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Some simple flows with large density gradient layer

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SOLUTIONS of Navier-Stokes equations for a gas flow, describing a specific type of layers, are obtained for some simple geometrical situations (one-dimensional flow and flows with cylindrical and spherical symmetries). The layers, in which the density has a large gradient, separate flow regions of a small density gradient (in the layer a density profile has a characteristic S-shape). In contrary to the shock waves in the large density gradient layers, the flow velocity is much smaller than the velocity of sound and the pressure variations are small while the density and the temperature vary strongly. On the other hand, the existence of gas flow across the layer makes the considered layers different from the contact surfaces. The large density gradient layers have been observed experimentally in slow convective flows near a continuous optical discharge.

W pracy znaleziono rozwiązania równań Naviera-Stokesa opisujące dla kilku prostych sytuacji geometrycznych (przepływy jednowymiarowe, o symetrii walcowej i sferycznej) specyficzny typ warstw w przepływie gazu. Warstwy te, charakteryzujące się dużymi gradientami gęstości, rozdzielają obszary przepływu o małych gradientach gęstości (profil gęstości ma kształt krzywej logistycznej). Warstwy dużych gradientów gęstości różnią się od fal uderzeniowych małą prędkością przepływu ($M \ll 1$) i prawie stałym ciśnieniem, przy silnej zmienności gęstości i temperatury, zaś od powierzchni kontaktowej — niezerową składową prędkości prostopadłą do warstwy. Warstwy dużych gradientów gęstości były obserwowane doświadczalnie w powolnych przepływach konwekcyjnych, występujących wokół ciągłego wyładowania optycznego.

В работе найдены решения уравнений Навье-Стокса, описывающие, для нескольких простых геометрических ситуаций (одномерные течения, течения с цилиндрической и сферической симметриями), специфический тип слоев в течении газа. Эти слои, характеризующиеся большими градиентами плотности, разделяют области течения с малыми градиентами плотности (профиль плотности имеет форму логистической кривой). Слои больших градиентов плотности отличаются от ударных волн малой скоростью течения ($M \ll 1$) и почти постоянным давлением, при сильной переменности плотности и температуры, от контактной же поверхности — ненулевой составляющей скорости перпендикулярной к слою. Слои больших градиентов плотности наблюдались экспериментально в медленных конвекционных течениях, выступающих вокруг непрерывного оптического разряда.

1. Introduction

WHEN A LASER beam is focused in a very small gas region (especially in monatomic gas, e.g. argon), then in certain conditions a continuous optical discharge arises. Because of high temperature (10^4K or more) gas in the focus of the beam becomes plasma. The continuous optical discharge is accompanied by a slow convective flow of an electrically neutral gas in the vicinity of the plasma region. The most striking feature of this flow is the high ratio of temperature close to the plasma region to that far away. The regions of high and low temperatures are, according to experiments, separated by a layer of large density gradients [1, 2]. A picture of such a layer, obtained by means of the Schlieren method, is presented in Fig. 1. The layer is neither a constant surface, since gas crosses it, nor a shock

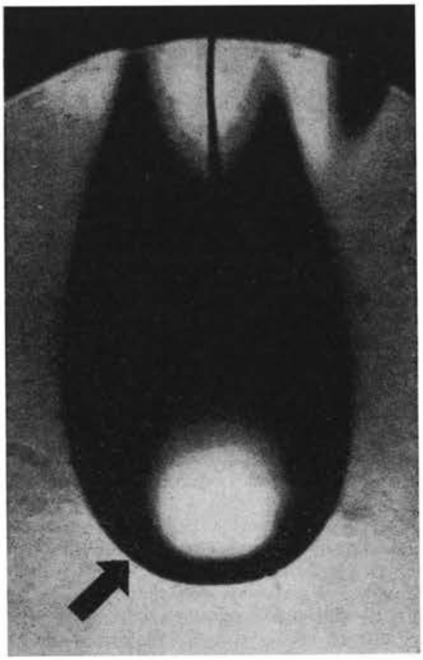

FIG. 1. A Schlieren photograph of continuous optical discharge. The large density gradient layer is indicated by an arrow. The photograph was obtained in the Fluid Mechanics Department of the Institute of Fundamental Technological Research in Warsaw.

wave since in the whole flow region the gas velocity is much smaller than the velocity of sound. To the author's best knowledge, the large density gradient leayers of the mentioned type have not been considered theoretically as yet.

