Arch. Mech., 33, 2, pp. 195–214, Warszawa 1981

Direct continuum model of an elastically-deformable polarizable and magnetizable body
II. Field equations

Z. BANACH (WARSZAWA)

WE CONSTRUCT by means of the Lagrangian function introduced in the paper [1] and the principle of stationary action stated in this article the equations of motion and fields and also the equations of conservation of energy, momentum and moment of momentum. Finally, we make a study of the relations of our theory with the local approaches.

Konstruuje się za pomocą wprowadzonej w pracy [1] funkcji Lagrange'a oraz zasady stacjonarnego działania ustalonej w tej pracy równania ruchu i pól oraz równania zachowania energii, pędu, momentu pędu. Rozpatruje się powiązania naszego modelu z lokalnymi metodami opisu.

Предложенная в работе [1] функция Лагранжа и принцип стационарного действия используются для построения уравнений сохранения энергии, импульса и момента импульса. Рассматривается связь предложенной модели с локальными методами описания.

1. Introduction

THE PURPOSE of this paper is to continue the investigations in our previous article [1]. The starting point of our discussion is the Lagrangian function and the principle of stationary action. In the equations of motion and fields and also in the equations of conservation of energy, momentum and moment of momentum some of the quantities (the first-order Cauchy stress tensor, the first order spin interaction stress tensor and so on) are expressed entirely in terms of the generalized densities, the potential functions of different types, the magnetic moment per unit mass and the electric moment per unit mass, respectively. It must be remarked that these mechanical quantities can be introduced to continuum mechanics in spite of the lack of special knowledge about the range of interactions between two particles. For the sake of simplicity we confine ourselves to the second-order theory. It is very interesting to look for the conditions which allow to make a transformation of the exact form of the equations of conservation into the form corresponding to the so-called second-order theory (for instance). The discussion of this problem as well as the discussion of the relations of our theory with a nonlocal continuum of EDELEN and ERINGEN [2, 3, 4] will be the subject of another paper.

The results of TIERSTEN [5] and COLLET [6] concerning ferromagnetic bodies under the quasi-stationary magnetic field are compared with those of ours.

2. An inhomogeneous principle of stationary action

The aim of this section is to establish an inhomogeneous principle of stationary action. We confine ourselves to the case of the magnetically saturated body which is of a particular interest. The saturation condition [5, 7] has the form

$$\mu^k_\alpha(X, t)\mu_{\alpha k}(X, t) = \mu^k_\alpha(x, t)\mu_{\alpha k}(x, t) = (\overset{m}{\mu}_\alpha)^2 = \text{const}, \tag{2.1}$$

where $\alpha = a, b$, and $\overset{m}{\mu}_\alpha = |\boldsymbol{\mu}_\alpha|$ denotes the saturation value of $\boldsymbol{\mu}_\alpha$. This assumption, although not valid when the critical phenomena are under consideration, is consistent with our reversible approach and can be used without doubt at temperatures much below the corresponding critical temperature. There exists a connection between angular momentum per unit mass in the α-subcontinuum $\mathbf{s}_\alpha$ on the one hand and the magnetic moment per unit mass in the α-subcontinuum $\boldsymbol{\mu}_\alpha$ on the other hand

$$\boldsymbol{\mu}_\alpha = \gamma_\alpha \mathbf{s}_\alpha, \tag{2.2}$$

where $\gamma_\alpha \equiv \frac{g_\alpha \mu_B}{\hbar}$ denotes the so-called gyromagnetic ratio, g_α is the magnetomechanical coefficient, μ_B is called the Bohr magneton, and $\hbar$ is the Planck constant [8, 9, 10]. Generally, the magnetomechanical coefficient g_α depends on the α-species ($\alpha = a, b$). If we assume that magnetic moments arise only from spin [10], then we have to write

$$g_a = g_b = g. \tag{2.3}$$

It results directly from Eq. (2.1) that in processes which occur in nature, the variation of the magnetic moment per unit mass in the α-subcontinuum $\boldsymbol{\mu}_\alpha$, denoted here by $\bar{\Delta}_0\boldsymbol{\mu}_\alpha$, during the time interval Δt has the form

$$(\bar{\Delta}_0 \mu_{\alpha t})(X, t; \Delta t) = -\epsilon_{ilm}\mu_{\alpha l}(X, t)\omega_{\alpha m}(X, t)\Delta t, \tag{2.4}$$

where $\boldsymbol{\omega}_\alpha(X, t)$ is the angular velocity of the magnetization vector $\boldsymbol{\mu}_\alpha(X, t)$ at X at time t; this means that for a virtual motion we can write, instead of Eq. (2.4),

$$(\bar{\delta}_0 \mu_{\alpha i})(X, t) = -\epsilon_{ilm}\mu_{\alpha l}(X, t)(\delta_0 \Theta_{\alpha m})(X, t), \tag{2.5}$$

where $\delta_0\boldsymbol{\Theta}_\alpha/\delta t$ denotes the axial vector which is not the time derivative of an actual vector function in the case of natural processes occurring in the α-subcontinuum. Furthermore, we must introduce obvious conditions following from Eq. (2.1):

$$\mu_{\alpha i}\dot{\mu}_{\alpha i} = 0, \quad \mu_{\alpha i}\mu_{\alpha i, K} = 0, \quad \mu_{\alpha i}\mu_{\alpha i, k} = 0. \tag{2.6}$$

Let us assume that $\bar{\delta}_0 \underset{(t_1, t_2)}{\overset{\Omega}{W}}$ denotes $\delta_0 \underset{(t_1, t_2)}{\overset{\Omega}{W}}$ after a replacement of $\delta_0\boldsymbol{\mu}_\alpha$ by $\bar{\delta}_0\boldsymbol{\mu}_\alpha$ ($\alpha = a, b$). An inhomogeneous principle of stationary action [2] has the form

$$\bar{\delta}_0 \underset{(t_1, t_2)}{\overset{\infty}{W}} + \sum_{(\alpha)} \int_{t_1}^{t_2} \delta_0 P_\alpha \, dt = 0, \tag{2.7}$$

where

$$\delta_0 P_\alpha = -(\gamma_\alpha)^{-1} \int_\infty dx \, \varrho_\alpha(x, t)\dot{\mu}_{\alpha i}(x, t)(\delta_0 \Theta_{\alpha i})(x, t). \tag{2.8}$$

The main difference between our inhomogeneous principle of stationary action and the so-called principle of stationary action is caused by the existence of the additional term $\sum_{(\alpha)} \int_{t_1}^{t_2} \delta_0 P_\alpha dt$ in the equality (2.7). The $\delta_0 P_\alpha$ is postulated by means of the formula (2.8). The principle (2.7) (in contrast to the principle of stationary action) allows one to obtain Eq. (3.4) governing the behaviour of the magnetic moment per unit mass.

A different approach to the same problem (the structure of the equation of balance of the electronic spin) is presented in the paper of S. KALISKI, Z. PŁOCHOCKI, D. ROGULA [23] and also in the article of M. LAX [24].

At first sight the principle of stationary action (2.7) leaves much to be desired on account of the variation $\delta_0 \overset{\infty}{\underset{(t_1, t_2)}{W}}$ of the action functional defined for the infinite spatial region ∞. On the other hand it must be noticed that $\delta_0 \overset{\infty}{\underset{(t_1, t_2)}{W}}$ exists because of a continuous transition of the body from the material state to the vacuum one. The above statement is sufficient (in the case of short range forces and in the case of the small radius of correlation between particles) to obtain the equations of motion and fields in the local form at each point x belonging to Ω where the homogeneity assumption is accepted from the principle (2.7) using only the well-known expression $\delta_0 \overset{\Omega}{\underset{(t_1, t_2)}{W}}$ for the finite region Ω.

3. Equations of motion and fields

In this section we study the structure of the equations of motion and fields. Let **E** and **B** be the electric field vector and the magnetic induction, respectively. There exists the relation between (φ, **A**) and (**E**, **B**)

$$
\begin{aligned}
E_l &\equiv -\varphi_{,l} - \frac{1}{c}\partial_t A_l, \\
B_l &\equiv \epsilon_{lln} A^n_{,l}.
\end{aligned} \tag{3.1}
$$

The independent equations of motion and fields are:

(i) balance of momentum, (ii) intramolecular force balance law, (iii) balance of the electronic spin, (iv) electromagnetic equations. In addition to the equations (i), (ii), (iii) and (iv), we must also require (v) continuity equations. It can be proved, using only the principle of stationary action (2.7), the defining relations (3.1) and results of the previous paper [1], that the equations (i), (ii), (iii), (iv) and (v) have the form

(i) Balance of momentum

$$\varrho(x,t)\ddot{x}_l = P_r E_{l,r} + \frac{1}{c}[\dot{\mathbf{x}} \times (P_r \mathbf{B}_{,r})]_l + \frac{\varrho}{c}(\dot{\mathbf{\Pi}} \times \mathbf{B})_l + \mathcal{M}_r B_{r,l} + f_l. \tag{3.2}$$

(ii) Intramolecular force balance law

$$\varrho(x,t)\frac{m_a m_b}{|q_a q_b|}\ddot{\Pi}_l = \varrho(x,t)\left[\mathbf{E} + \frac{1}{c}(\dot{\mathbf{x}} \times \mathbf{B})\right]_l + \varrho(x,t)^0 E_l + \varrho(x,t)^L E_l. \tag{3.3}$$

