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Optimization problems for elastic anisotropic bodies

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OPTIMIZATION problems for anisotropic bodies are considered in the paper. Certain variational problems connected with the rotation of matrices are discussed, and sufficient conditions of optimality are derived. The results are then used to determine the optimum distribution of moduli in anisotropic nonhomogeneous rods exhibiting maximum torsional rigidity. Optimization problems of anisotropic properties of elastic media are also considered in connection with two-dimensional problems of elasticity. Several examples are discussed dealing with optimization of the form of anisotropic bodies, and with simultaneous optimization of the form and properties of anisotropic materials.

W pracy rozważano zagadnienia optymalizacji dla ciał anizotropowych. Rozpatrzono pewne zadania wariacyjne związane z obrotem macierzy i otrzymano warunki wystarczające optymalizacji. Zastosowanie tych wyników pozwoliło uzyskać optymalny rozkład modułów w anizotropowych prętach sprężystych wykazujących maksymalną sztywność skręcania. Problemy optymalizacji anizotropowych własności ośrodków sprężystych rozważano również w odniesieniu do płaskich zagadnień teorii sprężystości. Przytoczono kilka przykładów dotyczących wyznaczenia optymalnej postaci ciał anizotropowych oraz równoczesnej optymalizacji postaci i własności ciał anizotropowych.

В данной работе обсуждаются постановки задач оптимизации анизотропных тел. Рассмотрены некоторые вариационные задачи, связанные с вращением матриц, и изучены достаточные условия оптимальности. С применением этих результатов найдены оптимальные распределения модулей в неоднородных анизотропных стержнях, обладающих максимальной жесткость при кручении. Вопросы оптимизации анизотропных свойств упругой среди рассмотрены также применительно к плоским задачам теории упругости. Приведены некоторые результаты, касающиеся отыскания оптимальной формы анизотропных тел и совместных задач оптимизации формы и анизотропных свойств.

1. Formulation of the optimization problems for anisotropic bodies

ELEMENTS of a structure acted on by external loads are usually subject to complex states of stress and strain, what means that the stresses and strains measured at different points and directions of the body are also different. This is the reason why anisotropic and nonhomogeneous materials are widely used in engineering practice. The fundamental idea of reinforcement consists in reducing the material in the unstressed parts of the body, in weakening the structure in "unemployed" parts and directions and, conversely, in strengthening these directions or portions which transmit the principal forces or contain dangerous stress concentrations.

The ability of modelling various type of structural anisotropy widens the range of applicability of structures made of composite materials and makes it possible to utilize the anisotropic properties of materials in elastic constructions. Such tasks like determination of the form of anisotropic bodies (made of materials with prescribed anisotropy), optimum distribution of elastic moduli in the deformed body, and simultaneous optimization of the form and internal structure of the body are of principal interest.

It should be mentioned that the formulation and solutions of the problems of optimization of reinforced structures presented in this paper are based on a purely phenomenological approach. The well-known equations of the theory of elasticity of anisotropic bodies are assumed to be satisfied. The elastic moduli A_{ijkl} which appear in the generalized Hooke's law

$$\sigma_{ij} = A_{ijkl}\,\varepsilon_{kl} \tag{1.1}$$

and in other equations governing the behaviour of structures are determined by the necessary tests performed on the reinforced materials. The stresses σ_{ij} and strains ε_{ij} represent second-order tensors, and the components A_{ijkl} form the fourth order tensor of elastic

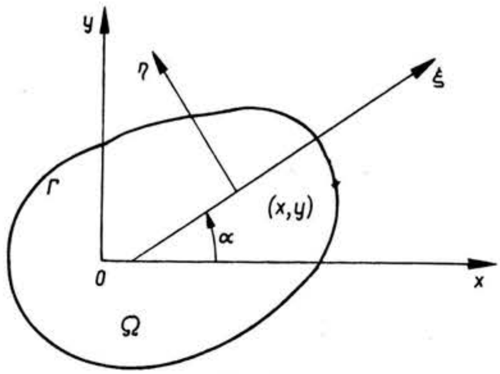

FIG. 1.

moduli. In the general case of anisotropic materials the number of independent moduli A_{ijkl} in Eq. (1.1) equals 21. If the structure of the anisotropic body reveals a certain type of symmetry, the elastic constants also possess certain symmetry properties, what is manifested at each point of the body by the existence of such directions which are equivalent from the point of view of elastic properties; consequently, the number of independent moduli A_{ijkl} is reduced.

It should be noted that, since the magnitudes A_{ijkl} form a fourth-order tensor, rotation of the coordinate axes makes it transform according to a linear law, the transformation coefficients being represented by products of the cosines $(n_{i'i}, \ldots, n_{l'l})$ of the angles made by new (primed) and original (unprimed) directions of the axes, $A_{i'j'k'l'} = A_{ijkl}n_{i'i}n_{j'j}n_{k'k}n_{l'l}$.

Various optimization problems may be formulated within the framework of the theory of anisotropic bodies [1–5]; some of them will be discussed here.

Determination of the optimum forms of elastic anisotropic bodies constitutes an essential generalization of the corresponding problems concerning isotropic bodies. Principal difficulties arising here are due to very complex forms of the equations governing the behaviour of anisotropic structures.

As an example of nonclassical formulation, the problem of optimum distribution of elastic moduli may be quoted. These problems will be treated in detail in this paper. For definiteness let us assume that the deformability of the structure will be minimized under the condition of its limited weight. Let us assume the body to consist of identical infinitesimal crystals arbitrarily oriented with respect to each other. The fact that the crystals are identical but arbitrarily oriented means that the positions of the axes of elastic symmetry with respect to a fixed Cartesian reference frame change with the position within the body, but the values of the elastic moduli measured along the axes of elastic symmetry remain unchanged. Let us denote the orientation of the axes of anisotropy at each point $x = \{x_1, x_2, x_3\}$ of the medium with respect to a fixed Cartesian coordinate system (x_1, x_2, x_3) by the angles $\alpha_1(x), \alpha_2(x), \alpha_3(x)$ representing the components of a vectorial function $\alpha(x)$, that is $\alpha(x) = \{\alpha_1(x), \alpha_2(x), \alpha_3(x)\}$. Let α_j $(j = 1, 2, 3)$ denote the angle made by the elastic symmetry axis x'_j and the fixed coordinate axis x_j. The problem of determination of the optimum orientation of the axes of anisotropy, i.e. finding the vector function $\alpha(x)$ from the condition of minimum compliance, is now reduced to

$$J_* = \min_\alpha J(\alpha). \tag{1.2}$$

Another optimization problem may be formulated by assuming that the anisotropy axes at each point of the body are fixed, while the values of the moduli corresponding to such axes represent the unknown functions sought for. Certain conditions may be imposed on the moduli resulting from the structure of the composite and from the mechanical properties of its components.