2. Assumptions

The main purpose of the present work is to analyse, in some very simple geometrical situations, the solutions of Navier-Stokes equations describing the large density gradient layers. For the sake of simplicity we shall limit ourselves to stationary flows depending on one space coordinate, with plane, cylindrical and spherical symmetry. Gas flow is directed from the low to the high temperature region. In particular, in spherical and cylindrical cases there is a radial inward flow of cold gas from infinity toward a hot sphere

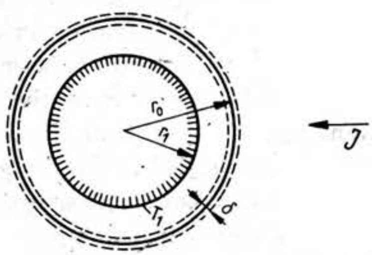

FIG. 2. General view of the layer in flows of cylindrical and spherical symmetry.

or cylinder on which positive heat sources and negative mass sources are distributed (Fig. 2). Analogically, in the case of plane symmetry gas flows in a half-space from infinity toward a given plane.

It is assumed that the gas is perfect and the specific heat of the gas and the Prandtl number are constant:

$$c_p,\ \Pr = \text{const}. \tag{2.1}$$

The coefficients of viscosity and heat conductivity are power functions of temperature with the same exponent n. The flow is then governed by the following equations:

$$\begin{aligned}
&(\varrho u r^{m-1})_r = 0,\\
&p_r = -\varrho u u_r - \frac{2}{3}\left[\frac{\mu}{r^{m-1}}(r^{m-1}u)_r\right]_r + 2(\mu u_r)_r + 2(m-1)\mu\left(\frac{u}{r}\right)_r,\\
&\varrho u c_p T_r = \frac{1}{r^{m-1}}(r^{m-1}\lambda T_r)_r + \mu\left[\frac{4}{3}\left(u_r - \frac{m-1}{2}\frac{u}{r}\right)^2 + (m-1)(3-m)\left(\frac{u}{r}\right)^2\right],\\
&\frac{\mu}{\mu_0} = \frac{\lambda}{\lambda_0} = \left(\frac{T}{T_0}\right)^n,\\
&p = R_g \varrho T,
\end{aligned} \tag{2.2}$$

where r denotes the space coordinate, ϱ — density, p — pressure, T — temperature, μ and λ — viscosity and heat conductivity coefficients, u — flow velocity, $\varkappa$ — adiabatic exponent. The constant m becomes 1, 2, 3 for plane, cylindrical, and spherical symmetry, respectively. The index r denotes differentiation with respect to the coordinate r.

Our interest is limited to the very slow, strongly subsonic flows with

$$M^2 \ll 1 \tag{2.3}$$

(M is the Mach number) and very strong variations of temperature. In such flows the pressure variations are small with respect to the temperature variations. As a consequence, the pressure is taken to be constant in the equation of state, while the density is inversely proportional to the temperature. The small flow velocity assumption makes it also possible to neglect the viscous dissipation term in the energy equation. Upon these assumptions Eqs. (2.2) may be replaced by Eqs. (2.4)

$$\begin{gathered}
(\varrho u r^{m-1})_r = 0,\\
\varrho u c_p T_r r^{m-1} = (r^{m-1}\lambda T_r)_r,\\
\frac{\mu}{\mu_0} = \frac{\lambda}{\lambda_0} = \left(\frac{T}{T_0}\right)^n,\\
\frac{\varrho}{\varrho_0} = \frac{T_0}{T},\\
p = \text{const}.
\end{gathered} \tag{2.4}$$

A specific property of the considered flow is that the fields of velocity, density and temperature do not depend on the momentum equation. As a result of the flow symmetry, the continuity equation has its integral, and hence the velocity may be expressed as a known function of density and coordinate. Should the density be constant, the velocity field would be determined by the equation of continuity. But the density varies as the inverse of the temperature. However, because of the specific form of the equation of state the energy equation does not include the pressure and can be integrated independently of the momentum equation, making it possible to determine the velocity, density, and temperature fields.

To solve the continuity and energy equations the following boundary conditions are assumed: $T = T_1$ for a given value of $r = r_1$, $T \to T_\infty$ for $r \to \infty$, and the constant and negative value J of the mass flux (equivalent to the boundary condition for u). For a given mass flux the general solution of the energy equation includes 2 independent constants. For that reason it is rather inconvenient to use 3 different parameters T_∞, T_1, r_1 in the boundary conditions, as we did. In Sect. 4 we shall introduce alternative, more convenient, boundary conditions, making use of certain properties of the solutions.

It will follow from further considerations that the assumed model loses its validity if the flow is continued too far on the hot side. That is the reason why we have introduced a certain surface $r = r_1$ bounding the flow. To give to this surface physical interpretation, we may imagine it as a boundary of a plasma region in a flow in the case of lack of gravitation (for $m = 1$ an upward flow in the presence of gravitation is admissible). A certain mechanism of sucking of gas inside the plasma region should be supposed. In another interpretation, valid for T_1 moderately large, the surface $r = r_1$ may be thought as a hot porous wall on which gas is sucked in.