(iii) Balance of the electronic spin

$$(3.4)\qquad (\gamma_\alpha)^{-1}\varrho_\alpha(x,t)\dot{\mu}_{\alpha i} = \varrho_\alpha(x,t)[\boldsymbol{\mu}_\alpha \times (\mathbf{B}+{}^{\mathrm{eff}}\mathbf{B}_\alpha)]_i; \qquad (\alpha = a, b).$$

(iv) Electromagnetic equations

$$(3.5)\qquad \begin{aligned} \nabla\times\mathbf{E} &= -\frac{1}{c}\,\partial_t\mathbf{B}, & \nabla\times\mathbf{H} &= \frac{1}{c}\,\partial_t\mathbf{D},\\ \operatorname{div}\mathbf{D} &= 0, & \operatorname{div}\mathbf{B} &= 0. \end{aligned}$$

(v) Continuity equations

$$(3.6)\qquad \partial_t n_{\alpha\beta} + \frac{\partial}{\partial x_p}\big(n_{\alpha\beta}v^p(x,t)\big) + \frac{\partial}{\partial y_p}\big(n_{\alpha\beta}v^p(y,t)\big) = 0, \qquad (\alpha,\beta = a, b),$$

where we introduced the following useful notations and abbreviations:

$$(3.7)\qquad {}^0E_i(x,t) \equiv -\frac{1}{2\varrho(x,t)}\sum_{(\alpha,\beta)}\int_\infty dy\,\varrho_{\alpha\beta}(x,y,t)\overset{i}{R}_{\alpha\beta}(x-y),$$

$$(3.8)\qquad {}^LE_i(x,t) \equiv -\frac{1}{\varrho(x,t)}\sum_{(\alpha,\beta)}\int_\infty dy\Big(\varrho_{\alpha\beta}(x,y,t)\overset{iq}{I}_{\alpha\beta}(x-y)\Pi_q(y,t) + \varrho_{\alpha\beta}(x,y,t)\overset{iq}{R}_{\alpha\beta}(x-y)\Pi_q(x,t)\Big),$$

$$(3.9)\qquad {}^{\mathrm{eff}}B_{\alpha i}(x,t) \equiv -\frac{1}{\varrho_\alpha(x,t)}\sum_{(\beta)}\int_\infty dy\Big(\varrho_{\alpha\beta}(x,y,t)\overset{iq}{J}_{\alpha\beta}(x-y)\mu_{\beta q}(y,t) + \varrho_{\alpha\beta}(x,y,t)\overset{iq}{C}_{\alpha\beta}(x-y)\mu_{\alpha q}(x,t)\Big),$$

$$(3.10)\qquad \begin{aligned} f_i(x,t) \equiv -\sum_{(\alpha,\beta)}\int_\infty dy\Big\{& n_{\alpha\beta}(x,y,t)\frac{\partial V_{\alpha\beta}}{\partial x_i} + (\varrho_{\alpha\beta}\Pi_p)_A(x,y,t)\frac{\partial \overset{p}{R}_{\alpha\beta}}{\partial x_i}\\ &+ (\varrho_{\alpha\beta}\Pi_p\Pi_q)(x,y,t)\frac{\partial \overset{pq}{I}_{\alpha\beta}}{\partial x_i} + (\varrho_{\alpha\beta}\Pi_p\Pi_q)_S(x,y,t)\frac{\partial \overset{pq}{R}_{\alpha\beta}}{\partial x_i}\\ &+ (\varrho_{\alpha\beta}\mu_{\alpha p}\mu_{\beta q})(x,y,t)\frac{\partial \overset{pq}{J}_{\alpha\beta}}{\partial x_i} + (\varrho_{\alpha\beta}\mu_{\alpha p}\mu_{\alpha q})_S(x,y,t)\frac{\partial \overset{pq}{C}_{\alpha\beta}}{\partial x_i}\Big\}, \end{aligned}$$

$$(3.11)\qquad \begin{aligned} \mathbf{D} &\equiv \mathbf{E}+4\pi\mathbf{P}, & \mathbf{H} &\equiv \mathbf{B}-4\pi\mathbf{M},\\ \mathbf{M} &\equiv \mathcal{M}-\frac{1}{c}\,\mathbf{v}\times\mathbf{P}, & \mathcal{M} &\equiv \sum_{(\alpha)}\varrho_\alpha\boldsymbol{\mu}_\alpha \equiv \varrho\boldsymbol{\mu}. \end{aligned}$$

In accordance with our previous remarks [1], we use either the usual Cartesian tensor notation or the direct dyadic notation. We have no possibility to discuss these equations in all details. Nevertheless, it must be remarked that the term $P_rE_{i,r}+(1/c)[\mathbf{v}\times(P_r\mathbf{B}_{,r})]_i + (\varrho/c)(\dot{\mathbf{\Pi}}\times\mathbf{B})_i + \mathcal{M}_rB_{r,i}$ in the right hand side of Eq. (3.2), and the term $[\mathbf{E}+(1/c)(\mathbf{v}\times\mathbf{B})]_i$ in the right hand side of Eq. (3.3) denote the body force caused by electromagnetic processes, and the effective electric field vector of electromagnetic nature (the local electromotive intensity), respectively. Our electromagnetic body force is identical to that of TIERSTEN and TSAI [11] but deduced from another point of view. At the same time, an equation very similar to our intramolecular force balance law (ii) has recently been obtained by MAUGIN [12–14], NELSON and LAX [15] but under significantly different assumptions. The equation (iii) has the accepted form but a detailed discussion of the structure of the effective magnetic induction ${}^{\mathrm{eff}}\mathbf{B}_\alpha$ must be postponed to another place. We believe that the similarities to the purely phenomenological theories will become then obvious. We take only note of the fact that the effective magnetic induction ${}^{\mathrm{eff}}\mathbf{B}_\alpha$ is, in accordance with the now well accepted results of the purely phenomenological theories [5, 6, 10, 11, 16, 17], to be decomposed into two parts, the isotropic exchange part and the anisotropic local part (see the last section of this paper). The equation $\mathbf{M} = \mathscr{M} - (1/c)(\mathbf{v} \times \mathbf{P})$ is not postulated but follows directly from our Lagrangian approach. The magnetizations $\mathscr{M}$ and $\mathbf{M}$ must be understood as magnetizations in the instantaneous local rest system of inertia and in our rest system of inertia, respectively [11, 18]. f_i, ${}^0E_i$, ${}^LE_i$, ${}^{\mathrm{eff}}B_{\alpha i}$ are associated with the strictly internal mechanical interactions and are expressed entirely in terms of the generalized densities, the intermolecular potentials of different types, the ionic polarization per unit mass and the magnetic moment per unit mass, respectively. Generally, Eqs. (3.2) – (3.4) tell us that the material fields $x(X, t)$, $\mathbf{\Pi}(X, t)$, $\boldsymbol{\mu}_\alpha(X, t)$ are subjected to the actions of external agencies (electromagnetic fields) and internal agencies of different types. There is no problem to show that the internal volume force is the divergence of the so-called nonsymmetric Cauchy stress tensor but we prefer another way of investigating this problem (see the next section). We refer the reader to ZORSKI's papers [19–20], in particular to his definition of the nonsymmetric Cauchy stress tensor.

It follows directly from Eq. (3.3) that if the external agencies (electromagnetic fields) vanish, the ionic polarization field still exists because of the term ${}^\circ E_i$. On the other hand, the microscopic characteristics of particles belonging to the a-subcontinuum and the b-subcontinuum as well as the macroscopic states of two continua described by the set of generalized densities are different. The main conclusion is that even if the electromagnetic field does not exist, the internal agencies influence on the behaviour of the a-subcontinuum and the b-subcontinuum in a different way. We are now in a position to look for the arbitrary mathematical assumption associated with the macroscopic states of two continua corresponding to the arbitrary (from the theoretical point of view) but useful (from the practical point of view) hypothesis that the term ${}^\circ E_i$ has no importance. Our assumption has the form

$$(3.12) \qquad \varrho_{\alpha\beta}(x, y, t) = \frac{1}{2}\,[\eta_{\alpha\beta}(x, y, t) + \eta_{\beta\alpha}(x, y, t)], \quad (\alpha, \beta = a, b),$$

where

$$\eta_{\alpha\beta}(x, y, t) = \eta_{\beta\alpha}(y, x, t).$$

The above arbitrary statement removes partly the differences between particles and can be understood as a specific kind of counting the average over types of particles. The physical sense have only the integrals over macroscopic volumes. For instance, we are interested in the following expression:

$$(3.13)\qquad \int\limits_{\Omega} dx\,\varrho(x,t)^{0}E_{l}(x,t) = -\frac{1}{2}\sum_{(\alpha,\beta)}\int\limits_{\Omega}\int\limits_{\infty} dx dy\,\varrho_{\alpha\beta}(x,y,t)\overset{i}{R}_{\alpha\beta}(x-y)$$

$$\cong -\frac{1}{2}\sum_{(\alpha,\beta)}\int\limits_{\Omega}\int\limits_{\Omega} dx dy\,\varrho_{\alpha\beta}(x,y,t)\overset{i}{R}_{\alpha\beta}(x-y)$$

$$= -\frac{1}{4}\sum_{(\alpha,\beta)}\int\limits_{\Omega}\int\limits_{\Omega} dx dy\,\varrho_{\alpha\beta}(x,y,t)[\overset{i}{R}_{\alpha\beta}(x-y)+\overset{i}{R}_{\alpha\beta}(y-x)] = 0.$$