Solution of these problems and determination of the optimum distribution of elastic moduli makes it possible to find the most suitable directions of the reinforcement, and to evaluate the quality of the structures traditionally used in practice. Even in cases in which the optimum structural anisotropy proves to be difficult to realize in practice, solutions of the optimization problems may be used to determine the limiting possibilities and the quasi-optimum reinforcement schemes.

Before passing to the discussion of particular optimization problems of anisotropic bodies, let us first consider the auxiliary problem dealing with the determination of optimum rotations of a given matrix.

2. On the problem of extremum connected with rotation of the matrix

Let us assume the system to be described by a scalar function φ and a square matrix T. The function $\varphi = \varphi(x, y)$ and the elements $t_{ij} = t_{ij}(x, y)$, $(i, j = 1, 2)$ of the matrix T are defined in the region Ω of the variables x, y. The function $\varphi = 0$ at the boundary Γ of Ω, and the matrix T satisfies the orthogonality conditions, i.e. $T^*T = E$, where T^* — transpose of the matrix and E — unit matrix. Let us define the functional J on the elements φ and T:

$$J(\varphi, T) = \iint_\Omega [(\nabla\varphi, T^*AT\nabla\varphi) - 2f\varphi]dxdy. \tag{2.1}$$

Here $f > 0$ — a prescribed function, ∇ — gradient, and the parentheses denote the scalar product.

In Eq. (2.1) the symbol A denotes a symmetric positive definite matrix with the elements $a_{ij} = a_{ij}(x, y)$, $(i, j = 1, 2)$. Positive definiteness of the matrix means that $a_{11} > 0$, $a_{11}a_{22} - a_{12}^2 > 0$ and that its eigenvalues are also positive $(\lambda_i > 0, i = 1, 2)$. Let us assume for the sake of simplicity that $\lambda_1(x, y) < \lambda_2(x, y)$.

The product of the matrices T^*AT appearing in Eq. (2.1) is now denoted by M, i.e. $M = T^*AT$. Note that the eigenvalues of M are identical with those of A.

Under the complementary conditions introduced above, let us consider the problem of minimization of the functional J with respect to φ and T,

$$\begin{gathered} J_* = \min_T \min_\varphi J(\varphi, T), \\ (\varphi)_\Gamma = 0, \qquad T^*T = E. \end{gathered} \tag{2.2}$$

Let us determine the neccessary conditions of optimization for the variational problem (2.1) and (2.2). To this end compare the function φ and matrix T with the value $\varphi+\delta\varphi$, $T(E+\delta T)$. The variated values satisfy the boundary condition and the orthogonality condition (2.2) provided the variation of φ at the boundary Γ vanishes, and δT is a skew-symmetric matrix. Using the equalities $(\delta\varphi)_\Gamma = 0$, $(\delta T)^* = -\delta T$, the first variation of the functional corresponding to the variations $\delta\varphi$ and δT is written in the form

$$\delta J = 2\iint\limits_\Omega \delta T\nabla\varphi, M\nabla\varphi)dxdy + 2\iint\limits_\Omega [(\nabla\delta\varphi, M\nabla\varphi) - f\delta\varphi][dxdy.$$

From the condition of the vanishing of δJ for an arbitrary scalar function $\delta\varphi$ equal to zero at Γ and an arbitrary skew-symmetrix matrix δT, we obtain the necessary conditions of extremum for φ and T. The condition of extremum of J with respect to φ has the form of the Euler equation $\operatorname{div}(M\nabla\varphi) = -f$. The necessary condition for J to assume extremum values with respect to T is easily found to be the collinearity of the vectors $\nabla\varphi$ and $M\nabla\varphi$. Consequently, the vector $\nabla\varphi$ is one of the eigenvectors of the matrix

$$M\nabla\varphi = \lambda_i \nabla\varphi. \tag{2.3}$$

On substituting the relation (2.3) into the Euler equation, we obtain the equations

$$\begin{gathered} \operatorname{div}(\lambda_i \nabla\varphi) = -f, \quad i = 1, 2, \\ \lambda_i = \frac{1}{2}\left[a_{11} + a_{22} + (-1)^i \sqrt{(a_{11}-a_{22})^2 + 4a_{12}^2}\right], \end{gathered} \tag{2.4}$$

which may be used to determine the stationary values of φ in the case of T being given by Eq. (2.3).

The elements of the orthogonal matrix T are now represented in the form $t_{11} = \cos\alpha$, $t_{12} = -t_{21} = \sin\alpha$, $t_{22} = \cos\alpha$, where α — the angle of rotation prescribed by the matrix T. Equation (2.3) is used to derive an explicit relation expressing the angle $\alpha = \alpha(x, y)$, in terms of $\varphi = \varphi(x, y)$. For definiteness let us assume that the vector $\nabla\varphi = \{\varphi_x, \varphi_y\}$ corresponds to the eigenvalue λ_i. But then the eigenvalues λ_j $(i \neq j)$ will correspond to the eigenvector $b = \{\varphi_y, -\varphi_x\}$. Let us perform scalar multiplication of both sides of the vector equation (2.3) by b; we obtain then $(b, M\nabla\varphi) = 0$. The relation contains two separate cases. The first one

$$\begin{gathered} \sin 2\alpha = Q, \quad Q = -\frac{2(a_{11}-a_{22})\varphi_x\varphi_y - 2a_{12}(\varphi_x^2 - \varphi_y^2)}{(\nabla\varphi)^2\sqrt{(a_{11}-a_{22})^2 + 4a_{12}^2}}, \\ \cos 2\alpha = P, \quad P = -\frac{(a_{11}-a_{22})(\varphi_x^2 - \varphi_y^2) + 4a_{12}\varphi_x\varphi_y}{(\nabla\varphi)^2\sqrt{(a_{11}-a_{22})^2 + 4a_{12}^2}} \end{gathered} \tag{2.5}$$

corresponds to the smaller eigenvalue λ_1. The second case $\cos 2\alpha = -P$, $\sin 2\alpha = -Q$ corresponds to the greater value λ_2. Thus the stationarity condition does not yield the unique method of determining the angle α and does not allow for the formulation of a closed boundary value problem necessary to find the values sought for. In order to find the unique dependence of α upon φ, it is necessary to consider the sign of the second variation of the functional subject to optimization, and to determine which of the two cases corresponds to the minimum of J.