3. Solutions of the continuity and energy equations

Upon integration the equation of continuity $(2.4)_1$ may be presented in the form

$$\varrho u = \frac{J}{\alpha_m r^{m-1}}, \tag{3.1}$$

where

$$\alpha_1 = 1,$$
$$\alpha_2 = 2\pi,$$
$$\alpha_3 = 4\pi.$$

To solve the energy equation it is convenient to introduce the dimensionless variables θ and R

$$\theta = \frac{T}{T_0}, \quad R = \frac{r}{L} = \frac{r c_p \varrho_0 u_0}{\lambda_0},$$

where

$$L = \frac{\lambda_0}{c_p \varrho_0 u_0}$$

is a characteristic length connected with heat conduction, and all symbols with an index 0 correspond to a certain characteristic point not specified as yet. It should be mentioned that for the inward flow both L and R are negative. Taking into account Eq. (3.1) and $(2.4)_4$, and integrating once the energy equation with the boundary condition at infinity, one obtains

$$\frac{d\theta}{dR} = \left(\frac{R_0}{R}\right)^{m-1} \frac{\theta-\theta_\infty}{\theta^n}. \tag{3.2}$$

Upon second integration one gets

$$\begin{aligned} R &= I(\theta) \qquad &&\text{for} \quad m = 1,\\ \frac{R}{R_0} &= \exp\left(\frac{I(\theta)}{R_0}\right) &&\text{for} \quad m = 2,\\ \frac{R}{R_0} &= \frac{1}{1-\dfrac{I(\theta)}{R_0}} &&\text{for} \quad m = 3, \end{aligned} \tag{3.3}$$

where

$$I(\theta) = \int_1^\theta \frac{\tau^n}{\tau-\theta_\infty}\,d\tau = \theta_\infty^n \int_{1/\theta_\infty}^{\theta/\theta_\infty} \frac{z^n}{z-1}\,dz = \theta_\infty^n\left[\bar{I}\left(\frac{\theta}{\theta_\infty}\right)-\bar{I}\left(\frac{1}{\theta_\infty}\right)\right]. \tag{3.4}$$

It was assumed that $\theta = 1$ for $R = 0$ if $m = 1$, and $\theta = 1$ for $R = R_0 \neq 0$ if $m \neq 1$. For $m = 1$ R_0 is not defined. Let us remark in this place that the solution (3.3) depends on two constants R_0 and θ_∞. The question of boundary conditions will be discussed later in Sect. 4. The indefinite integral $\bar{I}$ may be expressed analytically for any rational value of n, although a corresponding expression is rather complicated. Assuming $n = k/l$, where k and l are irreducible integers, we have the following formulas for even and odd values of 1, respectively:

$$\bar{I}(z) = \frac{z^n}{n} - (-1)^{k+1}\ln(x+1) + \ln(x-1) + \sum_{i=1}^{j-1} \cos(2in\pi)\ln\left(1-2x\cos\frac{i\pi}{j}+x^2\right)$$
$$-2\sum_{i=1}^{j-1}\sin(2in\pi)\,\mathrm{arc\,tg}\,\frac{x-\cos\dfrac{i\pi}{j}}{\sin\dfrac{i\pi}{j}} + \text{const}$$

for $l = 2j$, and

$$\bar{I}(z) = \frac{z^n}{n} + \ln(x-1) - (-1)^{k+1}\sum_{i=1}^{j}\cos\frac{(2i-1)k\pi}{2j+1}\ln\left(1+2x\cos\frac{2i-1}{2j+1}\pi+x^2\right)$$
$$-2(-1)^{k+1}\sum_{i=1}^{j}\sin\frac{(2i-1)k\pi}{2j+1}\,\mathrm{arc\,tg}\,\frac{x+\cos\dfrac{2i-1}{2j+1}\pi}{\sin\dfrac{2i-1}{2j+1}\pi} + \text{const}$$

for $l = 2j+1$, where $x = z^{1/l}$ and $k < l > 2$ (see [3]).

