. F. E. BROWDER, *Nonlinear equations of evolution*, Ann. of Math., **80**, 485–523, 1964.
3. M. A. BURKE and W. D. NIX, *A numerical study of necking in the plane tension test*, Int. J. Solids Structures, **15**, 379–393, 1979.
4. A. K. CHAKRABARTI and J. W. SPRETNAK, *Instability of plastic flow in the direction of pure shear, I. Theory, II. Experimental*, Metall. Trans., **6A**, 733–747, 1975.
5. R. L. COBLE, *A model for boundary diffusion controlled creep in polycrystalline materials*, J. Phys., **34**, 1679–1682, 1963.
6. B. D. COLEMAN and M. E. GURTIN, *Thermodynamics with internal state variables*, J. Chem. Phys., **47**, 597–613, 1967.
7. R. C. COOK and N. R. RISEBROUGH, *Superplasticity in a dilute zinc-aluminum alloy*, Scripta Metall., **2**, 487–490, 1968.
8. T. B. COX and J. R. LOW, *An investigation of the plastic fracture of AISI 4340 and 18 nickel — 200 grade maraging steel*, Metall. Trans., **5**, 1457–1470, 1974.
9. G. H. EDWARD and M. F. ASHBY, *Intergranular fracture during power-law creep*, Acta Metallurgica, **27**, 1505–1518, 1979.
10. R. L. FOSDICK and K. R. RAJAGOPAL, *Thermodynamics and stability of fluids of third grade*, Proc. R. Soc. Lond., A. **339**, 351–377, 1980.

11. A. K. GHOSH, *Tensile instability and necking in materials with strain hardening and strain-rate hardening*, Acta Metallurgica, **25**, 1413–1424, 1977.
12. A. K. GHOSH, *The influence of strain hardening and strain rate sensitivity on sheet metal forming*, J. Eng. Materials Tech. Trans. ASME, **99**, 264–274, 1977.
13. M. E. GURTIN, *Thermodynamics and stability*, Arch. Rational Mech. Anal., **59**, 63–96, 1975.
14. E. HILLE and R. S. PHILLIPS, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., **31**, Providence 1957.
15. C. HERRING, *Diffusional viscosity of a polycrystalline solid*, J. Appl. Phys., **21**, 437–445, 1950.
16. J. W. HUTCHINSON and K. W. NEALE, *Influence of strain-rate sensitivity on necking under uniaxial tension*, Acta Metallurgica, **25**, 839–846, 1977.
17. T. KATO, *Nonlinear semi-groups and evolution equations*, J. Math. Soc. Japan, **19**, 508–520, 1967.
18. G. LUMER and R. S. PHILLIPS, *Dissipative operators in a Banach space*, Pacific J. Math., **11**, 679–698, 1961.
19. G. J. MINTY, *Monotone nonlinear operators in Hilbert space*, Duke Math. J., **29**, 341–346, 1962.
20. I. MÜLLER, *On thermodynamics of a gas with vibrating molecules*, Ann. Ins. Henri Poincaré, **21**, 245–258, 1974.
21. I. MÜLLER, *Thermodynamics of mixture of fluids*, Journal de Mécanique, **14**, 267–303, 1975.
22. A. NEEDLEMAN, *A numerical study of necking in circular cylindrical bars*, J. Mech. Phys. Solids, **20**, 111–127, 1972.
23. A. NEEDLEMAN and J. R. RICE, *Limits to ductility set by plastic flow localization*, Brown University Technical Report, **57**, November 1977.
24. C. V. PAO, *The existence and stability of solutions of nonlinear operator differential equations*, Arch. Rational Mech. Anal., **35**, 16–29, 1969.
25. C. V. PAO and W. G. VOGT, *On the stability of nonlinear operator differential equations and applications*, Arch. Rational Mech. Anal., **35**, 30–46, 1969.
26. W. PAVINICH and R. RAJ, *Fracture at elevated temperature*, Metall. Trans., **8 A**, 1917–1933, 1977.
27. P. PERZYNA, *Thermodynamics of a unique material structure*, Arch. Mech., **27**, 791–806, 1975.
28. P. PERZYNA, *On material isomorphism in description of dynamic plasticity*, Arch. Mech., **27**, 473–484, 1975.
29. P. PERZYNA, *Coupling of dissipative mechanisms of viscoplastic flow*, Arch. Mech., **29**, 607–624, 1977.
30. P. PERZYNA, *Modified theory of viscoplasticity. Application to advanced flow and instability phenomena*, Arch. Mech., **32**, 403–420, 1980.
31. P. PERZYNA, *Stability phenomena of dissipative solids with internal defects and imperfections*, XVth IUTAM Congress, Toronto, August 1980; Theoretical and Applied Mechanics, Proc., ed. F. P. J. RIMROTT and B. TABARROK, North-Holland, Amsterdam [in print].
32. P. PERZYNA, *Necking phenomena in an elastic-viscoplastic body with internal defects and imperfections* Arch. Mech., (in print).
33. P. PERZYNA, *Physical model of an elastic-viscoplastic material with internal defects and inperfections*, Arch. Mech., (in print).
34. J. R. RICE, *The localization of plastic deformation*, IUTAM Congress, Delft, August 1976; Theoretical and Applied Mechanics, ed. W. T. KOITER, North-Holland, Amsterdam 1976, 207–220.
35. P. J. WRAY, *Strain-rate dependence of the tensile failure of polycrystalline material at elevated temperatures*, J. Appl. Phys., **40**, 4018–4029, 1969.
36. K. YOSIDA, *Functional analysis*, Springer, Berlin 1965.

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