It will be demonstrated that the functional J attains its minimum if the state (2.5), which corresponds to the smaller eigenvalue, is reached in the entire domain Ω. To this end let us write down the expression for the second variation of J resulting from varying φ and T under the conditions (2.2)

$$\delta^2 J = \iint_{\Omega} \{(\nabla\delta\varphi, M\nabla\delta\varphi) + (\delta T\nabla\varphi, M\delta T\nabla\varphi) + (M\nabla\varphi), \delta T\delta T\nabla\varphi) + 2(\delta T\nabla\varphi, M\nabla\delta\varphi) - 2(\nabla\delta\varphi, \delta TM\nabla\varphi)\}dxdy. \tag{2.6}$$

Let $\nabla\varphi$ — be the eigenvector corresponding to the eigenvalue λ_1. Then the orthogonal eigenvector $\delta T\nabla\varphi$ corresponds to the eigenvalue λ_2. If the matrix δT is represented in the form $\delta T = \delta\alpha B$, B denoting the skew-symmetric matrix with the elements $b_{11} = b_{22} = 0$, $b_{12} = -b_{21} = 1$, then $\delta T\delta T = -(\delta\alpha)^2 E$, $\delta T\nabla\varphi = \delta\alpha\{\varphi_y, -\varphi_x\}$. Using these relations, the expression (2.6) is transformed to

$$\delta^2 J = \iint_{\Omega} (p, Cp)dxdy, \quad p = \{\delta\alpha, \delta\varphi_x, \delta\varphi_y\}. \tag{2.7}$$

Here C — the symmetric square matrix 3×3 with the minors $\Delta_1, \Delta_2, \Delta_3$ of the first, second and third order are $\Delta_1 = (\lambda_2 - \lambda_1)(\nabla\varphi)^2$, $\Delta_2 = \lambda_1\Delta_1$, $\Delta_3 = \lambda_1\Delta_2$. The integrand in Eq. (2.7) represents a quadratic form of the components of $p = \{\delta\alpha, \delta\varphi_x, \delta\varphi_y\}$.

From the assumption of positive definiteness of the matrix A and the inequality $\lambda_1(x, y) < \lambda_2(x, y)$ it follows that all the minors are nonnegative ($\Delta_i \geqslant 0$, $i = 1, 2, 3$). The minors Δ_i vanish if $\nabla\varphi = 0$. The vanishing of the gradient over a finite subregion $\Omega_0 \subset \Omega$ contradicts Eq. (2.4) and hence the strong inequalities $\Delta_i > 0$ ($i = 1, 2, 3$) hold true almost everywhere on Ω. From these inequalities and the Silvester criterion it follows that the quadratic form (p, Cp) is positive definite and, consequently, $\delta^2 J > 0$. Thus if the function $\alpha(x, y)$ is related to φ by means of Eq. (2.5) in the region Ω (the case of minimum eigenvalue), then J attains its minimum.

3. On the optimum anisotropy of rods under torsion

The results obtained in Sect. 2 will be used to determine the optimum distribution field of moduli in twisted rods.

1) Let us consider the problem of torsion of an elastic anisotropic cylindrical rod. The rod is parallel to the z-axis in the rectangular coordinate system (x, y, z), and is subject to torsion about the axis by torques applied to its ends. The cross-section of the rod on the xy-plane is denoted by Ω, and the boundary of Ω — by Γ. The rod is assumed

to be rectilinearly anisotropic and to possess at each point a plane of elastic symmetry perpendicular to the axis z of the cylinder. Let us introduce the stress function $\varphi(x, y)$ related to the components of the stress tensor τ_{xz}, τ_{yz} and the rate of rotation per unit axial length θ according to the laws $\tau_{xz} = \theta\varphi_y$, $\tau_{yz} = -\theta\varphi_x$. In order to determine the stress function the variational principle is used (cf. [6]):

$$J = \iint\limits_{\Omega} (m_{11}\varphi_x^2 - 2m_{12}\varphi_x\varphi_y + m_{22}\varphi_y^2 - 4\varphi)dxdy \rightarrow \min_\varphi. \tag{3.1}$$

The minimum in Eq. (3.1) is sought for in the class of functions satisfying the condition $(\varphi)_\Gamma = 0$. The symbols m_{11}, m_{12}, m_{22} denote the deformation coefficients in the system (x, y, z). In addition to the system (x, y, z) introduce at each point $(x, y) \in \Omega$ another system ξ, η, ζ, the ζ axis being parallel to x, and the axis ξ — inclined by the angle $\alpha(x, y)$ to the x-axis (Fig. 1). In the system (ξ, η, ζ), the material of the rod is characterized by the deformation coefficients $a_{11}(x, y), a_{12}(x, y)$ $a_{22}(x, y)$ which are considered to be known functions of x and y. The coefficients m_{ij} are expressed in terms of a_{ij} by means of the relations [6]

$$\begin{aligned} m_{11} &= a_{11}\cos^2\alpha - a_{12}\sin 2\alpha + a_{22}\sin^2\alpha, \\ m_{22} &= a_{11}\sin^2\alpha + a_{12}\sin 2\alpha + a_{22}\cos^2\alpha, \\ m_{12} &= \frac{1}{2}(a_{11} - a_{22})\sin 2\alpha + a_{12}\cos 2\alpha, \end{aligned} \tag{3.2}$$

which may be represented in a matrix form $M = T^*AT$ (notations of Sect. 2).

Let us consider the function $\alpha(x, y)$ governing the orientation of the axes ξ, η, ζ and formulate the following optimization problem: determine the function $\alpha = \alpha(x, y)$ such that $\varphi(x, y)$ given by Eqs. (3.1) and (3.2) leads to the maximum torsional rigidity K of the rod (K equals twice the integral of φ over the region Ω). Let us note that if the material is locally orthotropic and the axes ξ, η, ζ coincide with the axes of orthotropy (in this case $a_{12} = 0$), the optimization problem consists in determining the optimum distribution of the angles of inclination of the orthotropy axes.

Taking into account the fact that for the function $\varphi(x, y)$ minimizing the functional (3.1) under the boundary condition $(\varphi)_\Gamma = 0$ the equality $J = -K$ holds true, the relation between J and K is written in the form $K = -\min_\varphi J$. With this in mind, the problem of maximization of the torsional rigidity K may be reduced to a consecutive evaluation of the minima of J with respect to φ and α, i.e. $K_* = \max_\alpha K = -\min_\alpha \min_\varphi J(\alpha, \varphi)$. In this manner, the problem of determining optimum orientations of the anisotropy axes is reduced to the solution of the variational problem considered in Sect. 2. Consequently, the results of Sect. 2 may be applied to the analysis of the problem of maximization of the torsional rigidity of rods.

2) In the general case, the determination of φ and α is reduced to the solution of the boundary value problem:

$$\begin{aligned} (\lambda_1\varphi_x)_x + (\lambda_1\varphi_y)_y &= -2, \quad (\varphi)_\Gamma = 0, \\ \lambda_1 &= \frac{1}{2}\left(a_{11} + a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}^2}\right) \end{aligned} \tag{3.3}$$

and to the computation of α according to Eq. (2.5). The boundary value problems of such kind have been solved for various methods of prescribing the coefficient λ_1 (in connection with the problems of torsion of nonhomogeneous rods) and are presented in [6, 7].

3) Let the deformation coefficients a_{ij} and, consequently, the eigenvalue λ_1, be independent of x, y. Then the equation of torsion of an optimum rod is reduced to the Poisson equation $\varphi_{xx}+\varphi_{yy} = -2\lambda_1^{-1}$ which describes, under the boundary condition $(\varphi)_\Gamma = 0$, the torsion of isotropic rods with the shear modulus $G = \lambda_1^{-1}$. Since the theory of torsion of isotropic homogeneous rods is well developed and solutions of the corresponding boundary value problems are known (in analytical or numerical forms) for most of the practically important cases of cross-section forms (cf. [8]), the method of reduction mentioned above makes possible the solution of the optimization problem posed in this section.