Generally, values of the viscosity exponent n lie between 1/2 and 1. For both these values of n there are much simpler formulas:

$$I = \theta - 1 + \theta_\infty \ln \frac{\theta - \theta_\infty}{1 - \theta_\infty} \qquad \text{for} \quad n = 1,$$

$$I = 2(\sqrt{\theta} - 1) + \sqrt{\theta_\infty} \ln \left(\frac{\sqrt{\theta} - \sqrt{\theta_\infty}}{\sqrt{\theta} + \sqrt{\theta_\infty}} \cdot \frac{1 + \sqrt{\theta_\infty}}{1 - \sqrt{\theta_\infty}} \right) \qquad \text{for} \quad n = \frac{1}{2}.$$

For the asymptotic cases $\theta^n \gg 1$ and $\theta^n - \theta^n_\infty \ll 1$ there are approximative formulas:

$$I \cong I^* = \frac{\theta^n}{n} \qquad \text{for} \quad \theta^n \gg 1, \tag{3.5}$$

$$I \cong I^{**} = \theta^n_\infty \ln(\theta^n - \theta^n_\infty) \qquad \text{for} \quad \theta^n - \theta^n_\infty \ll 1. \tag{3.6}$$

Knowing the field of temperature one may determine the fields of density and velocity:

$$\varrho = \frac{\varrho_0}{\theta}, \tag{3.7}$$

$$u = \frac{J\theta}{\alpha_m r^{m-1} \varrho_0}. \tag{3.8}$$

4. Structure of the layer

A density profile (3.7), resulting from the solution discussed in the previous section, has a form of a logistic curve with a characteristic deflection point (Fig. 3) which will be referred to by the previously introduced index 0. As it follows from the boundary conditions, ϱ tends to a constant value ϱ_∞ when $r \to \infty$. In a flow with plane symmetry there is another asymptotic value of ϱ for $r \to -\infty$; in this case $\varrho \to 0$. Thus the regions of

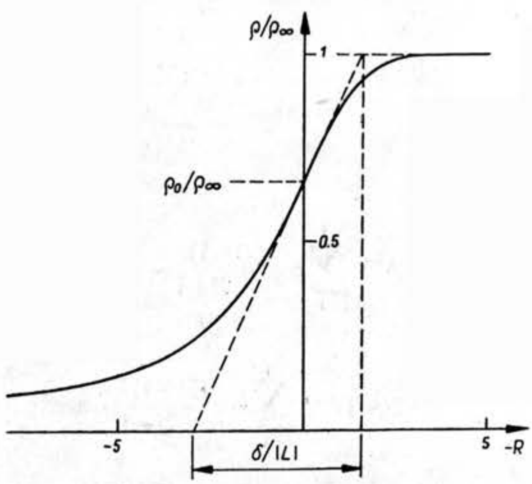

FIG. 3. Profile of density in a plane layer for $n = 1$.

a very small gradient of density are separated by the layer of a large gradient near the deflection point. For cylindrical and spherical symmetry $r \geqslant 0$ and $\varrho \to 0$ as $r \to 0$. In these cases we shall say that a layer of large density gradient exists if a thickness δ defined with the aid of a derivative in the deflection point is small with respect to the layer radius r_0

$$\delta = \frac{\varrho_\infty}{\left(\frac{d\varrho}{dr}\right)_{r=r_0}} = -\frac{L}{\theta_\infty\left(\frac{d}{dR}\frac{1}{\theta}\right)_{R=R_0}} \ll r_0. \tag{4.1}$$

From the equality $d^2\varrho/dr^2 = 0$, which is fulfilled in the deflection point, the value of ϱ in this point was found to be

$$\frac{\varrho_0}{\varrho_\infty} = \frac{1+n}{2+n} + \frac{m-1}{(2+n)R_0}. \tag{4.2}$$

It follows from Eq. (4.2) that for plane symmetry

$$\frac{\varrho_0}{\varrho_\infty} = \frac{1+n}{2+n}, \tag{4.3}$$

which means that for n in the range from 1/2 to 1, ϱ_0/ϱ_∞ lies between 3/5 and 2/3. For other symmetries the formula (4.3) is approximately valid if $|R_0| \gg 1$. The layer thickness is then found as

$$\delta = -\frac{(2+n)^2}{1+n} L. \tag{4.4}$$

From Eq. (4.4) it follows that the layer thickness is of the same order as the characteristic length L connected with heat conduction; in other words, the condition for the existence of a large density gradient layer, which was previously formulated as $\delta \ll r_0$, is equivalent to $|R_0| = |r_0/L| \gg 1$. Let us remark that R_0, the value of the nondimensional coordinate R in the deflection point, may be interpreted as a Peclet number

$$R_0 = \mathrm{Pe} = \frac{c_p \varrho_0 u_0 r_0}{\lambda_0}.$$

This interpretation is not valid, however, for the plane symmetry flow because in this case r_0 has not any physical meaning.

It was mentioned in Sect. 2 that another, more convenient form of boundary conditions would be introduced. Instead of r_1 and the corresponding temperature T_1, it is sufficient to give the layer radius r_0. Thus three parameters of the boundary conditions: T_∞, T_1, r_1 are replaced by two: T_∞ and r_0 because the temperature T_0 is determined by T_∞. For the plane flow the solution depends in fact on one parameter T_∞ because the layer radius has not any physical meaning in this case. The initial value of R for integrating Eq. (3.2) was taken arbitrarily as 0; another initial value of R would only translate the origin of the coordinate system.