At the same time, we have

$$\int\limits_{\Omega}\varrho(x,t)^{L}E_{l}(x,t)dx \neq 0.$$

4. Principles of conservation

Although there are in principle no difficulties to obtain the local statements of balance of total momentum, total moment of momentum and total energy in the most general form, it is definitely convenient if we take only into account the generalized forces being the first and the second-order volume integrals with respect to z (see Eqs. (4.5), (4.6), (4.14), (4.15), (4.16) and (4.17)). It is entirely consistent with COLLET's approach [6] requiring only the first and the second spatial derivatives of generalized velocity fields $\mathbf{v}$, $\dot{\mathbf{\Pi}}$, $\dot{\boldsymbol{\mu}}_{\alpha}$ to be important (the second-order theory). In spite of the lack of special knowledge about the range of interactions under consideration desired in classical approaches, we introduce concepts of the first-order Cauchy stress tensor, the second-order Cauchy stress tensor, the first-order ionic polarization stress tensor, the second-order ionic polarization stress tensor, the first-order spin interaction stress tensor, and the second-order spin interaction stress tensor, respectively. We are looking now for the conservation laws which are not new physical laws but result directly from the equations of motions and fields. On the other hand, it is interesting to see that these conservation laws can also be obtained by means of the Lagrangian approach. This problem has been extensively considered by ROGULA [21] for the abstract case of a system composed of a continuous local body with some internal degrees of freedom of material points of the body and certain external fields. It is now well known that the action functional has invariance properties with respect to certain transformations of space-time [21–22]. In the case of an inhomogeneous variational principle we must then write

$$(4.1)\qquad \delta \overset{\Omega}{\underset{(t_1,t_2)}{W}} = \delta_0 \overset{\Omega}{\underset{(t_1,t_2)}{W}} + \delta_1 \overset{\Omega}{\underset{(t_1,t_2)}{W}} = 0$$

if these transformations of space-time are taken into account. It is to be noted that no particular assumptions about the time interval (t_1, t_2), the spatial region Ω, the range of intermolecular potentials are necessary.

1. Translation of space coordinates. We obtain

$$(4.2)\qquad \begin{gathered} \delta t = 0,\quad \delta X_K = 0,\quad \delta x_l = \varepsilon_l,\quad \delta\Pi_l = 0,\quad \delta\mu_{\alpha l} = 0,\\ \delta\varphi = 0,\quad \delta A^k = 0,\quad \delta_0 x_l = \varepsilon_l,\quad \delta_0\Pi_l = 0,\quad \delta_0\mu_{\alpha l} = 0,\\ \delta_1\varphi = -\varphi_{,l}\varepsilon_l,\quad \delta_1 A^k = -A^k_{,l}\varepsilon_l, \end{gathered}$$

where $\alpha = a, b$ and ε_l denotes the infinitesimal transformation parameter.

I. The principle of conservation of total momentum

$$\varrho(x,t)\ddot{x}_i + \frac{1}{4\pi c}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})_i + \frac{\partial}{\partial x_r}\left[-(t_{ir} + \overset{EM}{T}_{ir})\right] = 0, \tag{4.3}$$

where

$$t_{ir} = {}^{(1)}t_{ir} + {}^{(2)}t_{irs,s}, \tag{4.4}$$

$${}^{(1)}t_{ir}(x,t) = \frac{1}{2}\sum_{(\alpha,\beta)}\int_{\infty} dz z^r \overset{i}{h}_{\alpha\beta}(x, x-z, t), \tag{4.5}$$

$${}^{(2)}t_{irs}(x,t) = \frac{1}{4}\sum_{(\alpha,\beta)}\int_{\infty} dz z^r z^s \overset{i}{h}_{\alpha\beta}(x, x-z, t), \tag{4.6}$$

$$\begin{aligned}\overset{i}{h}_{\alpha\beta}(x,y,t) &= n_{\alpha\beta}(x,y,t)\frac{\partial V_{\alpha\beta}}{\partial x_i} + (\varrho_{\alpha\beta}\Pi_p)_A(x,y,t)\frac{\partial \overset{p}{R}_{\alpha\beta}}{\partial x_i} \\ &+ (\varrho_{\alpha\beta}\Pi_p\Pi_q)(x,y,t)\frac{\partial \overset{pq}{I}_{\alpha\beta}}{\partial x_i} + (\varrho_{\alpha\beta}\Pi_p\Pi_q)_S(x,y,t)\frac{\partial \overset{pq}{R}_{\alpha\beta}}{\partial x_i} \\ &+ (\varrho_{\alpha\beta}\mu_{\alpha p}\mu_{\beta q})(x,y,t)\frac{\partial \overset{pq}{J}_{\alpha\beta}}{\partial x_i} + (\varrho_{\alpha\beta}\mu_{\alpha p}\mu_{\alpha q})_S(x,y,t)\frac{\partial \overset{pq}{C}_{\alpha\beta}}{\partial x_i},\end{aligned} \tag{4.7}$$

$$\overset{EM}{T}_{ir} = \frac{1}{4\pi}\left\{4\pi P_r\Omega_i + E_rE_i + B_rB_i - 4\pi B_r\mathcal{M}_i - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2 - 8\pi\mathcal{M}\cdot\mathbf{B})\delta_{ir}\right\}, \tag{4.8}$$

$$\Omega_i = E_i + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_i. \tag{4.9}$$

${}^{(1)}t_{ir}$ is the first order Cauchy stress tensor, ${}^{(2)}t_{irs}$ is the second-order Cauchy stress tensor, t_{ir} should be interpreted as the nonsymmetric Cauchy stress tensor, and $\overset{EM}{T}_{ir}$ denotes the so-called Maxwell stress tensor. For the first time the same Maxwell stress tensor was introduced to continuum mechanics by TIERSTEN and TSAI [11]. Our uniform variational approach is a very strong confirmation of their results. Nevertheless, the Cauchy stress tensor by no means can be understood as a sum of the Cauchy stress tensors defined for the a-subcontinuum and the b-subcontinuum, respectively. It is strictly connected with the interactions between two continua. It should be obvious, after using Eq. (3.6), that the Cauchy stress tensor depends nonlocally on the displacement vector, the ionic polarization per unit mass, and the magnetic moment per unit mass through a single volume integral (see also ZORSKI [19–20] and ERINGEN [4]).

2. Rotation of space coordinates. We have directly

$$\begin{gathered}\delta t = 0, \quad \delta X_K = 0, \quad \delta x_i = -\epsilon_{ijm}x_j\varepsilon_m, \quad \delta\Pi_i = -\epsilon_{ijm}\Pi_j\varepsilon_m, \\ \delta\mu_{\alpha i} = -\epsilon_{ijm}\mu_{\alpha j}\varepsilon_m, \quad \delta_0 x_i = \delta x_i, \quad \delta_0\Pi_i = \delta\Pi_i, \quad \delta_0\mu_{\alpha i} = \delta\mu_{\alpha i}, \\ \delta\varphi = 0, \quad \delta A_i = -\epsilon_{ijm}A_j\varepsilon_m, \quad \delta_1\varphi = \epsilon_{kjm}\varphi_{,k}x_j\varepsilon_m, \\ \delta_1 A_i = \epsilon_{kjm}A_{i,k}x_j\varepsilon_m - \epsilon_{ijm}A_j\varepsilon_m,\end{gathered} \tag{4.10}$$

where $\alpha = a, b$.

After very tedious calculations, we obtain:

II. The principle of conservation of total moment of momentum

$$(4.11)\quad \varrho(x,t)\frac{d}{dt}\left\{\epsilon_{ijk}x_j v_k + \frac{m_a m_b}{|q_a q_b|}\epsilon_{ijk}\Pi_j\dot{\Pi}_k + \sum_{(\alpha)}(\gamma_\alpha)^{-1}\frac{C_{(\alpha\neq e)}}{(1+C_{(\alpha\neq e)})}\mu_{\alpha i}\right\}$$
$$+\frac{1}{4\pi c}\frac{\partial}{\partial t}\{\epsilon_{ijk}x_j(\mathbf{E}\times\mathbf{B})_k\}+\frac{\partial}{\partial x_r}\Big\{-\epsilon_{ijk}x_j(t_{kr}+\overset{EM}{T}_{kr})$$
$$-\epsilon_{ijk}\Pi_j D_{kr}-\epsilon_{ijk}\sum_{(\alpha,\beta)}\mu_{\alpha j}\overset{kr}{F}_{\alpha\beta}-\epsilon_{ijk}\Big(x_{j,s}{}^{(2)}t_{krs}+\Pi_{j,s}{}^{(2)}D_{krs}+\sum_{(\alpha,\beta)}\mu_{\alpha j,s}{}^{(2)}\overset{krs}{F}_{\alpha\beta}\Big)\Big\}=0,$$

where

$$(4.12)\qquad D_{kr} = {}^{(1)}D_{kr}+{}^{(2)}D_{krs,s},$$

$$(4.13)\qquad \overset{kr}{F}_{\alpha\beta} = {}^{(1)}\overset{kr}{F}_{\alpha\beta}+{}^{(2)}\overset{krs}{F}_{\alpha\beta,s},$$

$$(4.14)\qquad {}^{(1)}D_{kr}(x,t) = \frac{1}{2}\sum_{(\alpha,\beta)}\int_{\infty} dz\, z^r \overset{k}{d}_{\alpha\beta}(x,x-z,t),$$

$$(4.15)\qquad {}^{(2)}D_{krs}(x,t) = \frac{1}{4}\sum_{(\alpha,\beta)}\int_{\infty} dz\, z^r z^s \overset{k}{d}_{\alpha\beta}(x,x-z,t),$$