4) Let us consider the problem of optimization for a rod made of a locally orthotropic material: $a_{11} = 1/G_1, a_{22} = 1/G_2, a_{12} = 0$, where G_1, G_2 — shear moduli. In this case the distribution of the angles of inclination of the orthotropy axes is given by the formula

$$\alpha = \frac{1}{2}\operatorname{arctg}\mu, \quad \mu = 2\varphi_x\varphi_y/(\varphi_x^2-\varphi_y^2), \tag{3.4}$$

$$\alpha = \frac{1}{2}\operatorname{arctg}\mu+\frac{\pi}{2}. \tag{3.5}$$

Angle α is found from Eq. (3.4) provided $\mu \geqslant 0, \varphi_x\varphi_y \geqslant 0$ or $\mu < 0, \varphi_x\varphi_y < 0$. Equation (3.6) is used to determine α if $\mu > 0, \varphi_x\varphi_y < 0$ and if $\mu < 0, \varphi_x\varphi_y > 0$.

It is easily demonstrated (cf. [4]) that each point $(x, y) \in \Omega$, the η-axis corresponding to the greater shear modulus G_1 is tangent to the φ contour line, while the ξ-axis connected with the smaller modulus G_2 is orthogonal to this line.

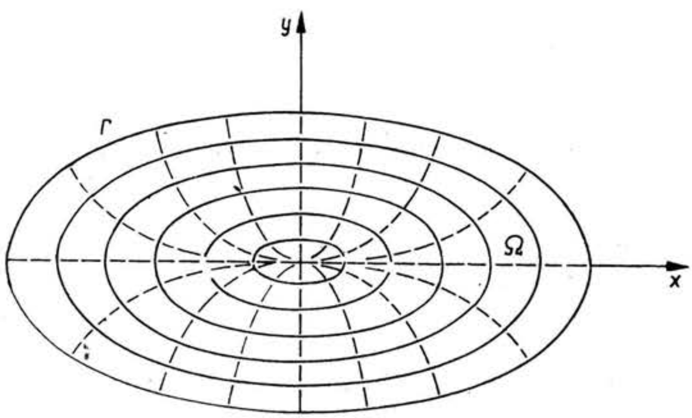

FIG. 2.

In Figs. 2 and 3 are shown the examples of solutions of the optimization problems concerning rods with elliptical and square cross-sections. Solid and dashed curves represent the respective families of contour lines of the stress function and of the orthogonal lines. The η-axes with the greatest shear modulus G_1 are tangent to the lines belonging to the

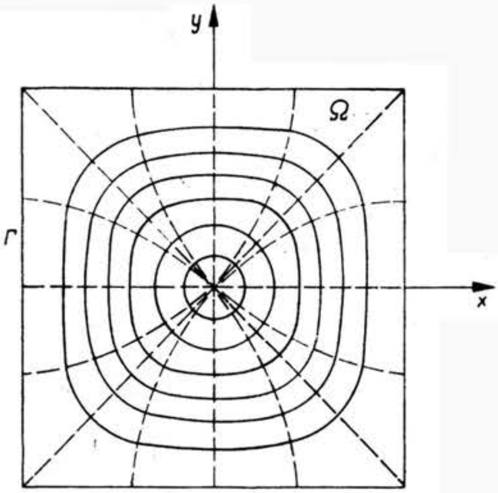

FIG. 3.

first family and the axes ξ (with the shear modulus G_2) are tangent to the lines of the second family.

Let us estimate the efficiency of optimization. To this end let us equate the rigidity K_* of the optimum rod to the rigidity of the homogeneous isotropic rod having the same cross-sectional area Ω and the shear modulus $G_c = (G_1+G_2)/2$. The gain in rigidity due to optimization is independent of the form of the cross-section and equals

$$\frac{K_*-K_c}{K_c} = \frac{G_1-G_2}{G_1+G_2}. \tag{3.6}$$

From Eq. (3.6) it is evident that the relative gain changes between 0—100% with the ratio of moduli G_1/G_2 changing from 0 to ∞.

4. Optimization of anisotropic properties of elastic media in plane elasticity problems

Let us consider the problems of optimization of anisotropic characteristics of elastic bodies in the cases of plane strain and plane stress problems.

1) Let us consider a plane problem of the theory of elasticity written in the rectangular coordinates x, y and concerning the equilibrium of an elastic anisotropic body, loaded by the forces q_x, q_y on one part Γ_1 of the contour and clamped on another part Γ_2. The material properties will be assumed to be constant in the direction of z perpendicular to the plane (xy), and the strain $\varepsilon_2 = 0$ (plane strain). The elastic medium is also assumed to be locally orthotropic, the axes of orthotropy being ξ and η. The position of the axes of orthotropy ξ, η relative to x, y at a point with the coordinates x, y is given by the angle $\alpha = \alpha(x, y)$(α — angle made by the axes x and ξ). The constants of orthotropy $A^0_{11}, A^0_{12}, A^0_{13}, A^0_{22}, A^0_{23}, A^0_{33}, A^0_{44}, A^0_{55}, A^0_{66}$ are given. Equilibrium of the elastic body under the boundary conditions prescribed above is characterized by the variational principle

(4.1)
$$\Pi = \iint_{\Omega} f dx dy - \int_{\Gamma_1} (uq_x + vq_y) ds \to \min_{u,v},$$
$$f = \frac{1}{2}(A_{11}\varepsilon_x^2 + A_{22}\varepsilon_y^2 + A_{66}\gamma_{xy}^2) + A_{12}\varepsilon_x\varepsilon_y + A_{16}\varepsilon_x\gamma_{xy} + A_{26}\varepsilon_y\gamma_{xy}.$$

Here u, v — displacements along the axes x and y, and $\varepsilon_x, \ldots, \varepsilon_{xy} = 1/2\gamma_{xy}, \sigma_x, \ldots, \tau_{xy}$ denote the components of the strain and stress tensors. The elastic moduli A_{ij} in a fined coordinate system x, y are expressed in terms of the constants A_{ij}^0 prescribed in the system ξ, η by the known transformation formulae [9, 10]

(4.2)
$$\begin{aligned} A_{11} &= C_1\cos^4\alpha + C_2\sin^4\alpha + C_3, & A_{22} &= C_1\sin^4\alpha + C_2\cos^4\alpha + C_3,\\ A_{12} &= (C_1+C_2)\sin^2\alpha\cos^2\alpha + A_{12}^0, & A_{16} &= \sin\alpha\cos\alpha(C_2\sin^2\alpha - C_1\cos^2\alpha).\\ A_{26} &= \sin\alpha\cos\alpha(C_2\cos^2\alpha - C_1\sin^2\alpha), & A_{66} &= (C_1+C_2)\sin^2\alpha\cos^2\alpha + A_{66}^0 \end{aligned}$$

in which $C_1 = A_{11}^0 - A_{12}^0 - 2A_{66}^0$, $C_2 = A_{22}^0 - A_{12}^0 - 2A_{66}^0$, $C_3 = A_{12}^0 + 2A_{66}^0$. The magnitudes A_{ij} are functions of the angle α, $A_{ij} = A_{ij}(\alpha)$, and Π is a functional of $\alpha(x, y)$. Observe that the minimum with respect to u and v in Eq. (4.1) is sought for in the class of functions $u(x, y), v(x, y)$ satisfying the kinematical conditions at Γ_2. The boundary conditions at Γ_1 are known to be natural for the functional (4.1) and they do not need to be satisfied in advance.