The density profiles for spherical flow for different values of n and R_0 are shown in Fig. 4. As opposed to the density profiles, the profiles of temperature and velocity have no deflection point. Both these parameters increase more and more as r decreases and tend to infinity while $r \to 0$ (if $m \neq 1$) or $r \to -\infty$ (if $m = 1$). The examples of temperature and

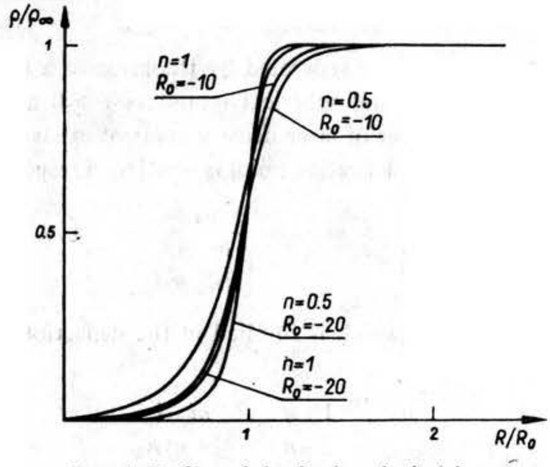

FIG. 4. Profiles of density in spherical layers.

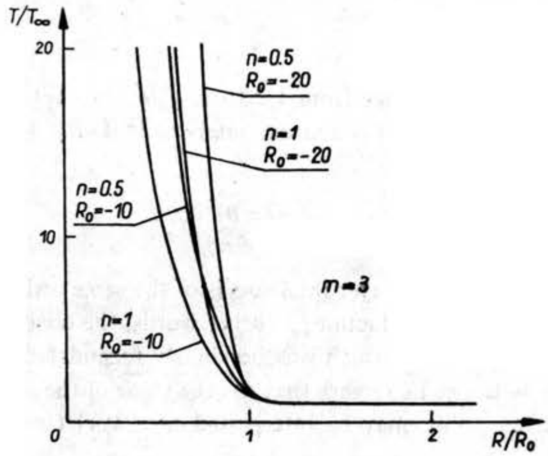

FIG. 5. Profiles of temperature in spherical layers.

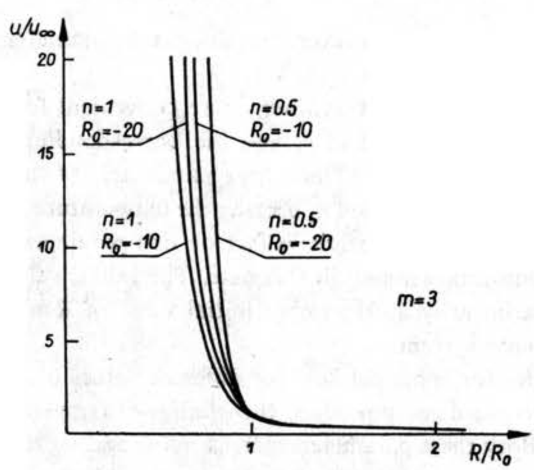

FIG. 6. Profiles of velocity in spherical layers.

velocity profiles are shown in Figs. 5 and 6. The convective energy flux E behaves in a way similar to that of temperature:

$$E_c = Jc_p T$$

equally as the heat flux Q

$$Q = Jc_p(T_\infty - T)$$

since they both depend linearly on the temperature and tend to + and − infinity, respectively, for $r \to 0$ ($r \to -\infty$ for $m = 1$), the full energy flux E ($E = E_c + Q$) being constant. For plane symmetry the velocity is proportional to the temperature. In other cases it increases downstream even more sharply than the temperature. It is evident that the flow cannot continue up to $r = 0$ (for $m \neq 1$) but should be bounded by a surface with mass and heat sources distributed on it. In particular, a position of this surface should be assumed such that the Mach number on it would be much less than 1, otherwise the assumed model would not be valid.