$$(4.16)\qquad {}^{(1)}\overset{kr}{F}_{\alpha\beta}(x,t) = \frac{1}{2}\int_{\infty} dz\, z^r \overset{k}{f}_{\alpha\beta}(x,x-z,t),$$

$$(4.17)\qquad {}^{(2)}\overset{krs}{F}_{\alpha\beta}(x,t) = \frac{1}{4}\int_{\infty} dz\, z^r z^s \overset{k}{f}_{\alpha\beta}(x,x-z,t),$$

$$(4.18)\quad \overset{k}{d}_{\alpha\beta}(x,y,t) = \frac{1}{2}\overset{k}{R}_{\alpha\beta}(x-y)\varrho_{\alpha\beta}(x,y,t)+\overset{kp}{I}_{\alpha\beta}(x-y)\Pi_p(y,t)\varrho_{\alpha\beta}(x,y,t)$$
$$+\overset{kp}{R}_{\alpha\beta}(x-y)\Pi_p(x,t)\varrho_{\alpha\beta}(x,y,t),$$

$$(4.19)\quad \overset{k}{f}_{\alpha\beta}(x,y,t) = \overset{kp}{J}_{\alpha\beta}(x-y)\mu_{\beta p}(y,t)\varrho_{\alpha\beta}(x,y,t)+\overset{kp}{C}_{\alpha\beta}(x-y)\mu_{\alpha p}(x,t)\varrho_{\alpha\beta}(x,y,t).$$

$$^{(1)}D_{kr},\ ^{(2)}D_{krs},\ D_{kr},\ ^{(1)}\overset{kr}{F}_{\alpha\beta},\ ^{(2)}\overset{krs}{F}_{\alpha\beta}\quad \text{and}\quad \overset{kr}{F}_{\alpha\beta}$$

may be referred to, respectively, as the first-order ionic polarization stress tensor, the second-order ionic polarization stress tensor, the ionic polarization stress tensor, the first-order spin interaction stress tensor, the second-order spin interaction stress tensor, the spin interaction stress tensor. It follows directly from Eq. (4.11) that the electric moment per unit mass $\mathbf{\Pi}$ and the magnetic moment per unit mass $\boldsymbol{\mu}_\alpha$ in the α-subcontinuum experience couples caused respectively by an ionic polarization traction vector $\mathbf{D}$ and a spin traction vector $\mathbf{F}_\alpha$ acting across surfaces of the ionic continuum. The spin traction vector $\mathbf{F}_\alpha$ is associated with interaction between the α-magnetic subcontinuum and the ionic subcontinua (see the second term on the right hand side of Eq. (4.19)) and also with interacion between the α-magnetic subcontinuum and the spins belonging to the same α-magnetic subcontinuum or to the different magnetic subcontinuum (see the first term on the right hand side of Eq. (4.19)). The latter interaction is either isotropic (see Eq. (4.6) in our previous paper [1]) and then the spatial nonuniformities in the magnetic fields are important (see also [10]) or anisotropic and then the spatial distribution of magnetization can be neglected. From the formal point of view exactly the same remarks can be adopted for the ionic polarization traction vector $\mathbf{D}$. On account of no saturation condition it is difficult, however, to consider phenomena arising from the spatial nonuniformities in the ionic polarization field. To complete the above discussion we give the expressions for the ionic polarization traction vector $\mathbf{D}$ and the spin traction vector $\mathbf{F}_\alpha$, respectively,

$$\begin{aligned} D_k &= n_r D_{kr}, \\ \overset{k}{F}_\alpha &= \sum_{(\beta)} n_r \overset{kr}{F}_{\alpha\beta} \quad (\alpha = a, b), \end{aligned} \tag{4.20}$$

where n_r denotes the unit vector normal to the infinitesimal element of surface under consideration.

3. Time translation. The infinitesimal time transformation has the following form:

$$\begin{gathered} \delta t = -\varepsilon, \quad \delta X_K = 0, \quad \delta x_i = 0, \quad \delta \Pi_i = 0, \quad \delta \mu_{\alpha l} = 0, \\ \delta\varphi = 0, \quad \delta A_k = 0, \quad \delta_0 x_l = v_l\,\varepsilon, \quad \delta_0 \Pi_l = \dot{\Pi}_l\,\varepsilon, \\ \delta_0 \mu_{\alpha l} = \dot{\mu}_{\alpha l}\,\varepsilon, \quad \delta_1 \varphi = (\partial_t \varphi)\,\varepsilon, \quad \delta_1 A_k = (\partial_t A_k)\,\varepsilon, \end{gathered} \tag{4.21}$$

where $\alpha = a, b$ and ε denotes the infinitesimal transformation parameter.

III. The principle of conservation of total energy

$$\begin{aligned} &\frac{\partial}{\partial t}\left\{\varrho K + \varrho e - \varrho \boldsymbol{\mu}\cdot\mathbf{B} + \frac{1}{8\pi}(\mathbf{E}^2+\mathbf{B}^2)\right\} \\ &\quad + \frac{\partial}{\partial x_r}\left\{-t_{kr}v_k - D_{kr}\dot{\Pi}_k - \sum_{(\alpha,\beta)} \overset{kr}{F}_{\alpha\beta}\dot{\mu}_{\alpha k} - {}^{(2)}t_{krs}v_{k,s} - {}^{(2)}D_{krs}\dot{\Pi}_{k,s}\right. \\ &\quad \left. - \sum_{(\alpha,\beta)} {}^{(2)}\overset{krs}{F}_{\alpha\beta}\dot{\mu}_{\alpha k,s} + \frac{c}{4\pi}(\mathbf{E}\times\mathbf{H})_r + (\varrho K + \varrho e)v_r - (\varrho\mathbf{\Pi}\cdot\mathbf{E} + \varrho\boldsymbol{\mu}\cdot\mathbf{B})v_r\right\} = 0, \end{aligned} \tag{4.22}$$

where

$$K = \frac{1}{2}\mathbf{v}^2 + \frac{m_a m_b}{2|q_a q_b|}\dot{\mathbf{\Pi}}^2, \tag{4.23}$$

$$\begin{aligned} e(x,t) = \frac{1}{2\varrho(x,t)} \sum_{(\alpha,\beta)} \int_\infty dy\,\{a_{\alpha\beta}(x,y,t) \\ + (\overset{k}{d}_{\alpha\beta}\Pi_k)_S(x,y,t) + (\overset{k}{f}_{\alpha\beta}\mu_{\alpha k})_S(x,y,t)\}, \end{aligned} \tag{4.24}$$

$$a_{\alpha\beta}(x,y,t) = n_{\alpha\beta}(x,y,t)\,V_{\alpha\beta}(|x-y|), \tag{4.25}$$

$$(\overset{k}{d}_{\alpha\beta}\Pi_k)_S(x,y,t) = \frac{1}{2}\{\overset{k}{d}_{\alpha\beta}(x,y,t)\Pi_k(x,t) + \overset{k}{d}_{\alpha\beta}(y,x,t)\Pi_k(y,t)\}, \tag{4.26}$$

$$(\overset{k}{f}_{\alpha\beta}\mu_{\alpha k})_S(x,y,t) = \frac{1}{2}\{\overset{k}{f}_{\alpha\beta}(x,y,t)\mu_{\alpha k}(x,t) + \overset{k}{f}_{\alpha\beta}(y,x,t)\mu_{\alpha k}(y,t)\}, \tag{4.27}$$

where K is the kinetic energy per unit mass and e is the internal energy per unit mass. The internal energy of the material subbody occupying the spatial region Ω, in accordance with the usual physical interpretation of the internal energy per unit mass, has the form

$$(4.28)\qquad E_{\Omega}^{\infty} = \int_{\Omega} \varrho(x,t)\, e(x,t) dx = \frac{1}{2} \sum_{(\alpha,\beta)} \int_{\Omega} \int_{\infty} dx dy \{a_{\alpha\beta}(x,y,t) + (\overset{k}{d}_{\alpha\beta} \Pi_k)_S (x,y,t) + (\overset{k}{f}_{\alpha\beta}\, \mu_{\alpha k})_S (x,y,t)\}.$$

The above statement may be understood as stating that the energy of the material subbody occupying the spatial region Ω depends on the environment ∞ of the material subbody as well. This fact follows directly from our Lagrangian approach and is a very strong confirmation of Edelen's suggestion on the same subject [2, 3]. Equation (4.22) states that the increase in the total energy of an infinitesimal fixed volume region of space associated with the material body and electromagnetic fields is due partly to the flux of energies of different types across the boundary, and partly to the works done by the generalized forces. Our equation of balance of energy is in many respects parallel to that of TIERSTEN and TSAI [11]. There is, however, one serious difference caused by the additional term $\varrho\boldsymbol{\mu} \cdot \mathbf{B}$. We believe that our result is correct on account of the uniform Lagrangian approach. From the physical point of view the term $\varrho\boldsymbol{\mu} \cdot \mathbf{B}$ states that coupling exists between the material body and the electromagnetic fields. It must be remarked that if the theory is treated in the frame of quasi-magnetostatics and the ferro-magnetic body is under consideration, the equation of balance of total energy (4.22) can be easily reduced, in contrast to that of Tiersten and Tsai, to the following well accepted form [5]:

$$(4.29)\qquad \frac{d}{dt} \int_{\Omega} dx \left\{\varrho K + \varrho e + \frac{1}{8\pi} \mathbf{H}^2\right\} = \int_{\partial\Omega} dS \left\{(t_{kr} n_r) v_k + (F_{kr} n_r) \dot{\mu}_k - \frac{c}{4\pi} (\mathbf{E}\times\mathbf{H})_r n_r + \frac{1}{8\pi} \mathbf{H}^2 v_r n_r + \mathbf{M}\cdot\mathbf{H} v_r n_r\right\},$$

where

$$K \to \frac{1}{2}\mathbf{v}^2, \qquad e \to e - 2\pi(\varrho)^{-1}\mathbf{M}^2, \qquad t_{kr} \to t_{kr} + 2\pi \mathbf{M}^2 \delta_{kr}.$$

It is assumed here that the tensors ${}^{(2)}t_{krs}$, ${}^{(2)}\overset{krs}{F}$ are unimportant.