The work done by external forces applied to the contour will be assumed as the optimization criterion property:

(4.3)
$$J(\alpha) = \frac{1}{2}\int_{\Gamma_1} (uq_x + vq_y) ds.$$

Consider now the optimization problem of this functional and seek the corresponding distribution of the angles of inclination of the orthotropy axes at each point ($\alpha = \alpha(x, y)$) to the fixed coordinate system x, y. As it has been mentioned before, the functional J is called the deformability (compliance) of the elastic body.

2) It is known that in the standard motion of determining the optimization conditions, use is made of the differential constraints (in our case — the equilibrium equations in the displacements u and v). However, in the problem under consideration, owing to the fact that the equations of equilibrium written in the displacements represent Euler's equations for the functional (4.1), the problem of optimization may be reformulated to follow the approach used in the preceding section, and to eliminate the differential relations from further consideration. To this end Clapeyron's theorem is used and the transformations are performed: $J_* = \min_\alpha(-\min_{u,v}\Pi) = -\max_\alpha \min_{u,v}\Pi$. Thus the original problem is reduced to the determination of a maximum. To simplify the procedure of deriving the stationarity conditions of Π with respect to α and to shorten the transformations, let us introduce at each point x, y a system of principal axes of strain X, Y and denote the strain tensor components by $\varepsilon_X, \varepsilon_Y, \varepsilon_{XY} (\varepsilon_{XY} = 0)$. Let ψ and χ denote the respective angles made by the axes X and ξ, x and X so that $\psi = \alpha - \chi$ (Fig. 4). The strains $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$, and $\varepsilon_X, \varepsilon_Y, \varepsilon_{XY}$ are related to each other by the known transformation formulae

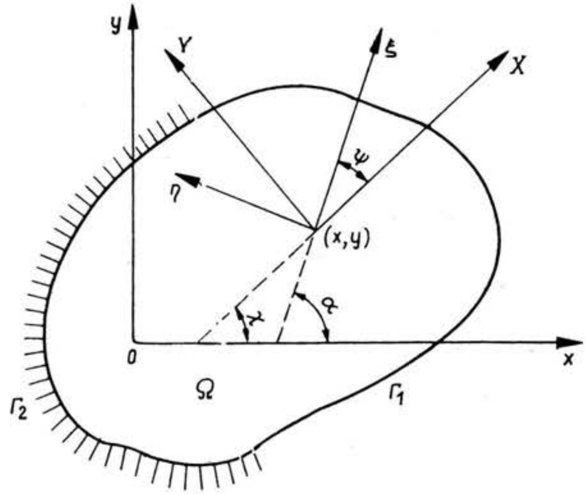

FIG. 4.

$$\varepsilon_X = \varepsilon_x \cos^2\chi + \varepsilon_y \sin^2\chi + \gamma_{xy} \sin\chi\cos\chi,$$
(4.4)
$$\varepsilon_Y = \varepsilon_x \sin^2\chi + \varepsilon_y \cos^2\chi - \gamma_{xy} \sin\chi\cos\chi,$$
$$\varepsilon_{XY} = \frac{1}{2}(\varepsilon_y - \varepsilon_x)\sin 2\chi + \frac{1}{2}\gamma_{xy}\cos 2\chi = 0.$$

The expression (4.1) for f is transformed to

$$f = \frac{1}{2} A_{11}(\psi)\varepsilon_X^2 + A_{12}(\psi)\varepsilon_X\varepsilon_Y + \frac{1}{2} A_{22}(\psi)\varepsilon_Y^2 = N\cos^4\psi + Q\cos^2\psi + R,$$

(4.5) $$N \equiv \frac{1}{2}(C_1 + C_2)(\varepsilon_X - \varepsilon_Y)^2, \quad Q \equiv (\varepsilon_X - \varepsilon_Y)(C_1\varepsilon_Y - C_2\varepsilon_X),$$

$$R \equiv \frac{1}{2} A_{22}^0 \varepsilon_X^2 + A_{12}^0 \varepsilon_X \varepsilon_Y + \frac{1}{2} A_{11}^0 \varepsilon_Y^2,$$

where ε_X, ε_Y are defined by the relations (4.4). Hence the functional Π is represented by means of Eqs. (4.2), (4.4) and (4.5) in terms of two angles ψ and χ satisfying the relation $\psi + \chi = \alpha$. In the derivation of the necessary condition of extremum, it is required that the first variation of the functional Π corresponding to the variations $\delta\alpha$, δu, δv should vanish. Let us note that variation of Π with respect to u, v and α should be done under the assumption that the variables are independent (like in all cases of functionals depending on vector functions [11]). Consequenly, the angle χ which enters the expressions (4.4) and (4.5) and is calculated from the third relation (4.4) as a function of the strain components $\mathrm{tg}\,2\chi = \gamma_{xy}/(\varepsilon_x - \varepsilon_y)$, may also be considered as independent of α. Thus in writing down the first variation we put $\partial f/\partial\psi = \partial f/\partial\alpha$. The condition of stationarity with respect to α takes the form $\sin 2\psi(2N\cos^2\psi + Q) = 0$. This condition contains three different schemes of orientation of the axes:

(4.6)
$$\text{(I)}\cos\psi = 0; \quad \text{(II)}\sin\psi = 0;$$
$$\text{(III)}\cos^2\psi = -Q/2N \quad \text{for} \quad 0 \leqslant -Q/2N \leqslant 1.$$

The orientation according to the third form is possible only under the condition of satisfying the inequality in Eq. (4.6). To clarify this condition, let us represent f in the form of a second-order trinomial $F = Nt^2+Qt+R$ of the variable $t = \cos^2\psi$. Since t varies in the interval $0 \leqslant t \leqslant 1$, the extremum of f with respect to t may be furnished either at the boundary points $t = 0$, $t = 1$ (what corresponds to cases (I) and (II)), or at an internal point. The inequality (4.6) expresses the condition of t belonging to the interval [0, 1].

3) Let us apply the conditions (4.6) to a particular case of optimization of the anisotropic properties of an elastic plane containing a circular hole. For definiteness it will be assumed that $A^0_{11} > A^0_{22}$. The region Ω has the form $r \geqslant a$, $0 < \theta \leqslant 2\pi$, where a — radius of the hole, and r, θ — polar coordinates of the system located in the center of the hole. The constant normal stresses p are applied to the boundary Γ $(r = a)$, i. e. $\sigma_r = p$, $\tau_{r\theta} = 0$.