5. Global characteristics of the layer

In this section we shall examine the influence of the boundary conditions and the material constants on geometrical characteritics of the layer, i.e. its thickness and the distance between the layer and the boundary surface. First we shall show that some simplifications are possible because Δr, the distance between the layer (deflection point) and the boundary surface is of the same order as the layer thickness δ. For Δr we have

$$\Delta r = \begin{cases} -r_1 & \text{for} \quad m = 1, \\ r_0 - r_1 & \text{for} \quad m \neq 1. \end{cases}$$

A ratio $\delta/\Delta r$ may be obtained from Eqs. (3.3) and (4.4)

$$\frac{\delta}{\Delta r} = -\frac{\delta}{r_1} = \frac{(n+2)^2}{n+1} \cdot \frac{1}{I_1} \qquad \text{for} \quad m = 1,$$

(5.1)
$$\frac{\delta}{\Delta r} = \frac{\delta}{r_0 - r_1} = -\frac{(n+2)^2}{n+1} \cdot \frac{1}{(1 - \exp(I_1/R_0))R_0} \qquad \text{for} \quad m = 2,$$

$$\frac{\delta}{\Delta r} = \frac{\delta}{r_0 - r_1} = \frac{(n+2)^2}{n+1}\left(\frac{1}{I_1} - \frac{1}{R_0}\right) \qquad \text{for} \quad m = 3.$$

Values of $\delta/\Delta r$ for the plane flow are shown in Fig. 7 as a function of the temperature ratio T_1/T_∞ (for different values of n). It is seen that $\delta/\Delta r$ decreases when the temperature ratio and n increase, but even for such a high temperature ratio as $T_1/T_\infty = 30$ it is of the order 1. The same goes for cylindrical and spherical flows. If

(5.2)
$$|R_0| \gg I_1,$$

the formulas $(5.1)_2$ and $(5.1)_3$ reduce to Eqs. $(5.1)_1$. Otherwise $\delta/\Delta r$ is even greater. As a consequence we shall put $r_0 \cong r_1$ (for $m \neq 1$). This assumption is not valid only for a very large temperature ratio which is difficult to achieve physically because of gas ioni-

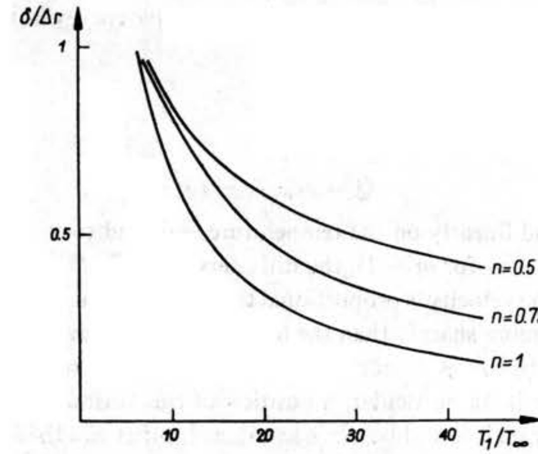

FIG. 7. Ratio of the layer thickness δ to the distance between the layer and the surface of the temperature T_1 (for a plane layer).

sation. For the sake of simplicity the fulfillment of Eq. (5.2) will be assumed. Now L, δ and Δr may be simply expressed by the boundary conditions and material constants:

$$L = \frac{\lambda_0}{c_p \varrho_0 u_0} = \left(\frac{2+n}{1+n}\right)^n \frac{\alpha_m \lambda_\infty r_1^{m-1}}{c_p J},$$

(5.3)
$$\delta = -\frac{(2+n)^{2+n}}{(1+n)^{1+n}} \frac{\alpha_m \lambda_\infty r_1^{m-1}}{c_p J},$$

$$\Delta r = -\left(\frac{2+n}{1+n}\right)^n \frac{\alpha_m \lambda_\infty r_1^{m-1}}{c_p J} \cdot I_1.$$

Some of the parameters (these which determine L) act on the layer by the effect of dilatation or compression of the scale, i.e. they increase or decrease in the same ratio as the thicknes of the layer δ and its distance Δr from the boundary surface. In particular, the rise of the mass flux makes the layer more compact while the rise of r_1 or λ_∞ have an opposite influence. In different way acts the temperature ratio which influence on Δr but not on δ. When T_1/T_∞ increases, the layer thickness remains constant, but the layer moves apart from the boundary surface. Similar is the effect of the heat flux because

$$\frac{Q_1}{|E|} \cong \frac{T_1}{T_\infty}.$$

As an example we shall discuss argon. It is assumed that this gas is in normal conditions at infinity and the radius of the boundary sphere is $r_1 = 1$ mm. In this case

$$\delta \cong \frac{3 \cdot 10^{-4}}{|J|} \frac{\text{gcm}}{\text{s}}.$$

In particular, for $J = -10^{-2}$ g/s there is $\delta \cong 0.03$ cm.