5. Localization

It is proposed now to study relations of our theory with the local approaches. The starting point of this discussion are the results of TIERSTEN [5] and COLLET [6] concerning ferromagnetic bodies under the quasi-stationary magnetic field.

5.1. General considerations

The main results of TIERSTEN [5] and COLLET [6], which will be the subject of our present study, have the form

$$t_{ij,j}+M_jH^i_{,j} = \varrho\dot{v}_i, \tag{5.1}$$

$$\epsilon_{ijk}\mu_jH^{\rm eff}_k = (\gamma)^{-1}\dot{\mu}_i, \tag{5.2}$$

$$\sigma_{ij}\hat{D}_{ij}+\mu_{ijk}K_{ijk}-\varrho\,{}^LH_i\hat{m}_i+{}^L\mathscr{H}_{ij}\hat{\mathscr{M}}_{ij} = \varrho\dot{e}, \tag{5.3}$$

where

$$t_{ij} = \sigma_{ij}-\mu_{ijk,k}+\varrho\,{}^LH_{[i}\mu_{j]}-{}^L\mathscr{H}_{[i|k|}\mu_{j],k}, \tag{5.4}$$

$$H^{\rm eff}_i = H_i+{}^LH_i+(\varrho)^{-1L}\mathscr{H}_{ij,j}, \tag{5.5}$$

$$\hat{D}_{ij}\equiv v_{(i,j)},\quad \Omega_{ij}\equiv v_{[i,j]},\quad K_{ijk}\equiv v_{i,jk}, \tag{5.6}$$

$$\hat{m}_i\equiv\dot{\mu}_i-\Omega_{ij}\mu_j,\quad \hat{\mathscr{M}}_{ij}\equiv(\dot{\mu}_i)_{,j}-\Omega_{ik}\mu_{k,j}. \tag{5.7}$$

$\sigma_{ij} = \sigma_{ji}$ is a first-order intrinsic stress tensor, $\mu_{ijk} = \mu_{ikj}$ is a second-order intrinsic stress tensor, ${}^LH_i$ is the magnetic anisotropy field, ${}^L\mathscr{H}_{ij}$ is the magnetic exchange tensor (we adopt here the original terminology of COLLET [6] and TIERSTEN [5]). The above equations are valid without studying any particular constitutive equations. Nevertheless, if the constitutive equations are introduced, we must write

$$\epsilon_{pij}\,{}^L\mathscr{H}_{ik}\mu_{j,k} = 0. \tag{5.8}$$

In Tiersten's theory this condition is a consequence of the invariance of the exchange energy in a rigid rotation of the entire spin continuum with respect to the lattice contiuum (we adopt here the original TIERSTEN's terminology [5]).

We are now in a position to start our discussion. Equation (5.1) is the direct consequence of Eq. (3.2). Indeed, in the ferromagnetic case we obtain very easily

$$\begin{aligned}P_rE_{i,r}+\frac{1}{c}[\mathbf{v}\times(P_r\mathbf{B}_{,r})]_i+\frac{\varrho}{c}(\dot{\mathbf{\Pi}}\times\mathbf{B})_i+\mathscr{M}_rB_{r,i}+f_i &= M_rB_{r,i}+f_i\\ &= M_rH_{r,i}-4\pi M_rM_{r,i}+f_i = M_rH_{i,r}-4\pi M_rM_{r,i}+f_i\\ &= M_rH_{i,r}+(t_{ir}-2\pi\mathbf{M}^2\delta_{ir})_{,r},\end{aligned} \tag{5.9}$$

where

$$t_{ir} = {}^{(1)}t_{ir}+{}^{(2)}t_{irs,s}, \tag{5.10}$$

$${}^{(1)}t_{ir}(x,t) = \frac{1}{2}\int\limits_{\infty} dz\,z^r\overset{i}{h}(x,x-z,t), \tag{5.11}$$

$${}^{(2)}t_{irs}(x,t) = \frac{1}{4}\int\limits_{\alpha\ \ d} z\,z^rz^s\overset{i}{h}(x,x-z,t), \tag{5.12}$$

$$\overset{i}{h}(x,y,t) = n_2(x,y,t)\frac{\partial V}{\partial x_i}+(\varrho_2\mu_p\mu_q)(x,y,t)\frac{\partial\overset{pq}{J}}{\partial x_i}+(\varrho_2\mu_p\mu_q)_S(x,y,t)\frac{\partial\overset{pq}{C}}{\partial x_i}. \tag{5.13}$$

The study of Eq. (5.2) is much more complicated. Equation (3.4) can be easily simplified to the following form:

$$\epsilon_{ijk}\mu_j(H_k+{}^{\rm eff}B_k) = (\gamma)^{-1}\dot{\mu}_i, \tag{5.14}$$

where

$$(5.15)\qquad {}^{\mathrm{eff}}B_k(x,t) \equiv -\frac{1}{\varrho(x,t)}\int_{\infty} dy\big(\varrho_2(x,y,t)\overset{kq}{J}(x-y)\mu_q(y,t) + \varrho_2(x,y,t)\overset{kq}{C}(x-y)\mu_q(x,t)\big).$$

In view of Eqs. (4.6) and (4.7) in the previous paper [1], Eq. (5.15) can be written in a more useful form:

$$(5.16)\qquad {}^{\mathrm{eff}}B_k(x,t) \equiv -\frac{1}{\varrho(x,t)}\int_{\infty} dy\,\varrho_2(x,y,t)\bar{J}(|x-y|)\mu_k(y,t) - \frac{1}{\varrho(x,t)}\int_{\infty} dy\,\varrho_2(x,y,t)\bar{C}(|x-y|)\mu_k(x,t) + {}^{L}H_k(x,t),$$

where

$$(5.17)\qquad {}^{L}H_k(x,t) \equiv -\frac{1}{\varrho(x,t)}\int_{\infty} dy\,\varrho_2(x,y,t)(y^k-x^k)(y^q-x^q)\,[\hat{J}(|x-y|)\mu_q(y,t) + \hat{C}(|x-y|)\mu_q(x,t)].$$

It must be remarked that the second term on the right of Eq. (5.16) is parallel to the magnetic moment per unit mass $\boldsymbol{\mu}$; this means that we are interested only in the first term on the right hand side of Eq. (5.16). Using the simple rules of differentiation, we obtain

$$(5.18)\qquad -\frac{1}{\varrho(x,t)}\epsilon_{ijk}\int_{\infty} dy\,\varrho_2(x,y,t)\bar{J}(|x-y|)\mu_j(x,t)\mu_k(y,t)$$

$$= \frac{1}{\varrho(x,t)}\epsilon_{ikj}\int_{\infty} dy\bar{J}(|x-y|)\frac{1}{2}\,[\varrho_2(y,x,t)\mu_k(y,t)\mu_j(x,t) - \varrho_2(x,y,t)\mu_k(x,t)\mu_j(y,t)]$$

$$\cong \frac{1}{\varrho(x,t)}\epsilon_{ikj}\frac{\partial}{\partial x_r}\left\{\frac{1}{2}\int_{\infty} dz\,z^r\bar{J}(|z|)\varrho_2(x,x-z,t)\mu_k(x,t)\mu_j(x-z,t)\right\}$$

$$= \frac{1}{\varrho(x,t)}\epsilon_{ikj}\frac{\partial}{\partial x_r}\left\{\mu_k(x,t)\left[\frac{1}{2}\int_{\infty} dz\,z^r\bar{J}(|z|)\varrho_2(x,x-z,t)\big(\mu_j(x-z,t)-\mu_j(x,t)\big)\right]\right\}.$$

It is convenient to introduce the following abbreviation:

$$(5.19)\qquad {}^{L}\mathscr{H}_{jr}(x,t) \equiv -\frac{1}{2}\int_{\infty} dy(y^r-x^r)\bar{J}(|y-x|)\varrho_2(x,y,t)\big(\mu_j(y,t) - \mu_j(x,t)\big) \cong -\frac{1}{2}\int_{\infty} dy(y^r-x^r)(y^p-x^p)\bar{J}(|y-x|)\varrho_2(x,y,t)\mu_{j,p}.$$