Due to axial symmetry, the angles of inclination of the axes of orthotropy α and the radial displacements u are independent of θ, i. e. $\alpha = \alpha(r)$, $u = u(r)$. Tangential stresses, shear strains and hoop displacements vanish: $\tau_{r\theta} = 0, \gamma_{r\theta} = 0, v = 0$. The principal axes of the strain tensor at each point r, θ of the region Ω have the radial and circumferencial directions. The minimized functional (4.3) evaluated along the contour Γ will be proportional to the radial displacement $u(a)$ of the contour points, and its value $J = 2\pi r\, p\, u(a)$ is assumed as the stiffness measure.

In the solution of the problem of stiffness optimization let us first consider the case when for all points of the region Ω the same scheme of orientation of the axes takes place. Let $\cos\psi = 0$ in Ω; this corresponds to the case when the axis of orthotropy with the greatest modulus A^0_{11} is oriented in the circumferential direction, and the axis with the smallest modulus A^0_{22} — in the radial direction. It follows that $f = 1/2 A^0_{22}\varepsilon_r^2 + A^0_{12}\varepsilon_r\varepsilon_\theta + 1/2 A^0_{11}\varepsilon_\theta^2$ and the equilibrium equation takes the form $u_{,rr}+u_{,r}/r-\varkappa^2 u/r^2 = 0$ where $\varkappa = \sqrt{A^0_{11}/A^0_{22}} > 1$. Integrating the equation and determining the integration constants from the boundary conditions $\sigma_r = p$ and the condition at infinity $\sigma_r = 0$, we obtain

$$u = \frac{pa^{\varkappa+1}}{\gamma r^{\varkappa}}, \quad J = \frac{2\pi a^2 p^2}{\gamma}, \quad \gamma = \sqrt{A^0_{11}A^0_{22}} - A^0_{12}. \tag{4.7}$$

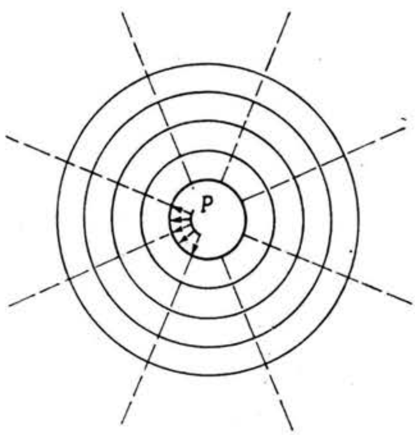

FIG. 5.

Solid and dashed lines in Fig. 5 indicate the respective directions with greater and smaller moduli.

If for $r \geqslant a$ $\sin\psi = 0$, the orthotropy axis with the greater modulus A^0_{11} has a radial direction. Directions corresponding to the greater and smaller moduli are shown in Fig. 6

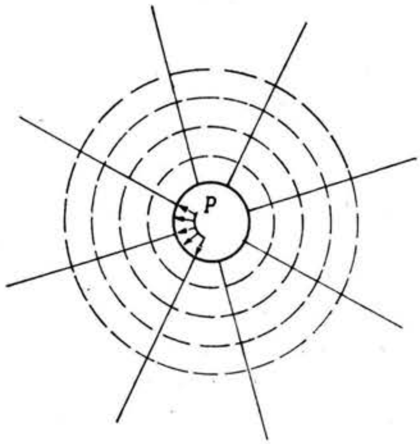

FIG. 6.

by solid and dashed lines. In this case $f = 1/2A^0_{11}\varepsilon_r^2 + A_{12}\varepsilon_r\varepsilon_\theta + 1/2\,A_{22}\varepsilon_\theta^2$, and the equation describing radial displacements has the form $u_{rr} + u_r/r - k^2 u/r^2 = 0$ with $k = 1/\varkappa < 1$. Integrating the equation of equilibrium and using the boundary conditions $\sigma_r = p$ at $r = a$ and $\sigma_r = 0$ at $r = \infty$, we obtain

$$(4.8) \qquad u = \frac{pa^{k+1}}{\gamma r^k}, \quad J = \frac{2\pi a^2 p^2}{\gamma}.$$

Comparison of Eqs. (4.7) and (4.8) yields the conclusion that the schemes of orientation of the orthotropy axes (I) and (II) lead to the same value of the functional subject to optimization.

The third scheme of orientation of the axes when $\cos^2\psi = -Q/2N$ in Ω, leads to the following equation of equilibrium: $u_{rr} + u_r/r - u/r^2 = 0$. The distribution of radial displacements satisfying this equation and the boundary conditions yields the values of the optimized functional and of the angle ψ,

$$(4.9) \qquad u = \frac{pa^2}{2A^0_{66} r}, \quad J = \frac{\pi a^2 p^2}{A^0_{66}}, \quad \cos^2\psi = \frac{1}{2}.$$

In this case the axes of orthotropy are tangent to the lines shown in Fig. 7. Comparison of the values of J in Eqs. (4.7)—(4.9) leads to the conclusion that if

$$(4.10) \qquad \sqrt{A^0_{11}A^0_{22}} - A^0_{12} > 2A^0_{66},$$

then the greatest stiffeness is obtained in the cases of orientation schemes (I) and (II). If, however, the inequality sign in Eq. (4.10) is reversed, the smaller value of J is obtained in the case of the orientation scheme (III).

The stationary solutions derived above were obtained under the assumption that each

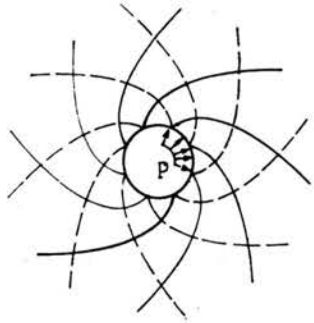

FIG. 7.

of the orientation schemes was realized in the entire region Ω. It may happen, however, that the optimum solution consists of regions with various stationary schemes of distribution of the moduli. A combination of schemes (I)–(III) is possible when the region Ω is divided into annular subregions bounded by circles $r = r_i$, in which different orientation schemes are realized. The solution of this problem leads to problems with unknown boundaries across which the stresses σ_r are continuous and the derivatives u_r suffer discontinuities. Application of the Weierstrass–Erdmann conditions at $r = r_i$ enables us, after elementary but rather tedious transformations, to show that if the elasticity moduli $A^0_{11}, A^0_{22}, A^0_{66}$ satisfy the inequality (4.10), then the "welds" of materials with different orientations of the orthotropy axes do not lead to the value of J smaller than that calculated from Eqs. (4.8) and (4.9). If the inequality (4.10) is not satisfied, combination of the discussed regions does not allow for reaching the value of J smaller than that following from Eq. (4.9). It is noted that in the case of "welding" of an arbitrary number of regions (I) and (II) (in which the orientation schemes (I) and (II) are valid), the functional proves to be equal to the value J calculated from Eqs. (4.7) and (4.8), i.e. it remains the same as in the case when in the entire region Ω any of those schemes is realized. Consequently, fulfilling of the inequality (4.10) implies the optimum orientation schemes (I) and (II), and in the opposite case — scheme (III).