6. The pressure

In the previous considerations the pressure was taken to be constant, and a solution for u, ϱ, T was found. This solution, being an exact solution of Eq. (2.4) may be interpreted as a first approximation solution of Eq. (2.2) for the square of the Mach number being a small parameter. To find how the pressure varies one should obtain a second approximation solution, i.e. to integrate the equation $(2.2)_2$ with the right side determined from the first approximation. This equation has the form

$$(6.1)\qquad \frac{d}{dR}\frac{p-p_\infty}{p_\infty} = \varkappa M_0^2\frac{\theta-\theta_\infty}{\theta^n}\left(\frac{R_0}{R}\right)^{3(m-1)}\left\{\left(\frac{4}{3}\Pr-1\right)\right.$$
$$\left.+(m-1)\left(\frac{R_0}{R}\right)^{2-m}\frac{\theta^n}{R_0}\left[\frac{\theta}{\theta-\theta_\infty}-\Pr\left(\frac{8}{3}+2n\right)\right]\right\}.$$

It may be seen that the pressure derivative in Eq. (6.1) is proportional to the small parameter M_0^2. Without solving this equation we shall indicate some of its consequences. The formula (6.1) becomes particularly simple for $m = 1$. In this case a sign of the derivative exhibits a change for $\Pr = 3/4$. It follows then that for $\Pr < 3/4$ gas moves in a direction of smaller p. In the reverse case gas moves against the pressure. The last result follows from the fact that the viscous forces, which are determined by the continuity and energy equations, have the opposite direction with respect to the pressure forces; and for $\Pr > 3/4$ they prevail over the pressure forces. For $\Pr = 3/4$ the pressure is constant in the flow. For the cylindrical and spherical flows the formula for the pressure derivative is more complicated. The second component in braces is much smaller than the first one if T is not very close to T_∞, and Pr is not very close to $3/4(|\theta^n/R_0| \ll 1$ as a result of Eq. (5.2)). The sign of the pressure derivative is then determined by the first component, analogically as in the plane flow (depending on Pr). However, far enough from the layer, where T is very close to T_∞, the second component becomes predominant and gas moves in a direction of smaller p independently of Pr.

7. The range of validity of the model

In the previous considerations we made various assumptions about orders of magnitude of some parameters and expressions. Now we shall analyse thoroughly these assumptions and their consequences. We have neglected the viscous dissipation term in the energy equation and we have assumed pressure in the equation of state to be constant. These assumptions may be expressed in the form of strong inequalities (7.1)–(7.2) which should be valid in the whole flow region:

$$(7.1)\qquad \beta = \frac{\phi}{\varrho u c_p T_r} \ll 1,$$

$$(7.2)\qquad \gamma = \frac{T}{p}\frac{dp}{dT} \ll 1,$$

where ϕ is a viscous dissipation term omitted in the right side of Eq. $(2.4)_2$

(7.3)
$$\phi = \mu\left[\frac{4}{3}\left(u_r - \frac{m-1}{2}\frac{u}{r}\right)^2 + (m-1)(3-m)\left(\frac{u}{r}\right)^2\right].$$

Next, we have assumed another strong inequalities about some constants:

(7.4)
$$M_1^2 \ll 1,$$

(7.5)
$$\left|\frac{r_0}{L}\right| \gg 1, \quad \text{for } m \neq 1,$$

(7.6)
$$\frac{\Delta r}{r_0} = \left|\frac{LI_1}{r_0}\right| \ll 1, \quad \text{for } m \neq 1,$$

(7.7)
$$\frac{T_1}{T_\infty} \gg 1.$$

As a consequence of the inequality (7.4), the Mach number is small in the whole flow region because M is a decreasing function of r:

$$M^2 = M_1^2 \frac{T}{T_1}\left(\frac{r_1}{r}\right)^{2(m-1)}.$$

The inequality (7.4) is closely connected with inequalities (7.1)–(7.2). In the following we shall consider this connection more profoundly. The inequality (7.5) is a condition of existence of large density gradient layer, while the inequality (7.6) and (7.7) were assumed for the sake of simplicity.

Taking into account the results of Sect. 3, in particular Eqs. (3.1) and (3.2), one may express β and γ by means of R and θ:

$$\beta = \frac{\varkappa-1}{\varkappa}\frac{M^2\mathrm{Pr}}{R_0^{2(m-1)}}\left(\frac{R_0}{R}\right)^{m-1}\frac{\theta-\theta_\infty}{\theta}\left\{\frac{4}{3}\left[1-\frac{3}{2}(m-1)\left(\frac{R_0}{R}\right)^{2-m}\cdot\right.\right.$$
$$\left.\left.\cdot\frac{\theta^n}{R_0}\frac{\theta}{\theta-\theta_\infty}\right]^2 + (m-1)(3-m)\left(\frac{R_0}{R}\right)^{2(2-m)}\left(\frac{\theta^n}{R_0}\right)^2\left(\frac{\theta}{\theta-\theta_\infty}\right)^2\right\},$$

$$\gamma = \varkappa M^2\left\{\left(\frac{4}{3}\mathrm{Pr}-1\right)+(m-1)\left(\frac{R_0}{R}\right)^m\cdot\left(\frac{\theta^n}{R_0}\right)\left[\frac{\theta}{\theta-\theta_\infty}-\mathrm{Pr}\left(\frac{8}{3}+2n\right)\right]\right\}.$$