It follows directly from Eqs. (5.14), (5.16)–(5.17), (5.18) and (5.19) that we can write

$$(5.20)\qquad \epsilon_{ijk}\mu_j(H_k + {}^{\mathrm{eff}}B_k) = \epsilon_{ijk}\mu_j\big(H_k + {}^{L}H_k + (\varrho)^{-1\,L}\mathscr{H}_{kr,r}\big) = \epsilon_{ijk}\mu_j H_k^{\mathrm{eff}} = (\gamma)^{-1}\dot{\mu}_i,$$

where

$${}^{\mathrm{eff}}B_k \neq H_k^{\mathrm{eff}} - H_k.$$

The condition (5.8) is satisfied automatically as a consequence of the definition (5.19) of the magnetic exchange tensor. To continue our discussion it must be noticed that in the local phenomenological approaches the magnetic moment per unit mass $\boldsymbol{\mu}$ measured at the point x defines the magnetic anisotropy field evaluated at the same point x. The expression (5.17) can be easily simplified with the help of the above remark; hence ${}^{L}H_k$ reduces to

$$(5.21)\qquad H_k(x,t) \equiv -\frac{1}{\varrho(x,t)}\int\limits_{\infty} dy\,\varrho_2(x,y,t)(y^k-x^k)(y^q-x^q)\,[\hat{J}(|x-y|) + \hat{C}(|x-y|)]\,\mu_q(x,t).$$

We assume, in accordance with the local phenomenological theories, that if the anisotropic interactions are under consideration, the spatial distribution of magnetization is, in contrast to the isotropic exchange interactions, to be unimportant. In consistence with the above statement, we must write (see Eq. (5.13))

$$(5.22)\qquad \begin{aligned} (\varrho_2\mu_p\mu_q)(x,y,t)\frac{\partial}{\partial x_l}\overset{pq}{J} &\cong (\varrho_2\mu_p\mu_p)(x,y,t)\frac{\partial}{\partial x_l}\bar{J}(|y-x|) \\ &+\varrho_2(x,y,t)\mu_p(x,t)\mu_q(x,t)\frac{\partial}{\partial x_l}[(y^p-x^p)(y^q-x^q)\hat{J}(|y-x|)], \\ (\varrho_2\mu_p\mu_q)_S(x,y,t)\frac{\partial}{\partial x_l}\overset{pq}{C} &\cong (\overset{m}{\mu})^2\varrho_2(x,y,t)\frac{\partial}{\partial x_l}\bar{C}(|y-x|) \\ &+\varrho_2(x,y,t)\mu_p(x,t)\mu_q(x,t)\frac{\partial}{\partial x_l}[(y^p-x^p)(y^q-x^q)\hat{C}(|y-x|)]. \end{aligned}$$

If the relations (5.22) are accepted, the decomposition (5.4) follows immediately. The first-order intrinsic stress tensor σ_{ij} and the second-order intrinsic stress tensor μ_{ijk} can be expressed entirely in terms of the generalized density, intermolecular potentials and the magnetic moment per unit mass. We obtain without special conceptual difficulties the following formulae:

$$(5.23)\qquad \sigma_{ij}(x,t) = {}^{(1)}\sigma_{ij}(x,t)+{}^{(2)}\sigma_{ij}(x,t),$$

$$(5.24)\qquad {}^{(1)}\sigma_{ij}(x,t) = \frac{1}{2}\int\limits_{\infty} dz\,\frac{z_iz_j}{|z|}\Big\{\Psi'(|z|)+\Phi'(|z|)z^pz^q\mu_p(x,t)\mu_q(x,t) - \frac{1}{2}\bar{J}'(|z|)[\mu_p(x-z,t)-\mu_p(x,t)]\,[\mu_p(x-z,t)-\mu_p(x,t)]\Big\}\varrho_2(x,x-z,t),$$

$$(5.25)\qquad {}^{(2)}\sigma_{ij}(x,t) = -\varrho^{L}H_{(i}\mu_{j)},$$

$$(5.26)\qquad \mu_{ijk}(x,t) = {}^{(1)}\mu_{ijk}(x,t)+{}^{(2)}\mu_{ijk}(x,t),$$

$$(5.27)\qquad {}^{(1)}\mu_{ijk}(x,t) = -\frac{1}{4}\int\limits_{\infty} dz\,\frac{z_iz_jz_k}{|z|}\Big\{\Psi'(|z|)+\Phi'(|z|)z^pz^q\mu_p(x,t)\mu_q(x,t) - \frac{1}{2}\bar{J}'(|z|)[\mu_p(x-z,t)-\mu_p(x,t)][\mu_p(x-z,t)-\mu_p(x,t)]\Big\}\varrho_2(x,x-z,t),$$

$$(5.28)\qquad {}^{(2)}\mu_{ijk}(x,t) = -\varrho^L H_{jk}\mu_i,$$

$$(5.29)\qquad {}^L H_{jk}(x,t) = \frac{1}{2\varrho(x,t)}\int_\infty dz\, z^j z^k \Phi(|z|) z^q \mu_q(x,t)\varrho_2(x,x-z,t),$$

where

$$\Psi(|z|) = \frac{1}{m_a^2} V(|z|) + \overset{m}{(\mu^2)}\big(\overline{C}(|z|) + \overline{J}(|z|)\big),$$

$$\Phi(|z|) = \hat{C}(|z|) + \hat{J}(|z|),$$

$$(5.30)\qquad \Psi'(|z|) = \frac{d\Psi(|z|)}{d(|z|)}, \quad \Phi'(|z|) = \frac{d\Phi(|z|)}{d(|z|)},$$

$$\overline{J}'(|z|) = \frac{d\overline{J}(|z|)}{d(|z|)}.$$

${}^L H_{jk}$ can be understood as a second-order magnetic anisotropy field. The energy per unit mass has the following form:

$$(5.31)\qquad e(x,t) = \frac{1}{2\varrho(x,t)}\int_\infty dy\Big\{\varrho_2(x,y,t)\Psi(|x-y|) + \varrho_2(x,y,t)\Phi(|x-y|)(x^p-y^p)$$

$$\times (x^q-y^q)\mu_p(x,t)\mu_q(x,t) - \frac{1}{2}\varrho_2(x,y,t)\overline{J}(|x-y|)[\mu_p(y,t)-\mu_p(x,t)][\mu_p(y,t)-\mu_p(x,t)]\Big\}.$$

If the range of interactions is small enough, we can write

$$(5.32)\qquad 0 = \int_\infty dy \frac{\partial}{\partial y_r}\Big\{\Big[\varrho_2(x,y,t)\Psi(|x-y|) + \varrho_2(x,y,t)\Phi(|x-y|)(x^p-y^p)(x^q-y^q)$$

$$\times \mu_p(x,t)\mu_q(x,t) - \frac{1}{2}\varrho_2(x,y,t)\overline{J}(|x-y|)[\mu_p(y,t)-\mu_p(x,t)][\mu_p(y,t)-\mu_p(x,t)]\Big]v_r(y,t)\Big\}.$$

In view of Eqs. (3.6) and (5.32), we obtain

$$(5.33)\qquad \varrho(x,t)\dot{e}(x,t) = \frac{\partial}{\partial t}[\varrho(x,t)e(x,t)] + \frac{\partial}{\partial x_r}[\varrho(x,t)e(x,t)v_r(x,t)] =$$

$$\frac{1}{2}\int_\infty dy\,\varrho_2(x,y,t)\Big\{\frac{\partial\Psi}{\partial y_i} + \frac{\partial\Phi}{\partial y_i}(x^p-y^p)(x^q-y^q)\mu_p(x,t)\mu_q(x,t) - 2\Phi(|x-y|)(x^q-y^q)$$

$$\times \mu_q(x,t)\mu_i(x,t) - \frac{1}{2}\frac{\partial\overline{J}}{\partial y_i}[\mu_p(y,t)-\mu_p(x,t)][\mu_p(y,t)-\mu_p(x,t)]\Big\}\big(v_i(y,t)-v_i(x,t)\big)$$

$$+ \int_\infty dy\,\varrho_2(x,y,t)\Phi(|x-y|)(x^p-y^p)(x^q-y^q)\mu_p(x,t)\dot{\mu}_q(x,t) + \frac{1}{2}\int_\infty dy\,\varrho_2(x,y,t)$$

$$\times \overline{J}(|x-y|)[\dot{\mu}_p(x,t)\mu_p(y,t) + \mu_p(x,t)\dot{\mu}_p(y,t)] \cong (\sigma_{ij}(x,t) + \varrho^L H_{[i}\mu_{j]})v_{i,j} + \mu_{ijk}K_{ijk} - \varrho^L H_i\mu_i$$

$$+\frac{1}{2}\int\limits_{\infty} dy\,\varrho_2(x,y,t)\bar{J}(|x-y|)[\dot{\mu}_p(x,t)\mu_p(y,t)+\mu_p(x,t)\dot{\mu}_p(y,t)] = \sigma_{ij}\hat{D}_{ij}+\mu_{ijk}K_{ijk}$$

$$-\varrho^L H_i\hat{m}_i+\frac{1}{2}\int\limits_{\infty} dy\,\varrho_2(x,y,t)\bar{J}(|x-y|)[\dot{\mu}_p(x,t)\mu_p(y,t)+\mu_p(x,t)\dot{\mu}_p(y,t)].$$

Using the saturation condition (2.1), we obtain directly

$$\dot{\mu}_p(x,t)\mu_p(y,t)+\mu_p(x,t)\dot{\mu}_p(y,t) = -[\mu_p(y,t)-\mu_p(x,t)][\dot{\mu}_p(y,t)-\dot{\mu}_p(x,t)]. \tag{5.34}$$

It follows from Eq. (5.34) that

$$\frac{1}{2}\int\limits_{\infty} dy\,\varrho_2(x,y,t)\bar{J}(|x-y|)[\dot{\mu}_p(x,t)\mu_p(y,t)+\mu_p(x,t)\dot{\mu}_p(y,t)] \cong {}^L\mathscr{H}_{ij}\dot{\mu}_{i,j} = {}^L\mathscr{H}_{ij}\hat{\mathscr{M}}_{ij}. \tag{5.35}$$