4) In the solution of two-dimensional optimization problems the numerical method of consecutive optimizations was used ([12, 13]). Computations were made for rectangular regions ($0 \leqslant x \leqslant a$, $-b/2 \leqslant y \leqslant b/2$) under various boundary loads. The region was mapped onto a unit square Ω and the solution was derived for several values of the parameter $\lambda = b/a$. Elastic moduli of glass laminates were used as the physical constants. The optimum distribution of the angles of inclination of the orthotropy axes $\alpha(x, y)$ are shown in Figs. 8 and 9. The lines tangent to the solid curves indicate the directions with the greatest modulus.

The distribution of α shown in Fig. 8 corresponds to the case of a plate clamped along the edge $x = 0$, $-1/2 \leqslant y \leqslant 1/2$ and extended by forces $q_x = 1$, $q_y = 0$ along the edge $x = 1$, $-1/2 \leqslant y \leqslant 1/2$. The edges $y = \pm 1$, $0 \leqslant x \leqslant 1$ are free from loads, $q_x = q_y = 0$. The parameter λ equals 2. It is seen that the lines with the maximum elastic modulus over most of the plate are parallel to the external forces, and in the entire region the lines are symmetric about the x-axis.

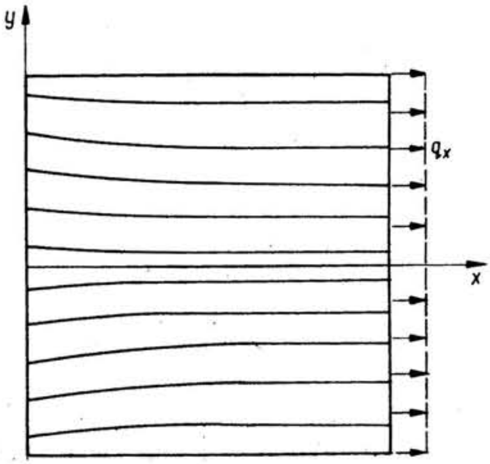

FIG. 8.

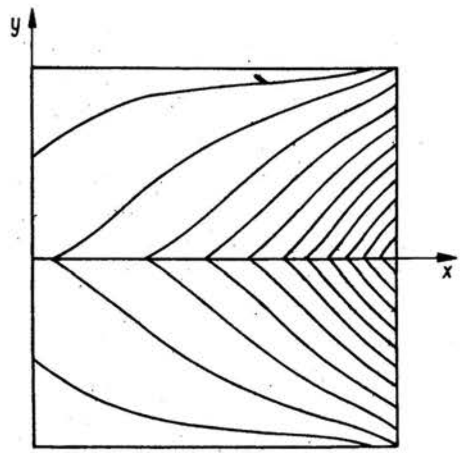

FIG. 9.

The distribution of angles $\alpha(x, y)$ in the case when loads $q_y = 0.001$, $q_x = 0$ are applied to the edge $x = 1$, $-1/2 \leqslant y \leqslant 1/2$ is shown in Fig. 9. The boundary conditions at the three remaining edges of the square are the same as in the preceding cases; the parameter λ equals 1. Orientation of the elastic moduli proves to be symmetric with respect to the line $y = 0$. Comparison of the optimum plates with the orthotropic plates (with the same value of λ) in which the orthotropy axis with the greatest modulus is parallel to the x-axis shows that with $\lambda = 1$ the relative gain in their stiffness due to optimization reaches 32%.

5. Choice of the form of anisotropic bodies and problems of simultaneous optimization of form and orientation of the axes of anisotropy

Problems of determination of the optimum forms of structural elements made of materials with prescribed anisotropic properties and the problems of simultaneous optimization of the form and orientation of the anisotropy axes are frequently encountered in practice; the problem will be explained on the basis of the example of seeking the maximum torsional rigidity of a rod.

1) The problem of maximization of the torsional rigidity K of a rod by means of a proper selection of its cross-sectional form (contour Γ) has the following formulation, $K_* = -\min_\Gamma \min_\varphi J$, where J is found from the formulae (3.1) and (3.2). Making use of the necessary condition for Γ to be optimum, of the isoperimetric condition and the equations and boundary conditions of the theory of torsion, we arrive at the boundary value problem of determining the stress function $\varphi(x, y)$ and the form of Γ,

$$\begin{gathered} a_{11}\varphi_{xx}-2a_{12}\varphi_{xy}+a_{22}\varphi_{yy} = -2, \quad (\varphi)_\Gamma = 0, \\ (a_{11}\varphi_x^2+a_{22}\varphi_y^2-2a_{12}\varphi_x\varphi_y)_\Gamma = \text{const}, \quad \operatorname{mes}\Omega = S. \end{gathered} \tag{5.1}$$

The solution of the optimization problem based on the relations (5.1) has the form

$$\begin{gathered} \varphi = \frac{1}{2(ab-c^2)}\left[\frac{\nu S}{\pi}-a_{22}x^2-2a_{12}xy-a_{11}y^2\right], \quad \nu = \sqrt{a_{11}a_{22}-a_{12}^2}, \\ \Gamma: a_{22}x^2+2a_{12}xy+a_{11}y^2 = \frac{\nu S}{\pi}, \quad K_* = \frac{S^2}{2\pi\nu}. \end{gathered} \tag{5.2}$$

In order to estimate the gain due to the optimization procedure, let us compare the obtained value of K_* with the rigidity of a rod with a circular cross-section $K_0 = S^2/\pi(a_{11}+a_{22})$ of the same cross-sectional area. We obtain $(K_*-K_0)/K_0 = (a_{11}+a_{22})/2\nu-1$.

For an orthotropic material $a_{12} = 0$, $a_{11} = 1/G_1$, $a_{22} = 1/G_2$, with G_1, G_2 denoting the shear moduli corresponding to the axes x and y. In this case

$$\frac{K_*-K_0}{K_0} = \frac{G_1+G_2}{2\sqrt{G_1G_2}}-1. \tag{5.3}$$

From Eq. (5.3) it is seen that the gain due to optimization increases for both the cases of $G_1/G_2 \to 0$ and $G_1/G_2 \to \infty$, what means that the relative gain increases with the increasing degree of anisotropy. The minimum gain equal to zero is obtained for $G_1 = G_2 = G$, that is for an isotropic material. In such a case $K = K_0 = GS^2/2\pi$, and the optimum cross-section has the form of a circle.