The last equalities become very simple for the plane flow. It is easily seen that the assumption of a small Mach number (7.4) is sufficient to provide that the conditions (7.1) and (7.2) are fulfilled in this case. For non-plane symmetries these conditions are rather complicated but they become simpler for the asymptotic cases $\theta \gg 1$ and $\theta-\theta_\infty \ll 1$. It may be seen that in the first case the condition (7.4) is sufficient too, as for the plane symmetry, while in the second case another condition (7.8) is needed:

(7.8)
$$M_0^2 \ll \theta-\theta_\infty.$$

The last inequality is bounding the flow field on the side of great radii. It results from the fact that the velocity and the pressure tend to their asymptotic values as a power of r, while temperature tends to the asymptotic value exponentially (that follows from Eq. (3.6)), hence far enough from the layer the pressure derivative is not small with respect to the temperature derivative. It should be mentioned, however, that for a slow enough flow the conditions (7.1) and (7.2) lose their validity in such a distance from the layer where the temperature is practically constant and very close to T_∞. In such a case it is justified to assume boundary conditions at infinity.

Now, going back to the conditions (7.4)–(7.7) let us observe that two of them containing r_0 are required only for non-plane flow, and the inequality (7.5) is in fact a consequence of the condition (7.6). Thus the conditions (7.4)–(7.7) reduce to the relations (7.9)–(7.10)

$$\left|\frac{r_0}{L}\right| \gg \left(\frac{T_1}{T_\infty}\right)^n \gg 1, \tag{7.9}$$

$$M_0^2 \ll 1, \tag{7.10}$$

where the left side inequality of the relation (7.9) is required only for $m \neq 1$. It is convenient to express the Mach number using the mass flux and other parameters:

$$M_1 = \frac{u_1}{a_1} = \frac{Ja_1}{\alpha_m \varkappa p r_1^{m-1}} = \frac{Ja_\infty}{\alpha_m \varkappa p r_1^{m-1}} \sqrt{\frac{T_1}{T_\infty}}. \tag{7.11}$$

Inserting the expressions $(5.3)_1$ and (7.11) for L and M_1 into the conditions (7.9) and (7.10), one obtains a double strong inequality for the mass flux J

$$\alpha_m \mu_\infty r_1^{m-2} I_1 \ll |J| \ll \frac{\alpha_m p r_1^{m-1}}{a_\infty} \sqrt{\frac{T_\infty}{T_1}} \quad \text{for} \quad m = 2, 3. \tag{7.12}$$

If the mass flux is too large, the condition of a small Mach number is not fulfilled; on the other hand, if it is too small, a layer of large density gradient does not appear. The right side of the relation (7.12) should be much greater than the left side of it, or r_1 should be large enough

$$r_1 \gg \bar{r} = \frac{a_\infty \mu_\infty}{p} I_1 \sqrt{\frac{T_1}{T_\infty}}.$$

As an example a value of $\bar{r}$ for a spherical flow of argon will be given. For normal conditions at infinity and a temperature ratio $T_1/T_\infty = 20$ it was obtained that $\bar{r}$ is of the order 10^{-4} cm. For other gases values of $\bar{r}$ are of the same order as for argon. For plane symmetry, the condition (7.13) should be fulfilled instead of the relation (7.12)

$$|J| \ll \frac{p}{a_\infty} \frac{T_\infty}{T_1}. \tag{7.13}$$

In this case there is only an upper limit for J which does not depend on r_1. For a flow of argon (with normal conditions at infinity and $T_1/T_\infty = 20$)

$$|J| \ll 7 \text{ g/s}.$$

8. Final remarks

In the paper solutions of Navier-Stokes equations describing the large density gradient layers have been obtained. In these layers the density profile has a characteristic S-shape (logistic curve) with a deflection point, the temperature and the velocity are increasing more and more sharply downstream and the pressure is almost constant. The solutions are based on the assumption of the inward (negative) mass flux and the outward (positive) heat flux (excluding the plane symmetry flow in which it was merely assumed that directions of both fluxes are opposite). Solutions obtained for other combinations of signs of Q and J do not descibe the layer of a large density gradient.

Of course, the flows considered depending on a one-space coordinate only are very simplified analogues of real three-dimensional flows, observed near a continuous optical discharge in which non-one-dimensional effects may be of great importance. Nevertheless it seems that the solutions obtained describe some important features of real layers of a large density gradient, and may be used as the basis for a more detailed analysis of more realistic flows.

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