It must be remarked that

$${}^L\mathscr{H}_{ij}\Omega_{ik}\mu_{k,j} = {}^L\mathscr{H}_{[i|j|}\mu_{k],j}\Omega_{ik} = 0. \tag{5.36}$$

5.2. Deformation measures

It will be convenient to summarize our previous formulae:

$$\sigma_{ij}(x,t) = {}^{(1)}\sigma_{ij}(x,t)+{}^{(2)}\sigma_{ij}(x,t), \tag{5.37}$$

$$\begin{aligned}{}^{(1)}\sigma_{ij}(x,t) = \frac{1}{2}\int\limits_{\infty} dz\,\frac{z_i z_j}{|z|}\Big\{&\Psi'(|z|)+\Phi'(|z|)z^p z^q\mu_p(x,t)\mu_q(x,t)\\ &-\frac{1}{2}\bar{J}'(|z|)[\mu_p(x+z,t)-\mu_p(x,t)][\mu_p(x+z,t)-\mu_p(x,t)]\Big\}\varrho_2(x,x+z,t),\end{aligned} \tag{5.38}$$

$${}^{(2)}\sigma_{ij}(x,t) = -\varrho^L H_{(i}\mu_{j)}, \tag{5.39}$$

$${}^L H_i(x,t) = -\frac{1}{\varrho(x,t)}\int\limits_{\infty} dz\,\Phi(|z|)z^i z^p\mu_p(x,t)\varrho_2(x,x+z,t), \tag{5.40}$$

$${}^L\mathscr{H}_{ij}(x,t) = -\frac{1}{2}\int\limits_{\infty} dz\,z^j\bar{J}(|z|)\big(\mu_i(x+z,t)-\mu_i(x,t)\big)\varrho_2(x,x+z,t), \tag{5.41}$$

$$\mu_{ijk}(x,t) = {}^{(1)}\mu_{ijk}(x,t)+{}^{(2)}\mu_{ijk}(x,t), \tag{5.42}$$

$$\begin{aligned}{}^{(1)}\mu_{ijk}(x,t) = \frac{1}{4}\int\limits_{\infty} dz\,\frac{z_i z_j z_k}{|z|}\Big\{&\Psi'(|z|)+\Phi'(|z|)z^p z^q\mu_p(x,t)\mu_q(x,t)\\ &-\frac{1}{2}\bar{J}'(|z|)[\mu_p(x+z,t)-\mu_p(x,t)][\mu_p(x+z,t)-\mu_p(x,t)]\Big\}\varrho_2(x;x+z,t),\end{aligned} \tag{5.43}$$

$${}^{(2)}\mu_{ijk}(x,t) = -\varrho^L H_{jk}\mu_i, \tag{5.44}$$

$${}^L H_{jk}(x,t) = -\frac{1}{2\varrho(x,t)}\int\limits_{\infty} dz\,z^j z^k\Phi(|z|)z^p\mu_p(x,t)\varrho_2(x,x+z,t), \tag{5.45}$$

$$(5.46)\quad e(x,t) = \frac{1}{2\varrho(x,t)} \int\limits_{\infty} dz \Big\{ \Psi(|z|) + \Phi(|z|) z^p z^q \mu_p(x,t) \mu_q(x,t)$$

$$- \frac{1}{2} \bar{J}(|z|)[\mu_p(x+z,t) - \mu_p(x,t)][\mu_p(x+z,t) - \mu_p(x,t)] \Big\} \varrho_2(x, x+z, t).$$

Since the range of interactions is assumed to be small enough, the undeformed distance $|Z| = |Y-X|$ between two particles is of the same order as the deformed distance $|z| = |x-y|$ between them. In this case we have with a sufficient approximation [7]

$$(5.47)\quad |z|^2 = (y^p - x^p)(y_p - x_p) \cong \left(x^p_{,K} Z^K + \frac{1}{2} x^p_{,KL} Z^K Z^L \right) \left(x_{p,N} Z^N + \frac{1}{2} x_{p,NM} Z^N Z^M \right)$$
$$\cong |Z|^2 + 2E_{KL} Z^K Z^L + F_{KLN} Z^K Z^L Z^N,$$

where

$$(5.48)\quad \begin{aligned} E_{KL} &\equiv \frac{1}{2}(x_{i,K} x^i_{,L} - \delta_{KL}), \\ F_{KLN} &\equiv x_{i,KL} x^i_{,N}. \end{aligned}$$

E_{KL}, F_{KLN} denote the deformation measures introduced by COLLET [6].

It follows from Eq. (5.47) that to the first order in E_{KL} and F_{KLN} we obtain

$$(5.49)\quad |z| = |Z| \left(1 + 2E_{KL} \frac{Z^K Z^L}{|Z|^2} + F_{KLN} \frac{Z^K Z^L Z^N}{|Z|^2} \right)^{\frac{1}{2}} \cong |Z| + E_{KL} \frac{Z^K Z^L}{|Z|}$$
$$+ \frac{1}{2} F_{KLN} \frac{Z^K Z^L Z^N}{|Z|} = |Z| + \Delta,$$

where

$$(5.50)\quad \Delta \equiv E_{KL} \frac{Z^K Z^L}{|Z|} + \frac{1}{2} F_{KLN} \frac{Z^K Z^L Z^N}{|Z|} \cong E_{KL} \frac{Z^K Z^L}{|z|} + \frac{1}{2} F_{KLN} \frac{Z^K Z^L Z^N}{|z|}.$$

Consider now the Taylor series of the potential function Ψ to the second order in E_{KL} and F_{KLN}, and the Taylor series of the potential functions Φ and $\bar{J}$ to the first order in E_{KL} and F_{KLN}

$$(5.51)\quad \Psi(|z|) = \Psi(|Z|) + \Psi'(|Z|)\Delta + \frac{1}{2} \Psi''(|Z|)\Delta\Delta = \Psi(|Z|) + \frac{Z^K Z^L}{|Z|} \Psi'(|Z|) E_{KL}$$
$$+ \frac{1}{2} \frac{Z^K Z^L Z^N}{|Z|} \Psi'(|Z|) F_{KLN} + \frac{1}{2} \frac{Z^K Z^L Z^P Z^Q}{|Z|^2} \Psi''(|Z|) E_{KL} E_{PQ}$$
$$+ \frac{1}{2} \frac{Z^K Z^L Z^P Z^Q Z^S}{|Z|^2} \Psi''(|Z|) E_{KL} F_{PQS} + \frac{1}{8} \frac{Z^K Z^L Z^N Z^P Z^Q Z^S}{|Z|^2} \Psi''(|Z|) F_{KLN} F_{PQS},$$

$$(5.52)\quad \Phi(|z|) = \Phi(|Z|) + \Phi'(|Z|)\Delta = \Phi(|Z|)$$
$$+ \frac{Z^K Z^L}{|Z|} \Phi'(|Z|) E_{KL} + \frac{1}{2} \frac{Z^K Z^L Z^N}{|Z|} \Phi'(|Z|) F_{KLN},$$

(5.53) $$\bar{J}(|z|) = \bar{J}(|Z|)+\bar{J}'(|Z|)\Delta = \bar{J}(|Z|)+\frac{Z^K Z^L}{|Z|}\bar{J}'(|Z|)E_{KL}+\frac{1}{2}\frac{Z^K Z^L Z^N}{|Z|}\bar{J}'(|Z|)F_{KLN}.$$

Since $\varrho_2(x, y, t)dy/\varrho(x, t) = \varrho_2^0(X, Y)dY/\varrho_0(X)$, the following relation can be accepted:

(5.54) $$e(X, t) = \frac{1}{2\varrho_0}\int_\infty dZ\Big\{\Psi(|z|)+\Phi(|z|)Z^K Z^L m_K m_L+\Phi(|z|)Z^K Z^L Z^N m_K m_{LN} + \frac{1}{4}\Phi(|z|)Z^K Z^L Z^N Z^M m_{KL}m_{NM}-\frac{1}{2}\bar{J}(|z|)Z^K Z^L G_{KL}\Big\}\varrho_2^0(X, X+Z),$$

where

(5.55) $$\begin{aligned} m_K &\equiv m_K(X, t) \equiv \mu_i(X, t)x^i_{,K},\\ m_{KL} &\equiv m_{KL}(X, t) \equiv \mu_i(X, t)x^i_{,KL},\\ G_{KL} &\equiv G_{KL}(X, t) \equiv \mu_{i,K}(X, t)\mu^i_{,L}(X, t)\end{aligned}$$

are consistent with the axiom of objectivity. For the sake of simplicity we confine ourselves to homogeneous centrosymmetric materials