2) Consider the problem of simultaneous optimization of the form of the region and of the angles of inclination $\alpha(x, y)$ of the anisotropy axes from the condition $K_* = \max_{\Gamma_0} \max_\alpha K$, where Γ_0 is the portion of the boundary of Ω sought for. Using the relations derived in Sect. 3, we arrive at the following formulation of the simultaneous optimization:

$$K_* = -\min_{\Gamma_0}\min_\alpha\min_\varphi J. \tag{5.4}$$

The functional J is determined from Eq. (3.1). The deformation coefficients a_{ij} are assumed to be independent of x, y. The minimum with respect to Γ_0 is obtained under the isoperimetric condition of a constant cross-sectional area of the rod, and the minimum with respect to φ is furnished in the class of fuctions satisfying the condition $(\varphi)_\Gamma = 0$. The "governing" parameters α and Γ_0 are calculated from the system of two necessary conditions of optimality consisting of Eq. (2.5) and the relation

$$(m_{11}\varphi_x^2-2m_{12}\varphi_x\varphi_y+m_{22}\varphi_y^2)_{\Gamma_0} = \lambda^2 \quad (\lambda^2 \text{ — a const}).$$

Analysis of the optimality conditions and of the fundamental relations of the problem makes it possible to transform the relation determining the contour Γ_0 to the following form:

$$(\nabla\varphi)^2_{\Gamma_0} = \lambda^2. \tag{5.5}$$

In this manner, the determination of $\varphi(x, y), \alpha(x, y)$ of the optimum cross-section of the rod is reduced to solving the boundary value problem $\varphi_{xx}+\varphi_{yy} = -2\lambda_1^{-1}, (\varphi)_\Gamma = 0$ with the additional condition (2.5), and in the case of orthotropic materials — by the formulae (3.4). If the region of the cross-section is simply connected and Γ_0 is understood as the boundary of $\Omega(\Gamma_0 = \Gamma)$ then in the case of a locally orthotropic rod the region Ω turns out to be a circle, and the axes of orthotropy at each point become parallel to the radius and to the circumferential direction.

In the general case in which a part of the boundary of Ω is prescribed and the other part is subject to optimization, the solution of the problem of simultaneous optimization is obtained directly, provided the solution of the auxiliary problem is found. This solution must consist in determining the cross-sectional form of a homogeneous isotropic rod (with shear modulus G) which exhibits the maximum torsional rigidity. In such a case the optimum form of Ω will be the same as in the auxiliary problem and, in the expression for the stress function $\varphi(x, y)$ governing the orthotropic medium, G must be replaced with G_1. The optimum distribution of the angles of orthotropy $\alpha(x, y)$ is determined by the stress function from Eq. (3.4).

6. Certain conclusions and remarks

The problems discussed above are connected with a new class of problems of optimization of the internal structure of elastic bodies. Particular attention is paid to the optimum orientation of the anisotropy axes. Analysis of the optimality conditions and of the stationary orientation schemes of the elastic moduli shows that the necessary conditions of extremum do not allow for a unique determination of the best orientation of the axes. Fundamental difficulties in the solution of optimization problems result from the necessity of considering various combinations of the stationary orientation schemes valid in separate subregions and comparison of the values assumed by the functional. Application of both the analytical and numerical methods of solution is not simple due to the existence of a large number of local maxima. However, the analytical and numerical results derived in the paper demonstrate the considerable effects which may be attained by means of optimization of anisotropic properties.

The solutions of optimization problems presented in this paper are based on the purely phenomenological approach and on the equations of the theory of elasticity of anisotropic bodies. The elastic moduli occurring in these equations are assumed to be known (from experiments). A more detailed analysis of the problems of optimization of constructions made of composite materials indicates the possibility of the prospective utilization of deformation and fracture mechanisms based on macrostructural properties. In this approach the mechanical characteristics of reinforced materials depend on the mechanical characteristics of the materials of the matrix and reinforcement, on the reinforcement coefficients and dimensions and on other macrostructural parameters. The advantages of such approaches lie in the possibility [14] of connecting the deformation and strength problems of elastic bodies, predicting the mechanical properties of composites on the basis of the mechanical properties of their components, solving the problems of optimum design of materials etc. The governing function of such processes of optimization of anisotropic properties may be represented by certain distributed parameters of the macrostructure. In fact, viewing the elastic moduli A_{ijkl} as certain averaged properties depending on the macro-structural parameters (concentration of materials, dimensions and position of reinforcing elements, etc.), the moduli may be assumed as the governing magnitudes. Such an approach to optimization problems enables us to take into account various structural and technological limitations and, as a result, the solutions may answer the questions of practical interest.

References

1. B. D. ANNIN, *Optimum design of elastic anisotropic nonhomogeneous bodies* [in Russian], Proc. 3rd National Congress on Theoretical and Applied Mechanics, Varna, Bulgaria 1977.
2. I. F. OBRAZCOV, V. V. VASILEV, V. A. BUNAKOV, *Optimum reinforcement of the shells of revolution of composite materials* [in Russian], Mašinostroenie, 1977.
3. N. V. BANIČUK, *On a certain problem of extremum for a system with distributed parameters and determination of optimum properties of elastic media* [in Russian], Dokl. AN SSSR, **242**, 5, 1978.
4. N. V. BANIČUK, *On the optimum anisotropy of twisted rods* [in Russian], Izv. AN SSSR, MTT, 4, 1978.
5. N. V. BANIČUK, *Optimalization of anisotropic properties of deformable media in plane problems of elasticity* [in Russian], Izv. AN SSSR, MTT, 1, 1979.
6. S. G. LEHNICKII, *Torsion of anisotropic and nonhomogeneous rods* [in Russian], Nauka, 1971.
7. V. A. LOMAKIN, *Theory of elasticity of nonhomogeneous bodies* [in Russian], Ed. Moscow Univ., 1976.
8. N. H. ARUTUNIAN, B. L. ABRAMIAN, *Torsion of elastic bodies* [in Russian], Fizmatgiz, 1963.
9. S. G. LEHNICKII, *Theory of elasticity of anisotoropic bodies* [in Russsian], Nauka, 1977.
10. A. K. MALMEISTER, V. P. TAMUŽ, G. A. TETERS, *Strength of rigid polymer materials* [in Russian], Zinatne, Riga 1965.
11. I. M. GELFAND, S. V. FOMIN, *Variational calculus* [in Russian], Fizmatgiz, 1961.
12. N. V. BANIČUK, V. M. KARTVELIŠVILI, A. A. MIRONOV, *Numerical solution of two-dimensional problems of optimization of elastic plates* [in Russian], Izv. AN SSSR, MTT, 1, 1977.
13. N. V. BANIČUK, V. M. KARTVELIŠVILI, A. A. MIRONOV, *On a numerical method of solving two-dimensional problems of optimization in elasticity* [in Russian], Proc. 5th Sov. Conf. on Numerical Methods of Solving Elasticity and Plasticity Problems, Part 2, VC SOAN SSSR, Novosibirsk 1978.
14. V. V. BOLOTIN, *Plane problem of the theory of elasticity for elements made of reinforced materials* [in Russian], Sb. Rasčety na Pročnost, Mašinostroenie, 12, 1966.

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