(5.56) $$n_2^0(X-Y) = n_2^0\big(-(X-Y)\big).$$

With the hypotheses (5.51)–(5.53) and (5.56), Eq (5.54) takes the form

(5.57) $$e(X, t) = e_0+\varrho_0^{-1}a_{KL}E_{KL}+\varrho_0^{-1}\Big\{\frac{1}{2}b_{KLPR}E_{KL}E_{PR}+\frac{1}{2}c_{KLPRSQ}F_{KLP}F_{RSQ}\Big\} + \varrho_0\Big\{\frac{1}{2}d_{KL}m_K m_L+\frac{1}{2}j_{KLPR}m_{KL}m_{PR}\Big\}+\frac{1}{2}\varrho_0 f_{KL}G_{KL}+\varrho_0\Big\{g_{KLPR}E_{KL}m_P m_R + i_{KLPRSQ}\Big[F_{KLP}m_R m_{SQ}+\frac{1}{2}E_{KL}m_{PR}m_{SQ}\Big]\Big\}+\varrho_0 h_{KLPR}E_{KL}G_{PR},$$

where

(5.58) $$\begin{aligned} e_0 &\equiv \frac{1}{2\varrho_0}\int_\infty dZ\varrho_2^0(Z)\Psi(|Z|),\\ a_{KL} &\equiv \frac{1}{2}\int_\infty dZ\varrho_2^0(Z)\frac{Z^K Z^L}{|Z|}\Psi'(|Z|),\\ b_{KLPR} &\equiv \frac{1}{2}\int_\infty dZ\varrho_2^0(Z)\frac{Z^K Z^L Z^P Z^R}{|Z|^2}\Psi''(|Z|),\\ c_{KLPRSQ} &\equiv \frac{1}{8}\int_\infty dZ\varrho_2^0(Z)\frac{Z^K Z^L Z^P Z^R Z^S Z^Q}{|Z|^2}\Psi''(|Z|),\\ d_{KL} &\equiv \frac{1}{(\varrho_0)^2}\int_\infty dZ\varrho_2^0(Z)Z^K Z^L\Phi(|Z|),\\ j_{KLPR} &\equiv \frac{1}{4(\varrho_0)^2}\int_\infty dZ\varrho_2^0(Z)Z^K Z^L Z^P Z^R\Phi(|Z|),\end{aligned}$$

(5.58) [cont.]
$$f_{KL} \equiv -\frac{1}{2(\varrho_0)^2}\int_{\infty} dZ\varrho_2^0(Z)\,Z^K Z^L \bar{J}(|Z|),$$

$$g_{KLPR} \equiv \frac{1}{2(\varrho_0)^2}\int_{\infty} dZ\varrho_2^0(Z)\,\frac{Z^K Z^L Z^P Z^R}{|Z|}\,\Phi'(|Z|),$$

$$i_{KLPRSQ} \equiv \frac{1}{4(\varrho_0)^2}\int_{\infty} dZ\varrho_2^0(Z)\,\frac{Z^K Z^L Z^P Z^R Z^S Z^Q}{|Z|}\,\Phi'(|Z|),$$

$$h_{KLPR} \equiv -\frac{1}{4(\varrho_0)^2}\int_{\infty} dZ\varrho_2^0(Z)\,\frac{Z^K Z^L Z^P Z^R}{|Z|}\,\bar{J}'(|Z|).$$

If in the natural state the initial stress vanishes, then

$$a_{KL} = 0. \tag{5.59}$$

The material coefficients $(b_{KLPR},\ c_{KLPRSQ})$, $(d_{KL},\ j_{KLPR})$, $(g_{KLPR},\ i_{KLPRSQ})$, f_{KL}, h_{KLPR} are called, according to accepted physical terminology, the elastic constants, magnetic anisotropy constants, magnetostrictive constants, exchange constant, exchangestrictive constant, respectively. The material coefficients have the symmetries in all indices. That property of our material coefficients is a consequence of the two-point interaction model and can be partly modified by introducing the model based on the three-point interactions [20]. Following the same line of arguments as given, we obtain immediately

$$\begin{aligned} {}^{(1)}\sigma_{ij}(X,t) = \frac{\varrho}{\varrho_0} x_{i,K} x_{j,L}\Big\{ & a_{KL} + b_{KLPR}E_{PR} + (\varrho_0)^2 g_{KLPR} m_P m_R \\ & + \frac{1}{2}(\varrho_0)^2 i_{KLPRQS} m_{PR} m_{QS} + (\varrho_0)^2 h_{KLPR} G_{PR}\Big\} \\ & + \frac{2\varrho}{\varrho_0} x_{i,KM} x_{j,L}\{c_{KMLPRQ}F_{PRQ} + (\varrho_0)^2 i_{KMLPRQ} m_P m_{RQ}\}, \end{aligned} \tag{5.60}$$

$$\begin{aligned} {}^{L}H_i(X,t) = & -\varrho_0 x^i_{,K}\{d_{KL}m_L + 2g_{KLPR}m_L E_{PR} + i_{KLMPRQ} m_{LM} F_{PRQ}\} \\ & -\varrho_0 x^i_{,KL}\{j_{KLPR} m_{PR} + i_{KLMPRQ}[F_{MPR} m_Q + E_{MP} m_{RQ}]\}, \end{aligned} \tag{5.61}$$

$${}^{L}\mathscr{H}_{ij}(X,t) = \varrho\varrho_0 x_{j,K}\mu_{i,L}\{f_{KL} + 2h_{KLPR}E_{PR}\}, \tag{5.62}$$

$${}^{(1)}\mu_{ijk}(X,t) = \frac{\varrho}{\varrho_0} x_{i,K} x_{j,L} x_{k,M}\{c_{KLMPRQ}F_{PRQ} + (\varrho_0)^2 i_{KLMPRQ} m_P m_{RQ}\}, \tag{5.63}$$

$${}^{L}H_{jk}(X,t) = -\varrho_0 x^j_{,K} x^k_{,L}\{j_{KLPR} m_{PR} + i_{KLMPRQ}[F_{MPR} m_Q + E_{MP} m_{RQ}]\}. \tag{5.64}$$

On the other hand, the following formulae must be accepted:

$${}^{(1)}\sigma_{ijk} = \varrho\left(\frac{\partial e}{\partial E_{KL}} x_{(i,K} + 2\frac{\partial e}{\partial F_{KML}} x_{(i,KM}\right) x_{j),L}, \tag{5.65}$$

$${}^{(1)}\mu_{ijk} = \varrho\frac{\partial e}{\partial F_{KLM}} x_{i,M} x_{(j,L} x_{k),K}, \tag{5.66}$$

$${}^{L}\mathscr{H}_{ij} = 2\varrho\frac{\partial e}{\partial G_{KL}}\mu_{i,L} x_{j,K}, \tag{5.67}$$

$$ {}^{L}H_i = -\left(\frac{\partial e}{\partial m_K} x_{i,K} + \frac{\partial e}{\partial m_{KL}} x_{i,KL}\right), \tag{5.68} $$

$$ {}^{L}H_{ik} = -\frac{\partial e}{\partial m_{KL}} x_{i,k} x_{k,L}. \tag{5.69} $$

The relations (5.65)–(5.67) are identical to those of COLLET [6]. Equation (5.68) is more general on account of the additional term $-\frac{\partial e}{\partial m_{KL}} x_{i,KL}$. The last equation (5.69) is a consequence of our approach and will not be discussed here because of no reference to it.

Acknowledgement

I would like to thank Professor H. ZORSKI for many helpful suggestions and discussions.

References

1. Z. BANACH, *Direct continuum model of an elastically-deformable polarizable and magnetizable body. Part I. Lagrangian function*, Arch. Mech., **32**, 1, 55, 1981.
2. D. G. B. EDELEN, *Nonlocal field theories*, in: Continuum Physics, IV, p. 75, (edited by A. C. ERINGEN), Academic Press, New York 1976.
3. D. G. B. EDELEN, in: Nonlocal theories of material systems, (edited by D. ROGULA), Ossolineum Press, Warsaw 1976.
4. A. C. ERINGEN, *Nonlocal polar field theories*, in: Continuum Physics, IV, p. 205, (edited by A. C. ERINGEN), Academic Press, New York 1976.
5. H. F. TIERSTEN, J. Math. Phys., **5**, 1298, 1964.
6. B. COLLET, Intern. J. Eng. Sci, **16**, 349, 1978.
7. W. F. BROWN, *Magnetoelastic interactions*, Springer Verlag, New York 1966.
8. R. S. WADAS, *Magnetism in spinels garnets and perovskites*, PWN, Warsaw 1974.
9. У. СЛИКТЕР, *Основы теории магнитного резонанса*, Мир, Москва 1967.
10. G. A. MAUGIN, J. Math. Phys., **17**, 1727, 1976.
11. H. F. TIERSTEN and C. F. TSAI, J. Math. Phys., **13**, 361, 1972.
12. G. A. MAUGIN, Arch. Mech., **28**, 679, 1976.
13. G. A. MAUGIN, Arch. Mech., **29**, 143, 1977.
14. G. A. MAUGIN, Arch. Mech., **29**, 251, 1977.
15. D. F. NELSON, M. Lax, Phys. Rev., B, **13**, 1785, 1976.
16. H. F. TIERSTEN, J. Math. Phys., **6**, 779, 1965.
17. G. A. MAUGIN, *Micromagnetism*, in: Continuum Physics, III, p. 221, (edited by A. C. ERINGEN), Academic Press, New York 1976.
18. R. A. GROT, *Electromagnetic interactions*, in: Continuum Physics, III, p. 129, (edited by A. C. ERINGEN), Academic Press, New York 1976.
19. H. ZORSKI, Proc. Symp. Continuum Models of Discrete Systems, Waterloo, Canada 1977.
20. H. ZORSKI, Lett. Appl. Engng. Sci., **16**, 315, 1978.
21. D. ROGULA, Proc. Vibr. Probl., **7**, 337, 1966.

22. J. RZEWUSKI, *Field theory*, part 1, PWN, Warsaw 1964.
23. S. KALISKI, Z. PŁOCHOCKI, D. ROGULA, Bull. Acad. Polon. Sci., Série Sci. techn., **4**, 135, 1962.
24. M. LAX, Phys. Rev. B, **16**, 4936, 1977.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received January 23, 1980.